Improved Linear Programming Bounds for Antipodal Spherical Codes

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Abstract. Let $S \subset [-1, 1)$. A finite set $\mathcal{C} = \{x_i\}_{i=1}^M \subset \Re^n$ is called a *spherical S-code* if $\|x_i\| = 1$ for each i, and $x_i^T x_j \in S$, $i \neq j$. For $S = [-1, .5]$ maximizing $M = |\mathcal{C}|$ is commonly referred to as the *kissing number* problem. A well-known technique based on harmonic analysis and linear programming can be used to bound M. We consider a modification of the bounding procedure that is applicable to *antipodal* codes; that is, codes where $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$. Such codes correspond to packings of lines in the unit sphere, and include all codes obtained as the collection of minimal vectors in a lattice. We obtain nontrivial improvements in upper bounds for kissing numbers attainable by antipodal codes in dimensions up to $n = 24$. We also show that for $n = 4$, 6 and 7 the antipodal codes with maximal kissing numbers are essentially unique, and correspond to the minimal vectors in the laminated lattices Λ_n.

Keywords: Spherical codes, kissing problem, antipodal codes, line packing, linear programming bounds.

1 Introduction

Let $S \subset [-1, 1)$. A finite set $\mathcal{C} = \{x_i\}_{i=1}^M \subset \Re^n$ is called a *spherical S-code* if $\|x_i\| = 1$ for each i, and $x_i^T x_j \in S$, $i \neq j$. When $S = [-1, \cos\theta]$ the points of $\mathcal{C}$ are the centers of nonoverlapping *spherical caps* of angular diameter θ, and if $\theta = \pi/3$ the points of $\mathcal{C}$ are the centers of nonoverlapping spheres of radius 1/2, all of which touch the sphere of radius 1/2 centered at the origin. Maximizing the number of such spheres is commonly referred to as the *kissing number* problem in $\Re^n$.

There is a very large literature concerning spherical codes, and the related *Tammes problem*: find a $[-1, \cos\theta]$-code of given cardinality M that maximizes θ (see [3, Chapters 1,3] and references therein). In addition to their purely geometrical interest these problems have a number of significant applications, for example to the construction of constant-energy codes for a Gaussian communication channel [3, Chapter 3.1]. A fundamental problem connected with spherical codes is to bound $M = |\mathcal{C}|$ for a given S. An approach based on harmonic analysis and linear programming [6] allows for the computation of explicit bounds on M for fixed n, and also asymptotic bounds on the sizes of spherical codes and related sphere packings for large n. In [3, Chapters 13-14] this approach is applied with $S = [-1, 1/2]$ to obtain bounds on M for $n = 3, \ldots, 24$, and to give precise characterizations of spherical codes that solve the kissing number problem in dimensions 8 and 24. For recent results concerning spherical codes, the Tammes problem and the linear programming bounds see [3, pp. xxiii-xxv].

In this paper we consider a strengthening of the linear programming bound that is applicable when $\mathcal{C}$ is *antipodal*; that is, $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$. Antipodal codes include all codes obtained as the set of minimial vectors in a lattice, so the antipodal bound applies to the size of any such *lattice code* in $\Re^n$. An antipodal code can also be viewed as a packing of lines in the unit sphere, which is the lowest-dimensional case of the packings of subspaces, or *Grassmannian packings*, considered in [2].

In the next section we describe the linear programming bound of [6], and a modification that is valid for antipodal codes. In Section 3 the antipodal bound is applied in the case of $S = [-1, 1/2]$ to obtain bounds on the kissing number attainable by antipodal codes in dimensions $n = 3, \ldots, 24$. (For all such n *except* 13, 14 and 15 the highest known kissing number corresponds to an antipodal code.) We obtain nontrivial improvements in the upper bound on M for most n. In Section 4 we use the solutions of the linear programming problems to obtain additional results for certain dimensions. In particular we prove that for $n = 4$, 6 and 7 the antipodal codes that attain the maximal kissing number are essentially unique, and correspond to the minimial vectors in the laminated lattices Λ_n.

2 Linear programming bounds

Let $\mathcal{C} = \{x_i\}_{i=1}^M$ be a spherical S-code in $\Re^n$, $n \geq 3$. In this section we describe a well-known linear programming bound for the size M of such a code. The *distance distribution* of the code is the function $\alpha(\cdot) : [-1, 1] \to \mathbf{Z}_+$ defined as

$$\alpha(s) = \frac{|\{(i,j) : x_i^T x_j = s\}|}{M}.$$

It is then easy to see that

$$\alpha(1) = 1, \tag{1a}$$

$$\sum_{s \in S} \alpha(s) = M - 1, \tag{1b}$$

$$\alpha(s) \geq 0, \ \alpha \in S. \tag{1c}$$

Let $\Phi_k(\cdot)$, $k = 0, 1, \ldots$ denote the Gegenbauer, or ultraspherical, polynomials

$$\Phi_k(t) = \frac{P_k^{(\beta,\beta)}(t)}{\binom{k+\beta}{k}}, \tag{2}$$

where $P_k^{(\beta,\beta)}$ is the Jacobi polynomial with $\beta = (n-3)/2$ [1]. The normalization in (2) is chosen so that $\Phi_k(1) = 1$ for all k. Using techniques from harmonic analysis it can be shown [6], [3, Chapter 9, 13] that

$$1 + \sum_{s \in S} \alpha(s)\Phi_k(s) \geq 0, \quad k = 1, 2, \ldots. \tag{3}$$

By combining (1) and (3) a bound on M can be obtained via the linear programming problem

$$\begin{aligned} \text{LP} \qquad \max \quad & \sum_{s\in S} \alpha(s) \\ \text{s.t.} \quad & \sum_{s\in S} \alpha(s)\Phi_k(s) \geq -1, \quad k = 1, 2, \ldots, \\ & \alpha(s) \geq 0, \;\; s \in S. \end{aligned}$$

Note that LP has both an infinite number of variables and constraints. In practice a bound on M can be obtained by working with a finite number of constraints $k = 1, \ldots, K$, and using a feasible solution to the dual problem to bound the optimal value of LP.

Our interest here is in modifying the problem LP to obtain an improved bound when $\mathcal{C}$ is antipodal. In this case it is obvious that the distance distribution satisfies $\alpha(s) = \alpha(-s)$, $s \in [-1, 1]$. Since the polynomials $\Phi_k(\cdot)$ are odd for k odd, it follows immediately that the constraints (3) are satisfied with equality for all odd k. Let $S_+ = S \cap [0, 1]$, and $S_{++} = S \cap (0, 1]$. Since $\Phi_k(\cdot)$ are even for k even, the constraints (3) for even k can be written

$$2 + \alpha(0)\Phi_{2k}(0) + 2 \sum_{s\in S_{++}} \alpha(s)\Phi_{2k}(s) \geq 0, \quad k = 1, 2, \ldots$$

A bound for M, the size of the code, can then be based on the linear programming problem

$$\begin{aligned} \text{LP+} \qquad \max \quad & \alpha(0) + 2 \sum_{s\in S_{++}} \alpha(s) \\ \text{s.t.} \quad & \alpha(0)\Phi_{2k}(0) + 2 \sum_{s\in S_{++}} \alpha(s)\Phi_{2k}(s) \geq -2, \quad k = 1, 2, \ldots, K, \\ & \alpha(s) \geq 0, \;\; s \in S_+. \end{aligned}$$

(If $0 \notin S$ the variable $\alpha(0)$ is omitted in LP+.) The dual of LP+ is the problem

$$\begin{aligned} \text{LD+} \qquad \min \quad & 2 \sum_{k=1}^{K} f_{2k} \\ \text{s.t.} \quad & \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s) \leq -1, \quad s \in S_+, \\ & f_{2k} \geq 0, \;\; k = 1, \ldots, K. \end{aligned}$$

In practice it may be impossible to solve LD+ exactly due to the infinite number of constraints. However by solving an approximation of LD+ using a finite set of points $s_1, s_2, \ldots s_N$ we can obtain values f_{2k}, $k = 1, \ldots, K$, so that

$$1 + \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s) \leq \epsilon, \quad s \in S_+, \tag{4}$$

where $0 \leq \epsilon < 1$. A bound on the size of the code is then given as follows.

Lemma 1 *Let f_{2k}, $k = 1, \ldots, K$ be nonnegative numbers satisfying* (4)*. If $\mathcal{C}$ is an antipodal spherical code, then $M = |\mathcal{C}| \leq 2 + 2(\sum_{k=1}^{K} f_{2k})/(1-\epsilon)$.*

Proof: Since $\mathcal{C}$ is antipodal, the identities (1a) and (1b) imply that $M \leq v(\text{LP+}) + 2$, where $v(\text{LP+})$ denotes the solution objective value in LP+. By weak duality [4] $v(\text{LP+}) \leq 2(\sum_{k=1}^{K} f_{2k})$, where f_{2k}, $k = 1, \ldots, K$ is feasible in LD+. But if f_{2k}, $k = 1, \ldots, K$ are nonnegative and satisfy (4), then $f_{2k}/(1-\epsilon)$, $k = 1, \ldots, K$ are feasible in LD+. □

3 Bounds on kissing numbers

We now consider the bound of Lemma 1 applied to the case of $S = [-1, .5]$, often referred to as the kissing number problem. In this case $S_+ = [0, .5]$. For $n = 3, 4, \ldots, 24$ we solve the approximation of LD+ obtained using $K = 6$, and the constraints generated by $\{s_j\}_{j=1}^{2001}$, $s_j = .00025(j-1)$. Let f_{2k}, $k = 1, \ldots, K$ be the solution of this linear programming problem, and let $\Phi(s) = 1 + \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s)$. To obtain the value ϵ required for the bound in Lemma 1 we use the following simple technique. Let j be such that $\Phi'(s_j) > 0$, $\Phi'(s_{j+1}) < 0$. Let $d_2 = \max\{\Phi''(s_j),\ \Phi''(s_{j+1})\}$. Then $\Phi''(s) \leq d_2 < 0$, $s \in [s_j, s_{j+1}]$, assuming that $\Phi''(\cdot)$ is negative and monotonic on this interval, which is easily checked. It follows that for $0 \leq \delta \leq .00025$,

$$\Phi(s_j + \delta) \leq \Phi(s_j) + \delta\Phi'(s_j) + \frac{\delta^2}{2} d_2,$$

from which we obtain an upper bound of the form

$$\epsilon = \Phi(s_j) - \frac{\Phi'(s_j)}{2d_2}.$$

In Table 1 we give the new antipodal bounds obtained as described above, as well as the original LP bounds and highest known kissing numbers, from [3]. The antipodal bounds are rounded down to the next even integer, since M must be even for an antipodal code. (The bounds for $n = 5$, 10 and 14 are further reduced by 2, as described in detail in Section 4.) For the purposes of bounding the size of an antipodal code the original LP bounds that are odd could also be reduced by one, but we retain the odd values here to provide agreement

Table 1: Best known kissing numbers and LP bounds

n	Best Known	Lattice?	Original Bound	Antipodal Bound
3	12	yes	13	12
4	24	yes	25	24
5	40	yes	46	40‡
6	72	yes	82	72
7	126	yes	140	126
8	240	yes	240	240
9	306	no†	380	366
10	500	no†	595	548‡
11	582	no†	915	820
12	840	no†	1416	1228
13	1130	no	2233	1866
14	1582	no	3492	2938‡
15	2564	no	5431	4962
16	4320	yes	8313	8160
17	5346	yes	12215	11478
18	7398	yes	17877	16122
19	10668	yes	25901	22724
20	17400	yes	37974	32340
21	27720	yes	56852	46878
22	49896	yes	86537	70164
23	93150	yes	128096	111126
24	196560	yes	196560	196560

†antipodal ‡see Section 4

with [3]. As described above $K = 6$ was used in the formulation of the problem used to obtain these bounds, but $f_{10} = f_{12} = 0$ in the solution for all n except for $n = 3$.

As can be seen in Table 1 the antipodal bounds are tight for dimensions 3, 4, 5, 6, 7, 8 and 24. The tight bounds for dimensions 8 and 24 are to be expected since the original LP bounds are tight, and the maximal kissing numbers are attained by lattice codes [3, Chapter 13]. The tight bounds for $n = 3$, 4, 5, 6 and 7 provide a new proof for the known result [8, 9] that the laminated lattices Λ_n have the highest possible kissing numbers for lattices in these dimensions. It is known that the maximal kissing number for $n = 3$ is 12 [7]. For $n = 4$, 5, 6 and 7 the tight bounds show that higher kissing numbers in these dimensions can only come from codes that are *not* antipodal; this is obvious for $n = 4$ from the original LP bound of

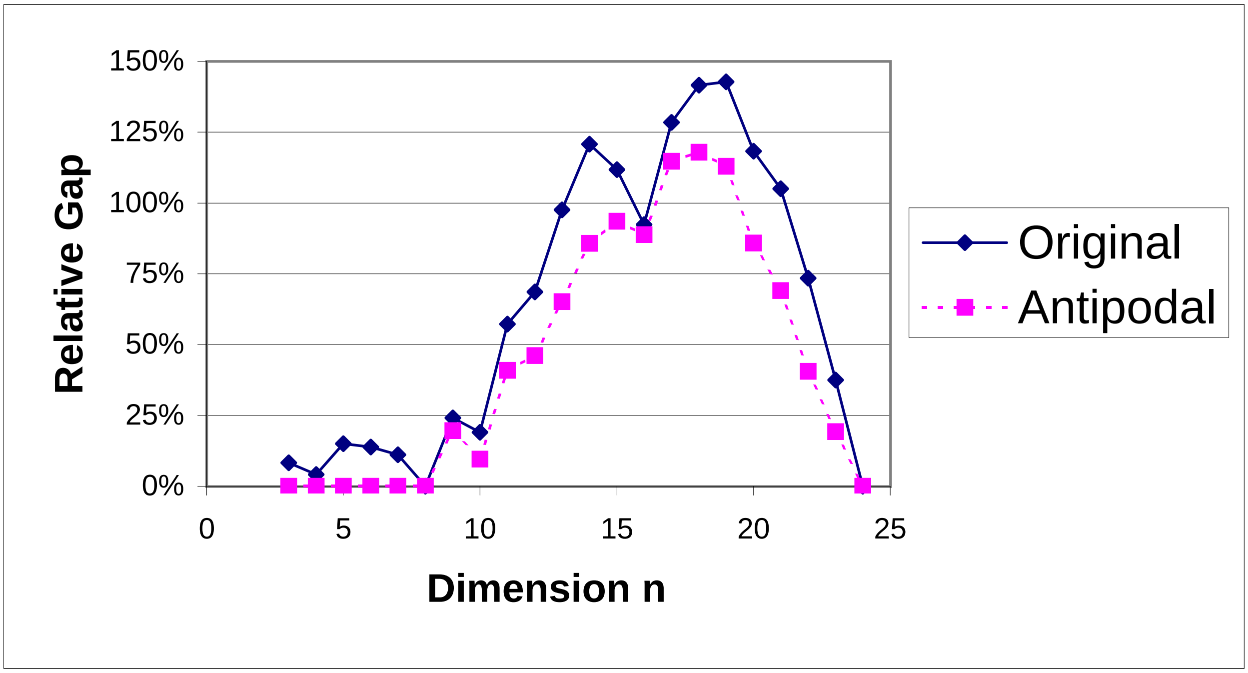

Figure 1: Comparison of original and antipodal LP bounds

25, but is to our knowledge a new result in dimensions 5, 6 and 7.

In Figure 1 we compare the relative "gaps" between the original and antipodal bounds and the highest known kissing number, for dimensions $3, 4, \ldots, 24$. In addition to bringing the gap to zero in dimensions 3-7, the antipodal bounds provide substantial reductions for some n between 9 and 23. In particular for $n = 10$, 22 and 23 the gap between the original upper bound and the best known code is reduced by almost 50%. For n between 9 and 23 the gap is reduced by an average of about 28%.

For some n the solutions of the discretized version of LD+ are particularly well structured, allowing for additional analysis that can either demonstrate that the code attaining the bound is essentially unique, or in fact cannot exist. We pursue this topic in detail in the next section for $n = 4$, 5, 6, 7, 10 and 14.

4 Uniqueness or nonexistence of certain antipodal codes

In this section we show that:

- For $n = 4$, 6 and 7 the *only* antipodal codes that attain the maximal possible kissing number correspond to orthogonal transformations of the set of minimal vectors of the laminated lattices Λ_n.

- For $n = 5$, 10 and 14 there is *no* antipodal code that attains the bound from LD+, and therefore this bound can be reduced by 2.

In all cases the analysis uses explicit rational coefficients f_{2k} suggested by the solution of LD+. For $n \leq 8$ it is known that Λ_n is the unique lattice with maximal density, and that Λ_n achieves the maximal kissing numbers for a lattice code [3, Chapter 1, Section 1.5]. Our method of proving the uniqueness of these codes in dimensions 4, 6 and 7 is similar to that used to prove that for $n = 8$ the minimal vectors from E_8 are the essentially unique code with kissing number 240 [3, Chapter 14.2, Theorem 7].

For a code $\mathcal{C} = \{x_i\}_{i=1}^M$, let $\alpha_i(s) = |\{j : x_i^T x_j = s\}|$. A code is called *distance invariant* if $\alpha_i(s)$ is independent of i for all s, and in this case $\alpha_i(s) = \alpha(s)$ for all i and s.

Lemma 2 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 4$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 24$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has $\alpha(0) = 6$, $\alpha(1/2) = \alpha(-1/2) = 8$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm 1/2, \pm 1\}$.*

Proof: For $n = 4$ the solution of LD+ has $f_2 = 6$, $f_4 = 5$, and a bound $2 + 2(f_2 + f_4) = 24$. Let

$$\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) = 16s^2\left(s^2 - \frac{1}{4}\right).$$

Then $\Phi(s) \leq 0$ for $s \in S_+$, with roots at 0 and 1/2. It follows from the complementary slackness property [4] that if $\mathcal{C}$ is an antipodal code with $M = 24$, then the distance distribution for $\mathcal{C}$ must satisfy $\alpha(s) = 0$, $s \notin \{0, \pm 1/2, \pm 1\}$, and in addition

$$\begin{aligned} \alpha(0)\Phi_2(0) + 2\alpha(1/2)\Phi_2(1/2) &= -2 \\ \alpha(0)\Phi_4(0) + 2\alpha(1/2)\Phi_4(1/2) &= -2. \end{aligned} \tag{5}$$

The unique solution of (5) is $\alpha(0) = 6$, $\alpha(1/2) = 8$. From (5) and the fact that the code is antipodal, $\mathcal{C}$ is a 5-design in $\Re^4$ [5]. Since $\mathcal{C}$ is also an S-code with $|S| = 4$, [5, Theorem 7.4] implies that $\mathcal{C}$ is distance invariant. □

Theorem 3 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 4$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 24$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_4 = D_4$.*

Proof: Let $\{x_i\}_{i=1}^{24}$ be the elements of $\mathcal{C}$, and define the lattice L consisting of points of the form

$$\sum_{i=1}^{24} \sqrt{2}\, a_i x_i, \quad a_i \in \mathbf{Z},\ i = 1, \ldots, 24.$$

It is then easy to show that L is an even integral lattice. Since L is generated by vectors of squared-norm 2, Witt's theorem [3, Chapter 4.3] implies that L is a direct sum of root lattices that are isometric with either A_n, $n \geq 1$ or D_n, $n \geq 4$. The only lattice of this form with at least 24 minimal vectors is D_4. □

In dimensions 6 and 7 very similar analysis obtains the following results.

Lemma 4 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 6$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 72$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has $\alpha(0) = 30$, $\alpha(1/2) = \alpha(-1/2) = 20$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm 1/2, \pm 1\}$.*

Proof: Similar to the proof of Lemma 2, using $f_2 = 14$, $f_4 = 21$. □

Theorem 5 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 4$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 24$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_6 = E_6$.*

Proof: Similar to the proof of Theorem 5, but with the additional root lattice E_6. □

Lemma 6 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 7$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 126$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has $\alpha(0) = 60$, $\alpha(1/2) = \alpha(-1/2) = 32$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm 1/2, \pm 1\}$.*

Proof: Similar to the proof of Lemma 2, using $f_2 = 234/11$, $f_4 = 448/11$. □

Theorem 7 *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 7$ and $S = [-1, .5]$ has $M = |\mathcal{C}| = 126$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_7 = E_7$.*

Proof: Similar to the proof of Theorem 5, but with the additional root lattices E_6 and E_7. □

Before proceeding we note that the bounds from LD+ for antipodal codes in dimensions 4, 5 (see below), 6 and 7 attain the "special bound" for antipodal codes described in [5, Example 8.4].

Next we turn to the non-existence results. For $n = 5$ the solution of LD+ produces $f_2 = 28/3$, $f_4 = 32/3$, and a bound of $2 + 2(f_2 + f_4) = 42$. Define the polynomial

$$\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) = 28s^2\left(s^2 - \frac{1}{4}\right).$$

Clearly $\Phi(s) \leq 0$ for $s \in S_+$, with roots at 0 and 1/2. The solution of LP+ must satisfy (5), which has a unique solution $\alpha(0) = 72/5$, $\alpha(1/2) = 64/5$. However $\mathcal{C}$ attaining this bound must be distance invariant, exactly as shown for dimensions 4, 6 and 7 above, so $\alpha(s)$ must be integral for all s. Therefore no antipodal code with $M = 42$ can exist. Since for an antipodal code M must be an even integer the bound of 42 can be reduced to 40.

For $n = 10$ the argument is similar. The solution of LD+ produces $f_2 = 30$, $f_4 = 368/3$, $f_6 = 364/3$, with a bound $2 + 2(f_2 + f_4 + f_6) = 550$. Define

$$\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) + f_6\Phi_6(s) = \frac{12544}{33}\left(s^2 - \frac{1}{4}\right)\left(s^2 - \frac{1}{56}\right)^2.$$

Then $\Phi(s) \leq 0$ for $s \in S_+$, with roots at 1/2 and $1/\sqrt{56} \approx .13363$. The optimal solution of LP+ must satisfy

$$\begin{aligned}
\alpha(1/2)\Phi_2(1/2) + \alpha(1/\sqrt{56})\Phi_2(1/\sqrt{56}) &= -1 \\
\alpha(1/2)\Phi_4(1/2) + \alpha(1/\sqrt{56})\Phi_4(1/\sqrt{56}) &= -1 \\
\alpha(1/2)\Phi_6(1/2) + \alpha(1/\sqrt{56})\Phi_6(1/\sqrt{56}) &= -1,
\end{aligned} \tag{6}$$

which has a unique solution $\alpha(1/\sqrt{56}) = 2352/13$, $\alpha(1/2) = 1210/13$. From (6) and the fact that $\mathcal{C}$ is antipodal, $\mathcal{C}$ is a 7-design in $\Re^6$, and is also an S-code with $|S| = 5$. From [5, Theorem 7.4] $\mathcal{C}$ is distance invariant, so $\alpha(s)$ must be integral for all s. Therefore no antipodal code with $M = 550$ can exist, and the bound can be reduced to 548.

For $n = 14$, from the solution of LD+ the polynomial $\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) + f_6\Phi_6(s)$ appears to have roots at 1/2 and 1/6. Using the computed bound of 2940, we conjecture that

$$\Phi(s) = \frac{10368}{5}\left(s^2 - \frac{1}{4}\right)\left(s^2 - \frac{1}{36}\right)^2,$$

from which we obtain $f_2 = 364/5$, $f_4 = 5811/11$, $f_6 = 47736/55$, and a bound of $2 + 2(f_2 + f_4 + f_6) = 2940$, as desired. The solution of LP+ must satisfy a system of equations of the form (6), but with $1/6$ replacing $1/\sqrt{56}$ throughout. The unique solution of this system is $\alpha(1/2) = 2275/8$, $\alpha(1/6) = 9477/8$. However $\alpha(s)$ must be integral for all s, exactly as in the case $n = 10$, so no antipodal code with $M = 2940$ can exist and the bound can be reduced to 2938.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Appl. Math Series 55, U.S. Dept. of Commerce, Washington DC, 1972.

[2] J.H. Conway, R.H. Hardin, and N.J.A. Sloane, "Packing lines, planes, etc.: packings in Grassmannian space," *Experimental Mathematics* 5 (1996), 139-159. See also `www.research.att.com/~njas/grass/`.

[3] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, third edition, Springer, New York, 1999.

[4] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, 1963.

[5] P. Delsarte, J.-M. Goethals, and J.J. Seidel, "Spherical codes and designs," *Geom. Dedicat.* 6 (1977), 363-388.

[6] G.A. Kabatiansky and V.I. Levenshtein, "Bounds for packings on a sphere and in space," *Problems of Information Transmission* 14 (1978), 1-17.

[7] J. Leech, "The problem of the thirteen spheres," *Math. Gazette* 40 (1956), 22-23.

[8] C. Musès, "The dimensional family approach in (hyper)sphere packing: a typological study of new patterns, structures, and interdimensional functions," *Applied Math. Computation* 88 (1997), 1-26.

[9] G.L. Watson, "One-class genera of positive quadratic forms," *J. London Math. Society* 38 (1963), 387-392.