On the min-cut max-flow ratio for multicommodity flows

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Abstract

In this paper we present a new bound on the min-cut max-flow ratio for multicommodity flow problems. We use a so-called aggregated commodity formulation and an optimal solution to its dual to show our main result. Currently, the best known bound for this ratio is proportional to $log(k)$ where k is the number of origin-destination pairs with positive demand. We show a new ratio that is proportional to $log(k^*)$ where k^* is the cardinality of the minimal vertex cover of the demand graph.

We therefore relate the min-cut max-flow ratio of a multicommodity flow problem to the number of source nodes instead of the number of origin destination pairs. This result appears to be more natural since a generalization of the min-cut max-flow theorem holds tight for flow problems with a single source and multiple sink nodes.

We also show a similar bound for the maximum multicommodity problem.

1 Introduction

In this paper we study multicommodity flow problems and present new bounds on the associated min-cut max-flow ratio. Starting with the pioneering work of Leighton and Rao [10] there has been ongoing research in the area of "approximate min-cut max-flow theorems" for multicommodity flows. We present a summary of previous work later in Section 1.3. We next state the well-known min-cut max-flow theorem and present an interpretation of it for flow problems with specified flow requirements. We then clarify what is meant by "minimum cut" and "maximum flow" for multicommodity flow problems.

Throughout the paper, we assume that the input graph is connected and has positive capacity on all edges.

1.1 Single commodity flows

Given an undirected graph $G = (V, E)$, edge capacities c_e for $e \in E$ and two special nodes $s,\ v \in V$, the well-known min-cut max-flow theorem [4] states that the value of the maximum flow from the *source* node s to the *sink* node v is equal to the capacity of the minimum cut:

$$\min_{S \subset V : s \in S, v \notin S} \left\{ \sum_{e \in \delta(S)} c_e \right\}$$

where $\delta(S) = \{e \in E\ :\ |e \cap S| = 1\}$. Let $t \in R_+$ be a specified flow requirement, then the min-cut max-flow theorem implies that t units of flow can be routed from s to v if and only if *the minimum cut-capacity to cut-load ratio* ρ^* where

$$\rho^* = \min_{S \subset V : s \in S, v \notin S} \left\{ \frac{\sum_{e \in \delta(S)} \ c_e}{t} \right\}$$

is at least 1.

It is possible to generalize this result to flows with a single source node and multiple sink nodes as follows: Given a source node s and a collection of sink nodes $v_q \in V \setminus \{s\}$ for $q \in Q$, it is possible to simultaneously route $t_q \in R_+$ units of flow from s to v_q for all $q \in Q$ if and only if $\rho^* \geq 1$ where

$$\rho^* = \min_{S \subset V : s \in S} \left\{ \frac{\sum_{e \in \delta(S)} \ c_e}{\sum_{q \in Q\ :\ v_q \notin S} t_q} \right\}.$$

This observation is the main motivation behind our study as it shows that a min-cut max-flow relationship holds tight for network flow problems (with specified flow requirements) as long as the sink nodes share a common source node. For the sake of completeness, we note that G is undirected, and thus the min-cut max-flow relationship also holds for a single sink node and multiple source nodes.

1.2 Multicommodity flows

A natural extension of this observation is to consider multicommodity flows, where a collection of pairs of vertices $\{s_q, v_q\},\ q \in Q$ together with a flow requirement t_q for each pair is provided. Let the minimum cut-capacity to cut-load ratio for multicommodity flows be similarly defined as

$$\rho^* = \min_{S \subset V} \left\{ \frac{\sum_{e \in \delta(S)} \ c_e}{\sum_{q \in Q : |S \cap \{s_q, v_q\}| = 1} t_q} \right\}.$$

In the remainder of the paper we refer to ρ^* as the minimum cut ratio. Clearly, it is possible to simultaneously route t_q units of flow from s_q to v_q for all $q \in Q$, only if $\rho^* \geq 1$. But the converse is not true ([13], [15]) and a simple counter example is the complete bipartite graph $K_{2,3}$ with unit capacity edges and unit flow requirements between every pair of nodes that are not connected by an edge.

For multicommodity flows, *metric inequalities* provide the necessary and sufficient conditions for feasibility, see [6], and [16]. More precisely, a given instance of the multicommodity flow problem is feasible if and only if the input data satisfies

$$\sum_{e \in E} w_e c_e \;\geq\; \sum_{q \in Q} dist(s_q, v_q) t_q$$

for all $w \geq 0$, where $dist(u, v)$ denotes the shortest path distance from u to v using w as edge weights. Notice that the above example with $K_{2,3}$ does not satisfy the metric inequality "generated" by $w_e = 1$ for all $e \in E$. It is easy to show that the condition $\rho^* \geq 1$ is implied by metric inequalities.

The *maximum concurrent flow* problem is the optimization version of the multicommodity flow feasibility problem, see [19] and [13]. Formally, given an undirected graph $G = (V, E)$, edge capacities c_e for $e \in E$ and a collection of pairs of vertices $\{s_q, v_q\}$, $q \in Q$ together with a flow requirement t_q for each pair, the objective is to find the maximum value of κ such that κt_q units of flow can be simultaneously routed from s_q to v_q for all $q \in Q$.

For a given instance of the multicommodity flow problem, let κ^* denote the value of the maximum concurrent flow. In other words, it is possible to simultaneously route κt_q units of flow from s_q to v_q for all $q \in Q$ if and only if $\kappa \leq \kappa^*$. Clearly the maximum concurrent flow value can not exceed the minimum cut ratio:

$$\rho^* \;\geq\; \kappa^*. \tag{1}$$

Our main purpose in this paper is show a reverse relationship between the minimum cut ratio and the maximum concurrent flow value. More precisely, we show that

$$\kappa^* \;\geq\; \frac{1}{c\,\lceil log\, k^* \rceil}\, \rho^* \tag{2}$$

where c is a constant and k^* is the cardinality of the minimal vertex cover for the demand graph. In other words, k^* is the size of the smallest set $K^* \subseteq V$ such that K^* contains at least one of s_q or v_q for all $q \in Q$. Throughout the paper, we assume that $k^* > 1$.

We call (2), an approximate min-cut max-flow theorem, as it relates the maximum (concurrent) flow of a multicommodity flow problem to the (scaled) capacity of the minimum cut. Combining (1) and (2) we can bound the min-cut max-flow ratio as follows:

$$c \lceil log\, k^* \rceil \;\geq\; \frac{\rho^*}{\kappa^*} \;\geq\; 1 \qquad (3)$$

As discussed in Section 1.1, this bound is $\rho^* = \kappa^*$ when $k^* = 1$.

1.3 Related work

Starting with the pioneering work of Leighton and Rao [10] there has been some interest in the so-called approximate min-cut max-flow theorems. The first such result in [10] shows that the upper bound in (3) is at most $O(log\,|V|)$ when $t_q = 1$ for all $q \in Q$. Later Klein, Agrawal, Ravi and Rao [9] extend this result to general t_q and show that the bound is $O(log\, C\; log\, D)$ where D is the sum of demands (i.e. $D = \sum_{q \in Q} t_q$) and C is the sum of capacities (i.e. $C = \sum_{e \in E} c_e$). Tragoudas [20] has improved this bound to $O(log\,|V|\; log\, D)$ and Garg, Vazirani and Yannakakis [5] has further improved it to $O(log\, k\; log\, D)$, where $k = |Q|$. Plotkin and Tardos [18] present the first bound that does not depend on the input data by showing that the upper bound in (3) is at most $O(log^2\, k)$. Finally Linial, London and Rabinovich [11] and Aumann and Rabani [1] independently show that the bound is at most $O(log\, k)$.

Our result improves this best known bound to $O(log\, k^*)$. To emphasize the difference between $O(log\, k)$ and $O(log\, k^*)$, we note that for an instance of the multicommodity flow problem with a single source node and $|V| - 1$ sink nodes, $k = |V| - 1$ whereas $k^* = 1$. In general, $k \geq k^* \geq k/|V|$.

We next present a linear programming formulation of the maximum concurrent flow problem using aggregate commodities. In Section 3 we show the $O(log\, k^*)$ bound and in Section 4, we discuss geometric implications of this result. Finally in Section 5, we show similar bounds for the so-called maximum multicommodity problem. In the linear programming formulation of the maximum multicommodity problem, an aggregate commodity may correspond to a vertex cover, or a “bipartite graph cover” of the demand graph.

2 Formulation

When formulating a multicommodity problem as a linear program, what is meant by a "commodity" can effect the size of the formulation significantly. Even though, this has been noticed and exploited by researchers interested in solving these linear programs (see, for example, [2] and [12]), it has been overlooked by researchers interested in the theoretical aspects of multicommodity flows. We next present a formulation for the concurrent flow problem using aggregate commodities. A commodity in this formulation aggregates all flow requirements with a common source node.

2.1 The concurrent flow problem

Given an undirected graph $G = (V, E)$ edge capacities c_e for $e \in E$ and flow requirements t_q for given pairs of vertices $\{s_q, v_q\}$, for all $q \in Q$, let T denote the corresponding flow requirement matrix. More precisely, $T_{[k,j]} = \sum_{q \in Q \,:\, s_q = k,\, v_q = j} t_q$ for all $k, j \in V$. We then define the set of "source nodes" $K \subseteq V$ to be $K = \{k \in V \;:\; \sum_{j \in V} T_{[k,j]} > 0\}$ and formulate the maximum concurrent flow problem as follows:

$$Maximize \quad \kappa$$

$$Subject\ to$$

$$\sum_{v:\{v,j\}\in E} f^k_{vj} - \sum_{v:\{j,v\}\in E} f^k_{jv} = \kappa\, T_{[k,j]} \qquad for\ all\ j \in V,\ k \in K\ with\ j \neq k$$

$$\sum_{v:\{v,k\}\in E} f^k_{vk} - \sum_{v:\{k,v\}\in E} f^k_{kv} = -\ \kappa \sum_{j\in V} T_{[k,j]} \qquad for\ all\ \ k \in K$$

$$\sum_{k\in K} \left(f^k_{jv} + f^k_{vj}\right) \leq c_{\{j,v\}} \qquad for\ all\ \{j,v\} \in E$$

$$\kappa \geq 0, \quad f^k_{jv} \geq 0 \qquad for\ all\ k \in K,\ and\ \{j,v\} \in E$$

where variable f^k_{vj} denotes the flow of commodity k from node i to node j, and variable κ denotes the value of the concurrent flow. We note that the original linear programming formulation of the maximum concurrent flow problem presented in [19] also uses aggregate commodities. We also note that using an aggregate flow vector f, it is easy to find disaggregated flows for node pairs (k, j) with $T_{[k,j]} > 0$. The disaggregation, however, is not necessarily unique.

2.2 A reformulation of the concurrent flow problem

To find the smallest set of commodities that would model the problem instance correctly, we do the following: Let $G^T = (V, E^T)$ denote the (undirected) demand graph where $E^T = \{\{i,j\} \in V \times V \ : \ T_{[i,j]} + T_{[j,i]} > 0\}$ and let $K^* \subseteq V$ be a minimal vertex cover of G^T. In other words, K^* is a smallest cardinality set that satisfies $\{i,j\} \cap K^* \neq \emptyset$ for all $\{i,j\} \in E^T$. We then modify the entries of the flow matrix T so that $T_{[k,j]} > 0$ only if $k \in K^*$. Note that this can be done without loss of generality since the capacity constraints in the formulation do not depend on the orientation of the flow.

Therefore, it is possible to formulate the maximum concurrent flow problem using $|K^*|$ commodities. We next slightly modify the formulation as follows:

$$
\begin{array}{ll}
\textit{Maximize} & \kappa \\
\textit{Subject to} & \\
\end{array}
$$

$$
\begin{array}{rcll}
\displaystyle\sum_{v:\{j,v\}\in E} f^k_{jv} - \sum_{v:\{v,j\}\in E} f^k_{vj} \ + \kappa\, T_{[k,j]} & \leq & 0 & \textit{for all } j \in V,\ k \in K^* \textit{ with } j \neq k \\
\displaystyle\sum_{k\in K} \left(f^k_{jv} + f^k_{vj}\right) & \leq & c_{\{j,v\}} & \textit{for all } \{j,v\} \in E \\
\kappa \ \textit{ free},\ f^k_{jv} & \geq & 0 & \textit{for all } k \in K^*, \textit{ and } \{j,v\} \in E
\end{array}
$$

where (i) we have deleted the flow balance equalities for the source nodes $k \in K^*$, (ii) changed the flow balance equalities for the remaining nodes to inequality, and (iii) relaxed the non-negativity requirement for κ. Note that these modifications do not affect the value of the optimal solution.

We next write the dual of this formulation:

$$
\begin{array}{ll}
\textit{Minimize} & \displaystyle\sum_{\{j,v\}\in E} c_{\{j,v\}}\ w_{\{j,v\}} \\
\textit{Subject to} &
\end{array}
$$

$$
\begin{array}{rcll}
\displaystyle\sum_{k\in K}\sum_{j\in V} T_{[k,j]}\ y^k_j & = & 1 & \\
y^k_v - y^k_j + w_{\{j,v\}} & \geq & 0 & \multirow{2}{*}{$\Big\}$ \textit{for all } $k \in K^*$, \textit{ and } $\{j,v\} \in E$} \\
y^k_j - y^k_v + w_{\{j,v\}} & \geq & 0 & \\
y^k_k & = & 0 & \textit{for all } k \in K^* \\
y^k_j & \geq & 0 & \textit{for all } j \in V,\ k \in K^* \textit{ with } j \neq k \\
w_{\{j,v\}} & \geq & 0 & \textit{for all } \{j,v\} \in E
\end{array}
$$

where we include dual variables y^k_k in the formulation even though there are no corresponding primal constraints. These variables are set the zero in a separate constraint. The main reason behind reformulating the primal problem and using redundant variables in the dual problem is to obtain a dual formulation that would have an optimal solution that satisfies the following properties.

Proposition 1 *Let $[\bar{y}, \bar{w}]$ be an optimal solution to the dual problem, and let $\hat{y} \in R^{|V|\times|V|}$ be the vector of shortest path distances (using $\bar{w}$ as edge weights) with $\hat{y}^k_j$ denoting distance from node k to j.*

(i) *For any $k \in K$ and $j \in V$, with $T_{[k,j]} > 0$, $\bar{y}^k_j$ is equal to $\hat{y}^k_j$.*

(ii) *For any $\{j, v\} \in E$, $\bar{w}_{\{j,v\}}$ is equal to $\hat{y}^j_v$.*

Proof. (i) For any $k \to j$ path $P = \{\{k, v_1\}, \{v_1, v_2\}, \ldots, \{v_{|P|-1}, j\}\}$ we have $\sum_{e \in P} w_e \geq \bar{y}^k_j$, implying $\hat{y}^k_j \geq \bar{y}^k_j$. If $\hat{y}^k_j > \bar{y}^k_j$ for some $k \in K$, $j \in V$ with $T_{[k,j]} > 0$, we can write $\sum_{k \in K} \sum_{j \in V} T_{[k,j]}\ \hat{y}^k_j = \sigma\ >\ \sum_{k \in K} \sum_{j \in V} T_{[k,j]}\ \bar{y}^k_j = 1$. Which implies that a new solution, with an improved objective function value, can be constructed by scaling $[\hat{y}, \bar{w}]$ by $1/\sigma$, a contradiction.

(ii) Clearly, $\bar{w}_{\{j,v\}} \geq \hat{y}^j_v$. If $\bar{w}_{\{j,v\}} > \hat{y}^j_v$, replacing $\bar{w}_{\{j,v\}}$ by $\hat{y}^j_v$ in the solution improves the objective function value, a contradiction (remember that $c_{\{j,v\}} > 0$ for all $\{j, v\} \in E$). ∎

Note that, using Proposition 1, it is possible to substitute some of the dual variables and therefore combine some of the primal constraints. We now express the maximum concurrent flow value using shortest path distances with respect to $\bar{w}$.

Corollary 2 *Let, κ^* be the optimal value of the primal (or, the dual) problem. Then,*

$$\kappa^* \quad = \quad \frac{\displaystyle\sum_{\{j,v\} \in E} c_{\{j,v\}}\ dist(j, v)}{\displaystyle\sum_{k \in K} \sum_{v \in V} T_{[k,v]}\ dist(k, v)} \tag{4}$$

where $dist(j, v)$ is the shortest path distance from node j to node v with respect to some edge weight vector ($\bar{w}$).

3 The min-cut max-flow ratio

We next argue that there exists a mapping $\Phi : V \rightarrow R_+^p$ for some p, such that $||\Phi(u)-\Phi(v)||_1$ is not very different from $dist(u,v)$ for node pairs $\{u,v\}$ that are of interest. We then substitute $||\Phi(u)-\Phi(v)||_1$ in place of $dist(u,v)$ in (4) and relate the new right hand side of (4) to the minimum cut ratio.

3.1 Mapping the nodes of the graph with small distortion

Our approach follows general structure of the proof of a related result by Bourgain [3] that shows that any $n-$point metric space can be embedded into l_1 with logarithmic distortion. We state this result more precisely in Section 4.

Given an undirected graph $G = (V,E)$, edge weights $w_e \geq 0$ for $e \in E$ and a set $K \subseteq V$ with $|K| > 1$ let $d(u,v)$ denote the shortest path distance from $u \in V$ to $v \in V$ using w as edge weights. For $v \in V$ and $S \subseteq K$ let $d(v,S) = \min_{k \in S}\{d(v,k)\}$ and define $d(v,\emptyset) = \sigma = \sum_{u \in V}\sum_{k \in K} d(u,k)$.

For any $j,t \geq 1$, let Q_j^t be random subset of K such that members of Q_j^t are chosen independently and with equal probability $P(k \in Q_j^t) = 1/2^t$ for all $k \in K$. Note that for all $j \geq 1$, Q_j^t has an identical probability distribution and $E[|Q_j^t|] = |K|/2^t$. For $m = \lceil log(|K|)\rceil$ and $L = 300 \cdot \lceil log(|V|)\rceil$, define the following (random) mapping $\Phi^R : V \rightarrow R_+^{mL}$

$$\Phi^R(v) = \frac{1}{L \cdot m}\begin{bmatrix} d(v,Q_1^1) & d(v,Q_1^2) & \dots & d(v,Q_1^m) \\ d(v,Q_2^1) & d(v,Q_2^2) & \dots & d(v,Q_2^m) \\ \vdots & \vdots & \ddots & \vdots \\ d(v,Q_L^1) & d(v,Q_L^2) & \dots & d(v,Q_L^m) \end{bmatrix}$$

Note that, $|d(u,S) - d(v,S)| \leq d(u,v)$ for any $S \subseteq V$, and therefore:

$$\begin{aligned} ||\Phi^R(u) - \Phi^R(v)||_1 &= \frac{1}{L \cdot m}\sum_{i=1}^{m}\sum_{j=1}^{L}\left|d(u,Q_j^i) - d(v,Q_j^i)\right| \\ &\leq \frac{1}{L \cdot m} \cdot L \cdot m \cdot d(u,v) \;=\; d(u,v) \end{aligned} \tag{5}$$

for all $u,v \in V$. We next bound $||\Phi^R(u) - \Phi^R(v)||_1$ from below.

Lemma 3 *For all $u \in K$ and $v \in V$ and for some $\alpha = O(log|K|)$ the following property*

$$||\Phi^R(u) - \Phi^R(v)||_1 \;\geq\; \frac{1}{\alpha} \cdot d(u,v)$$

holds simultaneously with positive probability.

Proof. For any $v \in V$ let $B(v,\delta) = \{k \in K : d(v,k) \leq \delta\}$ and $B^o(v,\delta) = \{k \in K : d(v,k) < \delta\}$, respectively, denote the collection of members of K that lie within the closed and open balls around v. We next define a sequence of δ's for pairs of nodes.

For any fixed $u \in K$ and $v \in V$ let

$$t^*_{uv} \quad = \quad \max\Big\{1, \Big\lceil log\Big(max\Big\{|B(u,d(u,v)/2)|, |B(v,d(u,v)/2)|\Big\}\Big)\Big\rceil\Big\}$$

and define

$$\delta^t_{uv} = \begin{cases} 0 & t = 0 \\ \max\{\delta \geq 0 \ : \ |B^o(u,\delta)| < 2^t \text{ and } |B^o(v,\delta)| < 2^t\} & t^*_{uv} > t > 0 \\ d(u,v)/2 & t = t^*_{uv} \end{cases}$$

We note that (i) $m = \lceil log(|K|)\rceil \geq t^*_{uv} > 0$, (ii) $|B^o(u,\delta^t_{uv})|, |B^o(v,\delta^t_{uv})| < 2^t$ for all $t \leq t^*_{uv}$ and (iii) $\max\{|B(u,\delta^t_{uv})|, |B(v,\delta^t_{uv})|\} \geq 2^t$ for all $t < t^*_{uv}$.

For a fixed $t \geq 0$, such that, $t < t^*_{uv}$, rename u and v as z_{max} and z_{other} so that $|B(z_{max},\delta^t_{uv})| \geq |B(z_{other},\delta^t_{uv})|$. Using $\frac{1}{e} \geq (1-\frac{1}{x})^x \geq \frac{1}{4}$, for any $x \geq 2$, we can write the following for any Q^{t+1}_j for $L \geq j \geq 1$:

$$\begin{aligned} P\Big(Q^{t+1}_j \cap B(z_{max},\delta^t_{uv}) = \emptyset\Big) &= \Big(1 - 2^{-(t+1)}\Big)^{|B(z_{max},\delta^t_{uv})|} &\leq (1-2^{-(t+1)})^{2^t} &\leq e^{-\frac{1}{2}} \\ P\Big(Q^{t+1}_j \cap B^o(z_{other},\delta^{t+1}_{uv}) = \emptyset\Big) &= \Big(1 - 2^{-(t+1)}\Big)^{|B^o(z_{other},\delta^{t+1}_{uv})|} &\geq (1-2^{-(t+1)})^{2^{t+1}} &\geq \frac{1}{4} \end{aligned}$$

Notice that $Q^{t+1}_j \cap B(z_{max},\delta^t_{uv}) \neq \emptyset$ implies that $d(z_{max},Q^{t+1}_j) \leq \delta^t_{uv}$, and similarly, $Q^{t+1}_j \cap B^o(z_{other},\delta^{t+1}_{uv}) = \emptyset$ implies that $d(z_{other},Q^{t+1}_j) \geq \delta^{t+1}_{uv}$. Using the independence of the two events (since the two balls are disjoint) we can now write:

$$P\Big(Q^{t+1}_j \cap B(z_{max},\delta^t_{uv}) \neq \emptyset \text{ and } Q^{t+1}_j \cap B^o(z_{other},\delta^{t+1}_{uv}) = \emptyset\Big) \geq \Big(1 - e^{-\frac{1}{2}}\Big) \times \frac{1}{4} \geq \frac{1}{11}$$

and therefore,

$$P\Big(\ \Big|d(z_{other},Q^{t+1}_j) - d(z_{max},Q^{t+1}_j)\Big| \geq \delta^{t+1}_{uv} - \delta^t_{uv}\Big) \geq \frac{1}{11}$$

or, equivalently,

$$P\Big(\ \Big|d(u,Q^{t+1}_j) - d(v,Q^{t+1}_j)\Big| \geq \delta^{t+1}_{uv} - \delta^t_{uv}\Big) \geq \frac{1}{11}$$

for all $t < t^*_{uv}$.

Let X^{tj}_{uv} be a random variable taking value 1 if $\left|d(u, Q^{t+1}_j) - d(v, Q^{t+1}_j)\right| \geq \delta^{t+1}_{uv} - \delta^t_{uv}$, and 0 otherwise. Note that for any fixed $u \in K$ and $v \in V$ if $\sum_{j=1}^{L} X^{tj}_{uv} \geq L/22$ (that is, at least one-half the expected number) for all $t < t^*_{uv}$, then we can write:

$$||\Phi^R(u) - \Phi^R(v)||_1 = \frac{1}{L \cdot m} \sum_{i=1}^{m} \sum_{j=1}^{L} \left|d(u, Q^i_j) - d(v, Q^i_j)\right|$$

$$\geq \frac{1}{L \cdot m} \sum_{i=1}^{t^*_{uv}} \frac{L}{22} \left(\delta^i_{uv} - \delta^{i-1}_{uv}\right) = \frac{1}{22m} \left(\delta^{t^*_{uv}}_{uv} - \delta^0_{uv}\right) = \frac{d(u,v)}{44m}.$$

We now use the Chernoff bound (see for example [14]) to claim that

$$P\Big(\sum_{j=1}^{L} X^{tj}_{uv} < \frac{1}{2} \times \frac{L}{11}\Big) < e^{-\frac{1}{2} \times \frac{1}{4} \times \frac{L}{11}}$$

for any $u \in K$, $v \in V$ and $t < t^*_{uv}$, which, in turn, implies that

$$P\Big(\sum_{j=1}^{L} X^{tj}_{uv} < \frac{L}{22} \quad for\ some\ u \in K,\ v \in V\ and\ t < t^*_{uv}\Big) < |K||V| \lceil log(|K|) \rceil e^{-L/88}$$

where the right hand side of the inequality is less than 1 for $L \geq 88(3 \cdot log(|V|)$. Therefore, with positive probability, $\sum_{j=1}^{L} X^{tj}_{uv} \geq \frac{L}{22}$ for all $u \in K,\ v \in V$ and $t < t^*_{uv}$, which implies that, with positive probability,

$$||\Phi^R(u) - \Phi^R(v)||_1 \geq \frac{d(u,v)}{44m}$$

for all $u \in K,\ v \in V$. ■

An immediate corollary of this result is the existence of a (deterministic) mapping with at most $log(k^*)$ distortion.

Corollary 4 *There exists a collection of sets $\bar{Q}^i_j \subseteq K$ for $m \geq i \geq 1$ and $L \geq j \geq 1$ such that the corresponding mapping $\Phi^D : V \rightarrow R^{mL}_+$ satisfies the following two properties:*

(i) $d(u,v) \geq ||\Phi^D(u) - \Phi^D(v)||_1$ *for all* $u, v \in V$

(ii) $d(u,v) \leq \alpha\ ||\Phi^D(u) - \Phi^D(v)||_1$ *for all* $u \in K$ *and* $v \in V$,

where $\alpha = c\ log|K|$ *for some constant c.*

3.2 Bounding the maximum concurrent flow value

Combining Corollary 2 and Corollary 4, we now bound the maximum concurrent flow value as follows:

$$\begin{aligned}
\kappa^* &= \frac{\sum_{\{u,v\}\in E} c_{\{u,v\}}\ dist(u,v)}{\sum_{k\in K}\sum_{v\in V} T_{[k,v]}\ dist(k,v)} \geq \frac{1}{\alpha} \times \frac{\sum_{\{u,v\}\in E} c_{\{u,v\}}\ ||\Phi^D(u) - \Phi^D(v)||_1}{\sum_{k\in K}\sum_{v\in V} T_{[k,v]}\ ||\Phi^D(k) - \Phi^D(v)||_1} \\
&= \frac{1}{\alpha} \times \frac{\sum_{i=1}^{m}\sum_{j=1}^{L}\left(\sum_{\{u,v\}\in E} c_{\{u,v\}}\ |d(u,\bar{Q}_j^i) - d(v,\bar{Q}_j^i)|\right)}{\sum_{i=1}^{m}\sum_{j=1}^{L}\left(\sum_{k\in K}\sum_{v\in V} T_{[k,v]}\ |d(k,\bar{Q}_j^i) - d(v,\bar{Q}_j^i)|\right)} \\
&\geq \frac{1}{\alpha} \times \frac{\sum_{\{u,v\}\in E} c_{\{u,v\}}\ |d(u,Q^*) - d(v,Q^*)|}{\sum_{k\in K}\sum_{v\in V} T_{[k,v]}\ |d(k,Q^*) - d(v,Q^*)|} \qquad (6)
\end{aligned}$$

for some set $Q^* \subseteq K$ such that $Q^* = Q_{j*}^{i*}$ for some $m \geq i^* \geq 1$ and $L \geq j^* \geq 1$. Note that, we have essentially bounded maximum concurrent flow value (from below) by a collection of cut ratios. We next bound it by the minimum cut ratio.

First, we assign indices to nodes in V so that $d(v_p, Q^*) \geq d(v_{p-1}, Q^*)$ for all $|V| \geq p \geq 2$, and let $x_p = d(v_p, Q^*)$. Next, we define $|V|$ nested sets $S_p = \{v \in V : x_v \leq x_p\}$ and the associated cuts $C_p = \{\{u,v\} \in E : |\{u,v\} \cap S_p| = 1\}$ and $T_p = \{(k,v) \in K \times V : |\{k,v\} \cap S_p| = 1\}$. we can now rewrite the summations in (6) as follows:

$$\begin{aligned}
\frac{1}{\alpha} \times \frac{\sum_{\{v_i,v_j\}\in E} c_{\{v_i,v_j\}}\ |x_i - x_j|}{\sum_{v_i\in K}\sum_{v_j\in V} T_{[v_i,v_j]}\ |x_i - x_j|} &= \frac{1}{\alpha} \times \frac{\sum_{p=2}^{|V|}(x_p - x_{p-1})\sum_{\{u,v\}\in C_p} c_{\{u,v\}}}{\sum_{p=2}^{|V|}(x_p - x_{p-1})\sum_{(k,v)\in T_p} T_{[k,v]}} \\
&\geq \frac{1}{\alpha} \times \frac{\sum_{\{u,v\}\in C_{p*}} c_{\{u,v\}}}{\sum_{(k,v)\in T_{p*}} T_{[k,v]}} \geq \frac{1}{\alpha}\ \rho^*
\end{aligned}$$

for some $p^* \in \{1, \ldots, |V|\}$. We have therefore shown that:

Theorem 5 *Given a multicommodity problem, let κ^* denote the maximum concurrent flow value, ρ^* denote the minimum cut ratio and k^* denote the cardinality of the minimal vertex cover of the associated demand graph. If $k^* > 1$, then*

$$c\ \lceil log\, k^* \rceil \ \geq\ \frac{\rho^*}{\kappa^*}\ \geq\ 1$$

for some constant c.

3.3 A Tight Example

We next formally state that there are problem instances for which the upper bound on the min-cut max-flow ratio is tight, up to a constant. This result is a relatively straight forward extension of [10] and we include it in here for the sake of completeness.

Lemma 6 *For any given $n, k^* \in Z_+$ with $n \geq k^*$, it is possible to construct an instance of the multicommodity problem with n nodes and k^* (minimal) aggregate commodities such that*

$$\frac{\rho^*}{\kappa^*} \quad \geq c \ \lceil log\, k^* \rceil$$

for some constant c.

Proof. We start with constructing a bounded-degree expander graph G^{k^*} with k^* nodes and $O(k^*)$ edges. See, for example, [14] for a definition, and existence of constant degree expander graphs. As discussed in [10], these graphs (with unit capacity for all edges and unit flow requirement between all pairs of vertices) provide examples with $\rho^*/\kappa^* \quad \geq c \ \lceil log\, k^* \rceil$ for some constant c. Note that the demand graph is complete and therefore any minimal vertex cover has size k^*.

We next augment G^{k^*} by adding $n - k^*$ new vertices and $n - k^*$ edges. Each new vertex has degree one and is connected to an arbitrary vertex of G^{k^*}. The new edges are assigned arbitrary capacities. The augmented graph, with the original flow requirements, has n nodes and satisfies $\rho^*/\kappa^* \quad \geq c \ \lceil log\, k^* \rceil$. ∎

4 Geometric interpretation

Both of the more recent studies (namely, [11] and [1]) that relate the min-cut max-flow ratio to the number of origin-destination pairs in the problem instance, take a geometric approach and base their results on the fact that a finite metric space can be mapped into a Euclidean space with logarithmic distortion. More precisely, they base their analysis on the following result that shows that n points can be mapped from l_∞^n to l_1^p with $O(log\, n)$ distortion (where l_b^a denotes R^a equipped with the norm $||x||_b = (\sum_{i=1}^a |x_i|^b)^{1/b}$).

Lemma 7 ([3], also see [11]) *Given n points $x_1, \ldots, x_n \in R^n$, there exists a mapping Φ : $R^n \rightarrow R^p$, with $p = O(log\, n)$, that satisfies the following two properties:*

$$
\begin{array}{llcll}
(i) & ||x_i - x_j||_\infty & \geq & ||\Phi(x_i) - \Phi(x_j)||_1 & \textit{for all } i, j \leq n \\
(ii) & ||x_i - x_j||_\infty & \leq \alpha & ||\Phi(x_i) - \Phi(x_j)||_1 & \textit{for all } i, j \leq n
\end{array}
$$

where $\alpha = c \; log\, n$ *for some constant* c. ∎

Using this result, it is possible to map the optimal dual solution of the disaggregated (one commodity for each source-sink pair) formulation to l_1^p with logarithmic distortion, see [11] and [1]. One can then show a $O(log\, k)$ bound by using arguments similar to the ones presented in Section 3.2.

We next give a geometric interpretation of Corollary 4 in terms of mapping n points from l_∞^m to l_1^p with logarithmic distortion with respect to a collection of "seed" points..

Lemma 8 *Given* n *points* $x_1, \ldots, x_n \in R^m$, *the first* $t \leq n$ *of which are special,* $t > 1$, *there exists a mapping* $\Phi : R^m \to R^p$ *with* $p = O(log\, n)$, *that satisfies the following two properties:*

$$
\begin{array}{llcll}
(i) & ||x_i - x_j||_\infty & \geq & ||\Phi(x_i) - \Phi(x_j)||_1 & \textit{for all } i, j \leq n \\
(ii) & ||x_i - x_j||_\infty & \leq \alpha & ||\Phi(x_i) - \Phi(x_j)||_1 & \textit{for all } i \leq t, \quad j \leq n
\end{array}
$$

where $\alpha = c \; log\, t$ *for some constant* c.

Proof. Let $G = (V, E)$ be a complete graph with n nodes where each node v_i is associated with point x_i for $i = 1, \ldots, n$. For $e = \{v_i, v_j\} \in E$, let $w_e = ||x_i - x_j||_\infty$ be the edge weight. Furthermore, let $d(u, v)$ denote the shortest path length between nodes $u, v \in V$ using w as edge weights. Note that

$$||x_i - x_j||_\infty \leq ||x_i - x_k||_\infty + ||x_k - x_j||_\infty$$

for any $i, j, k \leq n$ and therefore $d(v_i, v_j) = ||x_i - x_j||_\infty$ for all $i, j \leq n$. We can now use Corollary 4 to show the existence of a mapping $\Phi' : R^m \to R^q$ with $q = O(log\, n\, log\, t)$ that satisfies the desired properties.

To decrease the dimension of the image space, we scale Φ' by $\sqrt{Lm}$ to map the points $x_1, \ldots, x_n$ to l_2^q with $c' \; log\, t$ distortion. More precisely, we use $\Phi'' : R^m \to R^q$ where $\Phi''(x) = \sqrt{Lm}\; \Phi'(x)$. It is easy to see that:

$$
\begin{array}{ll}
(i) & ||\Phi''(x_i) - \Phi''(x_j)||_2 \;\leq\; \sqrt{(1/Lm) \sum_{k=1}^{m} \sum_{q=1}^{L} d(v_i, v_j)^2} \;=\; d(v_i, v_j) \;=\; ||x_i - x_j||_\infty, \\
(ii) & ||\Phi''(x_i) - \Phi''(x_j)||_2 \;\geq\; ||\Phi'(x_i) - \Phi'(x_j)||_1 \;\geq\; c' \; log\, t \; d(v_i, v_j) \;=\; c' \; log\, t \; ||x_i - x_j||_\infty.
\end{array}
$$

We can now use the following two facts (also used in [11],) to reduce the dimension of the image space to $O(log\, n)$: (i) For any $q \in Z_+$, n points can be mapped from l_2^q to l_2^p, where

$p = O(\log n)$ with constant distortion (see [7]), and (ii) For any $p \in Z_+$, l_2^p can be embedded in l_1^{2p} with constant distortion(see [17], Chapter 6) . ∎

We also note that for Lemma 8 (and Lemma 7), the mapping Φ actually satisfies: $||x' - x''||_\infty \geq ||\Phi(x') - \Phi(x'')||_1$ for all $x', x'' \in R^m$ (R^n).

5 Maximum multicommodity flows

The *"maximum multicommodity flow"* problem is a generalization of the (single commodity) maximum flow problem. Given an undirected graph $G = (V, E)$ with edge capacities c_e for $e \in E$, the objective here is to maximize the sum of flows that can be simultaneously sent between given pairs of vertices $\{s_q, v_q\}$, $q \in Q$. For this problem, the generalization of the minimum cut is the so-called minimum multicut, which is a collection of edges (of minimum total capacity) that separates s_q from v_q for all $q \in Q$.

We next present two formulations for this problem and describe new bounds on the ratio of the minimum multicut capacity to the maximum multicommodity flow.

5.1 The maximum multicommodity flow problem

As in Section 2.2, let $K^* \subseteq V$ be a minimal vertex cover of the demand graph $G^T = (V, E^T)$ where $E^T = \{\{s_q, v_q\} \in V \times V \ : \ q \in Q\}$ and let $T_k = \{v \in V \ : \ \{k, v\} \in E^T\}$ denote the set of sink nodes for $k \in K^*$. The problem can be formulated as follows:

$$
\begin{aligned}
&Maximize: \quad \sum_{k \in K^*} \sum_{j \in T_k} x_j^k \\
&Subject\ to: \\
&\sum_{v:\{j,v\} \in E} f_{jv}^k - \sum_{v:\{v,j\} \in E} f_{vj}^k + x_j^k \leq 0 \quad for\ all\ j \in V,\ k \in K^*\ with\ k \neq j \\
&\sum_{k \in K} \left(f_{jv}^k + f_{vj}^k \right) \leq c_{\{j,v\}} \quad for\ all\ \{j,v\} \in E \\
&x_j^k \geq 0, \quad f_{jv}^k \geq 0 \quad for\ all\ k \in K^*,\ and\ \{j,v\} \in E
\end{aligned}
$$

where, variable f_{jv}^k denotes the flow of commodity k from node j to node v and x_j^k denotes the total flow of commodity k that terminates at node j. The dual of this formulation is:

$$
\begin{aligned}
&\textit{Minimize} \quad \sum_{\{j,v\}\in E} c_{\{j,v\}}\; w_{\{j,v\}} \\
&\textit{Subject to} \\
&\qquad \left.\begin{aligned} y_v^k - y_j^k + w_{\{j,v\}} &\geq 0 \\ y_j^k - y_v^k + w_{\{j,v\}} &\geq 0 \end{aligned}\right\} \; \textit{for all } k \in K^*, \textit{ and } \{j,v\} \in E \\
&\qquad y_j^k \geq \begin{cases} 1 & \textit{for all } k \in K^*,\ j \in T_k \\ 0 & \textit{for all } k \in K^*,\ j \in V \setminus T_k \end{cases} \\
&\qquad y_k^k = 0 \quad \textit{for all } k \in K^* \\
&\qquad w_{\{j,v\}} \geq 0 \quad \textit{for all } \{j,v\} \in E
\end{aligned}
$$

where variable y_j^k can be interpreted as the shortest path distance from node k to node j using w as edge weights. Note that any feasible solution to the dual problem assigns weights to the edges in such a way that the shortest path distance from any $k \in K^*$ to any one of its sink nodes is at least 1.

We next state a $O(log\, k^*)$ bound on the associated min-cut max-flow ratio. This improves the previous best known bound of $O(log\, k)$, (where k denotes the number of origin-destination pairs) presented in Garg, Vazirani and Yannakakis [5].

Lemma 9 *Given a maximum multicommodity flow problem, let F^* denote the maximum total flow, $C(\Delta^*)$ denote the capacity of the minimum multicut and k^* denote the cardinality of the minimal vertex cover of the associated demand graph. If $k^* > 1$, then*

$$
c\,\lceil log\, k^* \rceil \;\; \geq \;\; \frac{C(\Delta^*)}{F^*} \;\; \geq \;\; 1
$$

for some constant c.

Proof. Clearly capacity of any multicut is an upper bound on the total flow implying $C(\Delta^*)/F^* \;\geq\; 1$. For the upper bound, we use the algorithm presented in Garg, Vazirani and Yannakakis [5] with the input set $V' = K^*$ and an optimal dual solution vector w^*. Given edge weights w, this (constructive) algorithm produces a multicut that separates any $k \in V'$ from vertices that have a shortest path distance of 1, or more from k. The multicut is guaranteed to have a capacity of at most $c\,\lceil log\,|V'| \rceil\,(\sum_{\{j,v\}\in E} c_{\{j,v\}}\; w_{\{j,v\}})$ for some constant c. In [5], the authors use this algorithm with $V' = \{s_q \;:\; q \in Q\}$ to prove a $log(k)$ bound. ∎

Also note that, if the w variables in the dual linear program are required to be integral, any feasible (integral) solution to the dual problem gives a multicut for the maximum multicommodity flow problem and the optimal solution gives a minimum multicut. Therefore, Lemma 9 implies that the integrality gap of this formulation of the minimum multicut problem is bounded by a factor of $O(log\, k^*)$.

5.2 A reformulation of the maximum multicommodity flow problem

A more compact formulation of the maximum multicommodity flow problem (i.e. a formulation with fewer variables) can be obtained by allowing a commodity to have multiple source nodes in addition to multiple sink nodes.

We define a *bipartite graph cover* of a graph to be a collection of subgraphs of the graph that satisfy the following two properties: (i) each subgraph is a complete bipartite graph, (ii) the edges of the subgraphs cover the edges of the graph. Notice that bipartite graph cover is a generalization of the vertex cover in the sense that given a vertex cover K, one one can construct a bipartite graph cover $\mathcal{B}$ with $|K| = |\mathcal{B}|$. We next formulate the problem using a bipartite graph cover of the demand graph.

As in Section 5.1, let $G^T = (V, E^T)$ be the demand graph where $E^T = \{\{s_q, v_q\} \in V \times V \ : q \in Q\}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{|\mathcal{B}|}\}$ be a bipartite graph cover of G^T where $B_k = (S_k, T_k, E_k)$ is a complete bipartite graph with $S_k, T_k \subseteq V$, $E_k \subseteq E^T$, and $\cup_k E_k = E^T$.

In the following reformulation, source nodes of a "commodity" k is denoted by S_k, and sink nodes by T_k. Let $B^* = \{1, 2, \ldots, |\mathcal{B}|\}$ be the index set for commodities.

$$Maximize: \quad \textstyle\sum_{k \in B^*} \sum_{j \in T_k} x_j^k$$

$$Subject\ to:$$

$$\begin{array}{lcll} \displaystyle\sum_{v:\{j,v\} \in E} f_{jv}^k - \sum_{v:\{v,j\} \in E} f_{vj}^k + x_j^k & \leq & 0 & for\ all\ \ k \in B^*,\ j \in V \setminus S_k \\ \displaystyle\sum_{B_k \in B^*} \left(f_{jv}^k + f_{vj}^k\right) & \leq & c_{\{j,v\}} & for\ all\ \{j, v\} \in E \\ x_j^k \geq 0, \quad f_{jv}^k & \geq & 0 & for\ all\ k \in B^*,\ and\ \{j, v\} \in E \end{array}$$

where, variable f_{jv}^k denotes the flow of commodity k from node j to node v and x_j^k denotes the total flow of commodity k that terminates at node j. Given an aggregate flow vector f, it is easy to find disaggregated flows by tracing each unit of x_j^k from node $j \in T_k$ to some $v \in S_k$. The disaggregation is not necessarily unique.

The dual of this formulation is:

$$
\begin{array}{llll}
\textit{Minimize} & \displaystyle\sum_{\{j,v\}\in E} c_{\{j,v\}}\; w_{\{j,v\}} & & \\
\textit{Subject to} & & & \\
& y_v^k - y_j^k + w_{\{j,v\}} & \geq & 0 \\
& y_j^k - y_v^k + w_{\{j,v\}} & \geq & 0
\end{array}
\left.\right\} \textit{for all } k \in B^*,\ \textit{and } \{j,v\} \in E
$$

$$
\begin{array}{lll}
y_j^k & \geq & \begin{cases} 1 & \textit{for all } k \in B^*,\ j \in T_k \\ 0 & \textit{for all } k \in B^*,\ j \in V \setminus T_k \end{cases} \\
y_j^k & = & 0 \quad \textit{for all } k \in B^*,\ j \in S_k \\
w_{\{j,v\}} & \geq & 0 \quad \textit{for all } \{j,v\} \in E
\end{array}
$$

where, variable y_v^k can be interpreted as the least shortest path distance between v and a member of S_k using w as edge weights.

If $|B^*| = 1$, the dual feasible set is integral (see [8], for example) and an optimal dual solution corresponds to a multicut of capacity equal to the maximum flow. It is also possible to see this by noticing that the problem can easily be transformed into a maximum flow problem with a single source node and a single sink node.

Based on this observation, we now relate the min-cut max-flow ratio to the size of the minimal bipartite graph cover of the demand graph.

Lemma 10 *Let F^* and $C(\Delta^*)$ be defined as in Theorem 9 and let $\mathcal{B}^*$ be a minimal bipartite graph cover of the demand graph in the sense that $k^{**} = |\mathcal{B}^*|$ is minimum, then*

$$k^{**} \quad \geq \quad \frac{C(\Delta^*)}{F^*}$$

Proof. Let $\mathcal{B}^* = \{B_1, B_2, \ldots, B_{k^{**}}\}$. We solve k^{**} maximum multicommodity problems, one for each $\mathcal{B}_i = \{B_i\}$, and obtain the maximum flow value F_i^* and the corresponding multicut Δ_i. Clearly, $F^* \geq F_i^* = C(\Delta_i)$, and $\sum_{i=1}^{k^{**}} C(\Delta_i) \geq C(\Delta^*)$. We can therefore write:

$$k^{**} \quad \geq \quad \sum_{i=1}^{k^{**}} \frac{F_i^*}{F^*} \quad = \quad \sum_{i=1}^{k^{**}} \frac{C(\Delta_i)}{F^*} \quad \geq \quad \frac{C(\Delta^*)}{F^*}.$$

∎

Depending on the problem instance, Lemma 10 can provide a tighter bound than Lemma 9. For example, consider an instance where $S_1 \subseteq V$, $S_2 = V \setminus S_1$ with $|S_1| = |S_2| = n/2$ and $E^T = \{\{s, v\} \in V \times V : s \in S_1,\ v \in S_2\}$. For this problem instance, the number of source-sink pairs is $k = n^2/4$, the size of the minimal vertex cover of G^T is $k^* = n/2$ and the size of the minimal bipartite graph cover of G^T is $k^{**} = 1$.

A remaining open question is whether or not one can show a $O(log\, k^{**})$ bound on the min-cut max-flow ratio for the maximum multicommodity flow problem. We were unable to prove or disprove such a bound.

6 Conclusion

In this paper we presented improved bounds on the min-cut max-flow ratio for the multicommodity flow and the maximum multicommodity flow problems. Our bounds are motivated by "compact" linear programming formulations based on covers of the demand graph. For both problems, our results suggest that the quality of the ratio depends on the the demand graph in a more structural way than the size of the edge set (i.e. number of origin-destination pairs).

To extend our approach to directed versions of the (maximum) multicommodity flow problems, one needs to find minimal covers of the "directed" demand graph in the following sense: The demand graph now has two nodes v' and v'' for each original node $v \in V$, and it has an undirected edge $\{u', v''\}$ if there is a flow requirement from node u to node v. A cover $\mathcal{C}$ of this undirected bipartite graph gives a linear programming formulation with $|\mathcal{C}|$ aggregate commodities and therefore provides a $|\mathcal{C}|$ bound on the min-cut max-flow ratio. Relating this ratio logarithmically to the number of aggregate commodities is an open problem.

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References

[1] Y. Aumann and Y. Rabani (1998): "Approximate min-cut max-flow theorem and approximation algorithm", *SIAM Journal on Computing*, 27, pp. 291–301.

[2] D. Bienstock, and O. Günlük (1995): "Computational experience with a difficult multicommodity flow problem", *Mathematical Programming*, 68, pp. 213–238.

[3] J. Bourgain (1985): "On Lipschitz embedding of finite metric spaces in Hilbert space", *Israel Journal of Mathematics*, 52, pp. 46–52.

[4] L. R. Ford, Jr. and D. R. Fulkerson (1962): Flows in Networks, Princeton University Press.

[5] N. Garg, V.V. Vazirani, and M. Yannakakis (1993): "Approximate max-flow min-(multi)cut theorems and their applications" in Proceedings of the 25th Annual ACM Symposium on Theory of Computing, pp. 698–707.

[6] M. Iri (1971): "On an extension of the max-flow min-cut theorem to multicommodity flows", *Journal of the Operations Research Society of Japan* 13, pp. 129–135.

[7] W. Johnson and J. Lindenstrauss (1984): "Extensions of Lipschitz mappings into a Hilbert space" *Contemporary Mathematics*, 26, pp. 189–206.

[8] A. V. Karzanov (1989): "Polyhedra related to undirected multicommodity flows", *Linear Algebra and its Applications*, 114-115, pp. 293–328.

[9] P. Klein, A. Agrawal, R. Ravi, and S. Rao (1990): "Approximation through multicommodity flow", in Proceedings of the 31st Annual IEEE Symposium on Foundations of Computer Science, pp. 726–737.

[10] F.T. Leighton and S. Rao (1998): "An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms", in Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science, pp. 422–431.

[11] N. Linial, E. London, and Y. Rabinovich (1995): "The geometry of graphs and some of its algorithmic applications", *Combinatorica*, 15, pp. 215–245.

[12] C. Lund, S. Phillips and N. Reingold, AT&T Research, Private communication.

[13] D. W. Matula (1985): "Concurrent flow and concurrent connectivity in graphs", in Graph Theory and its Applications to Algorithms and Computer Science, Wiley, New York, pp. 543–559.

[14] R. Motwani and P. Raghavan (1995): Randomized Algorithms, Cambridge University Press, Cambridge.

[15] H. Okamura and P. Seymour (1981): "Multicommodity flows in planar graphs", *Journal of Comput. Theory – Ser. B*, 31, pp. 75–81.

[16] K. Onaga, and O. Kakusho (1971): "On feasibility conditions of multicommodity flows in networks", *IEEE Transactions on Circuit Theory*, CT-18, 4, pp. 425–429.

[17] G. Pisier (1989): The Volume of Convex Bodies and Banach Space Geometry. Cambridge University Press, Cambridge.

[18] S. Plotkin and E. Tardos (1993): "Improved Bounds on the max-flow min-cut ratio for multi-commodity flows", *Proceedings 25'th Symposium on Theory of Computing* .

[19] F. Shahrokhi, and D. W. Matula (1990): "The maximum concurrent flow problem", *Journal of Association for Computing Machinery*, 37, pp. 318–334.

[20] S. Tragoudas (1996): " Improved approximations for the min-cut max-flow ratio and the flux", *Mathematical Systems Theory*, 29, pp. 157–167.