Stable Multi-Sets

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Abstract

In this paper we introduce a generalization of stable sets: stable multi-sets. A stable multi-set is an assignment of integers to the vertices of a graph, such that specified bounds on vertices and edges are not exceeded. In case all vertex and edge bounds equal one, stable multi-sets are equivalent to stable sets.

For the stable multi-set problem, we derive reduction rules and study the associated polytope. We state necessary and sufficient conditions for the extreme points of the linear relaxation to be integer. These conditions generalize the conditions for the stable set polytope. Moreover, the classes of odd cycle and clique inequalities for stable sets are generalized to stable multi-sets and conditions for them to be facet defining are determined.

The study of stable multi-sets is initiated by optimization problems in the field of telecommunication networks. Stable multi-sets emerge as an important substructure in the design of optical networks.

1 Introduction

Stable sets, also referred to as independent sets, cocliques, or set packings, are among the most studied structures in graph theory/combinatorial optimization. A stable set in a graph is a subset of the vertices, where no two vertices are adjacent. A generalization of the notion of stability in a graph is obtained by considering multi-sets of vertices, where the multiplicities of the elements (vertices) in the multi-set are bounded by vertex and edge weights. This generalization is called *stable multi-set* here. The study of stable multi-sets is motivated by an optimization problem from telecommunication industry. In the design of optical networks, structures like stable multi-sets play an important role. For more information how stable multi-sets are used in the optimization of optical networks, we refer to a future paper on this topic [9].

From a theoretical point of view, stable multi-sets play a role in a generalized vertex coloring problem. In (standard) vertex coloring, stable sets correspond to sets of vertices that can be

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colored with the same color. This property can be used to solve vertex coloring problems via integer programming [10]. Now, consider the generalization of coloring where each vertex has to be colored multiple times. The colors for a vertex need not to be different, but the number of times the same color is allocated, is bounded by vertex and edge values. Then, a stable multi-set corresponds to a multi-set of vertices that can be colored with the same color. Hence, a column generation approach like the one of Mehrotra and Trick [10] can be applied.

This paper serves as a first step in the development of polyhedral theory for stable multi-sets. Since the special case of stable sets has been studied very well, we start in Section 2 with revisiting the most important results for the stable set polytope that are relevant with this paper. Next, in Section 3, we formally define stable multi-sets and the associated optimization problem. Section 4 is devoted to rules to reduce the problem complexity. Our main purpose is to study the associated stable multi-set polytope, which is the topic of Section 5. We not only show that the well-known cycle and clique inequalities of the stable set polytope can be generalized to the stable multi-set polytope, but also prove necessary and sufficient conditions under which the polytope defined by the linear relaxation is integer. The paper is concluded in Section 6 with a comparison of the results obtained for stable multi-sets to stable sets and directions for further research.

In this paper we use the following (standard) notation. An undirected graph $G = (V, E)$ consists of a set of vertices V and a set of edges E. Throughout this paper, we assume that all graphs are simple, i.e., contain no loops and no multiple edges. Instead of edge $\{v, w\} \in E$, we usually use the short form vw. Let $N(v)$ denote the set of neighbors of $v \in V$, i.e., $N(v) := \{w \in V : vw \in E\}$. Moreover, for $W \subseteq V$, let $N(W) := \{v \in V \setminus W : vw \in E, w \in W\}$ be the set of vertices that W separates from the rest of the graph. Given a subset of vertices $S \subset V$, the subgraph of G induced by S is denoted by $G[S]$. For $v \in V$, let $e^v \in \{0,1\}^{|V|}$ denote the vertex unit vector defined by $e^v_v = 1$ and $e^v_w = 0$ for all $w \in V \setminus \{v\}$.

2 Preliminaries: the Stable Set Polytope

For stable sets, wide knowledge about the associated polytope has been collected over the years. We refer to Borndörfer [1] for a survey of these results in full detail. Here, we restrict ourselves to those results which are related to the results derived in this paper for the stable multi-set polytope. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The stable set polytope $T(G)$ is defined by

$$T(G) = \mathrm{conv}(\{t \in \{0,1\}^{|V|} : t_v \geq 0 \ \forall v \in V, t_v + t_w \leq 1 \ \forall vw \in E\}).$$

First of all, the polytope $T(G)$ is full dimensional and the *non-negativity constraints* $t_v \geq 0$ induce facets of it. Padberg [12] proved that the non-negativity constraints plus the *edge inequalities* $t_v + t_w \leq 1$ suffice to describe the stable set polytope completely if and only if the graph does not contain an odd cycle (i.e., G is bipartite) and no isolated vertices exist. Every odd cycle C leads to an *odd cycle inequality* $\sum_{v \in C} t_v \leq \lfloor |C|/2 \rfloor$ (cf. Padberg [11]). This inequality defines a facet of the polytope $T(G[C])$ if and only if the subgraph $G[C]$ is a hole. Grötschel et al. [8] showed that odd cycle inequalities can be separated in polynomial time.

Chvátal [3] introduced the concept of t-perfectness of a graph. A graph is t-perfect if the associated stable set polytope is completely described by the model inequalities and the odd cycle inequalities. By the *optimization = separation* theorem of Grötschel et al. [7], it follows that the stable set problem can be solved in polynomial time for t-perfect graphs. A full characterization of t-perfect graphs, however, is unknown. The class at least contains all graphs that do not contain a subdivision of K_4 such that all four cycles corresponding to triangles in K_4 are odd, i.e., the class of *strongly t-perfect* graphs (see Gerards and Schrijver [6]).

A *clique* in a graph G is a set Q of mutually adjacent vertices. Fulkerson [4] and Padberg [11] proved that the *clique inequality* $\sum_{v\in Q} t_v \leq 1$ defines a facet of $T(G)$ if and only if the clique is maximal. In case G is complete, Chvátal [2] showed that the maximum clique inequality defines the only non-trivial facet. The class of *perfect* graphs is defined by those graphs for which the stable set polytope is completely described by the non-negativity constraints and the clique inequalities. Although separation of clique inequalities is, in general, $\mathcal{NP}$-complete [5], the stable set problem for perfect graphs can be solved in polynomial time by the polynomial time separability of the *orthogonality* inequalities, a class of inequalities that contains the clique inequalities [8].

3 Problem Description

The concept of stable sets can be generalized by the introduction of (integer) weights on the vertices and edges of the graph. Now, we do not longer decide whether or not a vertex belongs to the (stable) set, but decide the number of times a vertex is in the (stable) *multi-set.*

Definition 3.1 *A multi-set S is a pair (V, t) consisting of a ground set V and a function $t : V \to \mathbb{Z}_0^+$, where $t(v)$ denotes the multiplicity of $v \in V$ in the multi-set S.*

In other words, multi-sets are sets that allow for the repetition of elements.

Definition 3.2 *Given a graph $G = (V, E)$, non-negative integers α_v associated with every vertex $v \in V$, and non-negative integers β_{vw} associated with every edge $vw \in E$, a stable multi-set (SMS) is a multi-set S defined by the multiplicity function $t : V \to \mathbb{Z}_0^+$ such that $t(v) \leq \alpha_v$ for all $v \in V$ and $t(v) + t(w) \leq \beta_{vw}$ for all $vw \in E$.*

Consider the example in Figure 1. At every vertex α_v is given, whereas β_{vw} is displayed at the edges. Possible stable multi-sets are for example $(0, 1, 4, 2, 6)$, $(2, 2, 3, 4, 4)$ or $(1, 3, 3, 3, 4)$.

The special case with $\alpha_v = 1$ for all $v \in V$ and $\beta_{vw} = 1$ for all $vw \in E$ represents the traditional stable set. Usually, we search for a stable set that is maximum with respect to some weight function. Hence, given weights c_v for all $v \in V$, the (weighted) stable multi-set problem (short: SMS problem) can be defined as follows.

<u>STABLE MULTI-SET PROBLEM</u>

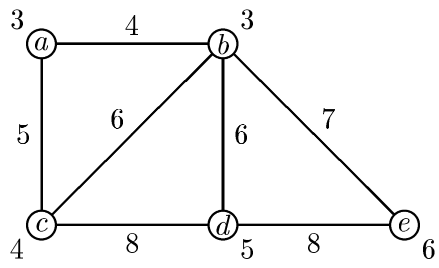

Figure 1: Example of a stable multi-set instance.

Input: A graph $G = (V, E)$, non-negative integers α_v and c_v associated with every vertex $v \in V$, non-negative integers β_{vw} associated with every edge $vw \in E$, and a positive integer $K \leq \sum_{v \in V} c_v \alpha_v$.

Question: Does there exist non-negative integers t_v such that $t_v \leq \alpha_v$ for all $v \in V$, $t_v + t_w \leq \beta_{vw}$ for all $vw \in E$, and $\sum_{v \in V} c_v t_v \geq K$?

For the example of Figure 1 and weight function $c = (1, 1, 1, 1, 1)$, the solution $(2, 2, 3, 4, 4)$ belongs to the set of optimal solutions (with value 15). Since the maximum (weighted) stable set problem is well known to be $\mathcal{NP}$-hard and occurs as a special case, the maximum (weighted) stable multi-set problem is also $\mathcal{NP}$-hard.

We can formulate the SMS problem with integer variables t_v as follows:

$$z = \max \sum_{v \in V} c_v t_v \tag{1}$$

$$\text{s.t.} \quad t_v + t_w \leq \beta_{vw} \qquad \forall vw \in E \tag{2}$$

$$t_v \leq \alpha_v \qquad \forall v \in V \tag{3}$$

$$t_v \in \mathbb{Z}_0^+ \qquad \forall v \in V \tag{4}$$

In the sequel, we refer to the inequalities (2) as the *edge inequalities*, whereas the *vertex inequalities* are given by (3). Together with the *non-negativity inequalities* $t_v \geq 0$, we will refer to them as the *model inequalities*.

In this paper, we primarily study the polytope associated with this formulation. However, by basic preprocessing it might be possible to reduce the graph and some of the bounds, and thus, to simplify the problem formulation. So we first point out these reduction rules before analyzing the polyhedral structure of (not further reducible) stable multi-set problems.

4 Reduction Rules for Stable Multi-Set Problems

We consider a given stable multi-set problem on a graph $G = (V, E)$ with vertex bounds α_v for all $v \in V$, edge bounds β_{vw} for all $vw \in E$, and objective values c_v for all $v \in V$. First of all we check whether $c_v > 0$ for all $v \in V$, since $c_v \leq 0$ implies $t_v = 0$, and the vertex can be removed from the problem. Secondly, we assume that $\alpha_v > 0$ for all $v \in V$ and $\beta_{vw} > 0$ for all $vw \in E$, because otherwise vertex v or both vertices v and w, respectively, can be deleted from the graph (with the appropriate vertex values fixed to 0). Next, if $\beta_{vw} \geq \alpha_v + \alpha_w$, inequality (2) will always be fulfilled so, edge vw can be removed from the instance. If $\alpha_v > \beta_{vw}$, then α_v can be set to β_{vw}. Hence, without loss of generality $\alpha_v \leq \min_{w \in N(v)} \beta_{vw}$. As a consequence, we may assume in the sequel that $0 < \max\{\alpha_v, \alpha_w\} \leq \beta_{vw} < \alpha_v + \alpha_w$ for all $vw \in E$.

Now, we can specify lower bounds on the variable values in any optimal solution independent of the objective function. Let $\gamma_v := \min_{w \in N(v)}\{\beta_{vw} - \alpha_w\}$ for $v \in V$.

Lemma 4.1 *Any optimal solution t^* meets $t^*_v \geq \gamma_v$ for all $v \in V$.*

In case there exist $\gamma_v > 0$, the values of the vectors α and β can be reduced using the lower bounds provided by Lemma 4.1 in order to simplify the problem representation.

Corollary 4.2 *For a graph $G = (V, E)$, let $P = (G, \alpha, \beta, c)$ define an SMS problem and set $\gamma_v = \min_{w \in N(v)}\{\beta_{vw} - \alpha_w\}$ for all $v \in V$. On the same graph, let $P' = (G, \alpha', \beta', c')$ be the SMS problem defined by $\alpha'_v = \alpha_v - \gamma_v$ for all $v \in V$, $\beta'_{vw} = \beta_{vw} - \gamma_v - \gamma_w$ for all $vw \in E$, and $c'_v = c_v$ for all $v \in V$.*
Then each optimal solution t' for P' corresponds to an optimal solution t for P by $t_v = t'_v + \gamma_v$. Moreover, if z' is the optimal value of P', then $z = z' + \sum_{v \in V} c_v \gamma_v$ is the optimal value for P.

Note that all previous assumptions are unaffected by this transformation, i.e., the condition $0 < \max\{\alpha'_v, \alpha'_w\} \leq \beta'_{vw} < \alpha'_v + \alpha'_w$ holds for all $vw \in E$. Furthermore, the reduced SMS problem P' can not be reduced again (in the same way), since $\gamma'_v = \min_{w \in N(v)}\{\beta'_{vw} - \alpha'_w\} = 0$ for all $v \in V$.

For the SMS problem of Figure 1, the reduced problem is given by Figure 2. In this case, $\gamma = (1, 1, 2, 2, 3)$.

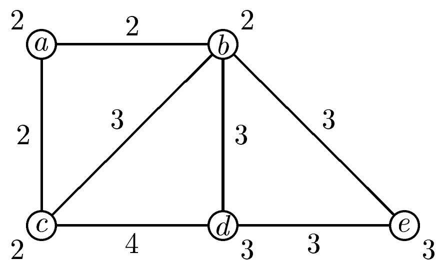

Figure 2: Reduced SMS problem of Figure 1.

Since the described transformation is straightforward and may reduce the problem size (or at least some values), it suffices to analyze the problem only for instances which are not further reducible this way.

5 The Stable Multi-Set Polytope

In this section, we study the stable multi-set polytope. In contrast to the well-studied stable set polytope, to our knowledge only Gerards and Schrijver [6] discuss a related polytope (in the context of the Edmonds-Johnson property).

An SMS polytope is defined by the triple (G, α, β) with $G = (V, E)$. We assume that none of the reduction rules described in the previous section can be applied anymore, i.e., we have $0 < \max\{\alpha_v, \alpha_w\} \leq \beta_{vw} < \alpha_v + \alpha_w$ for all $vw \in E$ and $\gamma_v = \min_{w \in N(v)}\{\beta_{vw} - \alpha_w\} = 0$ for all $v \in V$. For a graph $G = (V, E)$ and vectors $\alpha \in \mathbb{Z}_+^{|V|}$ and $\beta \in \mathbb{Z}_+^{|E|}$, let $T(G, \alpha, \beta)$ denote the set of all solutions to (2)–(4), i.e.,

$$T(G, \alpha, \beta) = \{t \in \mathbb{Z}^{|V|} : 0 \leq t_v \leq \alpha_v \ \forall v \in V,\ t_v + t_w \leq \beta_{vw} \ \forall vw \in E\}.$$

The convex hull of this set is denoted by

$$T_{IP}(G, \alpha, \beta) = \mathrm{conv}(T(G, \alpha, \beta)).$$

Moreover, by $T_{LP}(G, \alpha, \beta)$ we denote the polytope described by the linear relaxation of inequalities (2)–(4):

$$T_{LP}(G, \alpha, \beta) = \{t \in \mathbb{R}^{|V|} : 0 \leq t_v \leq \alpha_v \forall v \in V, t_v + t_w \leq \beta_{vw} \forall vw \in E\}$$

If there is no danger of confusion, we use T, T_{IP}, and T_{LP} as short version of $T(G, \alpha, \beta)$, $T_{IP}(G, \alpha, \beta)$, and $T_{LP}(G, \alpha, \beta)$, respectively.

5.1 Dimension and Trivial Facets

Let us now start the study of the SMS polytope T_{IP}. First of all, we state the dimension and the trivial facets of T_{IP}. Let $n = |V|$ denote the number of vertices.

Proposition 5.1 *(i) The dimension of T_{IP} equals the number of vertices,* $\dim(T_{IP}) = n$.

(ii) For all $v \in V$, the non-negativity inequality $t_v \geq 0$ defines a facet of T_{IP}.

(iii) For $v \in V$, the vertex inequality $t_v \leq \alpha_v$ defines a facet of T_{IP} if and only if $\beta_{vw} > \alpha_v$ for all $w \in N(v)$.

Proof.

(i) Since $\alpha_v > 0$ for all $v \in V$, T contains all unit vectors e^v for $v \in V$. Moreover, $0 \in T$. Obviously, these vectors are affinely independent.

(ii) Except for e^v, all vectors of the previous part satisfy $t_v = 0$ at equality.

(iii) We define n affinely independent solutions that satisfy $t_v = \alpha_v$. Let the solution t^v be defined by $t^v_v = \alpha_v$, $t^v_w = 0$ for $w \neq v$. Next, for all $w \in V$, $w \neq v$, we can define the solution t^w by $t^w_v = \alpha_v$, $t^w_w = 1$, $t^w_u = 0$ for $u \notin \{v, w\}$. These solutions are valid since $\alpha_w > 0$ and $\beta_{vw} > \alpha_v$ for $w \in N(v)$. Moreover, they are affinely independent.

In case there exists $w \in N(v)$ with $\beta_{vw} = \alpha_v$, then $t_v = \alpha_v$ implies $t_w = 0$. This yields $\dim(\{t \in T_{IP} : t_v = \alpha_v\}) \leq n - 2$, i.e., $t_v \leq \alpha_v$ can not define a facet of T_{IP}. ■

Completing the facets defined by the model inequalities, we state a similar result for the edge inequalities.

Proposition 5.2 *For $vw \in E$, the edge inequality $t_v + t_w \leq \beta_{vw}$ defines a facet of T_{IP} if and only if for all $u \in N(\{v,w\})$, there exist integers $\bar{x}_v \leq \alpha_v$ and $\bar{x}_w \leq \alpha_w$ with $\bar{x}_v + \bar{x}_w = \beta_{vw}$, $\bar{x}_v < \beta_{vu}$ if $u \in N(v) \setminus \{w\}$, and $\bar{x}_w < \beta_{wu}$ if $u \in N(w) \setminus \{v\}$.*

Proof. If $t_v + t_w \leq \beta_{vw}$ defines a facet of T_{IP}, there exist n affinely independent solutions of T_{IP} on the appropriate hyperplane. For each $u \in N(\{v,w\})$, there is at least one solution t^u with $t^u_u > 0$. Then $\bar{x}_v = t^u_v$ and $\bar{x}_w = t^u_w$ are integers with $\bar{x}_v + \bar{x}_w = \beta_{vw}$, and since $t^u_u > 0$, we have $\bar{x}_v \leq \beta_{uv} - t^u_u < \beta_{uv}$, if $u \in N(v) \setminus \{w\}$, and $\bar{x}_w \leq \beta_{uw} - t^u_u < \beta_{uw}$, if $u \in N(w) \setminus \{v\}$.

For sufficiency, we construct n affinely independent solutions satisfying $t_v + t_w = \beta_{vw}$. Since we have $\alpha_v + \alpha_w > \beta_{vw}$ by assumption, there must be integers $0 < \bar{x}_v \leq \alpha_v$ and $0 < \bar{x}_w \leq \alpha_w$ with $\bar{x}_v + \bar{x}_w = \beta_{vw}$. Then $t^1 = \bar{x}_v e^v + \bar{x}_w e^w$ is a solution of T_{IP}. Moreover, either $\bar{x}_v < \alpha_v$ or $\bar{x}_w < \alpha_w$. Without loss of generality, we assume $\bar{x}_v < \alpha_v$, and as a consequence, $\bar{x}_w = \beta_{vw} - \bar{x}_v > \beta_{vw} - \alpha_v \geq 0$ by the problem assumptions. Thus, the vector $t^2 = (\bar{x}_v + 1)e^v + (\bar{x}_w - 1)e^w$ is another solution of T_{IP}.

Now, let $u \in V \setminus \{v,w\}$ be arbitrarily chosen. If $u \notin N(\{v,w\})$, then we define $t^u = t^1 + e^u$ which is clearly also a solution of T_{IP}. If $u \in N(\{v,w\})$, then the solution $t^u = \bar{x}_v e^v + \bar{x}_w e^w + e^u$ is feasible for the properties of $\bar{x}_v$ and $\bar{x}_w$.

Summarized, the vectors t^1, t^2, and t^u for all $u \in V \setminus \{v,w\}$ are solutions of T_{IP} and satisfy $t_v + t_w \leq \beta_{vw}$ at equality. Since they are affinely independent, the edge inequality defines a facet of T_{IP}. ■

Note that in general, the inequalities $t_v \leq \alpha_v$ and $t_v + t_w \leq \beta_{vw}$ do not define facets for the stable set polytope. In this special case, it follows from Proposition 5.2 that $t_v + t_w \leq 1$ is a facet if and only if the vertices do not have a common neighbor, i.e., vw is a maximal clique (cf. Section 2). In the sequel, we will refer to the facets of Propositions 5.1 and 5.2 as the trivial facets.

5.2 Polynomial solvable cases

Next, we derive conditions under which the model inequalities describe all facets of T_{IP}, i.e., the special case $T_{LP} = T_{IP}$. For the stable set polytope, the edge inequalities (2) suffice to describe T_{IP} completely if and only if the graph does not contain an odd cycle (i.e., G is bipartite) and no isolated vertices exist (cf. Section 2). Since, T_{LP} also includes the vertex inequalities (3), the condition that the graph does not contain isolated vertices can be dropped. Hence, for the stable set polytope, $T_{IP} = T_{LP}$ if and only if G is bipartite.

For the stable multi-set polytope, $T_{IP} = T_{LP}$ may also hold for non-bipartite graphs. To state conditions that are necessary and sufficient for $T_{IP} = T_{LP}$, we first have to define the cycle inequalities that are valid for T_{IP}. For a cycle C in G, let $E(C)$ denote the edges on the cycle. Moreover, let $\beta(C) = \sum_{vw \in E(C)} \beta_{vw}$ denote the sum of the edge values on the cycle. A cycle C is called even (odd), if $|C|$ is even (odd), and even-valued (odd-valued), if

$\beta(C)$ is even (odd). For a cycle C in G, the *cycle inequality* is defined by

$$\sum_{v \in C} t_v \le \left\lfloor \tfrac{1}{2}\beta(C) \right\rfloor . \tag{5}$$

It is easy to see that all cycle inequalities (5) are valid for T_{IP}. Moreover, every even cycle inequality cannot be violated by any solution $t \in T_{LP}$.

Lemma 5.3 *Let C be an even cycle. Then the cycle inequality* (5) *is satisfied by all solutions $t \in T_{LP}$.*

Proof. Let the edges of the cycle be consecutively indexed, beginning with an arbitrary edge. Then the inequality defined by the sum of the edge inequalities of either the *even-indexed* edges in the cycle or the *odd-indexed* edges has a right hand side that is at least as strong as the right hand side of (5). Thus, the cycle inequality (5) is satisfied by all solutions, since all of them satisfy all edge inequalities. ■

Also every even-valued odd cycle inequality cannot be violated by any $t \in T_{LP}$.

Lemma 5.4 *Let C be an even-valued odd cycle. Then the cycle inequality* (5) *is satisfied by all solutions $t \in T_{LP}$.*

Proof. The cycle inequality (5) can be obtained by summing up all edge inequalities (2) half. ■

An even stronger result can be proved: if all odd cycles are even-valued, then the polytope is completely described by the model inequalities. This bases on a result of Gerards and Schrijver [6]. A *Chvátal-Gomory cut* for a linear system $Ax \le b$ is an inequality $cx \le \lfloor \delta \rfloor$, where c is integer, $c = \lambda A$ and $\delta = \lambda b$ for some vector $\lambda \ge 0$ of appropriate size. The *Chvátal-Gomory closure* of a polytope defined by the system $Ax \le b$ is the polytope defined by all Chvátal-Gomory cuts that can be derived from the system $Ax \le b$.

Proposition 5.5 (Gerards and Schrijver [6]) *Let A be the vertex-edge incidence matrix of a graph $\bar{G} = (\bar{V}, \bar{E})$, and $b \in \mathbb{Z}^{|E|}$, then the Chvátal-Gomory closure of the polytope defined by $Ax \le b$ has the same solution set as the system*

$$Ax \le b, \qquad \sum_{v \in C} x_v \le \left\lfloor \tfrac{1}{2} b(C) \right\rfloor \qquad \text{for all odd cycles } C \text{ in } \bar{G},$$

where $b(C) = \sum_{v \in C} b_v$.

Theorem 5.6 *If all odd cycles C in G are even-valued, then $T_{IP} = T_{LP}$.*

Proof. We prove the theorem by showing that the *Chvátal rank* (cf. [2, 13]) of the polytope T_{LP} is 0. The Chvátal rank of a polytope is defined as the minimum number of times *all* Chvátal-Gomory cuts have to be added to the set of linear inequalities in order to obtain a complete description of the integer polytope. The Chvátal rank of T_{LP} equals 0 if and only if $T_{IP} = T_{LP}$. Otherwise stated, we demonstrate that every Chvátal-Gomory cut is equivalent or dominated by a linear combination of the model inequalities (2)–(3).

To do so, we apply Proposition 5.5 to a slightly modified graph $\bar{G}$ where the vertex bounds are replaced by edge bounds. We define the graph $\bar{G} = (\bar{V}, \bar{E})$ by

$$\begin{aligned} \bar{V} &= V \cup \{\bar{v}, \bar{\bar{v}}\}, \\ \bar{E} &= E \cup \{v\bar{v} : v \in V\} \cup \{v\bar{\bar{v}} : v \in V\} \cup \{\bar{v}\bar{\bar{v}}\}, \end{aligned}$$

and

$$b_{vw} = \begin{cases} \beta_{vw} & \text{if } vw \in E, \\ \alpha_v & \text{if } v \in V, w = \bar{v} \vee w = \bar{\bar{v}}, \\ 0 & \text{if } v = \bar{v}, w = \bar{\bar{v}}. \end{cases}$$

Then every solution of T corresponds to a solution of $X := \{x \in \mathbb{Z}^{n+2} : Ax \leq b, x \geq 0\}$, and vice versa (note that $Ax \leq b$ implies $x_{\bar{v}} \leq 0$ and $x_{\bar{\bar{v}}} \leq 0$). Let $X_{LP} := \{x \in \mathbb{R}^{n+2} : Ax \leq b, x \geq 0\}$. To verify Chvátal rank 0 for T_{IP}, it suffices to prove that the Chvátal-Gomory cuts of rank 1 are satisfied by all solutions $x^* \in X_{LP}$. By applying the result of Gerards and Schrijver, all rank 1 Chvátal-Gomory cuts are given by the inequalities for the odd cycles $\bar{C}$ in $\bar{G}$. Now, we distinguish three cases where $|\bar{C} \cap \{\bar{v}, \bar{\bar{v}}\}| \in \{0, 1, 2\}$. If $\bar{C}$ is an odd cycle in G (i.e., $|\bar{C} \cap \{\bar{v}, \bar{\bar{v}}\}| = 0$), then $b(C)$ is even by the assumptions stated in the theorem, and the inequality is satisfied by every solution $x^* \in X_{LP}$ (see Lemma 5.4). Now, suppose $|\bar{C} \cap \{\bar{v}, \bar{\bar{v}}\}| = 1$, then we may assume without loss of generality that $\bar{v} \in \bar{C}$, $\bar{\bar{v}} \notin \bar{C}$. Let $\bar{C} = \{v_1, \ldots, v_n, \bar{v}\}$ (with n even). Then the odd cycle inequality reads

$$\sum_{i=1}^{n} x_{v_i} \leq \left\lfloor \tfrac{1}{2} \left(\sum_{i=1}^{n-1} \beta_{v_i v_{i+1}} + \alpha_{v_1} + \alpha_{v_n} \right) \right\rfloor . \tag{6}$$

Now, let $\bar{\bar{C}}$ be an even cycle with $\bar{\bar{C}} = \{v_1, \ldots, v_n, \bar{v}, \bar{\bar{v}}\}$. The even cycle inequality for this cycle is identical to (6) due to the setting of b_{vw}. As a consequence, by Lemma 5.3, inequality (6) cannot be violated. Finally, if $\bar{v}, \bar{\bar{v}} \in \bar{C}$, a similar ideas are applied. If $\bar{v}\bar{\bar{v}} \in E(C)$, we construct an even cycle by removing either $\bar{v}$ or $\bar{\bar{v}}$ from the cycle. By the definition of b_{vw} both cycles result in the same cycle inequality, which cannot be violated by any $x^* \in X_{LP}$ by Lemma 5.3. Remains the case $\bar{v}\bar{\bar{v}} \notin E(C)$. Let $C = \{\bar{v}, v_1, \ldots, v_p, \bar{\bar{v}}, v_{p+1}, \ldots, v_{2k+1}\}$ for some $1 \leq p \leq 2k$. Without loss of generality we may assume that p is odd. Then the cycle inequality is the sum of two other inequalities defined on the cycles $\{\bar{v}, v_1, \ldots, v_p, \bar{\bar{v}}\}$ and $\{\bar{v}, v_{p+1}, \ldots, v_{2k+1}\}$. The latter is even, and thus a linear combination of edge inequalities. This completes the proof. ■

Note that the condition of Theorem 5.6 is not necessary for $T_{IP} = T_{LP}$. Consider the instance in Figure 3. Although, it contains an odd cycle with odd value (i.e., 9), the polytope T_{IP} is completely described by the model inequalities. This is because the odd cycle inequality $x_u + x_v + x_w \leq 4$ is dominated by the sum of the model inequalities $x_u \leq 1$ and $x_v + x_w \leq 3$. For odd-valued odd cycles, this example precisely gives the condition under which an odd cycle inequality (5) cannot be violated. To state this result, as well as to prove it, we will frequently denote the vertices of an odd cycle C by $\{v_1, \ldots, v_{2k+1}\}$. In

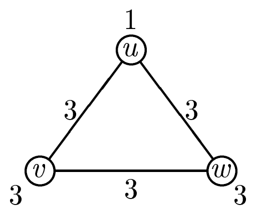

Figure 3: Example of an SMS instance with an odd-valued odd cycle and $T_{IP} = T_{LP}$.

general, the index j of a vertex v_j is calculated modulo $2k+1$, e.g., if $2k+1 < j \leq 4k+2$ then $j \equiv j - 2k - 1$. Moreover, we need the following lemma. The linear program in Lemma 5.7 determines the minimum right hand side of a cycle inequality (5) constructed as Chvátal-Gomory cut of the model inequalities.

Lemma 5.7 *Let $C = \{v_1, \ldots, v_{2k+1}\}$ be an odd cycle in G with $\beta(C)$ odd. Consider the linear program*

$$\min \quad z(\lambda, \mu) := \sum_{i=1}^{2k+1} \alpha_{v_i} \lambda_{v_i} + \sum_{i=1}^{2k+1} \beta_{v_i v_{i+1}} \mu_{v_i v_{i+1}} \tag{7}$$

$$s.t. \quad \lambda_{v_i} + \mu_{v_{i-1} v_i} + \mu_{v_i v_{i+1}} = 1 \qquad \forall i \in \{1, \ldots, 2k+1\} \tag{8}$$

$$\lambda_{v_i} \geq 0, \mu_{v_i v_{i+1}} \geq 0 \qquad \forall i \in \{1, \ldots, 2k+1\} \tag{9}$$

with optimal value z^*. If $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$, then there exists an integer vector (λ, μ) with $z(\lambda, \mu) = z^*$.

Proof. Let (λ, μ) be a solution with $z(\lambda, \mu) = z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$. Suppose $\lambda_{v_i} = 0$ for all $i = 1, \ldots, 2k+1$. Then the (unique) solution is given by $\mu_{v_i v_{i+1}} = \frac{1}{2}$ for all $i = 1, \ldots, 2k+1$ which contradicts $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$, since $\beta(C)$ is odd. So, at least one $\lambda_{v_i} > 0$. Next, suppose $\lambda_{v_i} \geq \lambda_{v_{i+1}} > 0$ for some $i \in \{1, \ldots, 2k+1\}$. Then the solution $(\bar{\lambda}, \bar{\mu})$ defined by

$$\bar{\lambda}_{v_j} = \begin{cases} \lambda_{v_i} - \lambda_{v_{i+1}} & \text{if } j = i, \\ 0 & \text{if } j = i+1, \\ \lambda_{v_j} & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_j v_{j+1}} = \begin{cases} \mu_{v_i v_{i+1}} + \lambda_{v_{i+1}} & \text{if } j = i, \\ \mu_{v_j v_{j+1}} & \text{otherwise,} \end{cases}$$

has objective value $z(\bar{\lambda}, \bar{\mu}) = z(\lambda, \mu) + \lambda_{v_{i+1}} (\beta_{v_i v_{i+1}} - \alpha_{v_i} - \alpha_{v_{i+1}}) < z(\lambda, \mu)$ by $\alpha_v + \alpha_w > \beta_{vw}$. So, without loss of generality $\lambda_{v_i} > 0$ implies $\lambda_{v_{i-1}} = \lambda_{v_{i+1}} = 0$. This idea can be generalized to obtain an integer optimal solution (λ, μ).

Let (i, j) be an index pair with $0 < \lambda_{v_i} \leq 1$, $0 < \lambda_{v_j} \leq 1$, $\lambda_{v_p} = 0$ for all $i < p < j$, and either λ_{v_i} or λ_{v_j} fractional. First, suppose $j - i = 2l$ for some $l > 0$. Moreover, without loss of generality let $\alpha_{v_i} + \sum_{p=1}^{l} \beta_{v_{i+2p-1} v_{i+2p}} \leq \alpha_{v_j} + \sum_{p=1}^{l} \beta_{v_{j-2p+1} v_{j-2p+2}}$ (otherwise i, j have to be exchanged, and the indexing direction must be reversed). Note that from the conditions of the sequence, it follows that $\mu_{v_{i+2p-2} v_{i+2p-1}} = \lambda_{v_j} + \mu_{v_j v_{j+1}}$, $\mu_{v_{i+2p-1} v_{i+2p}} = 1 - \lambda_{v_j} - \mu_{v_j v_{j+1}}$,

$p = 1, \ldots, l$, and $\lambda_{v_i} \leq 1 - \lambda_{v_j} - \mu_{v_j v_{j+1}}$. (Note that, from $\lambda_{v_i}, \lambda_{v_j} > 0$, it follows that $\lambda_{v_i}, \lambda_{v_j} < 1$.) Then the solution $(\bar{\lambda}, \bar{\mu})$ defined by

$$\bar{\lambda}_{v_q} = \begin{cases} \lambda_{v_i} + \lambda_{v_j} & \text{if } q = i, \\ 0 & \text{if } i < q \leq j, \\ \lambda_{v_q} & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_q v_{q+1}} = \begin{cases} \mu_{v_q v_{q+1}} - \lambda_{v_j} & \text{if } q < j,\ q - i \text{ even,} \\ \mu_{v_q v_{q+1}} + \lambda_{v_j} & \text{if } q < j,\ q - i \text{ odd,} \\ \mu_{v_q v_{q+1}} & \text{otherwise,} \end{cases}$$

is valid and has objective value

$$z(\bar{\lambda}, \bar{\mu}) = z(\lambda, \mu) + \lambda_{v_j}(\alpha_{v_i} + \sum_{p=1}^{l} \beta_{v_{i+2p-2} v_{i+2p-1}} - \alpha_{v_j} - \sum_{p=1}^{l} \beta_{v_{i+2p-1} v_{i+2p}}) \leq z(\lambda, \mu)$$

where the lower estimation comes from $\alpha_{v_i} + \sum_{p=1}^{l} \beta_{v_{i+2p-1} v_{i+2p}} \leq \alpha_{v_j} + \sum_{p=1}^{l} \beta_{v_{j-2p+1} v_{j-2p+2}}$. The number of fractional variables is reduced by at least one (namely λ_{v_j}).

If $j - i = 2l + 1$ for some $l > 0$, then either the solution $(\bar{\lambda}, \bar{\mu})$ or $(\tilde{\lambda}, \tilde{\mu})$ is at least as good as (λ, μ). Here, $(\bar{\lambda}, \bar{\mu})$ is defined by

$$\bar{\lambda}_{v_q} = \begin{cases} \lambda_{v_q} + \mu_{v_i v_{i+1}} & \text{if } q \in \{i, j\}, \\ 0 & \text{if } i < q < j, \\ \lambda_{v_q} & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_q v_{q+1}} = \begin{cases} 0 & \text{if } q < j,\ q - i \text{ even,} \\ 1 & \text{if } q < j,\ q - i \text{ odd,} \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

and $(\tilde{\lambda}, \tilde{\mu})$ by

$$\tilde{\lambda}_{v_q} = \begin{cases} \lambda_{v_q} - \lambda & \text{if } q \in \{i, j\}, \\ 0 & \text{if } i < q < j, \\ \lambda_{v_q} & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\mu}_{v_q v_{q+1}} = \begin{cases} \mu_{v_q v_{q+1}} + \lambda & \text{if } q < j,\ q - i \text{ even,} \\ \mu_{v_q v_{q+1}} - \lambda & \text{if } q < j,\ q - i \text{ odd,} \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

with $\lambda = \min\{\lambda_{v_i}, \lambda_{v_j}\}$. Again, the number of fractional variables (λ_v or μ_{vw}) is reduced by this construction. After repetition of this procedure a finite number of times, at most a single fractional λ_{v_i} variable remains. The existence of such a single fractional λ_{v_i} variable, however, leads to a contradiction. Let $0 < \lambda_{v_1} < 1$, whereas $\lambda_{v_i} \in \{0, 1\}$ for all $i \geq 2$. Then at least $\mu_{v_1 v_2}$ or $\mu_{v_{2k+1} v_1}$ is fractional as well. Without loss of generality, we can assume that $\mu_{v_1 v_2}$ is fractional. From $\lambda_{v_2} \in \{0, 1\}$ and $\mu_{v_1 v_2} > 0$ it follows that $\lambda_{v_2} = 0$ and $\mu_{v_2 v_3} = 1 - \mu_{v_1 v_2}$. Repeating this argument leads to $\lambda_{v_i} = 0$ for all $i \geq 2$, and $\mu_{v_{2k+1} v_1} = \mu_{v_1 v_2}$. It also holds that $\mu_{v_{2k+1} v_1} = 1 - \lambda_{v_1} - \mu_{v_1 v_2}$. This implies that $\mu_{v_{2i-1} v_{2i}} = \frac{1}{2} - \frac{1}{2}\lambda_{v_1}$ and $\mu_{v_{2i} v_{2i+1}} = \frac{1}{2} + \frac{1}{2}\lambda_{v_1}$ for $i = 1, \ldots, k$. However, then $(\lambda, \mu) = \lambda_{v_1}(\bar{\lambda}, \bar{\mu}) + (1 - \lambda_{v_1})(\tilde{\lambda}, \tilde{\mu})$ where $(\bar{\lambda}, \bar{\mu})$ and $(\tilde{\lambda}, \tilde{\mu})$ are defined by

$$\bar{\lambda}_{v_i} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_i v_{i+1}} = \begin{cases} 1 & \text{if } i \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\lambda}_{v_i} = 0 \text{ for all } i, \quad \text{and} \quad \tilde{\mu}_{v_i v_{i+1}} = \tfrac{1}{2} \text{ for all } i.$$

Since (λ, μ) is optimal, we get $z^* = z(\lambda, \mu) = z(\tilde{\lambda}, \tilde{\mu}) = \frac{1}{2}\beta(C)$, which contradicts $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$ for $\beta(C)$ odd.

So, after repeated application of the above described procedure, all λ_{v_i} variables are integer, with at least one non-zero. Consequently, all $\mu_{v_i v_{i+1}}$ variables are integer as well. ■

Theorem 5.8 *Let $C = \{v_1, \ldots, v_{2k+1}\}$ be an odd cycle in G with $\beta(C)$ odd. The odd cycle inequality* (5) *is dominated by a linear combination of the model inequalities if and only if*

$$\min_{i=1,\ldots,2k+1} \left\{ \alpha_{v_i} + \sum_{j=1}^{k} \beta_{v_{i+2j-1}v_{i+2j}} \right\} \leq \left\lfloor \tfrac{1}{2}\beta(C) \right\rfloor . \tag{10}$$

Proof. The proof of sufficiency is easy. Let p be an index for which the left hand side of (10) obtains its minimum. Then the inequalities $t_{v_p} \leq \alpha_{v_p}$, and $t_{v_{p+2j-1}} + t_{v_{p+2j}} \leq \beta_{v_{p+2j-1}v_{p+2j}}$ for all $j = 1, \ldots, k$ sum up to an inequality that is at least as strong as the odd cycle inequality (5).

Next, we show that (10) holds whenever the odd cycle inequality (5) is dominated, i.e., there exists a linear combination (λ, μ) of the model inequalities that results in the same left hand side as (5), but has a right hand side that is less than or equal to the right hand side of (5). The minimum right hand side of such a linear combination (λ, μ) can be calculated by the linear program (7)–(9), where λ_{v_i} corresponds to the inequality $t_{v_i} \leq \alpha_{v_i}$, and $\mu_{v_i v_{i+1}}$ to the inequality $t_{v_i} + t_{v_{i+1}} \leq \beta_{v_i v_{i+1}}$. We have to prove that (10) holds, whenever $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$. This is done by analyzing the optimal solution of (7)–(9).

By Lemma 5.7, we know that if $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$, then there exists an optimal solution (λ, μ), with λ and μ integer and at least one $\lambda_{v_i} > 0$. Now, let $I_\lambda = \{i : \lambda_{v_i} = 1\}$, and let $I_\mu = \{i : \mu_{v_i v_{i+1}} = 1\}$. Note that $|I_\lambda| = 2l + 1$ for some $l \geq 0$. If $l = 0$, then the proof is completed. Otherwise, suppose that, although $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$, (10) does not hold. Then for all $i \in \{1, \ldots, 2k+1\}$, $\alpha_{v_i} + \sum_{j=1}^{k} \beta_{v_{i+2j-1}v_{i+2j}} \geq \frac{1}{2}(\beta(C) + 1)$. Summation for all $i \in I_\lambda$ results in

$$\sum_{i \in I_\lambda} \alpha_{v_i} + l\beta(C) + \sum_{i \in I_\mu} \beta_{v_i v_{i+1}} \geq \frac{2l+1}{2} (\beta(C) + 1)$$

which yields

$$\sum_{i \in I_\lambda} \alpha_{v_i} + \sum_{i \in I_\mu} \beta_{v_i v_{i+1}} \geq \tfrac{1}{2} (\beta(C) + 1) + l \geq \tfrac{1}{2} (\beta(C) + 1) . \tag{11}$$

On the other hand, however, since (λ, μ) is optimal, it holds for every pair $i, j \in I_\lambda$ with $p \notin I_\lambda$ for all $i < p < j$ that the solution $(\tilde{\lambda}, \tilde{\mu})$ defined by

$$\tilde{\lambda}_{v_q} = \begin{cases} 0 & \text{if } q \in \{i, j\}, \\ \lambda_{v_q} & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\mu}_{v_q v_{q+1}} = \begin{cases} 1 - \mu_{v_q v_{q+1}} & \text{if } i \leq q < j \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

(i.e., an 0–1 exchange of the μ values between i and j) does not improve the solution value. Hence,

$$\alpha_{v_i} + \alpha_{v_j} + \sum_{p \in I_\mu : i < p < j} \beta_{v_p v_{p+1}} \leq \sum_{p \in I_\mu : i < p < j} \beta_{v_{p-1} v_p} + \beta_{v_{j-1} v_j} .$$

Summation for all pairs $i, j \in I_\lambda$ with $p \notin I_\lambda$ for all $i < p < j$ results in

$$2 \sum_{i \in I_\lambda} \alpha_{v_i} + \sum_{p \in I_\mu} \beta_{v_p v_{p+1}} \leq \sum_{p \in I_\mu} \beta_{v_{p-1} v_p} + \sum_{i \in I_\lambda} \beta_{v_{i-1} v_i} .$$

Addition of $\sum_{p\in I_\mu}\beta_{v_pv_{p+1}}$ to both sides gives

$$2\sum_{i\in I_\lambda}\alpha_{v_i}+2\sum_{p\in I_\mu}\beta_{v_pv_{p+1}}\le\beta(C).$$

Since the right hand side is odd by assumption, whereas the left hand side is even, it holds that

$$\sum_{i\in I_\lambda}\alpha_{v_i}+\sum_{p\in I_\mu}\beta_{v_pv_{p+1}}\le\tfrac{1}{2}\left(\beta(C)-1\right)$$

which contradicts to (11). So, (10) holds whenever $z^*\le\lfloor\frac{1}{2}\beta(C)\rfloor$. ∎

Now, Theorem 5.6 can be enhanced by combining the results from Lemma 5.4, Proposition 5.5 and Theorem 5.8.

Corollary 5.9 *The polytope T_{IP} is completely described by the model inequalities, i.e., $T_{IP}=T_{LP}$, if and only if for every odd cycle $C=\{v_1,\dots,v_{2k+1}\}$ in G either $\beta(C)$ is even or* (10) *is satisfied.*

5.3 Valid inequalities

Now that the case $T_{IP}=T_{LP}$ is characterized, we turn to the case $T_{IP}\neq T_{LP}$. Then, solving the linear relaxation of the integer programming formulation generally leads to a fractional solution. It is well known this relaxation can be tightened by the addition of *valid inequalities*. In this paper, we discuss two classes of valid inequalities, the *cycle inequalities* and the *clique inequalities*. The section is closed by a theorem to lift valid inequalities from non-reducible instances to reducible instances.

Cycle inequalities

In case condition (10) is not satisfied for an odd-valued odd cycle C, we now know that $T_{IP}\neq T_{LP}$. A natural question to be asked now is whether the cycle inequality (5) defines a facet of T_{IP}. For the stable set polytope, the odd cycle inequality defines a facet in case the graph is an odd cycle (cf. Section 2). For more general graphs, cycle inequalities have to be lifted to obtain a facet. Also for stable multi-sets we have to restrict ourselves to an odd cycle. Moreover, by Corollary 5.9, additional conditions are necessary. In the next proposition, we state conditions that are not only necessary but also sufficient for an odd cycle to be facet defining. Let G_C denote the subgraph restricted (not induced) to the cycle C, i.e., $G_C=(C,E(C))$.

Proposition 5.10 *Let $C=\{v_1,\dots,v_{2k+1}\}$ be an odd cycle in G with $\beta(C)$ odd. Then the odd cycle inequality* (5) *defines a facet of $T_{IP}(G_C,\alpha,\beta)$ if and only if*

$$\max_{j=1,\dots,2k+1}\left\{\sum_{p=1}^{k}\beta_{v_{j+2p-1}v_{j+2p}}\right\}\le\left\lfloor\tfrac{1}{2}\beta(C)\right\rfloor<\min_{j=1,\dots,2k+1}\left\{\alpha_{v_j}+\sum_{p=1}^{k}\beta_{v_{j+2p-1}v_{j+2p}}\right\}. \quad (12)$$

Proof. The odd cycle inequality (5) defines a facet if and only if there exist $2k+1$ affinely independent solutions that satisfy (5) with equality. Since the right hand side of (5) is violated by at most $\frac{1}{2}$, it follows that each of those solutions must satisfy all but one edge inequalities $t_{v_i} + t_{v_{i+1}} \leq \beta_{v_i v_{i+1}}$ with equality, whereas the remaining inequality has to have slack exactly one. Since C is an odd cycle (and thus the edge-node incidence matrix has full rank), only one such a solution can exist for each edge. As a consequence, we can identify the $2k+1$ solutions by the index i of the edge $v_i v_{i+1}$ that is not satisfied with equality by the specific solution. These solutions are denoted by t^i. Each solution t^i is uniquely defined by

$$t^i_{v_j} + t^i_{v_{j+1}} = \begin{cases} \beta_{v_j v_{j+1}} - 1 & \text{if } j = i, \\ \beta_{v_j v_{j+1}} & \text{otherwise,} \end{cases}$$

or equivalently

$$t^i_{v_j} = \tfrac{1}{2}\left(\beta^1_j - \beta^2_j + (-1)^{j-i}\right)$$

where

$$\beta^1_j = \sum_{p=0}^{k} \beta_{v_{j+2p} v_{j+2p+1}} \quad \text{and} \quad \beta^2_j = \sum_{p=1}^{k} \beta_{v_{j+2p-1} v_{j+2p}}.$$

To be feasible, these solutions have to satisfy (2)–(4). Integrality is clear since either β^1_j or β^2_j is odd (and the other even). The edge inequalities (2) are satisfied by definition. Hence, we only need to prove that for all solutions t^i

$$0 \leq t^i_{v_j} = \tfrac{1}{2}\left(\beta^1_j - \beta^2_j + (-1)^{j-i}\right) \leq \alpha_{v_j}$$

for all $j \in \{i+1, \ldots, i+2k+1\}$. Since for fixed j the difference $j - i$ is half the times even and half the times odd, this statement is equivalent to

$$1 \leq \beta^1_j - \beta^2_j \leq 2\alpha_{v_j} - 1$$

for all $j \in \{i+1, \ldots, i+2k+1\}$. Division by 2 and the addition of β^2_j to all parts leads to

$$\tfrac{1}{2} + \beta^2_j \leq \tfrac{1}{2}\beta(C) \leq \alpha_{v_j} + \beta^2_j - \tfrac{1}{2}$$

for all $j \in \{i+1, \ldots, i+2k+1\}$. Since $\beta(C)$ is odd, it follows that

$$\beta^2_j \leq \lfloor \tfrac{1}{2}\beta(C) \rfloor < \alpha_{v_j} + \beta^2_j$$

for all $j \in \{i+1, \ldots, i+2k+1\}$. Hence, the cycle inequality (5) defines a facet if and only if

$$\max_{j=1,\ldots,2k+1} \beta^2_j \leq \lfloor \tfrac{1}{2}\beta(C) \rfloor < \min_{j=1,\ldots,2k+1} \left\{\alpha_{v_j} + \beta^2_j\right\}$$

holds. ■

Gerards and Schrijver [6] proved that for strongly t-perfect graphs the model and odd cycle inequalities completely describe the stable multi-set polytope (i.e., the polytope has Chvátal rank 1). This result is independent of the vectors α and β. For actual values α_v and β_{vw}, stronger results for Chvátal rank 1 are likely to hold (cf. the results for Chvátal rank 0, Corollary 5.9). Such results are, however, unknown, even for the stable set case (t-perfect graphs are not fully characterized).

Moreover, the polynomial time separation algorithm for odd cycle inequalities proposed by Grötschel, Lovász and Schrijver [8], is not applicable to the separation of odd-valued odd cycles. The algorithm searches for an odd cycle with a specific property. In our case, however, odd cycles can have this property without violating the odd cycle inequality (i.e., when the cycle is even-valued or does not satisfy (10)). As a consequence, the question whether the separation of odd cycle inequalities for the SMS problem can be solved in polynomial time remains open.

Clique inequalities

A second class of inequalities that is well-known for the stable set polytope are the clique inequalities. For the stable multi-set polytope, we derive a similar result for cliques with uniform edge bounds $\beta_{vw} = \beta$ for all $vw \in E$ which we call *β-cliques.* For a clique $Q \subseteq V$ in the graph G, the *β-clique inequality* is defined by

$$\sum_{v \in Q} t_v \leq |Q| \lfloor \tfrac{1}{2}\beta \rfloor + (\beta \bmod 2) \tag{13}$$

First of all, we prove that in case of a complete graph $G = K_n$ and $\beta_{vw} = \beta$ for all $vw \in E$, the β-clique inequality is valid and defines a facet for suitable vertex bounds α_v.

Proposition 5.11 *Let $G = (V, E) = K_n$ for an integer $n \geq 2$, with $\beta_{vw} = \beta$ for all $vw \in E$ and $\alpha_v \geq \lceil \frac{1}{2}\beta \rceil$ for all $v \in V$. Then the β-clique inequality (13) with $Q = V$ is valid for T, and defines a facet of T_{IP} if and only if β is odd or $n = 2$.*

Proof. We prove the validity of (13) by induction on n. For $n = 2$, let $K_2 = \{v_1, v_2\}$, and consider the model inequality $t_{v_1} + t_{v_2} \leq \beta$. By $\beta = 2\lfloor \frac{1}{2}\beta \rfloor + (\beta \bmod 2)$, the inequality belongs to the class of clique inequalities (13).

Now, suppose the result holds for cliques of size $n-1$. Then for all $u \in V$, it holds (by assumption)

$$\sum_{v \in V \setminus \{u\}} t_v \leq (n-1) \lfloor \tfrac{1}{2}\beta \rfloor + (\beta \bmod 2).$$

Summation of these inequalities for all $u \in V$ yields

$$(n-1) \sum_{v \in V} t_v \leq n(n-1) \lfloor \tfrac{1}{2}\beta \rfloor + n(\beta \bmod 2).$$

Division by $n-1$ and rounding results in

$$\begin{aligned} \sum_{v \in V} t_v &\leq \left\lfloor n \lfloor \tfrac{1}{2}\beta \rfloor + (\beta \bmod 2) + \frac{1}{n-1}(\beta \bmod 2) \right\rfloor \\ &= n \lfloor \tfrac{1}{2}\beta \rfloor + (\beta \bmod 2) + \left\lfloor \frac{1}{n-1}(\beta \bmod 2) \right\rfloor . \end{aligned}$$

Because we can assume $n \geq 3$, we have $\frac{1}{n-1} \leq \frac{1}{2} < 1$. Moreover, from $(\beta \bmod 2) \in \{0, 1\}$, it follows $\lfloor \frac{1}{n-1}(\beta \bmod 2) \rfloor = 0$, which proves that (13) is valid for n.

Finally, we prove that (13) defines a facet of T_{IP}, if and only if β is odd or $n = 2$. If $G = K_2$, the conditions of Proposition 5.2 are satisfied, and so (13) defines a facet. If β is odd, then the n solutions t^v for all $v \in V$ defined by

$$t^v_u = \begin{cases} \lfloor \frac{1}{2}\beta \rfloor + 1 & \text{if } u = v, \\ \lfloor \frac{1}{2}\beta \rfloor & \text{otherwise} \end{cases}$$

are valid, satisfy the clique inequality (13) with equality, and are affinely independent. If neither β is odd nor $n = 2$, i.e., β is even and $n \geq 3$, then (13) is a linear combination of the model inequalities $t_v + t_w \leq \beta$ for all $vw \in E$, and thus cannot define a facet. ■

For more general graphs, the clique inequality is valid for all cliques that satisfy the described properties. To be facet defining in general graphs, an additional condition has to be satisfied. Since cliques of size 2 are already treated by Proposition 5.2, we restrict to cliques with at least 3 nodes.

Proposition 5.12 *Let $G = (V, E)$ be an arbitrary graph. Let $Q \subseteq V$, $|Q| \geq 3$, be a clique in G with $\beta_{vw} = \beta$ for all $v, w \in Q$, and $\alpha_v \geq \lceil \frac{1}{2}\beta \rceil$ for all $v \in Q$. Then,* (13) *defines a facet of T_{IP} if and only if β is odd, and for all $u \in N(Q)$, there exists $w \in Q$ with $w \notin N(u)$ or $\beta_{uw} \geq \lceil \frac{1}{2}\beta \rceil + 1$.*

Proof. We first prove sufficiency. By the described conditions, we have $|Q|$ affinely independent solutions t^v for $v \in Q$, defined by

$$t^v_u = \begin{cases} \lfloor \frac{1}{2}\beta \rfloor + 1 & \text{if } u = v, \\ \lfloor \frac{1}{2}\beta \rfloor & \text{if } u \in Q,\, u \neq v, \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, by assumption there is a node $v \in Q$ for each $w \in V \setminus Q$, such that $t^w = t^v + e^w$ is valid. These in total n vectors are affinely independent and satisfy (13) with equality, implying that (13) defines a facet of T_{IP}.

Necessity is shown indirectly. By Proposition 5.11, we know that for β even, the inequality (13) cannot define a facet. Moreover, if β is odd, but there exists $u \in N(Q)$ with $Q \subseteq N(u)$ and $\beta_{uw} \leq \lceil \frac{1}{2}\beta \rceil$ for all $w \in Q$, then all solutions satisfying (13) with equality also satisfy $t_u = 0$, which implies that (13) is a face of $t_u = 0$ and does not define a facet of T_{IP}. ■

Note that in case of a stable set polytope, Proposition 5.12 states the result of Fulkerson [4] and Padberg [11] that (13) defines a facet if and only if Q is a maximal clique.

In case G is complete and $\beta_{vw} = \beta$ for all $vw \in E$, T_{IP} is completely described by the clique inequalities. To prove this result we need some general properties of facet defining inequalities.

Lemma 5.13 *Let $\pi t \leq \pi_0$ be a valid inequality that defines a facet of T_{IP}. Then $\pi_0 \geq 0$, and if $\pi_0 = 0$, then $\pi t \leq 0$ is identical to a non-negativity inequality $t_v \geq 0$ for some $v \in V$. Moreover, if $\pi_0 > 0$, then $0 \leq \pi_v \leq \pi_0/\alpha_v$ for all $v \in V$.*

Proof. From $0 \in T$, it follows that $\pi_0 \geq 0$. If $\pi_0 = 0$, then the solution $t^v = e^v$, $v \in V$, implies that $\pi_v \leq 0$. But in this case, $\pi t \leq 0$ is a linear combination of the inequalities $t_v \geq 0$. Since both $t_v \geq 0$ and $\pi t \leq 0$ define facets of T_{IP}, the inequalities have to be identical.

Now, let $\pi_0 > 0$ and $\pi_v < 0$ for some $v \in V$. Since $\pi t \leq \pi_0$ defines a facet (and is not identical to a non-negativity constraint), there has to be a solution $\bar{t}$ with $\pi\bar{t} = \pi_0$ and $\bar{t}_v > 0$. Then the solution $\tilde{t}$ with $\tilde{t}_v = \bar{t}_v - 1$ and $\tilde{t}_w = \bar{t}_w$, for all $w \neq v$, however, violates $\pi t \leq \pi_0$. Finally, by $\alpha_v e^v \in T$ for all $v \in V$, $\pi_v \alpha_v \leq \pi_0$ for all $v \in V$. ∎

Theorem 5.14 *Let $G = (V, E) = K_n$ for an integer $n \geq 2$, with $\beta_{vw} = \beta$ for all $vw \in E$, β odd, and $\alpha_v \geq \lceil \frac{1}{2}\beta \rceil$ for all $v \in V$. Then T_{IP} is completely described by the model inequalities and the k-clique inequalities* (13) *for all $k \leq n$.*

Proof. We have to prove that each inequality $\pi t \leq \pi_0$ that defines a facet of T_{IP} is in fact identical to one of the inequalities describing the polytope P,

$$P = \left\{ \begin{array}{lll} t \in \mathbb{R}^n : & t_v \leq \alpha_v & v \in V, \\ & t_v + t_w \leq \beta_{vw} & vw \in E, \\ & \sum_{v \in S} t_v \leq |S| \lfloor \frac{1}{2}\beta \rfloor + 1 & S \subseteq V, |S| \geq 3 \end{array} \right\}.$$

First of all, without loss of generality, we assume that all coefficients $\pi_0 \in \mathbb{Z}_0^+$ and $\pi_v \in \mathbb{Z}_0^+$, $v \in V$. By Lemma 5.13, we can restrict us to $\pi_0 > 0$, and it holds that $\pi_v \geq 0$ for all $v \in V$. Moreover, in case $\pi_v = 0$ for all vertices but one $v \in V$, the inequality must be identical to the inequality $t_v \leq \alpha_v$. So, in the sequel we may assume at least two positive coefficients π_v. We also may assume that at least two different positive coefficients exist, otherwise the inequality has to be identical to an edge or clique inequality.

Now, let $u = \arg\max_{v \in V} \pi_v$. It holds that either $\pi_u > \sum_{v \in V \setminus \{u\}} \pi_v$ or $\pi_u \leq \sum_{v \in V \setminus \{u\}} \pi_v$. Suppose that $\pi_u > \sum_{v \in V \setminus \{u\}} \pi_v$ holds. Since $\pi t \leq \pi_0$ defines a facet not identical to $t_u \leq \alpha_u$, there has to be a solution $\bar{t} \in T_{IP}$ with $\pi\bar{t} = \pi_0$ and $\bar{t}_u \leq \alpha_u - 1$. Now, construct a new solution $\tilde{t}$ by $\tilde{t}_u = \bar{t}_u + 1$, $\tilde{t}_v = \max\{0, \bar{t}_v - 1\}$ for all $v \neq u$. Obviously $\tilde{t} \in T_{IP}$, but $\pi\tilde{t} > \pi_0$, which implies that $\pi t \leq \pi_0$ cannot be a valid inequality in this case.

So, it holds that $\pi_u \leq \sum_{v \in V \setminus \{u\}} \pi_v$. We construct a non-negative combination of the inequalities describing P, which is equivalent to or dominates $\pi t \leq \pi_0$. On the one hand, if $\pi_u = \sum_{v \in V \setminus \{u\}} \pi_v$, we can combine edge inequalities $t_u + t_w \leq \beta$ to an equivalent (or dominating) inequality. On the other hand, if there exists a $v \in V \setminus \{u\}$ with $\pi_v = \pi_u$, a combination of clique inequalities gives the desired result. In general, we have to combine these two approaches. To do so, let $V^{(0)} = V \setminus \{u\}$ and $\mu^{(0)} = 0$. Moreover, for $i \geq 1$ let $V^{(i)} = \{v \in V^{(i-1)} : \pi_v > \mu^{(i-1)}\}$ and $\mu^{(i)} = \min_{v \in V^{(i)}} \pi_v$. Now, consider the values $f(i) := \sum_{v \in V^{(i)}} \pi_v - \pi_u$ and $g(i) := (|V^{(i)}| - 1)\mu^{(i)}$. An increase of i to $i+1$ results in a decrease $f(i)$ by $|V^{(i)} \setminus V^{(i+1)}|\mu^{(i)}$, whereas $g(i)$ decreases by $|V^{(i)} \setminus V^{(i+1)}|\mu^{(i)} - (|V^{(i+1)}| - 1)(\mu^{(i+1)} - \mu^{(i)})$. So, the $f(i)$ decreases at least as fast as $g(i)$. Moreover, let q be the smallest integer such that $V^{(q+1)} = \emptyset$. Then $f(q) = |V^{(q)}|\mu^{(q)} - \pi_u \leq g(q)$, whereas $f(0) \geq 0 = g(0)$. As a consequence, there exists a minimal $p \in \{0, \ldots, q\}$ such that $f(p) \leq g(p)$.

Note that, $|V^{(p)}| \geq 1$, since $p \leq q$. Now, suppose $|V^{(p)}| = 1$. By definition of p, we have $f(p) \leq g(p)$ and $f(p-1) > g(p-1)$. With $|V^{(p)}| = 1$ we have $\sum_{v \in V^{(p)}} \pi_v = \mu^{(p)}$

and $\sum_{v\in V^{(p-1)}} \pi_v = (|V^{(p-1)}|-1)\mu^{(p-1)} + \mu^{(p)}$. This yields $\mu^{(p)} - \pi_u \leq 0$ and $(|V^{(p-1)}|-1)\mu^{(p-1)} + \mu^{(p)} - \pi_u > (|V^{(p-1)}|-1)\mu^{(p-1)}$, but both cannot be true. Hence, we have $|V^{(p)}| \geq 2$.

Now, consider the non-negative combination of the inequalities

$$\sum_{v\in V^{(i)}\cup\{u\}} t_v \leq (|V^{(i)}|+1)\lfloor \tfrac{1}{2}\beta \rfloor + 1 \quad \text{with weight } \mu^{(i)} - \mu^{(i-1)}, \text{ for } 1 \leq i < p,$$

$$\sum_{v\in V^{(p)}\cup\{u\}} t_v \leq (|V^{(p)}|+1)\lfloor \tfrac{1}{2}\beta \rfloor + 1 \quad \text{with weight } \frac{\sum_{v\in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}|-1)} - \mu^{(p-1)},$$

$$t_u + t_w \leq \beta \quad \text{with weight } \pi_w - \frac{\sum_{v\in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}|-1)}, \text{ for all } w \in V^{(p)}.$$

This yields to the inequality

$$\begin{aligned} \pi t \leq\ & \sum_{i=1}^{p-1} (\mu^{(i)} - \mu^{(i-1)}) \left\{ (|V^{(i)}|+1)\lfloor \tfrac{1}{2}\beta \rfloor + 1 \right\} \\ &+ \left(\frac{\sum_{v\in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}|-1)} - \mu^{(p-1)} \right) \left\{ (|V^{(p)}|+1)\lfloor \tfrac{1}{2}\beta \rfloor + 1 \right\} \\ &+ \sum_{w\in V^{(p)}} \left(\pi_w - \frac{\sum_{v\in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}|-1)} \right) \beta \end{aligned} \tag{14}$$

which has the same left hand side as the inequality $\pi t \leq \pi_0$. On the other hand, the solution $\bar{t}$ defined by $\bar{t}_u = \lfloor \frac{1}{2}\beta \rfloor + 1$, $\bar{t}_v = \lfloor \frac{1}{2}\beta \rfloor$ for all $v \neq u$ satisfies (14) at equality, because it satisfies all combined inequalities at equality. So, π_0 has to be at least as large as the right hand side of (14). As a consequence, $\pi t \leq \pi_0$ is identical to or dominated by a non-negative combination of the inequalities describing P. ■

As mentioned in Section 2, Chvátal [2] showed for the stable set polytope that the maximum clique inequality defines the only non-trivial facet if G is complete. By Proposition 5.12, we know that for $\beta > 1$ (and odd), in fact all clique inequalities are necessary.

For arbitrary β_{vw}, a closed formula for the right hand side of the clique inequalities becomes difficult to determine. In principle, the best right hand side for a clique inequality can be found by Chvátal-Gomory rounding (this procedure was also used in the proof of Proposition 5.11). The major obstacle, however, is that it is unclear which combination of vertex, edge and clique inequalities (on sub-cliques) produces the smallest right hand side. Consider the clique in Figure 4(a). The example contains 5 odd cycles with right hand side $\lfloor \frac{2*5+9}{2} \rfloor = 9$ as well as 5 odd cycles with right hand side $\lfloor \frac{5+2*9}{2} \rfloor = 11$. Chvátal-Gomory rounding delivers a right hand side for the 5 vertex clique inequality of respectively $\lfloor \frac{5*9}{3} \rfloor = 15$ and $\lfloor \frac{5*11}{3} \rfloor = 18$. Combination of all edges with value $\beta_{vw} = 5$ and Chvátal-Gomory rounding results in a right hand side of $\lfloor \frac{5*5}{2} \rfloor = 12$.

By definition, all three variants lead to a valid clique inequality. Only the clique inequality with minimum right hand side defines a facet of T_{IP}. In general, this need not to be the case, not even for maximal cliques. Consider the instance in Figure 4(b). The best right hand side of 12 for a 5 clique is derived by, for example, the combination of the odd cycle inequalities for the cycles $\{a,c,e\}$, $\{b,c,e\}$, $\{b,d,e\}$, $\{a,b,d\}$, and $\{a,c,d\}$. But, also the combination of the edge inequality $t_a + t_b \leq 5$ with the odd cycle inequality $t_c + t_d + t_e \leq 7$ results in a right hand side of 12. Hence, the 5 vertex clique cannot define a facet of T_{IP} although the clique is maximal.

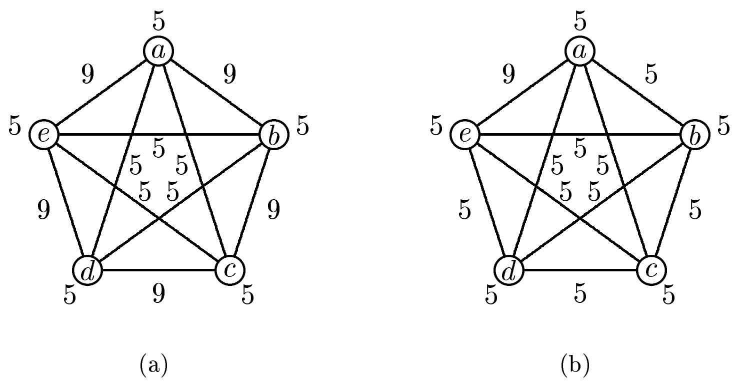

Figure 4: Examples of SMS cliques.

Lifting valid inequalities to reducible cases

In Section 4, we have seen several rules to reduce the problem size. The most significant rule deals with reducing the α and β values. We close this section by a Proposition that tells us how valid inequalities can be lifted in the reverse direction from a non-reducible instance to a reducible one.

Proposition 5.15 *Let $T_{IP}(G,\alpha,\beta)$ be a stable multi-set polytope and $u \in V$. Moreover, on the same graph G let $T_{IP}(G,\alpha^*,\beta^*)$ be a second stable multi-set polytope defined by*

$$\alpha^*_v = \begin{cases} \alpha_v + 1 & \text{if } v = u, \\ \alpha_v & \text{otherwise} \end{cases} \quad \text{and} \quad \beta^*_{vw} = \begin{cases} \beta_{vw} + 1 & \text{if } v = u \text{ or } w = u, \\ \beta_{vw} & \text{otherwise.} \end{cases}$$

If $\pi t \leq \pi_0$ with $\pi_u > 0$ defines a facet of $T_{IP}(G,\alpha,\beta)$, then $\pi t \leq \pi_0 + \pi_u$ defines a facet of $T_{IP}(G,\alpha^*,\beta^*)$.

Proof. Let t^* be a solution of $T_{IP}(G,\alpha^*,\beta^*)$ with $t^*_u > 0$. Then $t^* - e^u$ is a solution of $T_{IP}(G,\alpha,\beta)$, and as a consequence $\pi(t^* - e^u) \leq \pi_0$. So, $\pi t^* \leq \pi_0 + \pi_u$. If $t^*_u = 0$, then by $\alpha^*_v = \alpha_v \leq \beta_{uv} < \beta^*_{uv}$ for all $v \in N(u)$, we have that t^* is also a feasible solution for $T_{IP}(G,\alpha,\beta)$. So $\pi t^* \leq \pi_0 \leq \pi_0 + \pi_u$, and it follows that $\pi t \leq \pi_0 + \pi_u$ is valid for $T_{IP}(G,\alpha^*,\beta^*)$.

Now, by assumption let $\pi t \leq \pi_0$ be facet defining for $T_{IP}(G,\alpha,\beta)$. Thus, there exist $n = \dim(T_{IP}(G,\alpha,\beta))$ affinely independent solutions $t^1,\ldots,t^n$ of $T_{IP}(G,\alpha,\beta)$ that satisfy

	Stable set polytope	Stable multi-set polytope
Dimension	$\|V\|$	$\|V\|$
Trivial facets	$t_v \geq 0$	$t_v \geq 0$ $t_v \leq \alpha_v$ $t_v + t_w \leq \beta_{vw}$
Odd cycle inequalities	$\sum_{v \in C} t_v \leq \lfloor \frac{1}{2}\|C\| \rfloor$ facet for G_C	$\sum_{v \in C} t_v \leq \lfloor \frac{1}{2}\beta(C) \rfloor$ for $\beta(C)$ odd facet for G_C iff (12) holds
Clique inequalities	$\sum_{v \in Q} t_v \leq 1$ facet for Q maximal	$\sum_{v \in Q} t_v \leq \|Q\| \lfloor \frac{1}{2}\beta \rfloor + (\beta \mod 2)$ facet for $\beta_{vw} = \beta$ odd for all $vw \in E$
$T_{IP} = T_{LP}$	G bipartite	for all cycles C with $\|C\|$ odd, $\beta(C)$ even or (10) satisfied.
Complete description for K_n ($\beta_{vw} = \beta$ for all vw)	$\sum_{v \in V} t_v \leq 1$ $t_v \geq 0, \forall v \in V$	$\sum_{v \in S} t_v \leq \|S\| \lfloor \frac{1}{2}\beta \rfloor + 1, \forall S \subseteq V, \|S\| \geq 2$ $0 \leq t_v \leq \alpha_v, \forall v \in V$

Table 1: Similarities and differences for stable sets vs. stable multi-sets

$\pi t \leq \pi_0$ with equality. Then $t^1 + e^u, \ldots, t^n + e^u$ are n affinely independent solutions of $T_{IP}(G, \alpha^*, \beta^*)$ that satisfy $\pi t \leq \pi_0 + \pi_u$ with equality. ■

Note that the result does not hold in general for the reverse direction of Proposition 5.15, i.e., a decrease of the values α and β associated to some vertex u. Especially, solutions with $t_u^* = 0$ that satisfy $\pi t \leq \pi_0 + \pi_u$ at equality cannot be transformed to new solutions that satisfy $\pi t \leq \pi_0$ at equality.

6 Concluding Remarks

In this paper, we introduced and studied stable multi-sets. In particular, properties of the associated polytope were identified. In Table 1, the obtained results are compared with their analogies for the stable set polytope. Note that Table 1 does not show all conditions for the model inequalities to define facets. The table shows that many of the results obtained for stable sets have their counterpart for stable multi-sets.

As this paper is meant to be a starting point for studying the stable multi-set polytope, only a limited number of results for stable sets have been generalized, and many directions for further research exist. First of all, many other classes of valid inequalities are known for stable sets: wheel inequalities, web and anti-web inequalities, odd anti-hole inequalities, and blossom inequalities, to name a few. It is likely that many of them can be generalized to the stable multi-set polytope. Another opportunity is to study whether specific classes of inequalities are separable in polynomial time. As we pointed out, for the odd cycle inequalities, the polynomial time algorithm by Grötschel, Lovász and Schrijver [8] is not applicable anymore. For the separation of clique inequalities, the separation problem is even more complex, since no general right hand side is known.

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References

[1] R. Borndörfer. *Aspects of Set Packing, Partitioning, and Covering.* PhD thesis, Technische Universität Berlin, published at Shaker Verlag Aachen, 1998.

[2] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[3] V. Chvátal. On certain polytopes associated with graphs. *Journal of Computational Theory (B)*, 18:138–154, 1975.

[4] D. R. Fulkerson. Blocking and antiblocking pairs of polyhedra. *Mathematical Programming*, 1:168–194, 1971.

[5] M. R. Garey and D. S. Johnson. *Computers and intractability: a guide to the Theory of $\mathcal{NP}$-Completeness.* Freeman and Company, N.Y., 1979.

[6] A. M. H. Gerards and A. Schrijver. Matrices with the Edmonds-Johnson property. *Combinatorica*, 6(4):365–379, 1986.

[7] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

[8] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization.* Number 2 in Algorithms and Combinatorics. Springer-Verlag, 1988.

[9] A. M. C. A. Koster and A. Zymolka. The minimum converter wavelength assignment problem. ZIB-report in preparation, Konrad-Zuse-Zentrum für Informationstechnik Berlin, 2001.

[10] A. Mehrotra and M. A. Trick. A column generation approach for graph coloring. *INFORMS Journal on Computing*, 8:344–354, 1996.

[11] M. Padberg. On the facial structure of set packing polyhedra. *Mathematical Programming*, 5:199–215, 1973.

[12] M. Padberg. Covering, packing and knapsack problems. *Annals of Discrete Mathematics*, 4:265–287, 1979.

[13] A. Schrijver. *Theory of linear and integer programming.* Wiley, New York, 1986.