Minimum Risk Arbitrage with Risky Financial Contracts

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Abstract

For a set of financial securities specified by their expected returns and variance/covariances we propose the concept of a minimum risk arbitrage opportunity, characterize conditions under which such opportunities may exist, and those under which they fail to exist. We use conic duality and convex analysis to derive these characterizations. For practical computation a decidability result on the existence of an arbitrage opportunity is derived. Extension to the case of convex transaction costs is studied.

Keywords. Financial Securities, Arbitrage, Conic Duality, Conically Constrained Least Squares, Generalized Farkas Lemma, Second-order Cone Programming

1 Introduction and Background

Existence and exclusion issues of arbitrage in financial markets is a well studied area of mathematical finance which several research monographs and textbooks treat at different levels of detail; see e.g., [6, 9, 14].

The purpose of the present paper is (1) to introduce a novel arbitrage concept for risky financial contracts (or, securities for short) when the investor has access to the expected return and standard deviation data (or, perhaps an estimate thereof) of the securities, (2) to characterize the existence or non-existence thereof via generalized Farkas type results and tools of convex analysis, and, (3) to investigate relevant extensions. The main finance contribution of the paper is to propose a new type of arbitrage under

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partial probabilistic information, and to show that the proposed arbitrage model is computationally tractable as it involves the solution of convex, second-order cone programs that are routinely solved by polynomial interior point methods [10, 13]. In that sense, although it is a more general model than the classical discrete arbitrage model, the new model is, in theory and practice, no more difficult computationally than the classical theory which involves the use of linear programming duality.

Consider a set of n risky financial securities, and denote the return vector per dollar invested by $r \in \mathbf{R}^n$. Treating r as a random variable, let us denote its expected value by $\bar{r}$ and its $n \times n$ (symmetric, positive semidefinite) matrix of variance/covariances by Q. Now, define $x \in \mathbf{R}^n$ to represent a portfolio of the n securities. Then, for a given x the single period return of the portfolio is a random variable, namely, $r^T x$. If we denote the current prices of the securities using a vector $p \in \mathbf{R}^n$, the investor would make money if the following conditions were met, which we will refer to as an arbitrage situation, at the end of the single period:

There exists a portfolio x such that

$$\tilde{r}^T x \geq 0,\ p^T x < 0 \tag{1}$$

where $\tilde{r}$ is the realization of the random variable r revealed to the investor only at the end of the period. Now, let us act as a conservative investor who wants to guarantee "almost surely" an arbitrage situation at the end of a single period investment situation, and use the following principle that a random real is "never" less than its mean minus a positive scalar θ times its standard deviation. What would be then a safe version of the arbitrage situation of (1)? For such a risk averse investor, we would replace the conditions of (1) with the following:

There exists a portfolio x such that

$$\bar{r}^T x - \theta\sqrt{x^T Q x} \geq 0,\ \ p^T x < 0. \tag{2}$$

The engineering approach would be to choose $\theta = 2$, or $\theta = 3$. Intuitively, the larger the scalar θ, the smaller the risk. A weaker version of an arbitrage opportunity of this type is the following version of (2):

There exists a portfolio x such that

$$\bar{r}^T x - \theta\sqrt{x^T Q x} \geq 0,\ \ p^T x \leq 0. \tag{3}$$

To motivate the above development as in [2] let us assume that the returns $r_1, r_2, \ldots, r_n$ take values within the uncertainty intervals $\Delta_i = [\bar{r}_i - \sigma_i, \bar{r}_i + \sigma_i]$. Assume, furthermore, that r_i are mutually independent and symmetrically distributed in Δ_i with respect to the mean value $\bar{r}_i$. For a fixed choice of portfolio holdings x, the end-of-period return can be expressed as

$$R = \sum_{i=1}^{n} \bar{r}_i x_i + \varsigma,$$

where

$$\varsigma = \sum_{i=1}^{n} x_i (r_i - \bar{r}_i),$$

has zero mean and variance $\mathrm{Var}(\varsigma) = \sum_{i=1}^{n} x_i^2 \mathrm{E}\{(r_i - \bar{r}_i)^2\}$. Since the variance of r_i is bounded above by σ_i^2 one has $\mathrm{Var}(\varsigma) \leq V(x) \equiv \sum_{i=1}^{n} x_i^2 \sigma_i^2$. Therefore, one can say that typically the value of R will differ from the mean value of $\bar{r}^T x$ by a quantity proportional to $\sqrt{\mathrm{Var}(\varsigma)} \leq \sqrt{V(x)}$, variations on both sides being equally probable. Therefore, choosing a reliability coefficient θ and ignoring all events where the random return is less than $\bar{r}^T x - \theta\sqrt{V(x)}$, one arrives at the minimum risk arbitrage definitions introduced above. Notice that by ignoring the events where the total return is less than $\bar{r}^T x - \theta\sqrt{V(x)}$, one accepts the fact that $\mathrm{Prob}(\varsigma < -\theta\sqrt{V}) < e^{-\theta^2/2}$ as shown in [4]. The right-hand side is getting already quite small (in the order of 10^{-7} for $\theta = 6$) quickly with increasing values of θ.

Notice that, although nonlinear and not differentiable everywhere, (3) is a system of two convex inequalities.

The rest of this paper is organized as follows. In the next few paragraphs, we present some connections of the above arbitrage model to existing literature. In Section 2, we characterize the presence (absence) of such arbitrage opportunities via generalized Farkas type results. Section 3 gives a decidability result with important computational implications in that one can bypass the more technical generalized Farkas results of Section 2 to decide on the existence (or, non-existence) of minimum risk arbitrage. Finally, in Section 4 we develop an extension of the arbitrage model to the case where the investor faces nonlinear but convex transaction costs.

1.1 Connections to Previous Work

An interesting connection exists between the concepts we introduced above and the following well-known concepts, the Value-at-Risk formula [7], chance constrained optimization [17], and robust optimization paradigm of Ben-Tal and Nemirovski [2, 3]. In fact, the motivation for the present paper derived from the Ben-Tal-Nemirovski contributions. Let us begin by briefly reviewing the robust optimization idea. Our treatment in this section closely follows Section 2.6 of [10]. Assume that the vector r is a member of the ellipsoidal uncertainty set $\mathcal{E} = \{\bar{r} + Qu : \|u\|_2 \leq 1\}$ with Q an $n \times n$ symmetric and positive semidefinite matrix. In this case a "robust" version of the inequality $r^T x \geq 0$ is the following system

$$r^T x \geq 0, \text{ for all } r \in \mathcal{E}$$

which is equivalent to

$$\min_{r \in \mathcal{E}} r^T x \geq 0 = \min_{u:\|u\|_2 \leq 1} \bar{r}^T x + u^T Q^T x \geq 0.$$

The right hand side of the above is nothing else than the inequality

$$\bar{r}^T x - \|Qx\|_2 \geq 0$$

which is almost identical (up to a multiplicative factor in front of Q) to the first inequality in (3).

Assume now that the return vector is a Gaussian random vector, with mean $\bar{r}$ and covariance Q. If we require as in [17] that the inequality $r^T x \geq 0$ should hold with a confidence level exceeding η, where $\eta \geq 0.5$, i.e.,

$$\text{Prob}(r^T x \geq 0) \geq \eta.$$

Defining $u = r^T x$ one can normalize both sides of the inequality as follows:

$$\text{Prob}(\frac{u - \bar{u}}{\sqrt{\sigma}} \geq \frac{-\bar{u}}{\sqrt{\sigma}}) \geq \eta.$$

Since $\frac{u-\bar{u}}{\sqrt{\sigma}}$ is a zero mean unit variance Gaussian random variable the above probability constraint is simply equivalent to

$$\frac{-\bar{u}}{\sqrt{\sigma}} \leq \Phi^{-1}(1 - \eta)$$

where

$$\Phi(z) = \frac{1}{2\pi} \int_{-\infty}^{z} e^{-t^2/2} dt$$

is the CDF of a zero mean unit variance Gaussian random variable. Now, the above constraint is nothing other than

$$\bar{r}^T x + \Phi^{-1}(1-\eta)\|Qx\|_2 \geq 0$$

Since $\eta \geq 0.5$, $\Phi^{-1}(1-\eta)$ is a non-positive scalar. The close resemblance to the first inequality of (3) is now obvious. The above tail probability concepts are also reminiscent of the Value-at-Risk methodology used to limit the risk exposure of financial institutions [7].

2 Characterizing Minimum Risk Arbitrage

In this section we characterize the arbitrage opportunities introduced above, and absence thereof, using a tool known as conic duality theory in the optimization literature. First, we develop some preliminaries. The first inequality of the system (2) is equivalent to the following (after absorbing θ in to the square root for notational convenience, and denoting by $\bar{Q}$ the square root of Q, i.e., $\bar{Q} = Q^{\frac{1}{2}}$)

$$\|\bar{Q}x\|_2 \leq \bar{r}^T x \tag{4}$$

which is equivalent to say that

$$Ax \geq_{L^{n+1}} 0$$

where $A = \begin{pmatrix} \bar{Q} \\ \bar{r} \end{pmatrix}$, and the notation $Ax \geq_{L^{n+1}} 0$ denotes membership of Ax to the $(n+1)$-dimensional Lorenz cone, i.e., the closed, convex set $\{x \in \mathbf{R}^{n+1} : \sqrt{x_1^2 + x_2^2 + \ldots x_n^2} \leq x_{n+1}\}$. Therefore, we are dealing with the following system of two inequalities

$$Ax \geq_{L^{n+1}} 0, \;\; p^T x < 0. \tag{5}$$

We refer to the existence of a portfolio x such that (5) holds as a "minimum risk arbitrage opportunity". Similarly, we refer as a "weak minimum risk arbitrage opportunity" to the existence of a portfolio x such that the following conditions hold:

$$Ax \geq_{L^{n+1}} 0, \;\; p^T x \leq 0. \tag{6}$$

There are several sources dealing with conic linear systems. A recent and thorough reference is [12]. However, the most convenient for our purposes is [18]. The proofs of some our existence/non-existence results below are obtained, mutatis mutandis, using results from [18]. However, as [18] is an unpublished reference in the form of lecture notes, and our proofs involve some modifications, they are included to ensure completeness.

The following is useful in the proofs. The dual cone K^* to a cone K is defined as

$$K^* = \{x | x^T y \geq 0, \forall\, y \in K\}.$$

Notice that the closed, solid, convex and pointed cone L^{n+1} is also self-dual, i.e., $(L^{n+1})^* = L^{n+1}$. The summation operator $\oplus$ is defined as

$$A \oplus B = \{z | z = x + y \text{ with } x \in A, y \in B\}.$$

One has that $K_1^* \cap K_2^* = (K_1 \oplus K_2)^*$ and $(K_1 \cap K_2)^* = \mathrm{cl}(K_1 \oplus K_2)$; see Chapter 16 of Rockafellar [16].

In the results that follow, we will assume that

Assumption 1 *$p \in Range(A^T)$, i.e., there exists ξ such that $A^T\xi = p$.*

For convenience in the proofs, we will refer to the existence of a minimum risk arbitrage as the solvability of the system:

$$p^T x < 0,\; y = Ax,\; y \geq_{L^{n+1}} 0. \tag{7}$$

Let $\mathcal{L}$ denote the linear subspace $\{y : y = Ax\}$.

The results below characterize the existence of minimum risk arbitrage in terms of the solvability of a dual conic linear system in each case. This development extends the classical basic no-arbitrage valuation theory as in [6, 9, 14] which use linear programming duality and/or Farkas lemma and its variants to obtain corresponding results in the case where the return uncertainty is captured by a discrete probability distribution defined on a pay-off matrix.

Proposition 1 *(Existence of minimum risk arbitrage)*
There is a minimum risk arbitrage opportunity if and only if the system

$$A^T\lambda = p,\; \lambda \geq_{L^{n+1}} 0 \tag{8}$$

is strongly infeasible, i.e., if and only if

$$dist(\xi + \mathcal{L}^{\perp}, L^{n+1}) > 0, \tag{9}$$

where $A^T\xi = p$.

Proof: ($\Longrightarrow$) Let us pose (8) as the system

$$\lambda \geq_{L^{n+1}} 0, \ \lambda \in \mathcal{L}^{\perp} + \xi, \tag{10}$$

where $A^T\xi = p$ (c.f. Assumption 1). Suppose that there is a minimum risk arbitrage opportunity, but that (8) is not strongly infeasible. That is to say, assume the existence of $d \in \mathcal{L} \cap L^{n+1}$, i.e., the existence of x such that $Ax \geq_{L^{n+1}} 0$ with that $p^T x < 0$ (or, $\xi^T d < 0$ as $p = A^T \xi$). This implies that ξ is not an element of $(\mathcal{L} \cap L^{n+1})^*$. At the same time, (8) is not strongly infeasible so there is $\lambda^i \in L^{n+1}$ and $z^i \in \mathcal{L}^{\perp} + \xi$ such that

$$\|\lambda^i - z^i\| \to 0.$$

This means that $\xi \in (\mathcal{L} \cap L^{n+1})^*$, a contradiction.

($\Longleftarrow$) It suffices to show here that if there is no minimum risk arbitrage opportunity, then (8) cannot be strongly infeasible. No minimum risk arbitrage opportunity means that for any $d \in \mathcal{L} \cap L^{n+1}$ one has $p^T x \geq 0$. But, this is equivalent to saying that

$$\xi \in (\mathcal{L} \cap L^{n+1})^* = \mathrm{cl}(L^{n+1} \oplus \mathcal{L}^{\perp})$$

that is, there is $z^i \in L^{n+1}$ and $y^i \in \mathcal{L}^{\perp}$ such that

$$\|\xi - (z^i - y^i)\| \to 0,$$

which means that (8) is not strongly infeasible. ∎

To state the next result, we define the concept of an asymptotic minimum risk arbitrage opportunity as follows: Let y^i be a sequence such that $y^i \geq_{L^{n+1}} 0$. We say that an asymptotic minimum risk arbitrage opportunity exists if $\mathrm{dist}(y^i, \mathcal{L}) \to 0$ and $\limsup_i \xi^T y^i < 0$. Note that an asymptotic minimum risk arbitrage opportunity does not necessarily imply the existence of a minimum risk arbitrage opportunity, while the converse certainly holds.

Proposition 2 *(Absence of asymptotic minimum risk arbitrage)*
There is no asymptotic minimum risk arbitrage opportunity if and only if the system

$$A^T \lambda = p, \ \ \lambda \geq_{L^{n+1}} 0 \tag{11}$$

has a solution.

Proof: Suppose there exists an asymptotic minimum risk arbitrage opportunity, i.e., $\exists\, d^i \geq_{L^{n+1}} 0$ and $\exists\, l^i \in \mathcal{L}$ such that

$$\|d^i - l^i\| \to 0, \text{ and } \xi^T d^i \leq -\varepsilon < 0.$$

Now, consider any λ feasible for (11). Then, one has

$$(d^i - l^i)^T(\lambda - \xi) = (d^i)^T(\lambda - \xi) \geq \varepsilon > 0,$$

which is impossible since $\|d^i - l^i\| \to 0$.

For the converse, define $\tilde{L}^{n+1} = \mathbf{R}_+ \times L^{n+1}$ and

$$\tilde{\mathcal{L}} = \{\begin{pmatrix} \lambda_0 \\ \lambda \end{pmatrix} : -\lambda_0 \xi + \lambda \in \mathcal{L}^\perp\}.$$

Define also $\tilde{p} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and

$$\tilde{\mathcal{L}}^\perp = \{\begin{pmatrix} y_0 \\ y \end{pmatrix} : y_0 + \xi^T y = 0 \text{ and } y \in \mathcal{L}\}.$$

Now, use Proposition 1 which states that $\exists\, \tilde{d} \in \tilde{\mathcal{L}} \cap \tilde{L}^{n+1}$ with $\tilde{p}^T \tilde{d} < 0$ iff $(\tilde{p} + \tilde{\mathcal{L}}^\perp) \cap \tilde{L}^{n+1}$ is strongly infeasible. Note that the existence of $\tilde{d} \in \tilde{\mathcal{L}} \cap \tilde{L}^{n+1}$ is equivalent to the fact that (11) is feasible.

Suppose that $(\tilde{p} + \tilde{\mathcal{L}}^\perp) \cap \tilde{L}^{n+1}$ is not strongly infeasible, which implies the existence of a sequence $\begin{pmatrix} y_0^i \\ y^i \end{pmatrix}$ such that

$$\begin{pmatrix} 1 + y_0^i \\ y^i \end{pmatrix} \to \tilde{\mathcal{L}}^\perp.$$

This implies that for sufficiently large i one has

$$y^i \geq_{L^{n+1}} 0, \; y^i \to \mathcal{L}, \text{ and } \xi^T y^i \leq -1/2.$$

But, this is nothing other than an asymptotic minimum risk arbitrage opportunity. ∎

Corollary 1 *There is an asymptotic minimum risk arbitrage opportunity, but no minimum risk arbitrage opportunity if and only if the system*

$$A^T\lambda = p, \quad \lambda \geq_{L^{n+1}} 0 \tag{12}$$

is weakly infeasible, i.e., if and only if

$$dist(\xi + \mathcal{L}^\perp, L^{n+1}) = 0. \tag{13}$$

We say that there is no minimum risk arbitrage opportunity if for any x such that $0 \neq Ax \geq_{L^{n+1}} 0$ one has $p^T x > 0$. This definition of no-arbitrage corresponds to the weak form of the no-arbitrage condition in [15].

Proposition 3 *(Absence of minimum risk arbitrage: weak form)*
There is no minimum risk arbitrage opportunity nor weak minimum risk arbitrage opportunity if and only if the system

$$A^T\lambda = p, \quad \lambda \geq_{L^{n+1}} 0 \tag{14}$$

is strongly feasible, i.e., if and only if

$$(\xi + \mathcal{L}^\perp) \cap int\, L^{n+1} \neq \emptyset. \tag{15}$$

In the above proposition int L^{n+1} is the set $\{z \in \mathbf{R}^{n+1} : \sqrt{z_1^2 + z_2^2 + \dots z_n^2} < z_{n+1}\}$. We alternatively denote membership to this set as $z >_{L^{n+1}} 0$.
Proof: Let $x \in \text{int } L^{n+1}$. This is equivalent to saying that $\forall\, 0 \neq y \in L^{n+1}$ it follows that $x^T y > 0$.

Now, let $x \in (\xi + \mathcal{L}^\perp) \cap \text{int } L^{n+1}$. One can express such x as $x = \xi + z$, where $z \in \mathcal{L}^\perp$. Take any $d \neq 0$ such that $d \in \mathcal{L} \cap L^{n+1}$. Then, one has

$$0 < x^T d = (\xi + z)^T d = \xi^T d.$$

For the converse, suppose that $(\xi + \mathcal{L}^\perp) \cap \text{int } L^{n+1} = \emptyset$. Then, $\xi + \mathcal{L}^\perp$ and int L^{n+1} can be separated by hyperplane s such that

$$s^T x \geq 0,\ \forall\, x \in \text{int } L^{n+1},$$

and

$$s^T(\xi + y) \leq 0,\ \forall\, y \in \mathcal{L}^\perp.$$

But, since $-y \in \mathcal{L}^\perp$, one has $s^T\xi \leq 0$ and $s^T y = 0$ for all $y \in \mathcal{L}^\perp$. This implies that one has $s \in \mathcal{L} \cap L^{n+1}$ such that $s^T\xi \leq 0$, a contradiction. ∎

In accordance with [15] one can define the following strong form of the absence of minimum risk arbitrage. We say that there is no minimum risk arbitrage opportunity (strong form) if the optimization problem

$$\min_x \{p^T x | Ax \geq_{L^{n+1}} 0\} \tag{16}$$

has optimal value zero, and furthermore, the conic constraint holds as an inequality for all optimal solutions. We have immediately a sufficient condition.

Lemma 1 *(Absence of minimum risk arbitrage: strong form) There is no minimum risk arbitrage opportunity if the system*

$$A^T \lambda = p, \quad \lambda \geq_{L^{n+1}} 0 \tag{17}$$

is strongly feasible, i.e., if

$$(\xi + \mathcal{L}^\perp) \cap \mathit{int}\ L^{n+1} \neq \emptyset. \tag{18}$$

Proof: Observe that the conic system (17) is simply the dual of the problem (16), with an objective function identically zero. Since by assumption the dual problem is strongly feasible, and above bounded by zero trivially, the primal is solvable by Theorem 2.4.1 of [1], and strong duality is attained, and complementarity between $\lambda >_{L^{n+1}} 0$ and $Ax \geq_{L^{n+1}} 0$ implies that $Ax = 0$. ∎

3 Seeking a Minimum Risk Arbitrage

Although the results of Section 2 characterize the existence and non-existence conditions of a minimum risk arbitrage and of its variants in terms of the solvability of an alternative system, they do not immediately suggest a procedure to decide whether or not such arbitrage opportunities exist. One may have to try to solve both systems, one after the other, as second-order cone programs by interior point methods with polynomial computational complexity in the worst case; see [10]. Therefore, it may be worthwhile to determine in one shot which system is solvable while at the same time exhibiting a solution [8]. We will do so in the present section inspired by [8] by generalizing his results on linear programming duality. To this end, we begin with the following intermediate result.

Lemma 2 *1. System (8) is solvable if and only if the conically constrained linear least squares problem*

$$\min_{\lambda \in L^{n+1}} \frac{1}{2}\|A^T\lambda - p\|_2^2 \tag{19}$$

has optimal value 0.

2. *Problem (19) admits, in any case, at least one optimal solution* λ^D, *possibly non-unique. However, the image* $z^D = A^T\lambda^D$ *of any solution* λ^D *is unique.*

3. *The vector* λ^D *solves (19) optimally if and only if the following variational conditions hold:*

$$\lambda^D \geq_{L^{n+1}} 0, \quad Ar^D \geq_{L^{n+1}} 0, \quad \textit{and} \quad (\lambda^D)^T Ar^D = 0, \tag{20}$$

where $r^D = A^T\lambda^D - p$.

Proof: Part 1 is obvious. Since the set $Z = \{z = A^T\lambda : \lambda \geq_{L^{n+1}} 0\}$ is a closed convex set, Part 2 follows by direct application of the classical Minimum Distance to a Convex Set Theorem (Theorem 1, pp. 69–70) in Luenberger [11]. Using the same theorem, a necessary and sufficient condition that z^D be the unique minimizer in (19) is that $(p-z^D)^T(z-z^D) \leq 0$ for all $z \in Z$. This condition is equivalent to the condition that $(r^D)^T(z-z^D) \geq 0$ for all $z \in Z$. Take $z = 0$ since $0 \in Z$ in this inequality to obtain $(r^D)^T z^D \leq 0$. Take $z = 2z^D$ (as $2z^D \in Z$) in the inequality to get $(r^D)^T z^D \geq 0$. Therefore, we have $(r^D)^T z^D = 0$. Note that the inequality $(r^D)^T A^T(\lambda - \lambda^D) \geq 0$ for all $\lambda \geq_{L^{n+1}}$ implies $Ar^D \geq_{L^{n+1}} 0$. ∎

Proposition 4 *(Explicit decidability of minimum risk arbitrage)*
Let λ^D *be any optimal solution to the conically constrained least squares problem (19). Then the associated residual*

$$r^D = A^T\lambda^D - p$$

is unique, and $r^D = 0$ *if and only if* λ^D *solves (8). Otherwise, when* $r^D \neq 0$, *one has a minimum risk arbitrage opportunity in that* $Ar^D \geq_{L^{n+1}} 0$ *and* $p^T r^D = -\|r^D\|_2^2$.

Proof: The uniqueness of r^D directly follows from the previous lemma. To complete the proof note that

$$p^T r^D = (\lambda^D)^T A r^D - \|r^D\|_2^2 = -\|r^D\|_2^2,$$

by (20). ∎

In the light of the previous result, we can decide whether or not there is a minimum risk arbitrage opportunity by solving the conically constrained least squares problem (19). This can be achieved by the use of polynomial time interior point methods as detailed in the monograph [13]. Let λ^D be a solution to (19). If $A^T\lambda^D = p$ then we can conclude that there does not exist x such that

$$Ax \geq_{L^{n+1}} 0, \;\; p^T x < 0.$$

In other words, the problem

$$\min_x \{p^T x | Ax \geq_{L^{n+1}} 0\}$$

has optimal value zero, thus confirming weak form of no-arbitrage. Otherwise, the unique residual $r^D = A^T\lambda^D - p$ is satisfies

$$Ar^D \geq_{L^{n+1}} 0, \;\; p^T r^D < 0,$$

and thus is a minimum risk arbitrage opportunity.

4 Transaction Costs

In the present section we investigate an extension of the arbitrage concept to the case of convex transaction costs following Dermody and Prisman [5] who extended the no-arbitrage and valuation theory to markets with non-proportional transaction costs. In particular, they assume a convex transaction cost function f which is a function of the portfolio vector x such that $f(0) = 0$, i.e., no trading costs nothing, and $f(x) > 0$ for any $x \neq 0$.

Inspired by [5], we can develop a transaction cost version of the minimum risk arbitrage concept as follows:

There exists a portfolio x such that

$$\bar{r}^T x - \theta\sqrt{x^T Q x} \geq 0, \;\; p^T x + f(x) < 0 \qquad (21)$$

We have immediately a sufficient condition for no-arbitrage. We say that there is no minimum risk arbitrage opportunity in the presence of convex transaction costs if for any x such that $A^T x \geq_{L^{n+1}} 0$ one has $p^T x + f(x) \geq 0$. This corresponds to the weak form of no-arbitrage as in [15]. Note the difference between this definition and the one just before Proposition 3.

Lemma 3 *(Absence of minimum risk arbitrage with convex transaction costs) There is no minimum risk arbitrage opportunity in the presence of convex transaction costs if the system*

$$A^T \lambda = p, \quad \lambda \geq_{L^{n+1}} 0 \tag{22}$$

is feasible, i.e., if

$$(\xi + \mathcal{L}^{\perp}) \cap L^{n+1} \neq \emptyset. \tag{23}$$

Proof: Let $x \in (\xi + \mathcal{L}^{\perp}) \cap L^{n+1}$. One can express such x as $x = \xi + z$, where $z \in \mathcal{L}^{\perp}$. Take any d such that $d \in \mathcal{L} \cap L^{n+1}$. Then, one has

$$0 \leq x^T d = (\xi + z)^T d = \xi^T d.$$

Observing that $f(x) \geq 0$ for any portfolio x, the result follows. ∎

Clearly, the converse of the above result may fail to hold under further assumptions on f.

Inspired by Theorem 1 of [5] we can give a characterization (involving a weaker condition compared to the previous lemma) of the absence of an arbitrage opportunity in the presence of transaction costs. We need the following

Assumption 2 *There exists x such that $Ax >_{L^{n+1}} 0$,*

before we proceed.

Proposition 5 *(No-arbitrage in weak form in the presence of convex transaction costs)*
There is no minimum risk arbitrage opportunity (weak form) in the presence of convex transaction costs, if and only if there exists $\lambda \geq_{L^{n+1}} 0$ such that

$$A^T \lambda - p = v \text{ where } v \in \partial f(0), \tag{24}$$

or, equivalently, such that $Z \cap \partial f(0)$ is non-empty, where $Z = \{z | z = A^T \lambda - p;\ \lambda \geq_{L^{n+1}} 0\}$.

Proof: Dermody and Prisman [5] pose the absence of arbitrage using an optimization problem derived from (21). Similarly, one can write the no-arbitrage condition as:

$$\max_x \{-p^T x - f(x) | Ax \geq_{L^{n+1}} 0\} = 0. \tag{25}$$

Since we are dealing with a convex programming problem, under Assumption 2, and using Theorem 28.2 of [16], the problem (25) possesses a KKT vector. Since its optimal value is zero, the optimal value of the dual is also zero by Theorem 30.4 (d) and Theorem 30.3 (c) of [16]. Hence, the conditions of Theorem 30.5 of [16] are satisfied, and the dual of (25) given below has an optimal value of zero and possesses an optimal solution, $\lambda \geq_{L^{n+1}} 0$,

$$\min_{\lambda \geq_{L^{n+1}} 0} \max_{x : Ax \geq_{L^{n+1}} 0} \{-p^T x - f(x) + \lambda^T Ax\}. \tag{26}$$

This implies that some $\lambda^* \geq_{L^{n+1}} 0$ satisfies

$$\max_{x : Ax \geq_{L^{n+1}} 0} \{-p^T x - f(x) + (\lambda^*)^T Ax\} = 0, \tag{27}$$

which implies in turn

$$x^T (A^T \lambda^* - p) \leq f(x) \tag{28}$$

for every feasible x. But, this is precisely the subgradient inequality at 0. Therefore, $(A^T \lambda^* - p) \in \partial f(0)$ and $\lambda^* \geq_{L^{n+1}} 0$ satisfies conditions (24).

For the converse, it suffices to observe that (25) and (24) are in fact equivalent. This follows, mutatis mutandis, from [5] (pp. 71–72) and Lemma 1; see remarks after Lemma 4. ∎

The strong form of the no-arbitrage condition as in [15] in our context is

$$\max_x \{-p^T x - f(x) | Ax \geq_{L^{n+1}} 0\} = 0, \tag{29}$$

and, that the conic constraint holds as equality at all optimal solutions. A sufficient condition for this strong form no-arbitrage condition is obtained by direct extension of Lemma 2 of [5].

Lemma 4 *(No-arbitrage in strong form in the presence of convex transaction costs)*
There is no minimum risk arbitrage opportunity (strong form) in the presence of convex transaction costs if there exists $\lambda >_{L^{n+1}} 0$ *such that*

$$A^T \lambda - p = 0. \tag{30}$$

The proof follows, mutatis mutandis, from the proof of Lemma 1 of [5]. All that needs to be done is to modify all occurrences of $d \geq 0$ in the proof of Lemma 1 of [5] to $d \geq_{L^{n+1}} 0$, and all occurrences of $\max_x$ to $\max_{x:Ax \geq_{L^{n+1}} 0}$. Notice that the above lemma subsumes Lemma 1 as a special case where f is identically zero.

5 Concluding Remarks

In this paper we developed the minimum risk arbitrage concept, characterized the existence and absence of it and of its variants via conic duality, generalized Farkas type lemmas and convex analysis. We also offered a decidability result which allows one to detect arbitrage by solving a conically constrained least squares problem. We also considered the case of transaction costs.

Presently, we do not know whether the converse of Lemma 4 is true. However, we believe this is quite unlikely as second-order cone programs do not possess, in general, the strict complementarity property of linear programs. It would be interesting to investigate this issue in the future.

Another interesting extension of this paper would be to consider markets with frictions where investors face possibly nonlinear taxes. In particular, Prisman [15] extends the no-arbitrage concept to markets with frictions. A version of the above problem in our arbitrage context is certainly worth a separate investigation which will be undertaken subsequently. Yet, another avenue for further research is a multiperiod extension of the above results.

Finally, whether the proposed arbitrage model is of any practical value is for practitioners to judge.

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