Discrete convexity and unimodularity. I

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1 Introduction

In this article we introduce a theory of convexity for the lattice of integer points $\mathbb{Z}^n$, which we call a theory of *discrete convexity*.

What subsets $X \subset \mathbb{Z}^n$ could be called "convex"? One property seems indisputable: X should coincide with the set of all integer points of its convex hull $\mathrm{co}(X)$. Let us call such sets *pseudo-convex*. The resulting class $\mathcal{QC}$ of all pseudo-convex sets is closed under intersection. However, it falls short of a few other desirable properties for usual convex sets, in particular the separation property and stability under summation. But as we know, the separation property is a cornerstone of the classical theory of convex sets and functions and we wish our theory to have this property.

Discrete mathematics (see [3, 7, 6]) has a class of polytopes called *generalized polymatroids* (g-polymatroids). Now, we define a polymatroidal set (or *PM*-set) to be a (finite) set $X \subset \mathbb{Z}^n$ such that $X = \mathrm{co}(X) \cap \mathbb{Z}^n$ and the polytope $\mathrm{co}(X)$ is a generalized polymatroid. These sets turn out to have a few pleasant features and among them, that the sum of *PM*-sets is a *PM*-set, and that non-intersecting *PM*-sets can be separated by some linear functional.

We understand therefore, that, by narrowing our focus to the class of *PM*-sets, we end up with a "nicer" theory of discrete convexity. But then a new range of questions arises. Can we extend this class, without losing in the process the nice properties which precisely made us consider it at the very beginning? Do other classes exist which exhibit similar properties? If so, how are they to be constructed or described?

In this article, we decisively tackle these issues. In Section 2, we introduce the notion of a class of discrete convexity, specifically, a class of pseudo-convex subsets of $\mathbb{Z}^n$ for which the separation property holds. However, before going further, we notice that usual convexity is preserved under summation and under intersection. Requiring that a class of discrete convexity be closed under summation and under intersection leads to a poor in content theory (see Theorem 4). For example, the class of *PM*-sets, whilst closed under summation, is not closed under intersection. To obtain more interesting theory, we consider separately the following two groups of classes: *S*-classes, which are closed under summation, and *I*-classes, which are closed under intersection. As it turns out, these classes are dual in some sense.

As to what concerns the construction of *DC*-classes, the main point is we can translate the separation property of an S-class into an algebraic property, namely, the semigroup property of a collection of pure subgroups of $\mathbb{Z}^n$. We call such semigroups *pure systems*. We discuss their properties in Section 3. In Sections 4 and 5, we construct *S*-classes and *I*-classes by means of these pure systems. In particular, we obtain generalizations of Edmonds'polymatroid intersection theorem and the Hoffman-Kruskal theorem as consequences of these constructions.

The pure systems which correspond to classes of discrete convexity of finite subsets of $\mathbb{Z}^n$, bear a close relation to unimodular systems in $\mathbb{Z}^n$. The latter are invariant versions of totally unimodular matrices and we discuss their properties in Section 6.

Let us explain how to construct *S*-classes of discrete convexity via unimodular systems. Suppose that $\mathcal{R}$ is a unimodular system in $\mathbb{Z}^n$; a (finite) set $X \in \mathbb{Z}^n$ is called $\mathcal{R}$-convex if $X = \mathrm{co}(X) \cap \mathbb{Z}^n$ and each edge of the polytope $\mathrm{co}(X)$ is parallel to some vector $r \in \mathcal{R}$. We show that the class of $\mathcal{R}$-convex sets is an *S*-class. Moreover, this construction yields all *S*-classes of finite sets. In particular, the class of *PM*-sets corresponds to the following unimodular system, the root system $\mathbb{A}_n$ of vectors $\pm e_i$, $e_i - e_j$, $i, j = 1, \ldots, n$.

The *I*-classes result from a dual construction. A subset of $(\mathbb{Z}^n)^*$ is $*\mathcal{R}$-convex if it consists of integer solutions to a system of linear inequalities of the form $p(r) \leq b(r)$, $r \in \mathcal{R}$, where $p \in (\mathbb{Z}^n)^*$ and $b(r)$ are integers. The set of $*\mathcal{R}$-convex sets forms an *I*-class, and again all *I*-classes of finite sets are obtained through this construction.

In Section 7, we give "exterior" descriptions of both $\mathcal{R}$-convex and $*\mathcal{R}$-convex polyhedra. These descriptions are based on characterizing their supporting functions. In a companion article, we elaborate upon "interior" descriptions of such sets. We develop also the corresponding theory of discretely convex functions (noting that a similar theory for *PM*-sets was elaborated by Murota [11]).

Notations. We shell prefer to work here with abstract free abelian group of finite type M instead of $\mathbb{Z}^n$, since we do not want to peg our theory to some basis. Of course, M is isomorphic to $\mathbb{Z}^n$ for an appropriate n, but non-canonically. $V = M \otimes \mathbb{R} \cong \mathbb{R}^n$ denotes the ambient vector space. Points in M are called *integer* points of V. Given a subset $P \subset V$, we denote by $P(\mathbb{Z}) = P \cap M$ the set of integer points of P.

$M^* = \mathrm{Hom}(M, \mathbb{Z})$ denotes the group dual to M, $V^* = M^* \otimes \mathbb{R}$ is the dual of the vector space V. For $Q \subset V^*$, we put $Q(\mathbb{Z}) = Q \cap M^*$.

Let X, Y be subsets of V, then $X+Y = \{x+y,\ x \in X,\ y \in Y\}$ denotes the (Minkowski) sum of X and Y; $X - Y$ is understood in a similar fashion. Let $\mathrm{co}(X)$ denote the convex hull of X in V. $\mathbb{Z}(X)$ is the abelian subgroup in V generated by X, that is the set of linear combinations of the form $\sum_x m_x x$, where $x \in X$ and $m_x \in \mathbb{Z}$. The ($\mathbb{R}$-) vector subspace generated by X is denoted by $\mathbb{R}X$, and the convex cone by $\mathbb{R}_+X$. Then $\mathbb{R}_0X = \mathbb{R}(X - X)$ is the vector subspace parallel to X. Finally, $\mathbb{Z}_0X = \mathbb{Z}(X - X)$.

2 Discrete convexity: the basics

The issue here is to characterize the subsets X of the group M ($\cong \mathbb{Z}^n$) we would be willing to call "convex"? Our first requirement seems indisputable: suppose some integer point $m \in M$ can be written as a convex combination of points from X, then it should be in X. In other words, we should want the following equality

$$X = \mathrm{co}(X)(\mathbb{Z})$$

to hold. However, in order to make the next steps, we need to add some further requirements on the convex set $\mathrm{co}(X) \subset V$. The simplest one would be that $\mathrm{co}(X)$ is a polytope, this is equivalent to finiteness of the set X. We propose that the reader, at first reading, assume that this setting hold. However, sometimes we shall need to work with infinite sets. Then taking the convex set $\mathrm{co}(X)$ to be a polyhedron will meet our yearning for both generality and simplicity. Recall that a polyhedron is defined by the intersection of some finite collection of (closed) halfspaces of V. For example, linear sub-varieties of V, or polytopes (convex hull of some finite subset in V) are polyhedra. For more details about polyhedra, see [9] or [12].

Definition 1 *A subset $X \subset M$ is said to be* pseudo-convex *if $X = co(X)(\mathbb{Z})$ and $co(X)$ is a polyhedron.*

We denote by $\mathcal{QC}$ the set of pseudo-convex sets.

Definition 2 *A polyhedron $P \subset V$ is* rational *if it is given by a finite system of linear inequalities with rational (or integer) coefficients. A polyhedron P is* integral *if it is rational, and if every (non-empty) face of P contains an integer point.*

For example, a polytope is integral if and only if all its vertices are integer points.

Proposition 1 *Suppose $X \subset M$. The following assertions are equivalent:*

a) X is pseudo-convex;

b) $X = P(\mathbb{Z})$ for some integral polyhedron $P \subset V$;

c) X is the set of integral solutions of a finite system of linear inequalities with integer coefficients.

Proof. The implication $a) \Rightarrow b)$ is almost obvious; it suffices to take P to be $\mathrm{co}(X)$. The implication $b) \Rightarrow c)$ is obvious. Finally, implication $c) \Rightarrow a)$ is precisely Meyer's theorem (see, for example, [13], Theorem 16.1). Q.E.D.

Denote by $\mathcal{IPh}$ the class of all integral polyhedra in V. By Proposition 1, we have the natural bijection between the class $\mathcal{IPh}$ and the class $\mathcal{QC}$, which is given by the mappings $P \mapsto P(\mathbb{Z})$ and $X \mapsto \mathrm{co}(X)$. Both these classes are closed under integer translations ($X \to X + m$, $m \in \mathbb{Z}^n$), reflection ($X \to -X$), and faces ($X \to X \cap F$, where F is some face of the polyhedron $\mathrm{co}(X)$). Furthermore, the class $\mathcal{QC}$ is closed under intersection and does not closed under summation, whereas the class $\mathcal{IPh}$ is closed under summation and does not closed under intersection (the sum of two pseudo-convex sets need not be pseudo-convex, while the intersection of integral polyhedra need not be integral).

Now consider the following simple example in $\mathbb{Z}^2$. Let $X = \{(0,0),(1,1)\}$, $Y = \{(0,1),(1,0)\}$. Both X and Y are pseudo-convex. Despite that X and Y do not intersect, they can not be separated by a linear functional (or hyperplane).

This example points out that in order for a theory of discrete convexity to have the separation property, we need to consider narrower classes of subsets of M than the class $\mathcal{QC}$.

We say that a class $\mathcal{K} \subset \mathcal{QC}$ is *ample* if $\mathcal{K}$ is closed under a) integer translations, b) reflection and c) faces. In the same way we understand ampleness of a polyhedral class $\mathcal{P} \subset \mathcal{IPh}$.

Proposition 2 *Let $\mathcal{K} \subset \mathcal{QC}$ be an ample class. The following four properties of $\mathcal{K}$ are equivalent:*

(Add) for every $X, Y \in \mathcal{K}$ the sets $X \pm Y$ are pseudo-convex;

(Sep) if sets X and Y of $\mathcal{K}$ do not intersect, then there exists (integer) linear functional $p : V \longrightarrow \mathbb{R}$ such that $p(x) > p(y)$ for any $x \in X$, $y \in Y$;

(Int) if sets X and Y of $\mathcal{K}$ do not intersect, then the polyhedra co(X) and co(Y) do not intersect as well;

(Edm) for every $X, Y \in \mathcal{K}$ the polyhedron co$(X) \cap$ co(Y) is integral.

Proof.

$(Add) \Rightarrow (Sep)$. If X and Y have an empty intersection, then $0 \notin X - Y$. Since the set $X - Y$ is pseudo-convex, 0 does not belong to the polyhedron $\mathrm{co}(X - Y) = \mathrm{co}(X) - \mathrm{co}(Y)$. Hence there exists a linear (integral) functional $p : V \to \mathbb{R}$ which is strictly positive on $\mathrm{co}(X - Y)$. Therefore $p(x) > p(y)$ for $x \in X$ and $y \in Y$.

$(Sep) \Rightarrow (Int)$. This one is obvious.

$(Int) \Rightarrow (Add)$. Let us show that $X - Y$ is pseudo-convex. Since $\mathrm{co}(X - Y) = \mathrm{co}(X) - \mathrm{co}(Y)$ is a polyhedron, we need to prove that $X - Y = \mathrm{co}(X - Y) \cap M$. Suppose the integer point m lies in $\mathrm{co}(X - Y) = \mathrm{co}(X) - \mathrm{co}(Y)$. Then the polyhedra $\mathrm{co}(X)$ and $m + \mathrm{co}(Y) = \mathrm{co}(m + Y)$ intersect. Applying (Int) to the sets X and $m + Y$, we see that these sets also intersect, that is $m \in X - Y$.

$(Edm) \Rightarrow (Int)$. This implication is obvious.

$(Int) \Rightarrow (Edm)$. Suppose $X, Y \in \mathcal{K}$, $P = \mathrm{co}(X)$, $Q = \mathrm{co}(Y)$. We need to show that $P \cap Q$ is an integral polyhedron. Obviously $P \cap Q$ is rational. Therefore we need to establish that every (non-empty) face of $P \cap Q$ contains an integer point. We assume here, without loss of generality, that the face is minimal.

Suppose F is a minimal (non-empty) face of the polyhedron $P \cap Q$. Let P' (resp. Q') be a minimal face of P (resp. Q) which contains F. We claim that $F = P' \cap Q'$.

Projecting V along F, we may suppose additionally that F is of dimension 0. That is F consists of a single point, which is a vertex of $P \cap Q$. Suppose, on the contrary, that $P' \cap Q'$ contains some other point a. Since the point F is relatively interior both in P' and in Q', then F is an interior point of some segment $[a, b]$, lying in both P' and Q'. But in such a case the segment $[a, b] \subset P' \cap Q' \subset P \cap Q$, and F can not be a vertex of $P \cap Q$. Contradiction.

Thus, $F = P' \cap Q'$. Since our class $\mathcal{K}$ is closed under faces, the sets $P'(\mathbb{Z})$ and $Q'(\mathbb{Z})$ are members of $\mathcal{K}$. The property *(Int)* implies that the sets $P'(\mathbb{Z})$ and $Q'(\mathbb{Z})$ intersect. Because of this, F is an integer singleton. Q.E.D.

Definition 3 *An ample class* $\mathcal{K} \subset \mathcal{QC}$ *is a* class of discrete convexity *(DC-class) if it possesses anyone of the properties from Proposition 2.*

On the language of integral polyhedra, the definition of discrete convexity is formulated as follows. A class $\mathcal{P}$ of integral polyhedra is a *polyhedral class of discrete convexity* if it is ample and the following variant of Edmonds' condition holds.

(Edm') The intersection of any two polyhedra from $\mathcal{P}$ is an integral polyhedron (not necessarily in $\mathcal{P}$).

According to Proposition 2, the equivalent requirement is:

(Add') $(P + Q)(\mathbb{Z}) = P(\mathbb{Z}) + Q(\mathbb{Z})$ for every $P, Q \in \mathcal{P}$.

Let us now give a few examples.

Example 1. In the one-dimensional case, that is when $M \cong \mathbb{Z}$, the class $\mathcal{QC}$ of all pseudo-convex sets is a *DC*-class. This is not the case in higher dimensions of course.

The class of integral rectangles in the plane $\mathbb{R}^2$ is a *DC*-class. More generally, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are *DC*-classes in the groups M_1 and M_2, respectively, then the class of sets of the form $X_1 \times X_2$ with $X_i \in \mathcal{K}_i$, $i = 1, 2$, is a *DC*-class in $M_1 \times M_2$ as well.

Example 2. *Hexagons.* Consider now a more interesting class $\mathcal{H}$ of polyhedra in $\mathbb{R}^2$. This class consists of polyhedra defined by the inequalities $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, $c \le x_1 + x_2 \le d$, where a_1, a_2, b_1, b_2, c and d are integers. It is easy to check that this hexagon (generally speaking, this hexagon can be degenerated to a polyhedron with smaller number of edges) has integer vertices. Obviously, $\mathcal{H}$ is closed under integer translations, reflection, and faces. Since the intersection of hexagons yields a hexagon, we conclude that $\mathcal{H}$ is a polyhedral *DC*-class.

Example 3. *Base polyhedra.* This is one of the possible higher-dimensional generalizations of Example 3. Let N be a finite set, and $V = (\mathbb{R}^N)^*$, that is we interpret elements of V as measures on the set N. Recall, that a function $b : 2^N \to \mathbb{R} \cup \{+\infty\}$ is called *submodular* if for any S, $T \subset N$, the following inequality holds

$$b(S) + b(T) \ge b(S \cup T) + b(S \cap T).$$

The elements of V can be viewed as modular functions, i.e. functions which fulfill the above-written definition of submodularity with equality.

A *base polyhedron* is the polyhedron

$$B(b) = \{x \in V \mid x(S) \leq b(S),\ S \subset N,\ \text{ and } x(N) = b(N)\},$$

where b is a submodular function. Obviously, the class $\mathcal{B}$, which consists of base polyhedra with integer-valued b, is closed under integer translations and under reflection. One can show, that it is closed under faces, and hence, each base polyhedron has integral vertices. The well-known theorem by Edmonds [3] ensures that the property (*Edm*) obtains, and thus $\mathcal{B}$ is a polyhedral *DC*-class. The reader might find details of the proof of these properties of base polyhedra in [7], or see our Section 7.3.

Example 4. Here we give another high-dimensional generalization of Example 2. Let N be a finite set, and let $V^* = \mathbb{R}^N$ be the space of functions on N. Consider the class $\mathcal{L}$ of polyhedra in V^*, given by the inequalities of the form $a_i \leq x(i) \leq b_i$ and $a_{ij} \leq x(i) - x(j) \leq b_{ij}$, where $i, j \in N$, and a's and b's are integers. We claim that these polyhedra are integral. Indeed, their vertices are given by equalities of the form $x(i) = c_i$ and $x(i) - x(j) = c_{ij}$ where c's are integers. It is clear that x is an integer point.

Thus, the class $\mathcal{L}$ consists of integral polytopes. Since it is closed under intersection, the axiom (*Edm*') is satisfied automatically, and $\mathcal{L}$ is a polyhedral *DC*-class. □

We give a general construction of *DC*-classes in Sections 4 and 5.

In the classical context, convexity is preserved under summation and intersection. It would be natural therefore to require these properties for the discrete set-up. For example, both the classes of segments and hexagons and their products possess these properties. Moreover (see Theorem 4), these cases exhaust *DC*-classes, closed under both summation and intersection. The class $\mathcal{B}$ described in Example 3 is closed under summation, but not under intersection (if $|N| > 3$). Similarly, the class $\mathcal{L}$ described in Example 4 is closed under intersection, but not under summation (if $|N| > 2$). Therefore, when we consider classes closed under summation and classes closed under intersection separately, more interesting theory of discrete convexity is obtained.

Definition 4 *An ample class $\mathcal{K}$ of pseudo-convex sets is called an* S-class *if $X + Y \in \mathcal{K}$ for any $X, Y \in \mathcal{K}$.*

In particular, $X - Y \in \mathcal{QC}$ for any $X, Y \in \mathcal{K}$, and, thus, an S-class is a DC-class. However in order to characterize polyhedral S-classes, we have to require both that the class be closed under summation and the axiom *(Add')* be satisfied. Note that the intersection of two polyhedra of a polyhedral S-class is an integral polyhedron, but need not be a polyhedron of this class.

Definition 5 *An ample class $\mathcal{P}$ of integral polyhedra is called a* polyhedral I-class *if $P \cap Q \in \mathcal{P}$ for any $P, Q \in \mathcal{P}$.*

An I-class is a DC-class, since the axiom *(Edm')* holds. Let P and Q be polyhedra in an I-class, then $P(\mathbb{Z}) + Q(\mathbb{Z})$ is a pseudo-convex set, though $P + Q$ need not be a polyhedron of this class.

3 Homogenization of S-classes: Pure systems

Here we associate to any S-class of pseudoconvex sets a simpler class consisting of pure subgroups of M. A subgroup S of M is *pure* if $S = \mathbb{R}(S) \cap M$. In other words, a subgroup $S \subset M$ is pure iff S is pseudo-convex. Let us recall a few facts about pure subgroups.

1. A subgroup $S \subset M$ is pure iff the factor group M/S is torsion free.
2. A subgroup $S \subset M$ is pure iff the factor group M/S is free.
3. A subgroup $S \subset M$ is pure iff S is a direct summand of M, i.e., there exists a subgroup $T \subset M$ such that $M = S \oplus T$.
4. Let S be a pure subgroup of M, and let $f : M' \to M$ be a homomorphism of abelian groups; then $f^{-1}(S)$ is a pure subgroup of M'. In fact, the factor-group $M'/f^{-1}(S)$ is embedded in M/S, and, therefore, is torsion free. In particular, we have the following.
5. Let S be a pure subgroup of M; then, for any subgroup $M' \subset M$, the group $S \cap M'$ is a pure subgroup of M'.

In general case, the sum of pure subgroups of M need not be a pure subgroup of M.

Definition 6 *A set* $\mathcal{U}$ *of pure subgroups of* M *is called* pure system *if* $\mathcal{U}$ *is closed under summation, that is* $S + S' \in \mathcal{U}$ *for any* $S, S' \in \mathcal{U}$*. Subgroups* $S \in \mathcal{U}$ *(or, sometimes, the corresponding vector subspaces* $\mathbb{R}(S) \subset V$*) are called* flats *of the pure system* $\mathcal{U}$*.*

To any collection $\mathcal{K}$ of pseudo-convex sets, we associate the following collection of subgroups of M

$$\mathcal{U}(\mathcal{K}) = \{\mathbb{Z}_0(X), X \in \mathcal{K}\}.$$

We say that $\mathcal{U}(\mathcal{K})$ is the *homogenization* of $\mathcal{K}$.

Proposition 3 *Let* $\mathcal{K}$ *be an* S*-class of pseudo-convex subsets of* M*. Then* $\mathcal{U}(\mathcal{K})$ *is a pure system.*

Proof. Let us check that $\mathbb{Z}_0(X) = \mathbb{Z}(X - X)$ is a pure subgroup of M for any $X \in \mathcal{K}$. In fact, $\mathbb{R}_0(X)$ is equal to the union of polyhedra $[n]\mathrm{co}(X - X)$, $n = 1, 2, \ldots$, where $[n]X$ denotes the sum of n exemplars of X. Therefore, any point of $\mathbb{R}_0(X)$ is covered by some polyhedron $[n]\mathrm{co}(X - X)$. Accordingly to (Add), any integer point of such a polyhedron is equal to the sum of n integer points of $X - X$. Therefore, any integer point of $\mathbb{R}_0(X)$ belongs to $\mathbb{Z}_0(X)$, and, hence, $\mathbb{Z}_0(X)$ is pure.

Observe, that there holds $\mathbb{Z}_0(X) + \mathbb{Z}_0(Y) = \mathbb{Z}_0(X + Y)$. Thus, $\mathcal{U}(\mathcal{K})$ is closed under summation. Q.E.D.

In the next section we show how to dehomogenize a pure system $\mathcal{U}$, i.e. how to construct an S-class $\mathcal{P}h(\mathcal{U}, \mathbb{Z})$.

Let us consider the homogenization of the class $\mathcal{B}$ of base polyhedra from Example 3.

Example 5. Recall, that $V = (\mathbb{R}^N)^*$ is the space of measures on a finite set N. Let $B(b)$ be the base polyhedron defined by a submodular function $b : 2^N \to \mathbb{R} \cup \{+\infty\}$. It is easy to check that $-B(b)$ is the base polyhedron defined by the submodular function $b'(S) := b(N \setminus S) - b(N)$, $S \subset N$. Therefore, the base polyhedron $n(B(b) - B(b))$ is defined by the submodular function $b^n(S) = n(b(S) + b(N \setminus S) - b(N))$, $S \subset N$. Let $\mathcal{F}(b)$ be the set of subsets of N which satisfy the following equation $b(S) + b(N \setminus S) = b(N)$. Then $\emptyset$, $N \in \mathcal{F}(b)$, and $N \setminus S \in \mathcal{F}(b)$ with any $S \in \mathcal{F}(b)$. Submodularity of b implies that both $S \cap T$ and $S \cup T$ belong to $\mathcal{F}(b)$ with any S, $T \in \mathcal{F}(b)$. Thus, $\mathcal{F}(b)$ is a Boolean algebra. Since $\mathbb{R}_0(B(b)) = \mathbb{R}(B(b) - B(b)) = \cup_n B(b^n)$, we obtain that the linear subspace $\mathbb{R}_0(B(b))$ is defined by the function $b^\infty : 2^N \to \{0, +\infty\}$, $b^\infty(S) = \lim_{n\to\infty} b^n(S)$, $S \subset N$. Or, equivalently, $\mathbb{R}_0(B(b))$ is given by the following list of equations

$$x(S) = 0, \;\; S \in \mathcal{F}(b).$$

Therefore, flats of $\mathcal{U}(\mathcal{B})$ are in one-to-one correspondence with Boolean subalgebras of 2^N. Note, that a Boolean subalgebra can be viewed as an equivalence relation $\approx$ on N, or, equivalently, as a surjection $f : N \to N' = N/\approx$. Indeed, the inverse images of subsets $S' \subset N'$ form a Boolean subalgebra of 2^N.

Thus, the set of flats of $\mathcal{U}(\mathcal{B})$ is in one-to-one correspondence with the set of the equivalence relations on N. For an equivalence relation $\approx$, the corresponding flat $F(\approx)$ consists of measures $x \in V$ such that $x(S) = 0$ for each equivalence class S of the relation $\approx$. The dimension of this flat $F(\approx)$ is equal to $|N| - |N'|$.

Consider for instance one dimensional flats. These flats correspond to those equivalence relations, who possess a single class of equivalence of cardinality 2, all others ones being of cardinality 1. For example the flat $\mathbb{Z}(e_i - e_j)$ generated by the vector $e_i - e_j$, corresponds to the equivalence relation whose 2-element class of equivalence is $\{i, j\}$ (where (e_i), $i \in N$, denote the standard basis of V, else e_i is the Dirac measure at the point $i \in N$).

Similarly, flats of codimension 1 correspond to dichotomic equivalence relations (i.e. relations with only two classes of equivalence, say T and $N \setminus T$).

Let us point out that the one-dimensional flats $\mathbb{Z}(e_i - e_j)$, $i, j \in N$, generate the pure system $\mathcal{U}(\mathcal{B})$ as a semigroup. We denote by $\mathbb{A}(N)$ this pure system. □

Let us return to general pure systems.

Definition 7 *Let $\mathcal{U}$ and $\mathcal{U}'$ be pure systems in the groups M and M' respectively. A group homomorphism $\phi : M \to M'$ is called a* morphism of pure systems $\mathcal{U}$ *and* $\mathcal{U}'$ *if $\phi(S) \in \mathcal{U}'$ for any $S \in \mathcal{U}$. (Denote by $\phi : (M, \mathcal{U}) \to (M', \mathcal{U}')$ such a morphism.)*

Consider some examples of morphisms of pure systems.

Let $f : N \to N'$ be a mapping of sets. Then the natural homomorphism $f_* : (\mathbb{Z}^N)^* \to (\mathbb{Z}^{N'})^*$, is a morphism of the pure system $\mathbb{A}(N)$ in $\mathbb{A}(N')$.

Another example. Let $\mathcal{U}$ be a pure system in a group M, and $S \in \mathcal{U}$. Since S is a pure subgroup of M, the factor group $M' := M/S$ is free. Denote by $\pi_S : M \to M'$ the canonic projection, $\pi_S(T) = (T + S)/S$. We claim that the image of $\mathcal{U}$ under π_S forms the pure system $\mathcal{U}' := \{\pi_S(T), T \in \mathcal{U}\}$ in M'. Indeed, check that $\pi_S(T)$ is a pure subgroup of M' for $T \in \mathcal{U}$. We have $M'/\pi_S(T) = M/(S + T)$. Since $S + T$ is pure, $M'/\pi_S(T)$ is torsion free. Moreover, $\pi_S(T) + \pi_S(R) = \pi_S(T + R)$. Therefore, $\mathcal{U}'$ is a pure system, and, moreover, $\pi_S : (M, \mathcal{U}) \to (M', \mathcal{U}')$ is a morphism of pure systems.

There are several simple ways to construct pure systems. Projections are one of them, here are other two.

The first. Let $\mathcal{U}$ be a pure system in M and let $f : M' \to M$ be a homomorphism of groups. Consider the system

$$f^{-1}\mathcal{U} = \{f^{-1}(S) \text{ such that } S \in \mathcal{U} \text{ and } S \subset f(M')\}.$$

We assert that $f^{-1}\mathcal{U}$ is a pure system. We have pointed out that the inverse image of a pure subgroup is a pure subgroup. It remains to notice that $f^{-1}(S) + f^{-1}(S') = f^{-1}(S + S')$ for $S' \subset f(M')$. Obviously, f defines a morphism of pure systems $f^{-1}\mathcal{U}$ into $\mathcal{U}$.

The second. Let $\mathcal{U}$ and $\mathcal{U}'$ be two pure systems in M. Then the sum of $\mathcal{U}$ and $\mathcal{U}'$, $\mathcal{U} + \mathcal{U}' := \{S + S',\ S \in \mathcal{U}, S' \in \mathcal{U}'\}$, is a pure system if and only if all the sums $S + S'$, $S \in \mathcal{U}$, $S' \in \mathcal{U}'$, are pure subgroups of M.

Let $(M_1, \mathcal{U}_1)$ and $(M_2, \mathcal{U}_2)$ be two pure systems. Denote by i_1 and i_2 the embedding morphisms $i_1 : M_1 \to M_1 \times M_2$ and $i_2 : M_2 \to M_1 \times M_2$. Then $i_1(\mathcal{U}_1)$ and $i_2(\mathcal{U}_2)$ are two pure systems in $M_1 \times M_2$. Obviously, $i_1(\mathcal{U}_1) + i_2(\mathcal{U}_2)$ is a pure system in $M_1 \times M_2$. We call this sum of pure systems the *direct sum* and denote it by $\mathcal{U}_1 \oplus \mathcal{U}_2$.

Consider the following case of the second construction.

Proposition 4 *Let $\phi : (M, \mathcal{U}) \to (M', \mathcal{U}')$ be a morphism of pure systems. Then the sum of pure systems $\mathcal{U}$ and $\{0, \phi^{-1}(0)\}$ is a pure system in M.*

Proof. We have only to check the purity of subgroups of the form $S + \phi^{-1}(0)$, where $S \in \mathcal{U}$. Since $\phi^{-1}(\phi(S)) = S + \phi^{-1}(0)$, then the purity of $S + \phi^{-1}(0)$ follows from the purity of $\phi(S)$ and the property 4 of pure subgroups. Q.E.D.

Remark. Reduction by modulo 2 of any pure subgroup of M to $M \otimes \mathbb{F}_2$ keeps the rank ($\mathbb{F}_2$ denotes 2-elements field). Therefore, the reduction of any pure system consists of different subspaces of $M \otimes \mathbb{F}_2$. Hence any pure system is finite.

The finiteness of any pure system provides one more justification of our definition of pseudoconvex sets as sets of the form of integer points of integral polyhedra.

4 The construction of S-classes

In the previous section, we constructed pure systems via the homogenization of S-classes. Here we define the maximal S-class $\mathcal{P}h(\mathcal{U}, \mathbb{Z})$ of integral polyhedra given a pure system $\mathcal{U}$.

As a consequence, we receive a generalization of the Edmonds polymatroid intersection theorem.

Definition 8 *Let $\mathcal{U}$ be a pure system. A polyhedron P is said to be $\mathcal{U}$-*convex *(or $\mathcal{U}$-*polyhedron*) if, for any face F of P, $\mathbb{R}_0(F)$ is a flat of $\mathcal{U}$.*

We denote by $\mathcal{P}h(\mathcal{U})$ the set of $\mathcal{U}$-polyhedra and by $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ the set of integral $\mathcal{U}$-polyhedra. The class $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ is closed under (integer) translations, reflection and faces. Since the sum of flats is a flat, this class is closed under summation. The homogenization of $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ brings us back to $\mathcal{U}$. Clearly, if the homogenization of an ample class $\mathcal{P}$ is contained in $\mathcal{U}$, then $\mathcal{P} \subset \mathcal{P}h(\mathcal{U},\mathbb{Z})$.

We will show below that $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ is an S-class. But beforehand it would be convenient to demonstrate the following important surjectivity property.

Let $\phi : (M,\mathcal{U}) \to (M',\mathcal{U}')$ be a morphism of pure systems and let $\phi_{\mathbb{R}} = \phi \otimes \mathbb{R} : M \otimes \mathbb{R} \to M' \otimes \mathbb{R}$ be the corresponding linear map of vector spaces. If we associate to a polyhedron $P \subset V$ its image $\phi_{\mathbb{R}}(P) \subset V' = M' \otimes \mathbb{R}$, we obtain the map $\phi_{\mathbb{R}} : \mathcal{P}h(\mathcal{U},\mathbb{Z}) \to \mathcal{P}h(\mathcal{U}',\mathbb{Z})$.

Theorem 1 *Let $\phi : (M,\mathcal{U}) \to (M',\mathcal{U}')$ be a morphism of pure systems. Then, for any integral $\mathcal{U}$-polyhedron P, there holds*

$$\phi(P(\mathbb{Z})) = \phi_{\mathbb{R}}(P)(\mathbb{Z}).$$

Proof. We show that, for any integer point $m' \in \phi_{\mathbb{R}}(P)(\mathbb{Z})$, the intersection of P and the affine subspace $\phi_{\mathbb{R}}^{-1}(m')$ (in V) is an integral polyhedron. To see this, it is enough to show that the intersection of P and a flat of $\mathcal{U}$ is an integral polyhedron. In fact, choosing an appropriate translation, we may assume that $m' = 0$. By Proposition 4, we can add $\phi^{-1}(0)$ to $\mathcal{U}$ and have a larger pure system. Because of this, we may assume that $\phi^{-1}(0)$ is already a flat of $\mathcal{U}$.

Thus, let F be a flat of $\mathcal{U}$. Consider a vertex v of the intersection $P \cap F$. Let us show that v is integral. Replacing P by its minimal face, which contains v, we may assume that $P \cap F = v$. Now, if we replace P on its affine span aff(P), then we would have aff(P) $\cap F = v$. Thus, we may suppose that P is an integer translation of a flat G of $\mathcal{U}$.

By virtue of the equivalence of the properties (Int) and (Add), (see Proposition 2), we have only to check that $(F+G)(\mathbb{Z}) = F(\mathbb{Z}) + G(\mathbb{Z})$ for a pair of flats F and G of $\mathcal{U}$. But this follows from the definition of pure systems. Q.E.D.

As a consequence of this theorem, we obtain the following result.

Theorem 2 *Let $\mathcal{U}$ be a pure system. Then $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ is a S-class of integral polyhedra.*

Proof. The class $\mathcal{P}h(\mathcal{U},\mathbb{Z})$ is closed under integer translations, reflection, faces and summation. So we need only to check that (*Add'*) is true, i.e., for any pair of polyhedra P and $Q \in \mathcal{P}h(\mathcal{U},\mathbb{Z})$, we have

$$(P+Q)(\mathbb{Z}) = P(\mathbb{Z}) + Q(\mathbb{Z}).$$

Let us consider the homomorphism of summation:

$$\phi : M \times M \to M, \quad \phi(m, m') = m + m'.$$

Obviously, ϕ is a morphism of $(M \times M, \mathcal{U} \oplus \mathcal{U})$ to $(M, \mathcal{U})$. If P and $Q \in \mathcal{P}h(\mathcal{U},\mathbb{Z})$ then $P+Q$ is equal to the image of integral polyhedron $P \times Q \in \mathcal{P}h(\mathcal{U} \oplus \mathcal{U},\mathbb{Z})$. By virtue of Theorem 1, any integer point of $P+Q$ is the image of some integer point $(m, m') \in P \times Q$ and, hence, is equal to $m + m'$. Q.E.D.

Corollary 1 *Let $\mathcal{U}$ be a pure system. Then the intersection of any two integral $\mathcal{U}$-polyhedra is an integral polyhedron.*

5 The construction of I-classes

To construct I-classes, we proceed in a manner analogous to the construction of S-classes. First, we consider the homogenization of I-classes. It turns out that such a homogenization forms a collection of pure subgroups orthogonal to subgroups of some pure system. Next, we will show how these orthogonal systems are used to construct all I-classes. In particular, we obtain a generalization of the Hoffman-Kruskal theorem ([8]).

HOMOGENIZATION OF I-CLASSES

Definition 9 *A set $\mathcal{V}$ of pure subgroups of M is called a* pure $*$-system *if $\mathcal{V}$ is closed under intersection and the subgroup $S + S'$ is pure, for any $S, S' \in \mathcal{V}$.*

Let $\mathcal{L}$ be an I-class in M. As before, its homogenization is the collection

$$\mathcal{U}(\mathcal{L}) = \{\mathbb{Z}_0(S), S \in \mathcal{L}\}$$

of subgroups in M.

Proposition 5 *Let $\mathcal{L}$ be an I-class such that $A + A \in \mathcal{L}$ for any $A \in \mathcal{L}$. Then $\mathcal{U}(\mathcal{L})$ is a pure $*$-system.*

Proof. Let us check that $\mathbb{Z}_0(A)$ are pure subgroups of M, $A \in \mathcal{L}$. Since $A + A \in \mathcal{L}$ for any $A \in \mathcal{L}$, we may assume that the relative interior of $\mathrm{co}(A)$ contains an integer point. (Recall, that we denoted by $[n]A$ the sum of n exemplars of A.) Since $\mathbb{Z}_0(A) = \cup_{n \geq 1}[2^n](A - a)$ holds for any integer $a \in \mathrm{relint}(\mathrm{co}(A))$, we have only to check pseudo-convexity of sets $[2^n](A-a)$. But, $[2^n](A-a)$ belongs to $\mathcal{L}$ for any integer n and, therefore, is pseudo-convex.

Let us prove that the subgroup $\mathbb{Z}_0(A) + \mathbb{Z}_0(B)$ is pure for any $A, B \in \mathcal{L}$. Clearly we may assume that the relative interiors of both $\mathrm{co}(A)$ and $\mathrm{co}(B)$ contain 0. So we have to prove that $\mathbb{Z}(A) + \mathbb{Z}(B)$ is pure. This group is covered by the sets $[2^n]A + [2^n]B$, $n \geq 1$. Since $[2^n]A$ belongs to $\mathcal{L}$ as well as $[2^n]B$, the pseudo-convexity of $[2^n]A + [2^n]B$ follows from the property *(Add)*.

Finally, let us show that $\mathcal{U}(\mathcal{L})$ is closed under intersection. This holds, since, for any sets A, $B \in \mathcal{L}$ with $0 \in \mathrm{relint}(\mathrm{co}(A))$, $0 \in \mathrm{relint}(\mathrm{co}(B))$, we have the following sequence of equalities

$$\mathbb{R}(A) \cap \mathbb{R}(B) = \mathbb{R}(\mathrm{co}(A) \cap \mathrm{co}(B)) = \mathbb{R}(\mathrm{co}(A \cap B)) = \mathbb{R}(A \cap B).$$

Q.E.D.

We show now that pure $*$-systems are "orthogonal" to pure systems. Let S be a subgroup of M. The subgroup

$$S^\perp := \{p \in M^*, p(s) = 0 \ \forall \ s \in S\}$$

is said to be *orthogonal to* S. In other words, $S^\perp$ is the kernel of the homomorphism $M^* \to S^*$, dual to the inclusion $S \to M$. Observe, that a subgroup S is pure if and only if $S = S^{\perp\perp}$. For a collection $\mathcal{U}$ of subgroups of M, we denote by $\mathcal{U}^\perp = \{S^\perp, S \in \mathcal{U}\}$ the collection of orthogonal subgroups of M^*.

Proposition 6 *Let $\mathcal{U}$ be a pure system in M and let $\mathcal{V}$ be a pure $*$-system in M^*. Then $\mathcal{U}^\perp$ is a pure $*$-system in M^* and $\mathcal{V}^\perp$ is a pure system in M.*

Before proving this proposition, we introduce the notion of pure homomorphism generalizing the notion of pure subgroup. Let M and N be free abelian groups of finite type. We say that a homomorphism $f : M \to N$ is *pure* if the factor-group $N/f(M)$ is free. This means, of course, that the subgroup $f(M) \subset N$ is pure.

Lemma 1 *A homomorphism $f : M \to N$ is pure if and only if the dual homomorphism $f^* : N^* \to M^*$ is pure.*

Proof. Consider the canonic decomposition of $f : M \to N$ in two exact sequences

$$0 \to K \to M \to H \to 0, \quad \text{and} \quad 0 \to H \to N \to C \to 0.$$

Since f is pure, C is free. H is free as a subgroup of the free group N. Therefore, both the sequences are splitting. Hence the dual sequences

$$0 \to C^* \to N^* \to H^* \to 0, \quad 0 \to H^* \to M^* \to K^* \to 0$$

are exact (where $X^* = \mathrm{Hom}(X, \mathbb{Z})$). Since K^* is free, $f^* : N^* \to M^*$ is pure. Q.E.D.

Lemma 2 *Let F and G be pure subgroups of M. Then $F + G$ is a pure subgroup of M if and only if $F^\perp + G^\perp$ is a pure subgroup of M^*.*

Proof. The subgroup $F^\perp + G^\perp$ is the image of the summation homomorphism $\sigma : F^\perp \oplus G^\perp \to M^*$. Therefore, the purity of the subgroup $F^\perp + G^\perp$ is equivalent to the purity of the homomorphism σ. By virtue of Lemma 1, the purity of σ is equivalent to the purity of the dual homomorphism $\sigma^* : M \to (F^\perp \oplus G^\perp)^* = (F^\perp)^* \times (G^\perp)^*$. By identifying $(F^\perp)^*$ with M/F and $(G^\perp)^*$ with M/G, σ^* is identified with the natural homomorphism $M \to (M/F) \times (M/G)$. The cokernel of this homomorphism coincides with the cokernel of the map $F \times G \times M \to M \times M$, $(f, g, m) \to (f + m, g + m)$, that is isomorphic to $M/(F + G)$. Thus, we see that $M/(F + G)$ is free iff $F + G$ is a pure subgroup of M iff $F^\perp + G^\perp$ is a pure subgroups of M^*. Q.E.D.

Proof of Proposition 6. Since for any subgroups $F, G \subset M$ there holds $F^\perp \cap G^\perp = (F + G)^\perp$, the Proposition follows from Lemma 2. Q.E.D.

Dehomogenization of $*$-pure systems

Let $\mathcal{U}$ be a collection of pure subgroups of M. Then $\mathcal{U}^\perp := \{S^\perp \mid S \in \mathcal{U}\}$ is the collection of orthogonal linear subspaces (flats) in V^*. For any pair of flats $S \subset T$ of $\mathcal{U}^\perp$ such that S is of codimension 1 in T, we consider two closed halfspaces of T which S borders. Denote by $(S : T)^\pm$ these halfspaces of T.

We introduce the following set $\mathrm{Prim}(\mathcal{U})$, which consists of all flats of $\mathcal{U}^\perp$ and all "halfspaces" of the form $(S : T)^\pm$, for flats $S \subset T$ of $\mathcal{U}^\perp$ such that S is of codimension 1 in T. By taking finite intersections of translations of elements of this set, we get polyhedra in the dual space V^*.

Definition 10 *Let $\mathcal{U}$ be a collection of pure subgroups of M. A polyhedron Q in V^* is said to be $*\mathcal{U}$-*convex *(or $*\mathcal{U}$-*polyhedron*) if Q is defined by the intersection of translations of some finite subset of elements of Prim$(\mathcal{U})$.*

Denote by $\mathcal{P}h^*(\mathcal{U})$ the class of such polyhedra and by denote $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ the subclass of polyhedra, which are given by taking only integer translations of elements of Prim$(\mathcal{U})$. The class $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ is ample and closed under intersection. However, it may contain non-integral polyhedra.

Theorem 3 *The class $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ consists of integral polyhedra if and only if $\mathcal{U}$ is a pure system.*

Proof. Let $\mathcal{U}$ be pure. Then any vertex of a $*\mathcal{U}$-polyhedron is given by taking the intersection of some translations of some flats of $\mathcal{U}^\perp$. Since $\mathcal{U}^\perp$ is closed under intersection, we only need to check that the intersection of integer translations of any two flats $F, G \in \mathcal{U}^\perp$ is integral. Since $F(\mathbb{Z}) + G(\mathbb{Z})$ is pure (Lemma 2), i.e. $F(\mathbb{Z}) + G(\mathbb{Z}) = (F + G)(\mathbb{Z})$, then by virtue of equivalence of (*Int'*) and (*Add'*), we have that intersections of integer translations of F and G are integral.

Thus, any polyhedron of $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ has integral vertices, and, hence, as it is rational, it is integral.

Conversely, suppose that $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ consists of integral polyhedra. Then it is an I-class and $P + P \in \mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ for any $P \in \mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$. Obviously that the homogenization of $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ is exactly $\mathcal{U}^\perp$. By virtue of Proposition 6, we conclude that $\mathcal{U}$ is pure. Q.E.D.

Remark. When we consider special polyhedra of $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ which are given by taking intersections of translations of elements of Prim$(\mathcal{U})$ of the form of proper halfspaces of M^*, i.e., of the form $(S : T)^\pm$ with $T = M^*$, we obtain the Hoffmann–Kruskal theorem ([8]) as a particular case of Theorem 3.

Corollary 2 *Let $\mathcal{U}$ be a pure system. Then the class $\mathcal{P}h^*(\mathcal{U}, \mathbb{Z})$ is a (polyhedral) I-class. The same is true for the class $\mathcal{P}t^*(\mathcal{U}, \mathbb{Z})$ of integral $*\mathcal{U}$-polytopes.*

6 Unimodular systems

We have shown above, that pure systems play a crucial role in the description and construction of S-classes and I-classes of discrete-convex sets. In the sequel, we consider in more details those pure systems which contain "sufficiently many" one-dimensional flats. In terms of the corresponding S-classes, this means that these classes contain "sufficiently many" one-dimensional polyhedra, or that they contain "sufficiently many" polytopes (as, for example, the class $\mathcal{B}$ of base polyhedra from Example 3). In terms of I-classes, this means that there are "sufficiently many" polyhedra of codimension 1.

More exactly, we are interested in those pure systems $\mathcal{U}$ which are generated, as semigroups, by one-dimensional flats. When we deal with those systems, we do not need to introduce all flats, indeed it is enough to set-up the one-dimensional generators. One-dimensional pure subgroups of M have the form $\mathbb{Z}r$, where r is a primitive vector of M. This brings us to the following definition.

Definition 11 *A set $\mathcal{R} \subset M$ is called* unimodular *if, for any subset $B \subset \mathcal{R}$ the subgroup $\mathbb{Z}B \subset M$ is pure. A* unimodular system *is a pair $(M, \mathcal{R})$, where $\mathcal{R}$ is a unimodular set in M.*

We call *flats* (or *$\mathcal{R}$-flats*) subgroups $\mathbb{Z}B$, where $B \subset \mathcal{R}$. These flats form the pure system $\mathcal{U}(\mathcal{R})$. Thus, unimodular systems is another phrasing for pure systems generated by one-dimensional generators.

On the other hand, unimodular systems are closely related to totally unimodular matrices, that is matrices whose square submatrices have determinants 0 or ± 1. Suppose that a unimodular set $\mathcal{R}$ is of full dimension, or, equivalently, spans V. If we pick some basis $B \subset \mathcal{R}$ and represent vectors of $\mathcal{R}$ as linear combinations of the basis vectors, then the matrix of coefficients is totally unimodular. In particular, the coefficients of this matrix are either 0 or ± 1, which proves (again) the assertion about the finiteness of unimodular sets. Conversely, the columns of a totally unimodular $n \times m$ matrix yield a unimodular set in $\mathbb{Z}^n$. Thus unimodular systems are coordinate-free representations of totally unimodular matrices. The reader might find many other characterizations of totally unimodular matrices in [13].

Consider some important examples of unimodular systems.

Example 6. In Example 5, we introduced the pure system $\mathbb{A}(N)$, which is spanned by the one-dimensional flats $\mathbb{Z}(e_i - e_j)$, $i, j \in N$. Therefore, the set of vectors $e_i - e_j$, $i, j \in N$, is a unimodular set in $(\mathbb{Z}^N)^*$. Let us denote this system as well by $\mathbb{A}(N)$. Note that it is not of full dimension, since it spans the subspace $\{x, x(N) = 0\}$, which is orthogonal to the vector $\mathbf{1}_N \in \mathbb{R}^N$. We show further that the class $\mathcal{P}h(\mathbb{A}(N))$ coincides with the class of base polyhedra $\mathcal{B}$ from Example 3.

If we project the set $\mathbb{A}(N \cup \{0\}))$ along the axis $\mathbb{R}e_0$ onto the space $(\mathbb{R}^N)^*$, we obtain the full-dimensional unimodular system consisting of the vectors $\pm e_i$ and $e_i - e_j$, $i, j \in N$, in $(\mathbb{Z}^N)^*$. Of course, we could construct this system simply by adding the basic system $(\pm e_i, i \in N)$ to the system $\mathbb{A}(N)$. We denote this system by $\mathbb{A}_N$. We show that $\mathbb{A}_N$-polyhedra are precisely generalized polymatroids.

Sub-systems $\mathcal{R} \subset \mathbb{A}_N$ (more precisely, symmetrical sub-systems, which contain 0 and $-r$ for any $r \in \mathcal{R}$) are called *graphic* unimodular systems. In fact, to every such a system we can associate the (undirected) graph $G = (N, E)$ whose set of nodes is N and set of edges is $E = \{\{i, j\}, e_i - e_j \in \mathcal{R}\}$. Conversely, any graph (N, E) yields the sub-system $\mathcal{R} \subset \mathbb{A}(N)$ consisting of the vectors $e_i - e_j$, $\{i, j\} \in E$.

Example 7. To any graph G, one can associate another unimodular system, the *cographic* unimodular system $\mathbb{D}(G)$. It is located in the cohomology group $H^1(G, \mathbb{Z})$ of the graph G and consists of the cohomology classes $\pm[e]$, corresponding to oriented edges of the graph G. The proof of the unimodularity of the system $\mathbb{D}(G)$ is based on the fact that this system is, in some (matroidal) sense, dual to the graphic system associated with G.

The most interesting examples of cographic systems are related to cubic (or 3-valent) graphs. The simplest example of such graphs is the complete graph K_4 with 4 vertices. The corresponding system $\mathbb{D}(K_4)$ is isomorphic to $\mathbb{A}_3$. The bipartite graph $K_{3,3}$ yields a more interesting example. The system $\mathbb{D}(K_{3,3})$ consists of the 19 vectors in $\mathbb{R}^4$: $\{0$, $\pm e_i$, $i = 1, \ldots, 4$, $\pm(e_1 + e_2)$, $\pm(e_2 + e_3)$, $\pm(e_3 + e_4)$, $\pm(e_4 + e_1)$, $\pm(e_1 + e_2 + e_3 + e_4)\}$.

One can check that $\mathbb{D}(K_{3,3})$ is not a graphic system.

Example 8. There is an exceptional unimodular system in dimension 5, which is neither graphic no cographic, the system $\mathbb{E}_5$. It consists of the following 21 vectors: 0, $\pm e_i$, $i = 1, \ldots, 5$, $\pm(e_1 - e_2 + e_3)$, $\pm(e_2 - e_3 + e_4)$, $\pm(e_3 - e_4 + e_5)$, $\pm(e_4 - e_5 + e_1)$, $\pm(e_5 - e_1 + e_2)\}$.

According to the Seymour theorem [14], every unimodular system can be constructed via graphic and cographic systems and $\mathbb{E}_5$.

Many notions, defined previously for pure systems, can be transferred straightforwardly to unimodular systems. For example, a *morphism* of unimodular systems $(M, \mathcal{R}) \to (M', \mathcal{R}')$ is a homomorphism $\phi : M \to M'$ of abelian groups such that $\phi(\mathcal{R}) \subset \mathcal{R}'$.

The results of Sections 3 and 5 imply that pure systems, which are closed under intersection, give rise to DC-classes closed under summation and under intersection, and vice versa. The following theorem characterizes unimodular systems $\mathcal{R}$ for whose pure systems $\mathcal{U}(\mathcal{R})$ are closed under intersection.

Theorem 4 *Let $\mathcal{R}$ be a unimodular set such that the pure system $\mathcal{U}(\mathcal{R})$ is closed under intersection. Then $\mathcal{R}$ is the direct sum of copies of $\mathbb{A}_1$ and $\mathbb{A}_2$.*

Proof. The proof is by induction on the dimension of unimodular systems. The proposition is obvious in dimensions 1 and 2.

Consider the case of dimension 3. Assume $\mathcal{R}$ contains a flat S isomorphic to $\mathbb{A}_2$. Denote by e_1, e_2 and e_1+e_2 the vectors of $\mathcal{R}\cap S$. We claim that there is at most one more vector of $\mathcal{R}$ (up to collinearity). Suppose there are two non-collinear vectors. Clearly we may denote them by e_3 and e_1+e_3. Then, since $\mathcal{U}(\mathcal{R})$ is closed under intersection, e_2+e_3 and $e_1+e_2+e_3$ belong to $\mathcal{R}$. But this contradicts unimodularity of $\mathcal{R}$, and the claim is proven. Therefore, $\mathcal{R}$ is isomorphic to $\mathbb{A}_1 \oplus \mathbb{A}_2$.
One can similarly check that if $\mathcal{R}$ does not contain flats isomorphic to $\mathbb{A}_2$, then $\mathcal{R}$ is isomorphic to $\mathbb{A}_1 \oplus \mathbb{A}_1 \oplus \mathbb{A}_1$. Thus, in the 3-dimensional case, the proposition is verified.

General case. Let $\mathcal{U}(\mathcal{R})$ contain a flat S isomorphic $\mathbb{A}_2$. This means that S is a plane of V such that $\mathcal{R}\cap S \cong \mathbb{A}_2$. We will show that there exists a flat T of codimension 2 in V such that

$$\mathcal{R} = (\mathcal{R}\cap S)\cup(\mathcal{R}\cap T). \tag{1}$$

By induction $\mathcal{R}\cap T$ is equal to a sum of copies $\mathbb{A}_1$ and $\mathbb{A}_2$, and we have $\mathcal{R}\cap S \cong \mathbb{A}_2$, so if (1) is true, the proposition is also true.

Pick a flat T of $\mathcal{U}(\mathcal{R})$ of codimension 2 (in V) such that $T\cap S = 0$. Obviously such a flat exists.

Claim. $\mathcal{R}\subset S\cup T$.

In fact, consider the projection $\pi : V \to S$ with the kernel T (the projection along T). Then $\pi(\mathcal{R})$ is a unimodular system of S which contains $\mathcal{R}\cap S$. Because $\mathcal{R}\cap S \cong \mathbb{A}_2$ and the $\mathbb{A}_2$ is a maximal unimodular system, any vector $r\in\mathcal{R}$, which does not belong to $S\cup T$, is projected into some vector $r_1\in\mathcal{R}\cap S$. Therefore, we have $r-r_1\in T$. On the other hand, $r-r_1$ belongs to the flat $\mathbb{R}r+\mathbb{R}r_1$. Since $\mathcal{U}(\mathcal{R})$ is closed under intersection, the line $(\mathbb{R}r+\mathbb{R}r_1)\cap T$ is an one dimensional flat of $\mathcal{U}(\mathcal{R})$, and, hence, there exists a vector $r_2\in\mathcal{R}$, which spans this flat.

Now we consider the 3-dimensional subspace $S+\mathbb{R}r_2$ of V and the unimodular system $\mathcal{R}\cap(S+\mathbb{R}r_2)$. Obviously, the pure system of this unimodular system is closed under intersection. Therefore, $\mathcal{R} \cap (S + \mathbb{R}r_2)$ is isomorphic to $\mathbb{A}_2 \oplus \mathbb{A}_1$. Thus, there can be at most one generator outside of $\mathcal{R} \cap S$: the vector r_2. However, we have another one: the vector $r \neq \pm r_2$. A contradiction. Therefore $\mathcal{R} \subset S \cup T$ and the claim is proven.

Finally, suppose that $\mathcal{U}(\mathcal{R})$ contains no flats isomorphic to $\mathbb{A}_2$. In such a case, we assert that $\mathcal{R}$ equals the sum of n $(= \dim V)$ exemplars $\mathbb{A}_1$. Let $r_1, \ldots, r_n$ be linear independent elements of $\mathcal{R}$. We show that there holds $\mathcal{R} = \{\pm r_1, \ldots, \pm r_n\}$. Assume some $r \in \mathcal{R} \setminus \{\pm r_1, \ldots, \pm r_n\}$. Clearly we may assume that there holds $r = r_1 + \ldots + r_n$ (i.e. r does not belong to the coordinate hyperplanes). Let us consider the intersection of flats $\mathbb{R}r_1 + \mathbb{R}r_2$ and $\mathbb{R}r + \mathbb{R}r_3 + \ldots + \mathbb{R}r_n$. This intersection is a line $\mathbb{R}(r_1 + r_2)$ and it is a flat of $\mathcal{U}(\mathcal{R})$. Therefore, we have $r_1 + r_2 \in \mathcal{R}$ and, hence, $\{\pm r_1, \pm r_2, \pm(r_1 + r_2)\} \subset \mathcal{R}$, but $\{\pm r_1, \pm r_2, \pm(r_1 + r_2)\}$ is isomorphic to $\mathbb{A}_2$. A contradiction. Q.E.D.

Of course, the most interesting classes of discrete convexity are those that are maximal by inclusion. They correspond to maximal pure systems and maximal unimodular systems.

Definition 12 *A pure system $\mathcal{U}$ in M is said to be* maximal *if for any pure subgroup S of M, not of $\mathcal{U}$, the system $\mathcal{U} + \{0, S\}$ fails to be pure.*

A unimodular system is said to be maximal if it fits the above definition for pure subgroups of rank one.

$\mathbb{A}_n$ is the unique maximal unimodular system of dimension ≤ 3. In dimension 4, besides $\mathbb{A}_4$, there is another maximal unimodular system $\mathbb{D}(K_{3,3})$. In dimension 5, there are 4 non-isomorphic maximal unimodular systems; in dimension 6, there are 11. For more details, we refer to the article [4], which contains a complete description of maximal unimodular systems.

Remark. One can show that $\mathcal{U}(\mathbb{A}_n)$ is a maximal pure systems, but, $\mathcal{U}(\mathbb{D}(K_{3,3}))$ and $\mathcal{U}(\mathbb{E}_5)$ fail to be maximal. For example, the pure system $\mathcal{U}(\mathbb{E}_5)$ can be expanded by adding the group of integer points of the flat $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. In fact, let $\phi : \mathbb{Z}^5 \to \mathbb{Z}$ be the homomorphism sending $(x_1, x_2, x_3, x_4, x_5)$ to $x_1 + x_2 + x_3 + x_4 + x_5$. Then ϕ is a morphism of pure systems $(\mathbb{Z}^5, \mathcal{U}(\mathbb{E}_5))$ and $(\mathbb{Z}, \mathbb{A}_1)$, indeed. According to Proposition 4, we can add the kernel of this homomorphism to $\mathcal{U}(\mathbb{E}_5)$ and produce a larger pure system. Thus, $\mathcal{U}(\mathbb{E}_5)$ is not a maximal pure system. The pure system generated by $\mathbb{D}(K_{3,3})$ can be expanded by adding a group of rank 2.

Thus, the class of polyhedra $\mathcal{P}h(\mathcal{U}(\mathbb{A}_n), \mathbb{Z})$ is maximal, i.e. it can not be expanded by adding an integral polyhedron while conserving the property (Edm). Furthermore, for a maximal unimodular system $\mathcal{R}$, the class of polytopes $\mathcal{P}t(\mathcal{U}(\mathcal{R}),\mathbb{Z})$ is maximal in the class of polytopes, i.e. it can not be expanded by adding an integral polytope while conserving (*Edm*). However, the class of polyhedra $\mathcal{P}h(\mathcal{U}(\mathcal{R}),\mathbb{Z})$ need not be maximal in the class of polyhedra. For example, we can add the polyhedron $x_1+x_2+x_3+x_4+x_5=0$ to the class of polyhedra $\mathcal{P}h(\mathcal{U}(\mathbb{E}_5),\mathbb{Z})$ conserving the property (*Edm*). □

Let $\mathcal{R}$ be a unimodular system. Elements r of $\mathcal{R}$ are called *roots* of this system. They can be identified with morphisms of $\mathbb{A}_1$ to $\mathcal{R}$. Conversely, morphisms of $\mathcal{R}$ to $\mathbb{A}_1$ are called *coroots*. In other words, a coroot is a homomorphism of groups $\phi : M \to \mathbb{Z}$ such that $|\phi(r)| \leq 1$ for any root $r \in \mathcal{R}$. The set of coroots is denoted by $\mathcal{R}^*$.

A polyhedron is an $\mathcal{R}$-*polyhedron* if every of its face is parallel to some $\mathcal{R}$-flat. And identically for integral $\mathcal{R}$-polyhedra. Denote by $\mathcal{P}h(\mathcal{R})$ and $\mathcal{P}h(\mathcal{R},\mathbb{Z})$ these classes of polyhedra, respectively. A pseudo-convex set X in M is said to be $\mathcal{R}$-*convex set* if $\mathrm{co}(X)$ is a $\mathcal{R}$-polyhedron.

Notice that a polytope P is an $\mathcal{R}$-polytope if and only if each edge of P is parallel to some element of $\mathcal{R}$. For example, the sum of a collection of segments, each of them being parallel to some $r \in \mathcal{R}$, is an $\mathcal{R}$-polytope. Such polytopes are *zonotopes*. One can check that the class $\mathcal{P}t(\mathbb{E}_5)$ consists of zonotopes only.

A polyhedron (in V^*) is $*\mathcal{R}$-convex if and only if it is defined by the intersection of halfspaces of the form

$$H_r^+(a) = \{p \in V^*, p(r) \leq a\},$$

where $r \in \mathcal{R}$, $a \in \mathbb{R} \cup \{+\infty\}$. If all a's are integers, we have an integral $*\mathcal{R}$-polyhedron; we denote by $\mathcal{P}h^*(\mathcal{R})$ the class of $*\mathcal{R}$-polyhedra, and by $\mathcal{P}h^*(\mathcal{R},\mathbb{Z})$ the I-class of integral $*\mathcal{R}$-polyhedra.

Let us show that any $*\mathcal{R}$-polyhedron can be constructed by piecing together some "primitive" $*\mathcal{R}$-polytopes. In this context, it is useful to introduce the notion of dicing.

The *dicing* $\mathcal{D}(\mathcal{R})$ is the collection of hyperplanes $H_r(a) = \{v \in V^* \mid v(r) = a\}$, where $r \in \mathcal{R}$, $a \in \mathbb{Z}$. Connected components of $V^* \setminus \{\cup_{r\in\mathcal{R},\, a\in\mathbb{Z}} H_r(a)\}$ are said to be *regions* of the dicing. The regions are bounded iff $\mathcal{R}$ has full dimension. Faces of the closures of regions are said to be *chambers*. They form a decomposition of V^* in the sense that any two chambers intersect at a common face. (Some of these constructions are considered in details in [1].) Let q be a point in V^*. Then the minimal chamber containing q is

$$C(q) = \{p \in V^*,\, r(p) \geq \lfloor r(q) \rfloor,\ r \in \mathcal{R}\},$$

where $\lfloor \cdot \rfloor$ denotes the integral part of a real number. Chamber are integral $*\mathcal{R}$-polyhedron. They are "primitive" $*\mathcal{R}$-polyhedra, in the sense, that, for any $*\mathcal{R}$-polyhedron P, the intersection of P and a chamber is a face of this chamber, and each polyhedron of $\mathcal{P}h^*(\mathcal{R}, \mathbb{Z})$ is the union of some chambers of the dicing $\mathcal{D}(\mathcal{R})$.

Proposition 7 *A polyhedron Q belongs to $\mathcal{P}h^*(\mathcal{R}, \mathbb{Z})$ if and only if it is the union of some chambers of $\mathcal{D}(\mathcal{R})$.*

Proof. Let Q be an integral $*\mathcal{R}$-polyhedron. Then for any $q \in Q$, the chamber $C(q)$ is a subset of Q (see the explicit description of $C(q)$).
Conversely, suppose that the polyhedron Q is a union of chambers. Then any facet of Q is covered by facets of some chambers belonging to Q; and so, each facet of Q is defined by the equation $v(r) = a$ with some $r \in \mathcal{R}$ and $a \in \mathbb{Z}$. Q.E.D.

Example 9. Consider the *dicing star* $\mathbf{St}(\mathcal{R})$. It is formed by the union of those chambers of the dicing $\mathcal{D}(\mathcal{R})$, which contain the origin 0. In order to establish the convexity of $\mathbf{St}(\mathcal{R})$, we show that:

$$\mathbf{St}(\mathcal{R}) = \{p \in V^* \,|\, r(p) \leq 1\}.$$

For the time being, we call $\mathbf{St}'$ the polyhedron appearing on the right hand of the equality. Obviously any chamber which contains 0, belongs to $\mathbf{St}'$. Hence $\mathbf{St}(\mathcal{R}) \subset \mathbf{St}'$.

Conversely, let $p \in \mathbf{St}' \setminus \mathbf{St}(\mathcal{R})$. Assume we move from p to 0 along the segment $[0, p]$. For some t, $0 < t < 1$, the point tp will be on the boundary of $\mathbf{St}(\mathcal{R})$. Hence, there exists $r \in \mathcal{R}$ with $r(tp) = 1$. This implies that $r(p) = 1/t > 1$, a contradiction. □

Given the characterization of coroots, we see that integer points of $\mathbf{St}(\mathcal{R})$ are exactly the coroots. Thus

$$\mathbf{St}(\mathcal{R})(\mathbb{Z}) = \mathcal{R}^*.$$

Since $\mathbf{St}(\mathcal{R})$ is an integral polytope, we have

$$\mathbf{St}(\mathcal{R}) = \mathrm{co}(\mathcal{R}^*).$$

Moreover, 0 is the unique interior integer point of $\mathbf{St}(\mathcal{R})$.

We introduce one more useful structure connected to unimodular system $\mathcal{R}$, that of an arrangement $\mathcal{A}(\mathcal{R})$ or a fan $\Sigma(\mathcal{R})$ in V^*. The *arrangement* $\mathcal{A}(\mathcal{R})$ is the collection of hyperplanes (mirrors) $H_r(0) := r^{\perp}$, i.e., kernels of (non-zero) roots $r \in \mathcal{R}$. These hyperplanes divide V^* into a finite number of cones. These cones and their faces are called *cameras*. They form a *fan*, which we denote by $\Sigma(\mathcal{R})$. Of course, any camera of $\Sigma(\mathcal{R})$ is a $*\mathcal{R}$-polyhedron.

Let $\mathcal{R}$ be a full-dimensional unimodular system in M. Then the one-dimensional cameras (or rays) of the fan $\Sigma(\mathcal{R})$ correspond to the codimension 1 flats of $\mathcal{R}$. *Crossings* are primitive vectors of the group M^*, generating these cameras. Denote by $\mathcal{R}^\vee$ the set of crossings.

Lemma 3 *Let $\mathcal{R}$ be a unimodular set of full dimension. Then $\mathcal{R}^\vee \subset \mathcal{R}^*$.*

Proof. We need to show that $\xi(r) \in \{-1, 0, +1\}$ for any root $r \in \mathcal{R}$ and for any crossing $\xi \in \mathcal{R}^\vee$. The fact that ξ is primitive, means that $\xi : M \to \mathbb{Z}$ is surjective. Let us now pick $n-1$ linear independent roots $r_2, \ldots, r_n \in \mathcal{R}$ in the kernel of ξ. Therefore if a root r does not belong to the kernel, then the collection $r, r_2, \ldots, r_n$ is a basis of the group M, and, hence, $\xi(r) = \pm 1$. If r belongs to the kernel, then $\xi(r) = 0$. Q.E.D.

Remark. The convex hull of the roots, $\mathrm{co}(\mathcal{R})$, is $\mathcal{R}$-convex iff $\mathcal{R}^\vee = \mathcal{R}^*$. For example, $\mathcal{R}^\vee = \mathcal{R}^*$ holds with $\mathcal{R} = \mathbb{A}_n$, but does not hold with $\mathbb{D}(K_{3,3})$ or $\mathbb{E}_5$.

7 Exterior description of $\mathcal{R}$-polytopes and $*\mathcal{R}$-polytopes

In this section we characterize support functions of $\mathcal{R}$-polyhedra and the support functions of $*\mathcal{R}$-polyhedra. Let us recall, that support functions of base polyhedra are closely related to submodularity. Because of this, support functions of $\mathcal{R}$-polyhedra give rise to a generalization of submodularity.

7.1 Generalities

Recall that the support function of a (non-empty) closed convex set $A \subset V$ is the function $\phi(A; \cdot) : V^* \to \mathbb{R} \cup \{+\infty\}$ on the dual space V^* defined by the following formula

$$\phi(A; p) = \sup_{x \in A} p(x), \quad p \in V^*. \tag{2}$$

Let us work in a setting with convex and compact sets and polytopes in order to avoid messing up with infinite values. The support function is homogeneous and convex. Conversely, every homogeneous, convex function f on V^* is the support function of the *subdifferential* of f,

$$\partial(f) := \{x \in V \mid x(p) \leq f(p) \ \forall\, p \in V^*\}. \tag{3}$$

The set $\partial(f)$ is non-empty, convex, and compact; and the operations ϕ and ∂ are dual: $\partial(\phi(A)) = A$ and $\phi(\partial f) = f$ (see, for example, [12]).

Support functions of polytopes are characterized by a "piece-wise linearity" property. Specifically, the support function of a polytope coincides with some linear function on each cone of the normal fan.

A *fan* (in V^*) is a finite collection Σ of polyhedral cones possessing the following three properties: a) the cones $\sigma \in \Sigma$ cover V^*; b) every face of any $\sigma \in \Sigma$ is also in Σ; c) the intersection of two cones of Σ is a face of each of them. For example, in the previous section we defined the fan $\Sigma(\mathcal{R})$.

A convex function f on V^* is *compatible* with a fan Σ if f is linear on every cone σ from Σ. In this case, it is easy to show that $\partial(f)$ is a polytope. Moreover, let σ be a full-dimensional cone of the fan Σ; denote by x_σ a (unique) linear function on the space V^*, which coincides with f on the cone σ. Then x_σ (being considered as an element of V) is a vertex of the polytope $\partial(f)$. And all vertices of the polytope are of that form.

Conversely, the support function of any polytope P is compatible with the following fan $\mathcal{N}(P)$. Given a point $x \in P$, the following cone in the dual space V^*

$$\mathrm{Con}^*(P, x) = \{p \in V^*, p(x) \geq p(x') \quad \forall x' \in P\}$$

is said to be the *cotangent* cone to P at x. The collection of all cotangent cones $\mathrm{Con}^*(P, x)$, $x \in P$, forms the *cotangent fan* (or the *normal fan*) $\mathcal{N}(P)$ of the polytope P. For example, the cotangent fan of the zonotope $\sum_{r \in \mathcal{R}} \mathrm{co}(\{-r, r\})$ coincides with the arrangement fan $\Sigma(\mathcal{R})$.

Since the support function $\phi(P, \cdot)$ coincides with linear function $x : V^* \to \mathbb{R}$ on the cone $\mathrm{Con}^*(P, x)$, the support function is compatible with the fan $\mathcal{N}(P)$. The vertices of P corresponds to the cameras of full dimension of $\mathcal{N}(P)$, whereas the edges corresponds to the walls. If an edge is parallel to r then the corresponding wall is parallel to the "orthogonal" hyperplane $r^\perp$.

7.2 The case of $\mathcal{R}$-polytopes

Let P be an $\mathcal{R}$-polytope. It is clear that each cone of the normal fan $\mathcal{N}(P)$ is $*\mathcal{R}$-convex. This proves implication: $a) \Rightarrow b)$ of the following characterization of $\mathcal{R}$-convex polytopes.

Proposition 8 *Let $\mathcal{R}$ be a finite subset of V, and let $P \subset V$ be a polytope. The following assertions are equivalent:*

a) P is a $\mathcal{R}$-polytope;

b) the normal fan $\mathcal{N}(P)$ consists of $*\mathcal{R}$-convex cones;

c) the arrangement fan $\Sigma(\mathcal{R})$ is a refinement of the normal fan $\mathcal{N}(P)$;

d) the support function $\phi(P,\cdot)$ is compatible with the fan $\Sigma(\mathcal{N})$.

e) there exists a polytope P' such that $P+P'$ is a $\mathcal{R}$-zonotope.

Let us recall that a fan Σ is a *refinement* of a fan Σ' if any cone of Σ' is composed from cones of Σ.

Proof. The implications $b) \Rightarrow c) \Rightarrow d)$ are obvious. The equivalence $d)$ and $e)$ is a particular case of the following lemma (see, for example, [9]).

Lemma 4 *For polytopes P and Q the following assertions are equivalent:*

a) $\mathcal{N}(Q)$ is a refinement of $\mathcal{N}(P)$,

b) there exists a polytope P' such that $P+P'=kQ$, for some $k \geq 0$.

Finally, it is clear that a direct summand of an $\mathcal{R}$-polytope is a $\mathcal{R}$-polytope. This proves the implication $d) \Rightarrow a)$. Q.E.D.

Given a fan Σ in V^*, denote by $\mathcal{F}(\Sigma)$ the set of convex functions on V^*, which are compatible with the fan Σ. Thus, defining an $\mathcal{R}$-polytope or defining a function f from $\mathcal{F}(\Sigma(\mathcal{R}))$ are just one and the same thing.

Assume now that $\mathcal{R}$ is a full-dimensional unimodular system in the group M, and that $\mathcal{R}^\vee$ is the set of crossings. Then a function $f \in \mathcal{F}(\Sigma(\mathcal{R}))$ is uniquely determined by its restriction on $\mathcal{R}^\vee$, that is by the family of real numbers $(f(\xi), \xi \in \mathcal{R}^\vee)$. Thus, the polytope ∂f is defined by the intersection the the halfspaces $\{x \in V \mid \xi(x) \leq f(\xi)\}$, $\xi \in \mathcal{R}^\vee$,

$$\partial f = \{x \in V,\, \xi(x) \leq f(\xi)\, \forall \xi \in \mathcal{R}^\vee\}.$$

The values $f(\xi)$, $\xi \in \mathcal{R}^\vee$, being the restriction of a function of $\mathcal{F}(\mathcal{R})$, satisfy some kind of "submodularity" relations. These relations may be divided into two groups. The first group of relations addresses the functions' linearity on each cone of the fan. The second group of the relations yields convexity. Let us formulate these relations more explicitly:

I. Suppose that crossings $\xi_1, \ldots, \xi_m \in \mathcal{R}^\vee$ belong to a cone $\sigma \in \Sigma(\mathcal{R})$. Then any linear relation $\sum_i \alpha_i \xi_i = 0$ should imply the similar relation $\sum_i \alpha_i f(\xi_i) = 0$.

Of course, if every cone of the fan is simplicial (as in the case of $\mathbb{A}_n$), these relations disappear.

II. Suppose that we have two adjacent (full-dimensional) cones σ and σ' of the fan, which are separated by a wall τ. Let τ be spanned by the crossings $\xi_1, \ldots, \xi_m$, and let ξ, ξ' be crossings from σ, σ' respectively, which do not belong to the wall τ. Then any relation $\alpha\xi + \alpha'\xi' = \sum_i \alpha_i \xi_i$, where $\alpha, \alpha' > 0$, implies the relation $\alpha f(\xi) + \alpha' f(\xi') \geq \sum_i \alpha_i f(\xi_i)$.

According to Lemma 3, we can assume that $\alpha = \alpha' = 1$. But all the same, these relations do not look too inspiring. In effect, it is neither easy to provide a collection of numbers $(f(\xi), \xi \in \mathcal{R}^\vee)$ satisfying the relations I and II, nor easy to check that a given collection of numbers satisfies these relations. Nevertheless, in some cases we can define $\mathcal{R}$-polyhedra via inequalities

$$\xi(x) \leq b(\xi),\ \xi \in \mathcal{R}^\vee,$$

without caring about their "right parts".

Definition 13 *Let $\mathcal{R}$ be a unimodular system in M. A symmetric full-dimensional subset $\mathcal{Q} \subset \mathcal{R}^\vee$ is said to be a* laminarization *of $\mathcal{R}$ if $\mathcal{Q}^\vee \subset \mathcal{R}$.*

Proposition 9 *Let $\mathcal{Q}$ be a laminarization of a unimodular system $\mathcal{R}$. Then*

a) Q is a unimodular system in M^*;

b) any $*\mathcal{Q}$-polyhedron is $\mathcal{R}$-polyhedron;

c) any integral $*\mathcal{Q}$-polyhedron is an integral $\mathcal{R}$-polyhedron.

In other words, for any function $b : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the polyhedron in V, defined by the inequalities

$$\xi(x) \leq b(\xi),\ \xi \in \mathcal{Q},$$

is a $\mathcal{R}$-polyhedron (integral if b is integer valued).

Proof. Let us start by checking a). Let $\xi_1, \dots, \xi_n$ be in $\mathcal{Q}$, where $n = \mathrm{rk} M$. We should prove that $\xi_1, \dots, \xi_n$ form a basis of M^*. Denote by r_i a primitive element of M which is orthogonal to $\{\xi_j, j \neq i\}$. Each r_i, $i = 1, \dots, n$ is a $\mathcal{Q}$-crossing, hence, it is an element of $\mathcal{R}$. By unimodularity of $\mathcal{R}$, the collection $\{r_1, \dots, r_n\}$ forms a basis of M. By Lemma 3, $\xi_i(r_i) = \pm 1$. Therefore, $\{\xi_1, \dots, \xi_n\}$ is the dual basis to $\{r_1, \dots, r_n\}$, which proves a).

Let us show b). Since any $\mathcal{Q}$-crossing is in $\mathcal{R}$, then any $*\mathcal{Q}$-polytope is $\mathcal{R}$-polytope. The assertion about polyhedra follows from this, since any $*\mathcal{Q}$-polyhedron is approximated by $*\mathcal{Q}$-polytopes.

c) follows from a) and b). Q.E.D.

Example 10 (see also [7]). A family $\mathcal{T}$ of subsets of a finite set N is called *laminar* if for any A, $B \in \mathcal{T}$, either $A \subset B$, or $B \subset A$, or $A \cap B = \emptyset$ holds. Let $\mathcal{T}$ be a laminar family. We show that the set $\mathcal{Q}(\mathcal{T}) := \{\pm \mathbf{1}_S,\, S \in \mathcal{T}\} \subset \mathbb{R}^N$, is a laminarization of the system $\mathbb{A}_N$. (Adding, if necessary, singletons, we may assume that $\mathcal{Q}(\mathcal{T})$ is of full dimension.)

Consider $(n-1)$ linearly independent elements $\mathbf{1}_{S_i}$, $S_i \in \mathcal{T}$, $i = 1, \dots, n-1$. We have two different cases: $N \neq \cup_{i=1}^{n-1} S_i$ and $N = \cup_{i=1}^{n-1} S_i$. In the first case, there is exactly one element outside $\cup S_i$, say $k \in N$, therefore the kernel of $\mathbf{1}_{S_i}$, $i = 1, \dots, n-1$, is proportional to e_k. In the second case, since $\mathcal{T}$ is a laminar family and the kernel is one-dimensional, there exists a unique "non-separable pair" $k, l \in N$. This means that we cannot find a collection S_i, $i = 1, \dots, n-1$, which separates k and l ($k \in S_i$, $l \notin S_i$, or vice versa). Therefore the kernel is proportional to $e_k - e_l$. So, $\mathcal{Q}(\mathcal{T})^\vee \subset \mathbb{A}_N$.

In particular, for a laminar family $\mathcal{T}$ in N, the polyhedron defined by the inequalities

$$a(S) \leq x(S) \leq b(S), \quad S \in \mathcal{T},$$

is $\mathbb{A}_N$-polyhedron for any functions a, $b : \mathcal{T} \to \mathbb{R} \cup \{\infty\}$, and is integral $\mathbb{A}_N$-polyhedron for integer-valued a and b. $\square$

7.3 Base polytopes and generalized polymatroids

We show here that the class $\mathcal{B}$ of base polytopes (see Example 3) coincides with the class of $\mathbb{A}(N)$-polytopes (a similar assertion is also true for polyhedra; a proof, however, would involve support functions with infinite values), where $\mathbb{A}(N)$ is the unimodular system from Example 6.

Recall that the system $\mathbb{A}(N) \subset (\mathbb{R}^N)^*$ consists of differences $e_i - e_j$, $i, j \in N$. Consider now how the arrangement fan $\Sigma := \Sigma(\mathbb{A}(N))$ in the space $\mathbb{R}^N$ of functions on N looks like. Given the root $r = e_i - e_j$, the corresponding mirror $r^\perp$ consists of functions $p \in \mathbb{R}^N$ satisfying the relation $p(i) = p(j)$. This mirror divides the space of functions in two halfspaces $\{p : p(i) \geq p(j)\}$ and $\{p : p(i) \leq p(j)\}$. We see that cones of the fan Σ correspond to (weak) orders on N. If $\preceq$ is an order, then the corresponding cone $\sigma(\preceq)$ consists of monotone functions $p : (N, \preceq) \to (\mathbb{R}, \leq)$. For example, full-dimensional cones of Σ correspond to linear orderings; the line of constant functions $\mathbb{R}\mathbf{1}_N$ corresponds to the total indifference relation on N.

Since the system $\mathbb{A}(N)$ has codimension 1 in $(\mathbb{R}_N)^*$, the vertex of the fan Σ is not the point 0, but the line $\mathbb{R}\mathbf{1}_N$. For this reason, we are interested in two-dimensional cameras which act as crossings. Such a camera corresponds to a dichotomic order on N, which splits N into two classes S and $N \setminus S$ (S is different from $\emptyset$ and N). Therefore, this camera has the form $\mathbb{R}\mathbf{1}_N + \mathbb{R}_+\mathbf{1}_S$.

Suppose now that $f \in \mathcal{F}(\Sigma)$. Define the set-function $b : 2^N \to \mathbb{R}$, $b(S) = f(\mathbf{1}_S)$ for $S \subset N$. We assert that b is submodular. Indeed, let S and T be subsets of N. Then, by convexity of f,

$$b(S) + b(T) = f(\mathbf{1}_S) + f(\mathbf{1}_T) \geq 2f((\mathbf{1}_S + \mathbf{1}_T)/2).$$

On the other hand, since the points $\mathbf{1}_{S\cap T}$ and $\mathbf{1}_{S\cup T}$ belong to a cone of Σ, then

$$b(S \cap T) + b(S \cup T) = f(\mathbf{1}_{S\cap T}) + f(\mathbf{1}_{S\cup T}) = 2f((\mathbf{1}_{S\cap T} + \mathbf{1}_{S\cup T})/2).$$

Since

$$\mathbf{1}_S + \mathbf{1}_T = \mathbf{1}_{S\cap T} + \mathbf{1}_{S\cup T},$$

we have

$$b(S) + b(T) \geq b(S \cap T) + b(S \cup T),$$

that is b is submodular. Conversely, any submodular function b considered as a function on the set of vectors $\{\mathbf{1}_S,\ S \subset N\}$, has the unique extension $f = \tilde{b}$, compatible with the fan Σ, on whole $\mathbb{R}^N$. This extension coincides with the Choquet integral (see [2]) of the non-additive measure b, $\tilde{b}(p) = \int p db$. If b is submodular, then $\tilde{b}$ is convex (see [10]).

The corresponding polytope $\partial\tilde{b}$ is given by the following system of inequalities

$$\mathbf{1}_S(x) = x(S) \leq b(S), \quad S \subset N, \quad x(N) = b(N),$$

and is a base polytope. Thus, we prove

Proposition 10 *The class $\mathcal{P}t(\mathbb{A}(N))$ of $\mathbb{A}(N)$-polytopes coincides with the class of base polytopes.*

Of course, the class of $\mathbb{A}(N)$-polyhedra coincides with the class of base polyhedra, and the class of integral $\mathbb{A}(N)$-polyhedra coincides with the class of integral base polyhedra.

Generalized polymatroids.

In the same spirit, we can check that the class of generalized polymatroids in $(\mathbb{R}^N)^*$ coincides with the class of $\mathbb{A}_N$-polyhedra. The arrangement $\mathcal{A}(\mathbb{A}_N)$ consists of hyperplanes $p(i) = 0$, $i \in N$, and $p(i) = p(j)$, $i, j \in N$. The collection of vectors $\{\pm \mathbf{1}_S,\ S \subset N\}$ is the set of crossings. Cones of $\Sigma(\mathbb{A}_n)$ are in a one-to-one correspondence with pairs of orders $(\preceq_W, \preceq_{W'})$ on partitions (W, W') of N. These partitions derive from the partitions of coordinates in non-negative and negative parts; W denotes the non-negative coordinates of vectors of a cone, whereas W' denotes the negative ones.

Now let $f \in \mathcal{F}(\Sigma(\mathbb{A}_N))$. Consider the following two functions on 2^N: $a(S) := -f(-\mathbf{1}_S)$ and $b(S) := f(\mathbf{1}_S)$ for $S \subset N$. There are three kinds of relations between crossings: $\mathbf{1}_S + \mathbf{1}_T = \mathbf{1}_{S\cup T} + \mathbf{1}_{S\cap T}$, $-\mathbf{1}_S - \mathbf{1}_T = -\mathbf{1}_{S\cup T} - \mathbf{1}_{S\cap T}$, and

$$\mathbf{1}_S + (-\mathbf{1}_T) = \mathbf{1}_{S-T} + (-\mathbf{1}_{T-S}). \tag{4}$$

The first two yield submodularity of b and supermodularity of a, while the third yields the following inequalities

$$b(S) - a(T) = f(\mathbf{1}_S) + f(-\mathbf{1}_T) \geq f(\mathbf{1}_{S-T}) + f(-\mathbf{1}_{T-S}) = b(S-T) - a(T-S). \tag{5}$$

Thus, the pair (b, a) is a strong pair in the sense of [7]. The corresponding polyhedron ∂f is given by the inequalities

$$a(S) \leq x(S) \leq b(S),$$

where $S \subset N$ and, by definition, ∂f is a generalized polymatroid.

Conversely, we can extend any strong pair (b, a) to a function of $\mathcal{F}(\Sigma(\mathbb{A}_N))$ by performing analogues operation with that of the Choquet integration.

7.4 $*\mathcal{R}$-polytopes

Consider the $*\mathcal{R}$-convex polytope Q (in the dual space V^*). Its normal fan $\mathcal{N}(Q)$ is now in V. Since Q is defined by the inequalities $r(p) \leq a(r)$ where $a : \mathcal{R} \to \mathbb{R}$ is a function, the rays of the fan $\mathcal{N}(Q)$ take the form $\mathbb{R}_+ r$, $r \in \mathcal{R}$. The only requirement for a fan to be the normal fan of a $*\mathcal{R}$-convex polytope is that its rays be generated by some roots from $\mathcal{R}$. In other words, all cones of the fan $\mathcal{N}(Q)$ should be $\mathcal{R}$-convex.

Proposition 11 *A polytope $Q \subset V^*$ is $*\mathcal{R}$-convex if and only if the normal fan $\mathcal{N}(Q)$ consists of $\mathcal{R}$-convex cones.*

Proposition 8 and this Proposition 11 are doubtlessly dual; we discuss in more details this duality in [5].

Remark. Given two $*\mathcal{R}$-convex polytopes Q and Q', the "union" of the fans $\mathcal{N}(Q)$ and $\mathcal{N}(Q')$ (that is the common refinement $\mathcal{N}(Q) \wedge \mathcal{N}(Q')$), may enclose rays that are not in $\mathcal{R}$. Therefore, in general, the sum of $*\mathcal{R}$-polytopes need not be a $*\mathcal{R}$-polytope. However, if $Q + Q'$ is $*\mathcal{R}$-polytope, then both Q and Q' are $*\mathcal{R}$-polytopes. And more generally, if polytopes Q and Q' are such that there exists $*\mathcal{R}$-polytope Q'' whose normal fan $\mathcal{N}(Q'')$ is a refinement of $\mathcal{N}(Q)$ and $\mathcal{N}(Q')$, then the sum $Q + Q'$ is a $*\mathcal{R}$-polytope.

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