CentER

Discussion paper

No. 2001-26

ON CONES OF NONNEGATIVE QUADRATIC FUNCTIONS

By Jos F. Sturm and Shuzhong Zhang

April 2001

ISSN 0924-7815

On Cones of Nonnegative Quadratic Functions

Jos F. Sturm[*] Shuzhong Zhang[†]

March 31, 2001

Abstract

We derive LMI-characterizations and dual decomposition algorithms for certain matrix cones which are generated by a given set using generalized co-positivity. These matrix cones are in fact cones of non-convex quadratic functions that are nonnegative on a certain domain. As a domain, we consider for instance the intersection of a (upper) level-set of a quadratic function and a half-plane. We arrive at a generalization of Yakubovich's S-procedure result. As an application we show that optimizing a general quadratic function over the intersection of an ellipsoid and a half-plane can be formulated as SDP, thus proving the polynomiality of this class of optimization problems, which arise, e.g., from the application of the trust region method for nonlinear programming. Other applications are in control theory and robust optimization.

Keywords: LMI, SDP, Co-Positive Cones, Quadratic Functions, S-Procedure, Matrix Decomposition.

AMS subject classification: 90C22, 90C20.

JEL code: C61.

[*]Department of Econometrics, Tilburg University, The Netherlands. Part of this research was performed at the Department of Quantitative Economics, Maastricht University, The Netherlands, and as a visiting researcher at the Chinese University of Hong Kong.

[†]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong.

1 Introduction

In mathematics it is important to study functionals that are nonnegative over a given domain. As an example, the concept of duality is based on such a consideration and in convex analysis, the dual (polar) of a cone consists exactly of all the linear mappings that are nonnegative (nonpositive) over the cone itself. As another example, the positive semidefinite matrices are defined as the quadratic forms that are nonnegative over the whole Euclidean space. No doubt these are extremely important concepts. Recently, optimization with positive semi-definiteness restrictions (linear matrix inequalities), known as semidefinite programming, or SDP for short, received a lot of attention; see [13] and references therein.

In this paper, we shall apply the power of SDP to solve problems involving general quadratic functions. We first introduce the cones formed by quadratic functions that are nonnegative over a given region. Properties of such cones are discussed. In some special cases, we are able to characterize these cones using linear matrix inequalities (LMIs). The characterization leads us to solve several new classes of optimization problems, arising e.g. from the trust region method for nonlinear programming [9, 3]. The results also provide new tools for robust optimization [2, 1], in which the constraints can now depend in a quadratic fashion on the uncertain parameter.

Our results can be considered as extensions of Yakobuvich's S-procedure result [14], which characterizes quadratic functions that are nonnegative over the domain defined by another single quadratic function. References on quadratic systems and error bounds can be found in Luo and Sturm [4]. Some recent results on LMIs and nonnegativity expressed as *sum of squares* (SOS) can be found in Parrilo [7] and Nesterov [6].

An important concept that is used in our approach is 'co-positivity over a domain D', which reduces to the usual concept of co-positivity when D is the nonnegative orthant (i.e. $D = \Re^n_+$). When D is a polyhedral cone, we arrive at the generalized co-positivity concept of Quist et al. [8].

The organization of the paper is as follows. We introduce our definitions and notation concerning co-positivity with respect to a cone, cones of nonnegative quadratic functions on a specified domain, as well as the concept of homogenization in Section 2. Section 3 is devoted to a possible application of our analysis, namely non-convex quadratic optimization. We describe how general non-convex quadratic optimization problems can be reformulated as conic linear programming over cones of nonnegative quadratic functions. In Section 4 we investigate the cones that are obtained by homogenization of a domain that is given as the intersection of upper level sets of some quadratic functions. Then, in Section 5, two matrix decomposition results are proven in a constructive way. The results serve the purpose of characterizing, in terms of LMIs, cones of nonnegative quadratic functions for three different classes of domains of nonnegativity. The domains considered are defined either by a non-convex quadratic inequality, or an equality constraint in a strictly concave (or convex) quadratic function, or the combination of a covex quadratic inequality and a linear (affine) inequality. Based on the technique of semidefinite programming, these results imply among others the polynomial solvability of non-convex quadratic optimization problems over (unions of) these three classes of domains. We conclude the paper in Section 6. We want to remark that the material of Section 3 is merely an illustration, and the reader can skip this section if desired. After reading Section 2, it is possible to proceed immediately with Section 5, which includes our main results, and track back to the technical lemmas in Section 4 whenever they are referred to.

Notation. Given a set D in a Euclidean space, we let $\text{cone}(D)$ denote the convex cone consisting of all nonnegative combinations of elements of D. Similarly, we let $\text{conv}(D)$ denote the convex set consisting of all convex combinations of elements of D. If D is a cone, then $\text{conv}(D) = \text{cone}(D)$. We associate with a cone K in a Euclidean space the dual cone $K^* := \{y \mid x \bullet y \geq 0 \, \forall x \in K\}$, where '$\bullet$' denotes the standard inner product of the Euclidean space. We let $\mathcal{S}^{n\times n}$ denote the $n(n+1)/2$-dimensional Euclidean space of symmetric $n \times n$ matrices, with the standard inner product

$$X \bullet Y = \text{ tr } XY = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij},$$

for $X, Y \in \mathcal{S}^{n\times n}$. We let $\mathcal{S}_+^{n\times n}$ denote the cone of positive semidefinite matrices in $\mathcal{S}^{n\times n}$. Also, '$X \succ 0$' ('$X \succeq 0$') means that X is a symmetric positive definite (positive semidefinite) matrix.

2 Preliminaries

Let $D \subseteq \Re^n$ be a given set. Consider all symmetric matrices that are co-positive over D, i.e.

$$\mathcal{C}_+(D) := \{Z \in \mathcal{S}^{n\times n} \mid x^{\mathrm{T}} Z x \geq 0, \, \forall x \in D\}. \tag{1}$$

It is obvious that $\mathcal{C}_+(D)$ is a closed convex cone, and that

$$\mathcal{C}_+(D) = \mathcal{C}_+(D \cup (-D)). \tag{2}$$

We also have an obvious dual characterization of $\mathcal{C}_+(D)$, namely:

Proposition 1 *It holds that*

$$\mathcal{C}_+(D) = \left(\text{ cone} \left\{ yy^T \middle| y \in D \right\} \right)^*.$$

Proof. If $X \in \mathcal{C}_+(D)$ then by definition $0 \leq y^{\mathrm{T}} X y = X \bullet (yy^{\mathrm{T}})$ for all $y \in D$. Since the sum of nonnegative quantities is nonnegative, it follows that $X \bullet Z \geq 0$ whenever Z is a nonnegative combination of matrices in $\{yy^{\mathrm{T}} | y \in D\}$. This establishes $\mathcal{C}_+(D) \subseteq (\text{ cone}\{yy^{\mathrm{T}} | y \in D\})^*$. Conversely, if $X \bullet Z \geq 0$ for all $Z \in \text{ cone}\{yy^{\mathrm{T}} | y \in D\}$ then certainly $0 \leq X \bullet (yy^{\mathrm{T}}) = y^{\mathrm{T}} X y$ for all $y \in D$, and hence $X \in \mathcal{C}_+(D)$. **Q.E.D.**

Clearly, $\mathcal{C}_+(\Re^n) = \mathcal{S}_+^{n \times n}$ is the set of positive semidefinite matrices. In another well known case, where $D = \Re_+^n$, the set $\mathcal{C}_+(D)$ is called the *co-positive cone*. Testing whether a given matrix belongs to the co-positive cone is coNP-hard, i.e., testing whether it does not belong to the co-positive cone is NP-hard; see Murty and Kabadi [5]. We remark for general D that the validity of the claim '$Z \notin \mathcal{C}_+(D)$' can be certified by a vector $x \in D$ for which $x^{\mathrm{T}} Z x < 0$; this decision problem is therefore in NP, provided that '$x \in D$' is easy to check.

Two classical theorems from convex analysis are particularly worth mentioning in the context of this paper: the bi-polar theorem and Carathéodory's theorem [10]. The *bi-polar theorem* states that if $K \subseteq \Re^n$ is a convex cone, then $(K^*)^* = \text{ cl}(K)$, i.e. dualizing K twice yields the closure of K. *Carathéodory's theorem* states that for any set $S \subseteq \Re^n$ it holds that $x \in \text{conv}(S)$ if and only if there exist $y_1, y_2, \ldots, y_{n+1}$ such that $x = \sum_{i=1}^{n+1} \alpha_i y_i$ for some $\alpha_i \geq 0$ with $\sum_{i=1}^{n+1} \alpha_i = 1$.

Using the bi-polar theorem, it follows from Proposition 1 that $\mathcal{C}_+(D)^* = \text{ cl cone}\left\{ yy^{\mathrm{T}} \middle| y \in D \right\}$. The following lemma, which is based on Carathéodory's theorem, implies further that $\mathcal{C}_+(D)^* = \text{ cone}\{yy^{\mathrm{T}} \mid y \in \text{ cl}(D)\}$.

Lemma 1 *Let $D \subseteq \Re^n$. Then*

$$\text{cl cone}\{yy^T \mid y \in D\} = \text{ cone}\{yy^T \mid y \in \text{ cl}(D)\}.$$

Proof. Suppose that $Z \in \text{ cl cone}\{yy^{\mathrm{T}} \mid y \in D\}$ then $Z = \lim_{k \to \infty} Z_k$ for some $Z_k \in \text{ cone}\{yy^{\mathrm{T}} \mid y \in D\}$. Since the dimension of $\mathcal{S}^{n \times n}$ is $N := n(n+1)/2$, it follows from Carathéodory's theorem that for given Z_k there exists an $n \times (N+1)$ matrix Y_k such that $Z_k = Y_k Y_k^{\mathrm{T}}$, and each column of Y_k is a positive multiple of a vector in D. Furthermore, we have

$$\|Y_k\|_F^2 = \text{ tr } Y_k Y_k^{\mathrm{T}} = \text{ tr } Z_k \to \text{ tr } Z.$$

Therefore, the sequence $Y_1, Y_2, \ldots$ is bounded, and must have a cluster point Y^* for $k \to \infty$. Obviously, each column of Y^* is then a positive multiple of a vector in $\text{cl}(D)$, and since $Z = Y^*(Y^*)^{\mathrm{T}}$, it follows that $Z \in \text{ cone}\{yy^{\mathrm{T}} \mid y \in \text{ cl}(D)\}$. The converse relationship is trivial. **Q.E.D.**

By definition, $\mathcal{C}_+(D)$ consists of all quadratic forms that are nonnegative on D. We shall now consider the cone of all nonnegative quadratic functions (not necessarily homogeneous) that are nonnegative on D. Namely, we define

$$\mathcal{FC}_+(D) := \left\{ \begin{bmatrix} z_0, & z^{\mathrm{T}} \\ z, & Z \end{bmatrix} \middle| \; z_0 + 2z^{\mathrm{T}}x + x^{\mathrm{T}}Zx \geq 0, \, \forall x \in D \right\}. \tag{3}$$

For a quadratic function $q(x) = c + 2b^{\mathrm{T}}x + x^{\mathrm{T}}Ax$, we introduce its matrix representation, denoted by

$$\mathrm{M}(q(\cdot)) = \begin{bmatrix} c, & b^{\mathrm{T}} \\ b, & A \end{bmatrix}. \tag{4}$$

In this notation, $q(x) \geq 0$ for all $x \in D$ if and only if $\mathrm{M}(q(\cdot)) \in \mathcal{FC}_+(D)$.

In order to derive a dual characterization of the matrix cone $\mathcal{FC}_+(D)$, we need the concept of *homogenization.* Formally, for a set D, its homogenization is given by

$$\mathcal{H}(D) = \mathrm{cl}\left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \Re_{++} \times \Re^n \middle| \; x/t \in D \right\},$$

which is a closed cone (not necessarily convex). If D is a bounded set, then

$$\mathcal{H}(D) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \; t > 0, \, x/t \in \mathrm{cl}(D) \right\} \cup \{0\}. \tag{5}$$

Otherwise, this may not be true. A simple example is $D = [1, +\infty)$; in this case $\begin{bmatrix} 0, & 1 \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$. As another example, $\mathcal{H}(\Re^n) = \Re_+ \times \Re^n$. The following proposition states that the nonnegative quadratic functions on D and the nonnegative quadratic forms on $\mathcal{H}(D)$ are the same geometric objects, hence our interest in the concept of homogenization.

Proposition 2 *For any set $D \neq \emptyset$, it holds that*

$$\mathcal{FC}_+(D) = \mathcal{C}_+(\mathcal{H}(D)) = \mathcal{C}_+\left(\mathcal{H}(D) \cup (-\mathcal{H}(D))\right).$$

Proof. The second identity is a special case of relation (2). Furthermore, to see that $\mathcal{C}_+(\mathcal{H}(D)) \subseteq \mathcal{FC}_+(D)$, it suffices to observe that $x \in D$ implies $\begin{bmatrix} 1, & x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$ by definition of $\mathcal{H}(D)$. It remains to show that $\mathcal{FC}_+(D) \subseteq \mathcal{C}_+(\mathcal{H}(D))$.

Let $\begin{bmatrix} t, & x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$, i.e. there exist $t_k > 0$ and $x_k/t_k \in D$ such that $t = \lim_{k\to\infty} t_k$ and $x = \lim_{k\to\infty} x_k$. Any $\begin{bmatrix} z_0, & z^{\mathrm{T}} \\ z, & Z \end{bmatrix} \in \mathcal{FC}_+(D)$ necessarily satisfies

$$z_0 + 2z^{\mathrm{T}}(x_k/t_k) + (x_k/t_k)^{\mathrm{T}}Z(x_k/t_k) \geq 0,$$

or equivalently

$$z_0 t_k^2 + 2t_k z^{\mathrm{T}} x_k + x_k^{\mathrm{T}} Z x_k \geq 0.$$

By taking limits we get

$$z_0 t^2 + 2tz^{\mathrm{T}} x + x^{\mathrm{T}} Z x \geq 0,$$

which leads to the conclusion that

$$\begin{bmatrix} z_0, & z^{\mathrm{T}} \\ z, & Z \end{bmatrix} \in \mathcal{C}_+(\mathcal{H}(D)).$$

Q.E.D.

Combining Proposition 2 with Proposition 1, we arrive at the following corollary.

Corollary 1 *For any nonempty set D, it holds that*

$$\mathcal{FC}_+(D) = \text{conv}\{yy^T \mid y \in \mathcal{H}(D)\}^*.$$

Using Lemma 1 and the fact that $\mathcal{H}(D)$ is, by definition, a closed cone, we can dualize Corollary 1 to

$$\mathcal{FC}_+(D)^* = \text{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D)\}. \tag{6}$$

We remark from Proposition 2 that

$$\mathcal{FC}_+(\Re^n) = \mathcal{C}_+\left(\mathcal{H}(\Re^n) \cup (-\mathcal{H}(\Re^n))\right) = \mathcal{C}_+(\Re^{n+1}) = \mathcal{S}_+^{(1+n)\times(1+n)}. \tag{7}$$

In other words, the cone of $(n+1) \times (n+1)$ positive semidefinite matrices is equal to the cone of (matrix representations of) quadratic functions that are nonnegative on the entire domain $\Re^n$.

Another case that deserves special attention is the sphere with radius 1 centered at the origin,

$$B(n) := \{x \in \Re^n \mid \|x\| \leq 1\}.$$

Since this is a bounded set, we may apply (5) to conclude that

$$\mathcal{H}(B(n)) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \, \|x\| \leq t \right\} =: \text{SOC}(n+1).$$

We see that the homogenization of $B(n)$ is the Lorentz cone, or second order cone, denoted by $\text{SOC}(n+1)$. According to Corollary 1, it holds that

$$\mathcal{FC}_+(B(n)) = (\text{conv}\{yy^{\mathrm{T}} \mid y \in \text{SOC}(n+1)\})^*.$$

In Section 4, we will consider (among others) domains of the form $D = \{x | q(x) \geq 0\}$, where $q(\cdot)$ is a given quadratic function. Choosing $q(x) = 1 - x^{\mathrm{T}} x$ yields $D = B(n)$. In Section 5, we will see as a special case of Theorem 1 that

$$\mathrm{conv}\{yy^{\mathrm{T}} \mid y \in \mathrm{SOC}(n+1)\} = \{X \in \mathcal{S}^{(1+n)\times(1+n)} \mid X \succeq 0,\, J \bullet X \geq 0\},$$

where $J := \mathrm{M}(q(\cdot))$, i.e.

$$J = \begin{bmatrix} 1, & 0 \\ 0, & -I \end{bmatrix} \in \mathcal{S}^{(1+n)\times(1+n)}.$$

This will then easily lead to the relation

$$\mathcal{FC}_{+}(B(n)) = \{Z \mid Z - tJ \succeq 0,\, t \geq 0\},$$

which is known from Rendl and Wolkowicz [9] and Fu, Luo and Ye [3].

3 Global Non-Convex Quadratic Optimization

Consider the general non-convex quadratic optimization problem

$$(P) \quad \inf\{f(x) \mid x \in D\},$$

where $f(\cdot)$ is a (non-convex) quadratic function and $D \subset \Re^n$ is a possibly non-convex domain. Let N be an arbitrary positive integer. Then

$$\inf\{f(x) \mid x \in D\} = \inf\left\{ \sum_{i=1}^{N} t_i^2 f(x_i) \,\middle|\, \sum_{j=1}^{N} t_j^2 = 1 \text{ and } x_i \in D,\, i = 1,2,\ldots,N \right\}.$$

Namely, $f(x)$ with $x \in D$ can never be smaller than the right hand side, since one may set $x_i = x$ for all i. Conversely, $\sum_{i=1}^{N} t_i^2 f(x_i)$ can never be smaller than the left hand side since

$$\sum_{i=1}^{N} t_i^2 f(x_i) \geq \min_{j=1,2,\ldots,N}\{f(x_j)\} \cdot (\sum_{i=1}^{N} t_i^2) = \min_{j=1,2,\ldots,N}\{f(x_j)\} \geq \inf\{f(x) \mid x \in D\}.$$

Using the matrix representation of $f(\cdot)$, we have

$$t_i^2 f(x_i) = t_i^2 \begin{bmatrix} 1 \\ x_i \end{bmatrix}^{\mathrm{T}} \mathrm{M}(f(\cdot)) \begin{bmatrix} 1 \\ x_i \end{bmatrix} = y_i^{\mathrm{T}}\, \mathrm{M}(f(\cdot)) y_i, \quad y_i := \begin{bmatrix} |t_i| \\ |t_i| x_i \end{bmatrix}.$$

Obviously $x_i \in D$ implies $y_i \in \mathcal{H}(D)$. Conversely, we have for any $y = \begin{bmatrix} t, & \xi^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$ with $e_1^{\mathrm{T}} y = t > 0$, where e_1 denotes the first column of the identity matrix, that

$$y^{\mathrm{T}}\, \mathrm{M}(f(\cdot)) y = t^2 f(\xi/t) \geq t^2 \inf\{f(x) \mid x \in D\}.$$

By definition of $\mathcal{H}(D)$, it thus follows that if $\inf\{f(x) \mid x \in D\} > -\infty$ then

$$y \in \mathcal{H}(D) \Longrightarrow y^{\mathrm{T}}\, \mathrm{M}(f(\cdot))y \geq (e_1^{\mathrm{T}} y)^2 \inf\{f(x) \mid x \in D\}. \tag{8}$$

Therefore,

$$\inf\{f(x) \mid x \in D\} = \inf\left\{ \sum_{i=1}^{N} y_i^{\mathrm{T}}\, \mathrm{M}(f(\cdot))y_i \,\middle|\, \sum_{j=1}^{N} (e_1^{\mathrm{T}} y_j)^2 = 1 \text{ and } y_i \in \mathcal{H}(D),\, i = 1, 2, \ldots, N \right\}.$$

Since the above relation holds in particular for $N = 1 + n(n+1)/2$, it follows from Carathéodory's theorem that

$$\inf\{f(x) \mid x \in D\} = \inf\left\{ \mathrm{M}(f(\cdot)) \bullet Z \,\middle|\, z_{11} = 1 \text{ and } Z \in \mathrm{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D)\} \right\},$$

where $z_{11} = e_1^{\mathrm{T}} Z e_1$ denotes the $(1,1)$-entry of Z. Using also (6), we conclude that the non-convex problem (P) is equivalent to the convex problem (MP), defined as

$$(MP) \quad \inf\{\, \mathrm{M}(f(\cdot)) \bullet Z \mid Z \in \mathcal{FC}_+(D)^*,\, z_{11} = 1\}.$$

Notice that if $D \neq \emptyset$ then $z_{11} > 0$ for any Z in the relative interior of $\mathcal{FC}_+(D)^*$. It follows that (MP) has a feasible solution in the relative interior of $\mathcal{FC}_+(D)^*$. Hence (MP) satisfies the relative Slater condition, or interior point condition, which implies that there can be no duality gap, and that either (MP) is unbounded or the dual optimal value is attained [11]. Indeed, the dual of (MP) is (MD),

$$(MD) \quad \sup\{\phi \mid \, \mathrm{M}(f(\cdot)) - \phi e_1 e_1^{\mathrm{T}} \in \mathcal{FC}_+(D)\}.$$

Since $\mathrm{M}(f(\cdot)) - \phi e_1 e_1^{\mathrm{T}} = \mathrm{M}(f(\cdot) - \phi)$, we may rewrite (MD) as

$$\sup\{\phi \mid f(x) \geq \phi \text{ for all } x \in D\},$$

and it is clear the optimal value of (MD) is indeed equal to the optimal value of (MP).

In principle, the non-convex problem (P) and the the convex problem (MP) are completely equivalent. Namely, Carathéodory's theorem implies that if Z is a feasible solution for (MP) then there exist $y_i = \begin{bmatrix} t_i, & \xi_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$, $i = 1, 2, \ldots, N$ such that $Z = \sum_{i=1}^{N} y_i y_i^{\mathrm{T}}$. If there is an i such that $t_i = 0$ and $y_i^{\mathrm{T}}\, \mathrm{M}(f(\cdot))y_i < 0$ then (P) must be unbounded due to (8). Otherwise, we have

$$\mathrm{M}(f(\cdot)) \bullet Z \geq \min\{f(\xi_i/t_i) \mid i \text{ such that } t_i > 0\}.$$

Equality holds if and only if

$$\begin{cases} \mathrm{M}(f(\cdot)) \bullet Z = f(\xi_i/t_i) \text{ for all } i \text{ with } t_i > 0 \\ y_i^{\mathrm{T}}\, \mathrm{M}(f(\cdot))y_i = 0 \text{ for all } i \text{ with } t_i = 0. \end{cases}$$

This shows that if Z is an optimal solution to (MP) then the decomposition $Z = \sum_{i=1}^{N} y_i y_i^{\mathrm{T}}$ yields an optimal solution *for any* y_i with $e_1^{\mathrm{T}} y_i > 0$. Since $\sum_{i=1}^{N} (e_1^{\mathrm{T}} y_i)^2 = 1$, it yields *at least* one (global) optimal solution.

A solution to the dual problem (MD) can be used to certify *global* optimality in the primal problem (MP) or (P). We remark that the classical approach only yields *local* optimality conditions for (P). The fact that we can reformulate a general non-convex problem (P) into a convex problem (MP) does not necessarily make such a problem easier to solve. For example, we already encountered in Section 2 the NP-hard problem of deciding whether a matrix is in the complement of $\mathcal{FC}_+(\Re^n_+)$. Furthermore, Carathéodory's theorem states only the existence of a decomposition of Z; it is in general not clear how such a decomposition should be constructed. Indeed it is well known that problem (P) is NP-hard [12] in its general setting.

However, in all three cases that we will discuss in Section 5, namely,

1. $D = \{x \mid q(x) \geq 0\}$,
2. $D = \{x \mid q(x) = 0\}$ with $q(\cdot)$ strictly concave, and
3. $D = \{x \mid q(x) \geq 0, \text{ and } a^{\mathrm{T}} x \geq a_0\}$ with $q(\cdot)$ concave,

the optimization problem (MP) and its dual (MD) turn out to be *Semidefinite Programming* (SDP) problems for which polynomial-time and effective solution methods exist. And furthermore, we propose efficient algorithms to decompose matrices in the dual cone $\mathcal{FC}_+(D)^*$ as a sum of rank-1 solutions in $\mathcal{FC}_+(D)^*$. Therefore, once we find a (nearly) optimal Z solution to (MP) we will also have (nearly) optimal x solutions to (P). This is remarkable, since (P) has some nasty features: the optimal solution set of (P) can be disconnected, and, in cases 1) and 2), the quadratically constrained sets 'D' are not necessarily convex.

We remark that problem (MD) has only one variable, and only one conic constraint. In general however, a conic linear programming model has multiple variables and multiple conic constraints. The general framework allows for the optimal *design* of quadratic functions, and for robust optimization where the constraints depend on the uncertain parameters in a quadratic fashion. The dual matrix decomposition will then yield worst case scenarios for the optimal robust design.

4 Quadratically Constrained Sets

In this section we shall study the case when the domain D is defined by some quadratic (in)equalities. Our aim is to show that under certain conditions, $\mathcal{H}(D)$ or $\mathcal{H}(D) \cup (-\mathcal{H}(D))$ can then completely be characterized by homogeneous quadratic constraints. With such a characterization, it is then easy to check whether a given vector belongs to $\mathcal{H}(D)$; the claim that a matrix belongs to $\mathcal{FC}_+(D)^*$ can then in principle be certified due to (6) and Carathéodory's theorem.

As a first step, let us consider one quadratic function $q(x) = c + 2b^{\mathrm{T}}x + x^{\mathrm{T}}Ax$, and its upper level set

$$D = \{x \in \Re^n \mid q(x) \geq 0\}.$$

Obviously, $q(x) \geq 0$ for all $x \in D$, so that $\mathrm{M}(q(\cdot)) \in \mathcal{FC}_+(D)$. The following lemma characterizes the homogenized cone of D.

Lemma 2 *Consider a quadratic function $q(x) = c + 2b^T x + x^T Ax$ for which the upper level set $D = \{x \mid q(x) \geq 0\}$ is nonempty. It holds that*

$$\mathcal{H}(D) \cup (-\mathcal{H}(D)) = \left\{ \left[\begin{array}{c} t \\ x \end{array} \right] \middle| \; t^2 c + 2tb^T x + x^T Ax \geq 0 \right\}.$$

Proof. We remark first that $\mathrm{M}(q(\cdot)) \in \mathcal{FC}_+(D) = \mathcal{C}_+\left(\mathcal{H}(D) \cup (-\mathcal{H}(D))\right)$, where the identity follows from Proposition 2. Therefore,

$$\left[\begin{array}{c} t \\ x \end{array} \right] \in \mathcal{H}(D) \cup (-\mathcal{H}(D)) \Longrightarrow t^2 c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax \geq 0.$$

To show the converse, we consider a pair (t, x) in the set

$$\left\{ \left[\begin{array}{c} t \\ x \end{array} \right] \middle| \; t^2 c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax \geq 0 \right\}.$$

If $t > 0$, then $x/t \in D$ and so $\left[\begin{array}{cc} t, & x^{\mathrm{T}} \end{array} \right]^{\mathrm{T}} \in \mathcal{H}(D)$. If $t < 0$ then $(-x)/(-t) \in D$, and so $\left[\begin{array}{cc} t, & x^{\mathrm{T}} \end{array} \right]^{\mathrm{T}} \in -\mathcal{H}(D)$. It remains to consider the case $t = 0$. We have

$$0 \leq t^2 c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax = x^{\mathrm{T}}Ax = (-x)^{\mathrm{T}}A(-x).$$

Since $D \neq \emptyset$, there must exist $\bar{x}$ such that $q(\bar{x}) \geq 0$. Let $\epsilon \in \Re \setminus \{0\}$. Then

$$\epsilon^2 q((x + \epsilon\bar{x})/\epsilon) = \epsilon^2 q(\bar{x}) + 2\epsilon(b + A\bar{x})^{\mathrm{T}}x + x^{\mathrm{T}}Ax.$$

Therefore, if $(b + A\bar{x})^{\mathrm{T}}x \geq 0$ then $(x + \epsilon\bar{x})/\epsilon \in D$ for all $\epsilon > 0$ and hence $\left[\begin{array}{cc} 0, & x^{\mathrm{T}} \end{array} \right]^{\mathrm{T}} \in \mathcal{H}(D)$. And otherwise, i.e. $(b + A\bar{x})^{\mathrm{T}}x < 0$, we have $(x + \epsilon\bar{x})/\epsilon = (-x - \epsilon\bar{x})/(-\epsilon) \in D$ for all $\epsilon < 0$ and hence $\left[\begin{array}{cc} 0, & -x^{\mathrm{T}} \end{array} \right]^{\mathrm{T}} \in \mathcal{H}(D)$ so that $\left[\begin{array}{cc} 0, & x^{\mathrm{T}} \end{array} \right]^{\mathrm{T}} \in -\mathcal{H}(D)$. **Q.E.D.**

In the sequel of this section, we allow multiple quadratic constraints in the definition of D. In the next lemma, we impose a condition under which D must be bounded, and hence relation (5) applies.

Lemma 3 *Let* $q_i(x) = c_i + 2b_i^T x + x^T A_i x$, $i = 1, ..., m$. *Assume that*

$$D = \{x \mid q_i(x) \geq 0,\ i = 1, ..., m\} \neq \emptyset.$$

Suppose furthermore that there exist $y_i \geq 0$, $i = 1, ..., m$, *such that* $\sum_{i=1}^m y_i A_i \prec 0$. *In particular, this implies that* D *is a compact set. Then we have*

$$\mathcal{H}(D) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ t \geq 0,\ t^2 c_i + 2tb_i^T x + x^T A_i x \geq 0,\ i = 1, ..., m \right\}.$$

Proof. We first remark that $x \in D$ implies $\sum_{i=1}^m y_i q_i(x) \geq 0$. Since $\sum_{i=1}^m y_i q_i(x)$ is a strictly concave quadratic function, it follows that D is (contained in) a bounded set. Since $\sum_{i=1}^m y_i A_i \prec 0$ and $y_i \geq 0$ for all $i = 1, \ldots, m$, we also have the obvious implication

$$\min_{i=1,\ldots,m} x^{\mathrm{T}} A_i x \geq 0 \Longrightarrow x^{\mathrm{T}} (\sum_{i=1}^m y_i A_i) x \geq 0 \Longrightarrow x = 0. \tag{9}$$

Therefore, we have

$$\begin{aligned}
&\left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ t \geq 0,\ t^2 c_i + 2tb_i^{\mathrm{T}} x + x^{\mathrm{T}} A_i x \geq 0,\ i = 1, \ldots, m \right\} \\
= &\left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ t > 0,\ t^2 q_i(x/t) \geq 0,\ i = 1, \ldots, m \right\} \cup \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} \middle| \ x^{\mathrm{T}} A_i x \geq 0,\ i = 1, \ldots, m \right\} \\
= &\left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ t > 0,\ x/t \in D \right\} \cup \{0\} \\
= &\ \mathcal{H}(D),
\end{aligned}$$

where the last two steps follow from (9) and (5), respectively. **Q.E.D.**

Since an equality constraint can be represented by two inequalities, we arrive at the following corollary.

Corollary 2 *Let* $q_i(x) = c_i + 2b_i^T x + x^T A_i x$, $i = 1, ..., m + l$. *Assume that*

$$D = \{x \mid q_i(x) \geq 0,\ i = 1, ..., m \text{ and } q_j(x) = 0,\ j = m + 1, ..., m + l\} \neq \emptyset.$$

Suppose furthermore that there exist $y_i \geq 0$, $i = 1, \ldots, m + l$, *such that* $\sum_{i=1}^{m+l} y_i A_i \prec 0$. *Then we have*

$$\begin{aligned}
\mathcal{H}(D) = \{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ & t \geq 0,\ t^2 c_i + 2tb_i^T x + x^T A_i x \geq 0,\ i = 1, \ldots, m; \\
& t^2 c_j + 2tb_j^T x + x^T A_j x = 0,\ j = m + 1, \ldots, m + l\}.
\end{aligned}$$

The next lemma deals with a convex domain. In the presence of convexity, we no longer require D to be bounded. As a special case, it includes a domain defined by one concave and one linear inequality; this case will be studied in detail later.

Lemma 4 *Let $q_i(x) = c_i + 2b_i^T x + x^T A_i x$, $i = 1, \ldots, m$, be concave functions. Suppose that*

$$D = \{x \mid q_i(x) \geq 0,\, i = 1, \ldots, m\} \neq \emptyset.$$

We have

$$\mathcal{H}(D) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \; t \geq 0,\, t^2 c_i + 2tb_i^T x + x^T A_i x \geq 0,\, tc_i + 2b_i^T x \geq 0,\, i = 1, \ldots, m \right\}.$$

Proof. For $\begin{bmatrix} t, & x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$ we have a sequence $t_n > 0$, $x_n/t_n \in D$ with $(t_n, x_n) \to (t, x)$. The fact that $x_n/t_n \in D$ implies for all $i = 1, \ldots, m$ that

$$t_n > 0, \quad t_n^2 q_i(x_n/t_n) = t_n^2 c_i + 2t_n b_i^{\mathrm{T}} x_n + x_n^{\mathrm{T}} A_i x_n \geq 0 \tag{10}$$

and hence, using the concavity of $q_i(\cdot)$,

$$t_n c_i + 2b_i^{\mathrm{T}} x_n \geq -x_n^{\mathrm{T}} A_i x_n / t_n \geq 0. \tag{11}$$

By taking limits in the relations (10) and (11), we have

$$t \geq 0, \quad t^2 c_i + 2tb_i^{\mathrm{T}} x + x^{\mathrm{T}} A_i x \geq 0, \quad tc_i + 2b_i^{\mathrm{T}} x \geq 0, \quad i = 1, \ldots, m. \tag{12}$$

Conversely, assume that (12) holds. If $t > 0$ then (12) implies that $t^2 q_i(x/t) \geq 0$ for all $i = 1, \ldots, m$, so that $x/t \in D$ and hence $\begin{bmatrix} t, & x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$. Otherwise, i.e. if $t = 0$, then (12) implies that $x^{\mathrm{T}} A_i x \geq 0$ and $b_i^{\mathrm{T}} x \geq 0$ for all $i = 1, \ldots, m$. Since the A_i's are negative semidefinite, it further follows that $A_i x = 0$ for all $i = 1, \ldots, m$. Therefore, we have for $\bar{x} \in D$ and $\epsilon > 0$ that

$$q_i(\bar{x} + x/\epsilon) = q_i(\bar{x}) + 2b_i^{\mathrm{T}} x/\epsilon \geq 0 \text{ for all } i = 1, \ldots, m,$$

and hence $\begin{bmatrix} \epsilon, & x^{\mathrm{T}} + \epsilon \bar{x}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$. Letting $\epsilon \downarrow 0$, it follows that $\begin{bmatrix} 0, & x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{H}(D)$, as desired. **Q.E.D.**

Interestingly, $\mathcal{H}(D)$ in Lemma 4 admits a second order cone representation:

Lemma 5 *Let $q(x) = c + 2b^T x + x^T A x$ be a concave function. Then there must exist a matrix R such that $A = -R^T R$. Let r and n denote the number of rows and columns in R respectively. The following three statements for $t \in \Re$ and $x \in \Re^n$, (13), (14) and (15), are equivalent:*

$$t \geq 0,\, t^2 c + 2tb^T x + x^T A x \geq 0,\, tc + 2b^T x \geq 0 \tag{13}$$

$$(ct + 2b^T x + t) \geq \sqrt{(ct + 2b^T x - t)^2 - 4x^T A x} \tag{14}$$

$$\begin{bmatrix} c+1, & 2b^T \\ c-1, & 2b^T \\ 0, & 2R \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \in \mathrm{SOC}(r+2). \tag{15}$$

Proof. Statements (14) and (15) are obviously equivalent. To see that they are also equivalent with (13) we observe

$$ct + 2b^{\mathrm{T}}x + t \geq |ct + 2b^{\mathrm{T}}x - t| \iff t \geq 0,\, tc + 2b^{\mathrm{T}}x \geq 0.$$

Moreover, in general, for any $\alpha, \beta \in \Re$, we have

$$\alpha + \beta \geq |\alpha - \beta| \iff \alpha \geq 0,\, \beta \geq 0.$$

Therefore, we have

$$ct + 2b^{\mathrm{T}}x + t \geq |ct + 2b^{\mathrm{T}}x - t| \iff t \geq 0,\, tc + 2b^{\mathrm{T}}x \geq 0.$$

The correctness of the lemma is now easily verified. **Q.E.D.**

The advantage of having a second order cone formulation of $\mathcal{H}(D)$, $D = \{x \mid q(x) \geq 0\}$ with $q(\cdot)$ concave, is that we immediately also get a second order cone formulation of the dual cone, $\mathcal{H}(D)^*$. Namely, we have in general for a given $k \times n$ matrix B and a cone K that

$$Bx \in K^* \iff y^{\mathrm{T}} B x \geq 0 \,\forall y \in K \iff x \in \{B^{\mathrm{T}} y \mid y \in K\}^*, \tag{16}$$

i.e.

$$\{x \mid Bx \in K^*\} = \{B^{\mathrm{T}} y \mid y \in K\}^*; \tag{17}$$

when x is not a vector but a matrix, we may either use the above identity after vectorization, or interpret B^{T} as the adjoint of a linear operator B.

Corollary 3 *Let $q(x) = c + 2b^T x + x^T A x$ be a concave functions with $D = \{x \mid q(x) \geq 0\} \neq \emptyset$. Then*

$$\mathcal{H}(D) = \{x \mid Bx \in \mathrm{SOC}(2 + \operatorname{rank}(A))\} = \{B^T y \mid y \in \mathrm{SOC}(2 + \operatorname{rank}(A))\}^*,$$

where

$$B = \begin{bmatrix} c+1, & 2b^T \\ c-1, & 2b^T \\ 0, & 2R \end{bmatrix},$$

and R is a $\operatorname{rank}(A) \times n$ matrix such that $A = -R^T R$.

For the case that b is in the image of A, or equivalently $\sup\{q(x) \mid x \in \Re^n\} < \infty$, a more compact second order cone formulation is possible:

Lemma 6 *Let $q(x) = c + 2b^T x + x^T Ax$ be a concave function with $b = A\beta$. Let R be an $r \times n$ matrix such that $A = -R^T R$. Then $\max\{q(x) \mid x \in \Re^n\} = c - b^T\beta$. If $c - b^T\beta \geq 0$ then*

$$t \geq 0,\, t^2 c + 2tb^T x + x^T Ax \geq 0,\, tc + 2b^T x \geq 0$$

if and only if

$$\begin{bmatrix} \sqrt{c - b^T\beta}, & 0 \\ R\beta, & R \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \in \mathrm{SOC}(r+1). \tag{18}$$

5 Dual Matrix Decompositions

This section addresses the problem of computing the Carathéodory decomposition of a dual matrix solution $Z \in \mathcal{FC}_+(D)^*$ into rank-1 solutions $Z = \sum_{i=1}^{N+1} y_i y_i^{\mathrm{T}}$, $y_i \in \mathcal{H}(D)$. Moreover, we will use the decomposition algorithms in this section to obtain LMI characterizations of cones of non-negative quadratic functions over certain quadratically constrained regions.

5.1 One quadratic constraint

In this subsection we are concerned with a domain given by an upper level set of a single quadratic function. We shall first discuss a relatively simple matrix decomposition problem. A derivation of the LMI characterization of $\mathcal{FC}_+(D)$ follows thereafter.

As is well known, a matrix $X \in \mathcal{S}^{n\times n}$ is a positive semidefinite matrix of rank r if and only if there exist $p_i \in \Re^n$, $i = 1, 2, \ldots, r$, such that

$$X = \sum_{i=1}^{r} p_i p_i^{\mathrm{T}}.$$

Our new proposition is the following.

Proposition 3 *Let $X \in \mathcal{S}^{n\times n}$ be a positive semidefinite matrix of rank r. Let $G \in \mathcal{S}^{n\times n}$ be a given matrix. Then, $G \bullet X \geq 0$ if and only if there exist $p_i \in \Re^n$, $i = 1, 2, \ldots, r$, such that*

$$X = \sum_{i=1}^{r} p_i p_i^T \text{ and } p_i^T G p_i \geq 0 \text{ for all } i = 1, 2, \ldots, r.$$

The proof of this proposition is constructive. The crux of the construction is highlighted in the following procedure.

Procedure 1

Input: *$X, G \in \mathcal{S}^{n\times n}$ such that $0 \neq X \succeq 0$ and $G \bullet X \geq 0$.*

Output: *Vector $y \in \Re^n$ with $0 \leq y^T G y \leq G \bullet X$ such that $X - yy^T$ is a positive semidefinite matrix of rank $r-1$ where $r = \operatorname{rank}(X)$.*

Step 0 *Compute $p_1, \ldots, p_r$ such that $X = \sum_{i=1}^r p_i p_i^T$.*

Step 1 *If $(p_1^T G p_1)(p_i^T G p_i) \geq 0$ for all $i = 2, 3, \ldots, r$ then return $y = p_1$. Otherwise, let j be such that $(p_1^T G p_1)(p_j^T G p_j) < 0$.*

Step 2 *Determine α such that $(p_1 + \alpha p_j)^T G (p_1 + \alpha p_j) = 0$. Return $y = (p_1 + \alpha p_j)/\sqrt{1+\alpha^2}$.*

Lemma 7 *Procedure 1 is correct.*

Proof. If the procedure stops in Step 1 with $y = p_1$ then all the quantities $p_i^{\mathrm{T}} G p_i$, $i = 1, \ldots, r$ have the same sign. Furthermore, the sum of these quantities is nonnegative, since

$$\sum_{i=1}^r p_i^{\mathrm{T}} G p_i = G \bullet X \geq 0.$$

Therefore, $p_i^{\mathrm{T}} G p_i \geq 0$ for all $i = 1, 2, \ldots, r$. Moreover, $X - yy^{\mathrm{T}} = \sum_{i=2}^r p_i p_i^{\mathrm{T}}$ so that indeed $X - yy^{\mathrm{T}}$ is a positive semidefinite matrix of rank $r-1$.

Otherwise (i.e. the procedure does not stop in Step 1), the quadratic equation in Step 2 of Procedure 1 always has 2 distinct roots, because $(p_1^{\mathrm{T}} G p_1)(p_j^{\mathrm{T}} G p_j) < 0$. The definitions of α and y in Step 2 imply that $0 = y^{\mathrm{T}} G y \leq G \bullet X$. Moreover, by letting $u := (p_j - \alpha p_1)/\sqrt{1+\alpha^2}$, we have

$$X - yy^{\mathrm{T}} = uu^{\mathrm{T}} + \sum_{i \in \{2,3,\ldots,r\} \setminus j} p_i p_i^{\mathrm{T}},$$

which has rank $r-1$, establishing the correctness of Procedure 1. **Q.E.D.**

Proof of Proposition 3:

It is obvious that the statement holds true for a matrix X of rank 0. Assume now that such is true for any matrix X with $\operatorname{rank}(X) \in \{0, 1, \ldots, r\}$ for a certain $r \in \{0, 1, \ldots, n-1\}$. Consider $X \in \mathcal{S}_+^{n\times n}$ with $G \bullet X \geq 0$ and $\operatorname{rank}(X) = r+1$. Applying Procedure 1, and using Lemma 7, we can find y_1 such that

$$\operatorname{rank}(X - y_1 y_1^{\mathrm{T}}) = r,\ X - y_1 y_1^{\mathrm{T}} \succeq 0,\ 0 \leq y_1^{\mathrm{T}} G y_1 \leq G \bullet X.$$

By induction, we conclude that there exist $y_2, \ldots, y_{r+1}$ such that

$$X - y_1 y_1^{\mathrm{T}} = \sum_{i=2}^{r+1} y_i y_i^{\mathrm{T}}$$

where $y_i^{\mathrm{T}} G y_i \geq 0$, $i = 2, \ldots, r+1$. **Q.E.D.**

Proposition 3 can be readily extended to a more specific form, as shown in the following corollary.

Corollary 4 *Let $X \in \mathcal{S}^{n \times n}$ be a positive semidefinite matrix of rank r. Let $G \in \mathcal{S}^{n \times n}$ be a given matrix, and $G \bullet X \geq 0$. Then, we can always find $p_i \in \Re^n$, $i = 1, 2, \ldots, r$, such that*

$$X = \sum_{i=1}^{r} p_i p_i^T \text{ and } p_i^T G p_i = G \bullet X / r \text{ for } i = 1, 2, \ldots, r.$$

The key to note here is that if $p_i^{\mathrm{T}} G p_i = G \bullet X/r$ are not satisfied for all $i = 1, ..., r$, then there will always exist two indices, say i and j such that $p_i^{\mathrm{T}} G p_i < G \bullet X/r$ and $p_j^{\mathrm{T}} G p_j > G \bullet X/r$. Similar as in Procedure 1, we can always find α, such that $(p_i + \alpha p_j)^{\mathrm{T}} G (p_i + \alpha p_j) = G \bullet X/r$.

Below we shall use the decomposition result in Proposition 3 to get explicit representations of some non-negative quadratic cones. We will use the property that if K_1 and K_2 are two convex cones, then

$$K_1^* \cap K_2^* = (K_1 + K_2)^*, \tag{19}$$

where $K_1 + K_2 = \{x + y \mid x \in K_1, y \in K_2\}$; see Corollary 16.4.2 in Rockafellar [10]. In fact, (19) is a special case of (17) with $K = K_1 \times K_2$ and $B = \begin{bmatrix} I, & I \end{bmatrix}^{\mathrm{T}}$. Dualizing both sides of (19), we also have (using the bi-polar theorem)

$$(K_1^* \cap K_2^*)^* = \operatorname{cl}(K_1 + K_2). \tag{20}$$

Theorem 1 *Let $q : \Re^n \to \Re$ be a quadratic function, and suppose that the upper level set $D = \{x \mid q(x) \geq 0\}$ is nonempty. Then*

$$\operatorname{conv}\left\{ yy^T \,\middle|\, y \in \mathcal{H}(D) \right\} = \{X \succeq 0 \mid \ \mathrm{M}(q(\cdot)) \bullet X \geq 0\}. \tag{21}$$

The cone of quadratic functions that are nonnegative on D is therefore

$$\mathcal{FC}_+(D) = \{X \succeq 0 \mid \ \mathrm{M}(q(\cdot)) \bullet X \geq 0\}^* = \operatorname{cl}\{Z \mid Z - t\,\mathrm{M}(q(\cdot)) \succeq 0, \, t \geq 0\}. \tag{22}$$

Proof. Using Proposition 3 and Lemma 2 respectively, we have

$$\begin{aligned}\{X \succeq 0 \mid \mathrm{M}(q(\cdot)) \bullet X \geq 0\} &= \mathrm{conv}\left\{ yy^{\mathrm{T}} \middle| y^{\mathrm{T}}\, \mathrm{M}(q(\cdot)) y \geq 0 \right\} \\ &= \mathrm{conv}\left\{ yy^{\mathrm{T}} \middle| y \in \mathcal{H}(D) \cup (-\mathcal{H}(D)) \right\},\end{aligned}$$

and obviously $\mathrm{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D) \cup (-\mathcal{H}(D))\} = \mathrm{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D)\}$. This establishes (21).

Using Corollary 1 and relation (21), we have

$$\mathcal{FC}_+(D) = \mathrm{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D)\}^* = \{X \succeq 0 \mid \mathrm{M}(q(\cdot)) \bullet X \geq 0\}^*. \tag{23}$$

Applying (20), it further follows that

$$\begin{aligned}\mathcal{FC}_+(D) &= \mathrm{cl}\left(\mathcal{S}_+^{(1+n)\times(1+n)} + \{t\, \mathrm{M}(q(\cdot)) \mid t \geq 0\} \right) \\ &= \mathrm{cl}\{Z \mid Z - t\, \mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\}.\end{aligned}$$

Q.E.D.

We remark that in general, the set $\{Z \mid Z - t\, \mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\}$ is not necessarily closed. Consider for instance the function $q : \Re \to \Re$ defined as $q(x) = -x^2$, for which $D = \{x \mid q(x) \geq 0\} = \{0\}$. Clearly, the function $f(x) = x$ is nonnegative on D, but the 2×2 matrix

$$\mathrm{M}(f(\cdot)) - t\, \mathrm{M}(q(\cdot)) = \begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix} - t \begin{bmatrix} 0, & 0 \\ 0, & -1 \end{bmatrix}$$

is not positive semidefinite for any t. However, for any $\epsilon > 0$ and $t \geq 1/\epsilon$, we have

$$\begin{bmatrix} \epsilon, & 1 \\ 1, & 0 \end{bmatrix} - t \begin{bmatrix} 0, & 0 \\ 0, & -1 \end{bmatrix} \succeq 0.$$

Letting $\epsilon \downarrow 0$, we see in this case that $\mathrm{M}(f(\cdot))$ is (merely) a limit point of $\{Z \mid Z - t\, \mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\}$.

As a corollary to Theorem 1, we arrive at the following well known result from robust control, which is known as the S-procedure [14].

Corollary 5 *Let $f : \Re^n \to \Re$ and $q : \Re^n \to \Re$ be quadratic functions, and suppose that there exists $\bar{x} \in \Re^n$ such that $q(\bar{x}) > 0$. Let $D = \{x \mid q(x) \geq 0\}$. Then*

$$\mathcal{FC}_+(D) = \{Z \mid Z - t\, \mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\}.$$

This means that $f(x) \geq 0$ for all $x \in D$ if and only if there exists $t \geq 0$ such that $f(x) - tq(x) \geq 0$ for all $x \in \Re^n$.

Proof. Let $y := \begin{bmatrix} 1, & \bar{x}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and let

$$Z \in \operatorname{cl}\{Z \mid Z - t\,\mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\}.$$

Then there exist $Z_k \in \mathcal{S}^{(1+n)\times(1+n)}$ and $t_k \in \Re_+$ with $Z_k - t_k\,\mathrm{M}(q(\cdot)) \succeq 0$ and $Z_k \to Z$. We have

$$0 \leq y^{\mathrm{T}}(Z_k - t_k\,\mathrm{M}(q(\cdot)))y = y^{\mathrm{T}} Z_k y - t_k q(\bar{x}),$$

so that $0 \leq t_k \leq y^{\mathrm{T}} Z_k y / q(\bar{x})$. It follows that $\{t_k\}$ is bounded and hence it has a cluster point t such that $Z - t\,\mathrm{M}(q(\cdot)) \succeq 0$. This shows that

$$\{Z \mid Z - t\,\mathrm{M}(q(\cdot)) \succeq 0,\, t \geq 0\} \text{ is closed.} \tag{24}$$

By definition, $f(x) \geq 0$ for all $x \in D$ if and only if

$$\mathrm{M}(f(\cdot)) \in \mathcal{FC}_+(D). \tag{25}$$

Using (7), we know that $f(x) - tq(x) \geq 0$ for all $x \in \Re^n$ if and only if

$$\mathrm{M}(f(\cdot)) - t\,\mathrm{M}(q(\cdot)) \in \mathcal{FC}_+(\Re^n) = \mathcal{S}_+^{(1+n)\times(1+n)}. \tag{26}$$

Using Theorem 1 with (24), we have (25) if and only if (26) holds for some $t \geq 0$. **Q.E.D.**

The regularity condition that there exists $\bar{x} \in \Re^n$ such that $q(\bar{x}) > 0$ is equivalent to stating that $\mathrm{M}(q(\cdot))$ is not negative semidefinite. Namely, $q(x) \leq 0$ for all x if and only if $-q(\cdot)$ is nonnegative on the whole $\Re^n$, which holds if and only if $\mathrm{M}(q(\cdot)) \preceq 0$; see (7).

For the special case that $q(\cdot)$ is concave, the LMI representation of $\mathcal{FC}_+(D)$ as stated in Theorem 1 can also be found in Fu, Luo and Ye [3] and Rendl and Wolkowicz [9].

5.2 One quadratic equality constraint

The following proposition states a special case of Corollary 4.

Proposition 4 *Let $X \in \mathcal{S}^{n\times n}$ be a positive semidefinite matrix of rank r. Let $G \in \mathcal{S}^{n\times n}$ be a given matrix. Then, $G \bullet X = 0$ if and only if there exist $p_i \in \Re^n$, $i = 1, 2, \ldots, r$, such that*

$$X = \sum_{i=1}^{r} p_i p_i^T \text{ and } p_i^T G p_i = 0 \text{ for all } i = 1, 2, \ldots, r.$$

Similar to Theorem 1 we obtain from Proposition 4 and Corollary 2 the following result.

Theorem 2 *Let $q : \Re^n \to \Re$ be a strictly concave quadratic function, and suppose that the level set $D = \{x \mid q(x) = 0\}$ is nonempty. Then*

$$\operatorname{conv}\left\{ yy^T \,\middle|\, y \in \mathcal{H}(D) \right\} = \{ X \succeq 0 \mid \ \mathrm{M}(q(\cdot)) \bullet X = 0\}$$

and

$$\mathcal{FC}_+(D) = \ \mathrm{cl}\{Z \mid Z - t\,\mathrm{M}(q(\cdot)) \succeq 0,\, t \in \Re\}.$$

Corollary 6 *Let $f : \Re^n \to \Re$ and $q : \Re^n \to \Re$ be quadratic functions, and suppose that $q(\cdot)$ is strictly concave and that there exist $x^{(1)}, x^{(2)} \in \Re^n$ such that $q(x^{(1)}) > 0$ and $q(x^{(2)}) < 0$. Let $D = \{x \mid q(x) = 0\}$. Then $f(x) \geq 0$ for all $x \in D$ if and only if there exists $t \in \Re$ such that $f(x) - tq(x) \geq 0$ for all $x \in \Re^n$.*

The proof of the above result is analogous to the proof of Corollary 5.

Considering both Corollary 5 and Corollary 6 we remark that if a quadratic function $f(\cdot)$ is nonnegative on the level set $D = \{x \mid q(x) = 0\}$ of a strictly concave quadratic function $q(\cdot)$, then there cannot exist two solutions $x^{(1)}$ and $x^{(2)}$ such that $q(x^{(1)}) < 0$ and $q(x^{(2)}) > 0$, but $\max(f(x^{(1)}), f(x^{(2)})) < 0$.

5.3 One linear and one concave quadratic constraint

In this subsection we will deal with a domain defined by one linear and one concave quadratic constraint.

Let $q(x) = c + 2b^{\mathrm{T}}x + x^{\mathrm{T}}Ax$ be a concave quadratic function with a nonempty upper level set $D := \{x \mid q(x) \geq 0\}$. Because of the concavity of $q(\cdot)$, D is convex and hence $\mathcal{H}(D)$ is a convex cone. Using Lemma 4 we have

$$\mathcal{H}(D) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \,\middle|\, t \geq 0,\, t^2c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax \geq 0,\, tc + 2b^{\mathrm{T}}x \geq 0 \right\}.$$

Due to the concavity of $q(\cdot)$, it holds that $x^{\mathrm{T}}Ax \leq 0$ for all x and therefore

$$\begin{cases} t \geq 0,\, t^2c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax > 0 \Longrightarrow t > 0,\, tc + 2b^{\mathrm{T}}x > 0 \\ t > 0,\, t^2c + 2tb^{\mathrm{T}}x + x^{\mathrm{T}}Ax \geq 0 \Longrightarrow tc + 2b^{\mathrm{T}}x \geq 0. \end{cases} \tag{27}$$

Suppose that $a \in \Re^{1+n}$ and $X \in \mathcal{S}_+^{(1+n)\times(1+n)}$ are such that $Xa \neq 0$. Let U be a matrix of full column rank such that

$$X = UU^{\mathrm{T}}.$$

Then we have $Xa = U(U^{\mathrm{T}}a)$ so that

$$X - \frac{1}{a^{\mathrm{T}}Xa}(Xa)(Xa)^{\mathrm{T}} = U\left(I - \frac{1}{\|U^{\mathrm{T}}a\|^2}(U^{\mathrm{T}}a)(U^{\mathrm{T}}a)^{\mathrm{T}}\right)U^{\mathrm{T}}. \tag{28}$$

It is clear that the right hand side in the above equation is a positive semidefinite matrix of rank $r-1$, where $r = \operatorname{rank}(X)$. This fact is used in Lemma 8.

Procedure 2

Input: *$X \in \mathcal{S}^{(1+n)\times(1+n)}$, a concave quadratic function $q : \Re^n \to \Re$ and a vector $a \in \Re^{1+n}$ such that $X \succeq 0$, $\mathrm{M}(q(\cdot)) \bullet X \geq 0$, and $0 \neq Xa \in \mathcal{H}(D)$, where $D := \{x \mid q(x) \geq 0\}$.*

Output: *One of the two possibilities:*

- *Vector $y \in \mathcal{H}(D)$ with $0 \leq y^T \mathrm{M}(q(\cdot))y \leq \mathrm{M}(q(\cdot)) \bullet X$ and $a^T y \geq 0$ such that $X^{\text{new}} := X - yy^T$ is a positive semidefinite matrix of rank $r-1$ where $r = \operatorname{rank}(X)$, and $X^{\text{new}}a \in \mathcal{H}(D)$.*
- *Vector $0 \neq y \in \mathcal{H}(D)$ with $0 \leq y^T \mathrm{M}(q(\cdot))y \leq \mathrm{M}(q(\cdot)) \bullet X$ and $a^T y \geq 0$ such that $X^{\text{new}} := X - yy^T$ is a positive semidefinite matrix, and $X^{\text{new}}a \neq 0$ is on the boundary of $\mathcal{H}(D)$, i.e., $a^T X^{\text{new}} \mathrm{M}(q(\cdot))X^{\text{new}}a = 0$.*

Step 0 *Let $p_1 := Xa/\sqrt{a^T Xa}$ and compute $p_2, \ldots, p_r$ such that $X - p_1 p_1^T = \sum_{i=2}^r p_i p_i^T$ and that the first entry of p_i is non-negative, $i = 2, \ldots, r$.*

Step 1 *If $p_1^T \mathrm{M}(q(\cdot))p_1 \leq \mathrm{M}(q(\cdot)) \bullet X$ then return $y = p_1$. Otherwise, let $j \in \{2, 3, \ldots, r\}$ be such that $p_j^T \mathrm{M}(q(\cdot))p_j < 0$.*

Step 2 *Determine $\alpha > 0$ such that $(p_1 + \alpha p_j)^T \mathrm{M}(q(\cdot))(p_1 + \alpha p_j) = 0$. Let*

$$v = (p_1 + \alpha p_j)/\sqrt{1+\alpha^2} \text{ and } w(t) = Xa - t(a^T v)v.$$

Define γ_0 and γ_1 to be such that $w(t)^T \mathrm{M}(q(\cdot))w(t) = \gamma_0 - \gamma_1 t$ for all $t \in \Re$.

Step 3 *If $\gamma_1 > \gamma_0$ then let $y = \sqrt{\gamma_0/\gamma_1}\,v$, else let $y = v$.*

Lemma 8 *Procedure 2 is correct.*

Proof. By definition of $\mathcal{H}(D)$, $Xa \in \mathcal{H}(D)$ implies $(Xa)^{\mathrm{T}} \mathrm{M}(q(\cdot))Xa \geq 0$. Therefore, if Procedure 2 stops with $y = p_1 := Xa/\sqrt{a^{\mathrm{T}}Xa}$ in Step 1 then $y^{\mathrm{T}} \mathrm{M}(q(\cdot))y = (Xa)^{\mathrm{T}} \mathrm{M}(q(\cdot))Xa/(a^{\mathrm{T}}Xa) \geq 0$, and $\operatorname{rank}(X - yy^{\mathrm{T}}) = r - 1$ as stipulated by (28). Moreover, $a^{\mathrm{T}}y = \sqrt{a^{\mathrm{T}}Xa} \geq 0$ so that

$X^{\rm new} a = Xa - (a^{\rm T}y)y = Xa - Xa = 0 \in \mathcal{H}(D)$. Therefore, the procedure terminates correctly in Step 1.

Suppose now that the procedure does not stop at Step 1, i.e.

$$p_1^{\rm T}\ {\rm M}(q(\cdot))p_1 >\ {\rm M}(q(\cdot)) \bullet X \geq 0. \tag{29}$$

Using (27), it follows that the first entry of p_1 is (strictly) positive. Furthermore, since

$$\sum_{j=2}^{r} p_j^{\rm T}\ {\rm M}(q(\cdot))p_j =\ {\rm M}(q(\cdot)) \bullet X - p_1^{\rm T}\ {\rm M}(q(\cdot))p_1 < 0,$$

it also follows that there is indeed a $j \in \{2, 3, \ldots, r\}$ such that $p_j^{\rm T}\ {\rm M}(q(\cdot))p_j < 0$.

The quadratic equation in Step 2 of Procedure 2 always has one positive and one negative root, due to $p_1^{\rm T}\ {\rm M}(q(\cdot))p_1 > 0$ and $p_j^{\rm T}\ {\rm M}(q(\cdot))p_j < 0$. The procedure defines α to be the positive root. Because the first entry in p_j was made nonnegative in Step 0, it follows that $p_1 + \alpha p_j \in \Re_{++} \times \Re^n$. This further means that the first entry in $v := (p_1 + \alpha p_j)/\sqrt{1+\alpha^2}$ is positive. Moreover, $v^{\rm T}\ {\rm M}(q(\cdot))v = 0$ due to the definition of α. As can be seen from (27), these two properties of v imply that $0 \neq v \in \mathcal{H}(D)$. This proves that $0 \neq y \in \mathcal{H}(D)$ after termination in Step 3.

Using the definition of p_1 and $p_2, p_3, \ldots, p_r$, we have

$$(\sum_{i=2}^{r} p_i p_i^{\rm T})a = (X - p_1 p_1^{\rm T})a = Xa - Xa = 0.$$

This implies that $\sum_{i=2}^{r}(p_i^{\rm T}a)^2 = 0$ and hence $p_i^{\rm T}a = 0$ for all $i = 2, \ldots, r$. We further obtain that

$$a^{\rm T}v = \frac{a^{\rm T}(p_1 + \alpha p_j)}{\sqrt{1+\alpha^2}} = \frac{a^{\rm T}p_1}{\sqrt{1+\alpha^2}} = \sqrt{\frac{a^{\rm T}Xa}{1+\alpha^2}} > 0,$$

and hence $a^{\rm T}y \geq 0$ after termination in Step 3.

The scalars γ_0 and γ_1 in Step 2 are well defined, since $v^{\rm T}\ {\rm M}(q(\cdot))v = 0$ due to the definition of α. In fact, it is easily verified that $\gamma_0 = a^{\rm T}Xa > 0$ and $\gamma_1 = 2(a^{\rm T}v)(a^{\rm T}Xv)$. To simplify notations, we define

$$\tau := \begin{cases} 1, & \text{if } \gamma_1 \leq \gamma_0 \\ \gamma_0/\gamma_1, & \text{if } \gamma_1 > \gamma_0\ (> 0) \end{cases}$$

so that $y = \sqrt{\tau}v$. By definition of γ_0 and γ_1, we have

$$w(t)^{\rm T}\ {\rm M}(q(\cdot))w(t) = \gamma_0 + t\gamma_1 > 0 \text{ for } 0 \leq t < \tau. \tag{30}$$

Using (27), this implies by a continuity argument that $w(\tau) \in \mathcal{H}(D)$. However,

$$w(\tau) = Xa - \tau(a^{\rm T}v)v = Xa - (a^{\rm T}y)y = X^{\rm new}a,$$

so that $X^{\text{new}}a \in \mathcal{H}(D)$ as desired. Furthermore, we have

$$\tau < 1 \Longrightarrow 0 = w(\tau)^{\mathrm{T}}\, \mathrm{M}(q(\cdot))w(\tau) = a^{\mathrm{T}}X^{\text{new}}\, \mathrm{M}(q(\cdot))X^{\text{new}}a,$$

which means that $X^{\text{new}}a$ is on the boundary of $\mathcal{H}(D)$ if $\tau < 1$. Furthermore, $X^{\text{new}}a = w(\tau) \neq 0$ since $w(t) \neq 0$ for any $t \neq 1$.

It remains to verify that if $\tau = 1$ then $\operatorname{rank}(X - yy^{\mathrm{T}}) = r - 1$, where $r = \operatorname{rank}(X)$. We now introduce $u = (p_j - \alpha p_1)/\sqrt{1+\alpha^2}$, for which we have the obvious relation

$$uu^{\mathrm{T}} + vv^{\mathrm{T}} = p_1 p_1^{\mathrm{T}} + p_j p_j^{\mathrm{T}}.$$

Since $\tau = 1$ implies $y = v$, we therefore get

$$X - yy^{\mathrm{T}} = X - vv^{\mathrm{T}} = uu^{\mathrm{T}} + \sum_{i \in \{2,3,\ldots,r\} \setminus j} p_i p_i^{\mathrm{T}}.$$

It follows that $\operatorname{rank}(X - yy^{\mathrm{T}}) = r - 1$. **Q.E.D.**

We observe from Lemma 8 that if Procedure 2 does *not* reduce the rank of X then the vector $X^{\text{new}}a$ is nonzero and on the boundary of $\mathcal{H}(D)$. However, if we apply the procedure to X^{new} we find that $0 \neq p_1^{\text{new}} = X^{\text{new}}a/\sqrt{a^{\mathrm{T}}X^{\text{new}}a}$ and $p_1^{\mathrm{T}}\, \mathrm{M}(q(\cdot))p_1 = 0$. Therefore, the procedure exits at Step 1 to produce $X^{\text{final}} := X^{\text{new}} - p_1^{\text{new}}(p_1^{\text{new}})^{\mathrm{T}}$ with $X^{\text{final}}a = 0$. We will decompose X^{final} using Procedure 1. Based on this scheme, we arrive at the matrix decomposition result as stated in Proposition 5 below.

Proposition 5 *Let $q : \Re^n \to \Re$ be a concave quadratic function, $D = \{x \mid q(x) \geq 0\} \neq \emptyset$, $X \in \mathcal{S}_+^{(1+n)\times(1+n)}$ and $\mathrm{M}(q(\cdot)) \bullet X \geq 0$, and a vector $a \in \Re^{1+n}$ be such that $Xa \in \mathcal{H}(D)$. Then there exist y_i, $i = 1, \ldots, k$ for some $k \in \{r, r+1\}$ with $r = \operatorname{rank}(X)$, such that*

$$X = \sum_{i=1}^{k} y_i y_i^T$$

and $y_i \in \mathcal{H}(D)$ and $a^T y_i \geq 0$, $i = 1, \ldots, k$.

Proof. We distinguish three cases.

Case 1. If $Xa = 0$, then we invoke Procedure 1 to obtain $X = \sum_{i=1}^{r} y_i y_i^{\mathrm{T}}$ with $y_i \in \mathcal{H}(D)$ for all $i = 1, 2, \ldots, r$; see Proposition 3. Moreover, since $0 = a^{\mathrm{T}}Xa = \sum_{i=1}^{r}(a^{\mathrm{T}}y_i)^2$, it follows that $a^{\mathrm{T}}y_i = 0$ for $i = 1, 2, \ldots, r$. This shows that if $Xa = 0$ then the proposition holds with $k = r$.

Case 2. Consider now the case that $Xa \neq 0$ and applying Procedure 2 once on X does *not* reduce the rank. Apply Procedure 2 to obtain $y_1 \in \mathcal{H}(D)$ with $a^{\mathrm{T}} y_1 \geq 0$ such that

$$X^{\text{new}} := X - y_1 y_1^{\mathrm{T}} \succeq 0,\ X^{\text{new}} a \in \mathcal{H}(D),\ \ \mathrm{M}(q(\cdot)) \bullet X^{\text{new}} \geq 0.$$

If $\operatorname{rank}(X^{\text{new}}) = \operatorname{rank}(X) = r$ then $a^{\mathrm{T}} X^{\text{new}}\, \mathrm{M}(q(\cdot)) X^{\text{new}} a = 0$ and we can apply Procedure 2 on X^{new} to obtain $y_2 \in \mathcal{H}(D)$ with $a^{\mathrm{T}} y_2 \geq 0$ such that

$$X^{\text{final}} := X - y_1 y_1^{\mathrm{T}} - y_2 y_2^{\mathrm{T}} \succeq 0,\ X^{\text{final}} a = 0,\ \ \mathrm{M}(q(\cdot)) \bullet X^{\text{final}} \geq 0,$$

and $\operatorname{rank}(X^{\text{final}}) = r - 1$. Since Case 1 applies to X^{final}, we know that $X^{\text{final}} = \sum_{i=3}^{r+1} y_i y_i^{\mathrm{T}}$, with $y_i \in \mathcal{H}(D)$ and $a^{\mathrm{T}} y_i \geq 0$, $i = 3, 4, \ldots, r+1$. Hence, the proposition also holds true for Case 2 with $k = r + 1$.

Case 3. The remaining case is that $Xa \neq 0$ and applying Procedure 2 once on X reduces the rank. Since the rank is always nonnegative, we can reduce this case to either Case 1 or Case 2 by a recursive argument: we can now prove the proposition by induction on $\operatorname{rank}(X)$. Namely, suppose now that the proposition holds true for any matrix X with $\operatorname{rank}(X) \in \{0, 1, \ldots, r\}$ for a certain $r \in \{0, 1, \ldots, n\}$. Consider $X \in \mathcal{S}_+^{(n+1)\times(n+1)}$ with $\operatorname{rank}(X) = r + 1$, for which Procedure 2 yields a vector $y_1 \in \mathcal{H}(D)$ with $a^{\mathrm{T}} y_1 \geq 0$ such that

$$X^{\text{new}} := X - y_1 y_1^{\mathrm{T}} \succeq 0,\ X^{\text{new}} a \in \mathcal{H}(D),\ \ \mathrm{M}(q(\cdot)) \bullet X^{\text{new}} \geq 0,$$

and $\operatorname{rank}(X^{\text{new}}) = \operatorname{rank}(X) - 1 = r$. By induction, we conclude that there exist $y_2, \ldots, y_{k+1}$ for some $k \in \{r, r+1\}$ such that

$$X - y_1 y_1^{\mathrm{T}} = \sum_{i=2}^{k+1} y_i y_i^{\mathrm{T}}$$

where $y_i \in \mathcal{H}(D)$ and $a^{\mathrm{T}} y_i \geq 0$ for all $i = 2, \ldots, k+1$. **Q.E.D.**

Using similar reasoning as before, the above decomposition result implies an LMI characterization of $\mathcal{FC}_+(D)$.

Theorem 3 *Let $q : \Re^n \to \Re$ be a concave quadratic function, $a \in \Re^{n+1}$. Let*

$$D := \{x \mid q(x) \geq 0\}, \quad L := \{x \mid \begin{bmatrix} 1, & x^T \end{bmatrix} a \geq 0\}.$$

Suppose $D \cap L \neq \emptyset$. Then

$$\operatorname{conv}\left\{ yy^T \,\middle|\, y \in \mathcal{H}(D \cap L) \right\} = \left\{ X \in \mathcal{S}_+^{(1+n)\times(1+n)} \,\middle|\, \mathrm{M}(q(\cdot)) \bullet X \geq 0,\ Xa \in \mathcal{H}(D) \right\}.$$

Consequently, the cone of all quadratic functions that are nonnegative on D is

$$\begin{aligned} \mathcal{FC}_+(D) &= \left\{ X \in \mathcal{S}_+^{(1+n)\times(1+n)} \,\middle|\, \mathrm{M}(q(\cdot)) \bullet X \geq 0,\ Xa \in \mathcal{H}(D) \right\}^* \\ &= \mathrm{cl}\{Z \mid Z - (t\,\mathrm{M}(q(\cdot)) + a\psi^T + \psi a^T) \succeq 0,\ t \geq 0, \psi \in \mathcal{H}(D)^*\}. \end{aligned}$$

Proof. By Lemma 4 we know that

$$\mathcal{H}(D \cap L) = \{y \in \mathcal{H}(D) \mid a^{\mathrm{T}} y \geq 0\},$$

and so

$$\operatorname{conv}\left\{ yy^{\mathrm{T}} \;\middle|\; y \in \mathcal{H}(D \cap L)\right\} = \operatorname{conv}\left\{ yy^{\mathrm{T}} \;\middle|\; y \in \mathcal{H}(D),\, a^{\mathrm{T}} y \geq 0\right\}.$$

Suppose that X is a matrix in the above set, i.e. $X = \sum_{i=1}^{k} y_i y_i^{\mathrm{T}} \succeq 0$ with $y_i \in \mathcal{H}(D)$ and $a^{\mathrm{T}} y \geq 0$ for $i = 1, 2, \ldots, k$. Since $y_i \in \mathcal{H}(D)$, we certainly have $y_i^{\mathrm{T}}\, \mathrm{M}(q(\cdot)) y_i \geq 0$ and consequently

$$M(q(\cdot)) \bullet X = \sum_{i=1}^{k} y_i^{\mathrm{T}}\, \mathrm{M}(q(\cdot)) y_i \geq 0.$$

Moreover, $a^{\mathrm{T}} y_i \geq 0$ and $y_i \in \mathcal{H}(D)$ for all $i = 1, 2, \ldots, k$, and

$$Xa = \sum_{i=1}^{k} (a^{\mathrm{T}} y_i) y_i.$$

In other words, Xa is a nonnegative combination of vectors in the cone $\mathcal{H}(D)$, which implies that $Xa \in \mathcal{H}(D)$.

Conversely, for $X \succeq 0$ with $M(q(\cdot)) \bullet X \geq 0$ and $Xa \in \mathcal{H}(D)$, we know from Proposition 5 that $X = \sum_{i=1}^{k} y_i y_i^{\mathrm{T}}$ with $y_i \in \mathcal{H}(D)$ and $a^{\mathrm{T}} y \geq 0$ for $i = 1, 2, \ldots, k$. We conclude that

$$\operatorname{conv}\left\{ yy^{\mathrm{T}} \;\middle|\; y \in \mathcal{H}(D \cap L)\right\} = \left\{ X \in \mathcal{S}_{+}^{(1+n)\times(1+n)} \;\middle|\; \mathrm{M}(q(\cdot)) \bullet X \geq 0,\, Xa \in \mathcal{H}(D)\right\}. \tag{31}$$

Using Corollary 1 and (31), we have

$$\begin{aligned} \mathcal{FC}_{+}(\mathcal{H}(D \cap L)) &= \operatorname{conv}\{yy^{\mathrm{T}} \mid y \in \mathcal{H}(D \cap L)\}^{*} \\ &= \left\{ X \in \mathcal{S}_{+}^{(1+n)\times(1+n)} \;\middle|\; \mathrm{M}(q(\cdot)) \bullet X \geq 0,\, Xa \in \mathcal{H}(D)\right\}^{*}. \end{aligned} \tag{32}$$

We remark from (17) that

$$\{ay^{\mathrm{T}} + ya^{\mathrm{T}} \mid y \in \mathcal{K}\}^{*} = \{X \in \mathcal{S}^{n\times n} \mid Xa \in \mathcal{K}^{*}\}, \tag{33}$$

where the dual is taken in the Euclidean space $\mathcal{S}^{n\times n}$.

Applying (20) and (32)–(33), it follows that

$$\begin{aligned} \mathcal{FC}_{+}(D \cap L) &= \operatorname{cl}\left(\mathcal{S}_{+}^{(1+n)\times(1+n)} + \{t\, \mathrm{M}(q(\cdot)) \mid t \geq 0\} + \{a\psi + \psi a^{\mathrm{T}} \mid \psi \in \mathcal{H}(D)^{*}\}\right) \\ &= \operatorname{cl}\{Z \mid Z - (t\, \mathrm{M}(q(\cdot)) + a\psi^{\mathrm{T}} + \psi a^{\mathrm{T}}) \succeq 0,\, t \geq 0, \psi \in \mathcal{H}(D)^{*}\}. \end{aligned}$$

Q.E.D.

We recall from Corollary 3 that

$$\mathcal{H}(D) = \{x \mid Bx \in \mathrm{SOC}(2+\ \mathrm{rank}(A))\} = \{B^{\mathrm{T}}y \mid y \in \mathrm{SOC}(2+\ \mathrm{rank}(A))\}^*,$$

for a certain matrix B depending on A, b, c. Therefore, Theorem 1 characterizes $\mathcal{FC}_+(D \cap L)$ and its dual in terms of semidefinite and second order cone constraints. As a corollary to Theorem 1, we arrive at the following result.

Corollary 7 *Let $f : \Re^n \to \Re$ and $q : \Re^n \to \Re$ be quadratic functions, and $a \in \Re^{n+1}$. Suppose $q(\cdot)$ is concave and that there exists $\bar{x} \in \Re^n$ such that $q(\bar{x}) > 0$ and $\begin{bmatrix} 1, & \bar{x}^T \end{bmatrix} a > 0$. Let*

$$D := \{x \mid q(x) \geq 0\}, \quad L := \{x \mid \begin{bmatrix} 1, & x^T \end{bmatrix} a \geq 0\}.$$

Then

$$\mathcal{FC}_+(D \cap L) = \{Z \mid Z - (t\ \mathrm{M}(q(\cdot)) + a\psi^T + \psi a^T) \succeq 0,\ t \geq 0, \psi \in \mathcal{H}(D)^*\}.$$

This means that $f(x) \geq 0$ for all $x \in D \cap L$ if and only if there exists $t \geq 0$ and $\psi \in \mathcal{H}(D)^*$ such that

$$f(x) - tq(x) - (\begin{bmatrix} 1, & x^T \end{bmatrix} a)(\begin{bmatrix} 1, & x^T \end{bmatrix} \psi) \geq 0 \text{ for all } x \in \Re^n.$$

Proof. Let

$$Z \in \mathrm{cl}\{Z \mid Z - (t\ \mathrm{M}(q(\cdot)) + a\psi^{\mathrm{T}} + \psi a^{\mathrm{T}}) \succeq 0,\ t \geq 0, \psi \in \mathcal{H}(D)^*\}.$$

Then there exist $Z_k \in \mathcal{S}^{(1+n)\times(1+n)}$, $t_k \in \Re_+$ and $\psi_k \in \mathcal{H}(D)^*$ such that

$$Z_k - (t_k\ \mathrm{M}(q(\cdot)) + a\psi_k^{\mathrm{T}} + \psi_k a^{\mathrm{T}}) \succeq 0, \quad Z_k \to Z. \tag{34}$$

Let $y := \begin{bmatrix} 1, & \bar{x}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. Clearly, $a^{\mathrm{T}}y > 0$ and $y^{\mathrm{T}}\,\mathrm{M}(q(\cdot))y = q(\bar{x}) > 0$. Since $q(\bar{x}) > 0$ it follows that y is in the interior of $\mathcal{H}(D)$ and hence

$$\psi_k^{\mathrm{T}} y > 0, \text{ for all } 0 \neq \psi_k \in \mathcal{H}(D)^*. \tag{35}$$

Due to (34) we have

$$0 \leq y^{\mathrm{T}}(Z_k - t_k\ \mathrm{M}(q(\cdot)) - 2a\psi_k^{\mathrm{T}})y = y^{\mathrm{T}}Z_k y - t_k q(\bar{x}) - 2(a^{\mathrm{T}}y)(\psi_k^{\mathrm{T}}y).$$

Now using the fact that $q(\bar{x}) > 0$, $a^{\mathrm{T}}y > 0$ and $\psi_k^{\mathrm{T}}y \geq 0$, we obtain that

$$0 \leq t_k \leq y^{\mathrm{T}}Z_k y / q(\bar{x}), \quad 0 \leq \psi_k^{\mathrm{T}}y \leq y^{\mathrm{T}}Z_k y/(2a^{\mathrm{T}}y),$$

which shows that t_k and $\psi_k^{\mathrm{T}} y$ are bounded. Furthermore, y is in the interior of the (solid) cone $\mathcal{H}(D)$, so that the facts that $\psi_k^{\mathrm{T}} y$ is bounded and $\psi_k \in \mathcal{H}(D)^*$ implies that $\|\psi_k\|$ is bounded. Therefore, the sequences t_k and ψ_k have cluster points t and ψ respectively, and

$$Z - (t\,\mathrm{M}(q(\cdot)) + a\psi^{\mathrm{T}} + \psi a^{\mathrm{T}}) \succeq 0.$$

It follows that

$$\{Z \mid Z - (t\,\mathrm{M}(q(\cdot)) + a\psi^{\mathrm{T}} + \psi a^{\mathrm{T}}) \succeq 0,\, t \geq 0, \psi \in \mathcal{H}(D)^*\} \text{ is closed.} \tag{36}$$

The corollary now follows by the same argument as in the proof of Corollary 5. **Q.E.D.**

We remark that, using the problem formulation (MD) in Section 3, minimizing a quadratic function $f(\cdot)$ over the set $D \cap L$ can now be equivalently written as

$$\begin{array}{ll} \text{minimize} & \mathrm{M}(f(\cdot)) \bullet X \\ \text{subject to} & \mathrm{M}(q(\cdot)) \bullet X \geq 0 \\ & Xa \in \mathcal{H}(D) \\ & x_{11} = 1 \\ & X \succeq 0. \end{array}$$

This formulation, which is a *semidefinite programming* problem with the same optimal value as the original problem, is different from a straightforward semidefinite relaxation problem

$$\begin{array}{ll} \text{minimize} & \mathrm{M}(f(\cdot)) \bullet X \\ \text{subject to} & \mathrm{M}(q(\cdot)) \bullet X \geq 0 \\ & e_1^{\mathrm{T}} Xa \geq 0 \\ & x_{11} = 1 \\ & X \succeq 0. \end{array}$$

Notice that $e_1 a^{\mathrm{T}} + a e_1^{\mathrm{T}}$ is the matrix representation of the linear inequality, so that the above relaxation corresponds to applying the S-procedure with two quadratic constraints. This relaxation may admit a gap with the original problem. For instance, if $q(x) = 1 - x^2$, $a = \begin{bmatrix} 0, & 1 \end{bmatrix}^{\mathrm{T}}$ and $f(x) = 1 + x - x^2$, then the optimal solutions are $x = 1$ or $x = 0$ with value $f(x) = 1$. However, the optimal solution to the straightforward semidefinite relaxation is $X = I$ with value $\mathrm{M}(f(\cdot)) \bullet I = 0$. Indeed, $Xa = e_2 \notin \mathrm{SOC}(2)$ so that X cannot be decomposed as a convex combination of feasible rank-1 solutions.

6 Conclusion

The results claimed in Theorems 1, 2 and 3 are quite powerful. They characterize, using linear matrix inequalities, all the quadratic functions that are nonnegative over the respectively specified domains. If we decompose the dual optimal solution using procedures 1 and 2, we find that all the components y_i yield optimal solutions and directions to the (non-convex) quadratic optimization problem. To the best of our knowledge, such decomposition procedures have not been proposed before.

In trust region methods for nonlinear programming, one often needs to solve problems of type (P) in Section 3, where D is a unit ball. The problem is known to be solvable in polynomial time; for detailed discussions, see [15]. Our result extends the polynomial solvability property to a non-convex quadratic constraint (inequality or equality) and a non-convex quadratic objective. Another case that we can handle is a non-convex objective with a concave quadratic inequality constraint and an additional linear restriction. The complexity status of the problem to minimize a non-convex quadratic function over the intersection of two general ellipsoids is still an open problem in the study of trust region methods. However, our last application solves this problem for the special case where the two ellipsoids, or more generally, level sets of two concave quadratic functions, have the same geometric structure (may still be of very different sizes)[1]. Specifically, consider

$$\begin{array}{ll} \text{minimize} & q_0(x) \\ \text{subject to} & q_1(x) = x^{\mathrm{T}}Qx - 2b_1^{\mathrm{T}}x + c_1 \geq 0 \\ & q_2(x) = x^{\mathrm{T}}Qx - 2b_2^{\mathrm{T}}x + c_2 \geq 0. \end{array}$$

The key to note is that the feasible set of the above problem can be viewed as the union of two sets

$$\{x \mid x^{\mathrm{T}}Qx - 2b_1^{\mathrm{T}}x + c_1 \geq 0\} \cap \{x \mid 2(b_2 - b_1)^{\mathrm{T}}x + c_2 - c_1 \geq 0\}$$

and

$$\{x \mid x^{\mathrm{T}}Qx - 2b_2^{\mathrm{T}}x + c_2 \geq 0\} \cap \{x \mid 2(b_1 - b_2)^{\mathrm{T}}x + c_1 - c_2 \geq 0\}.$$

Minimizing an indefinite quadratic function $q_0(x)$ over each of these sets individually can be solved via an SDP formulation as shown in this paper. Hence, applying the method twice solves the whole problem.

References

[1] A. Ben-Tal and A.S. Nemirovsky. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[2] L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications*, 18(4):1035–1064, 1997.

[1]We thank Yinyu Yu for our discussions leading to this observation.

[3] M. Fu, Z.-Q. Luo, and Y. Ye. Approximation algorithms for quadratic programming. *Journal of Combinatorial Optimization*, 2(1):29–50, 1998.

[4] Z.-Q. Luo and J.F. Sturm. Error bounds for quadratic systems. In J.B.G. Frenk, C. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization*, pages 383–404. Kluwer Academic Publishers, 2000.

[5] K.G. Murty and S.N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Mathematical Programming*, 39:117–129, 1987.

[6] Y. Nesterov. Squared functional systems and optimization problems. In J.B.G. Frenk, C. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization*, pages 405–440. Kluwer Academic Publishers, 2000.

[7] P.A. Parrilo. On a decomposition of multivariable forms via LMI methods. Technical report, California Institute if Technology, Pasadena, CA, USA, 2000.

[8] A.J. Quist, E. de Klerk, C. Roos, and T Terlaky. Copositive relaxation for general quadratic programming. *Optimization Methods and Software*, 9:185–208, 1998. Special Issue Celebrating the 60th Birthday of Professor Naum Shor.

[9] F. Rendl and H. Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Mathematical Programming*, 77(2):273–299, 1997.

[10] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, NJ, 1970.

[11] J.F. Sturm. Theory and algorithms for semidefinite programming. In J.B.G. Frenk, C. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization*, pages 3–196. Kluwer Academic Publishers, 2000.

[12] S.A. Vavavis. *Nonlinear Optimization: Complexity Issues.* Oxford Science, New York, NY, USA, 1991.

[13] Henry Wolkowicz, Romesh Saigal, and Lieven Vandenberghe, editors. *Handbook on Semidefinite Programming.* Kluwer Academic Publishers, 2000.

[14] V.A. Yakubovich. S-procedure in nonlinear control theory. *Vestnik Leningradskovo Universiteta, Seriya Matematika*, pages 62–77, 1971. English translation in Vertnik Leningrad University 4, pp. 73 – 93, 1977.

[15] Y. Ye. *Interior Point Algorithms: Theory and Analysis.* John Wiley & Sons, New York, NY, USA, 1997.