An Inventory-Location Model: Formulation, Solution Algorithm and Computational Results

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ABSTRACT

We introduce a new distribution center (DC) location model that incorporates working inventory and safety stock inventory costs at the distribution centers. In addition, the model incorporates transport costs from the suppliers to the DCs that explicitly reflect economies of scale through the use of a fixed cost term. The model is formulated as a non-linear integer-programming problem. Model properties are outlined. A Lagrangian relaxation solution algorithm is proposed. By exploiting the structure of the problem we can find a low-order polynomial algorithm for the non-linear integer programming problem that must be solved in solving the Lagrangian relaxation subproblems. A number of heuristics are outlined for finding good feasible solutions. In addition, we describe two variable forcing rules that prove to be very effective at forcing candidate sites into and out of the solution. The algorithms are tested on problems with 88 and 150 retailers. Computation times are consistently below one minute and compare favorably with those of an earlier proposed set partitioning approach for this model (Shen, 2000; Shen, Coullard and Daskin, 2000). Finally, we discuss the sensitivity of the results to changes in key parameters including the fixed cost of placing orders. Significant reductions in these costs might be expected from e-commerce technologies. The model suggests that as these costs decrease it is optimal to locate additional facilities.

1. Introduction

Managing inventory has become a major challenge for firms as they simultaneously try to reduce costs and improve customer service in today's increasingly competitive business environment. Managing inventory consist of two critical tasks. First, we must determine the *number* of stocking locations or distribution centers to have. Second, we must determine the *amount* of inventory to maintain at each of the centers. Often these tasks are undertaken separately, resulting in a degree of suboptimization. In this paper, we outline an integrated approach to determining the number of distribution centers to establish and the magnitude of inventory to maintain at each center.

We argue that the importance of inventory management as outlined above has, if anything, been heightened by e-commerce. Whether it is business to consumer (B2C) or business to business (B2B), e-commerce end customers expect high levels of service and speedy deliveries. At the same time, many of the inputs to traditional inventory management decision-making are changing rapidly. E-commerce offers the hope of significantly reducing order costs thereby allowing smaller more frequent shipments. The model we propose below explicitly includes order shipment costs. As a result it is capable of estimating the impacts of sharply reduced ordering costs on the number of distributions centers, their locations, and the optimal inventory ordering policy. E-commerce can also reduce variability across the supply chain by making end customer orders visible throughout the chain. This visibility should reduce the bullwhip effect (Lee, Padmanabhan and Whang, 1997; Lee, 1996; Simchi-Levi, Kaminsky and Simchi-Levi, 2000). Our model does not directly account for these effects of e-commerce though the model does explicitly consider safety stock inventories at the distribution centers which are a function of the variability of customer demand.

This work was motivated by a study at a local blood bank conducted by two of the authors. The blood bank supplied roughly 30 hospitals in the greater Chicago area. Our focus was on the production and distribution of platelets, the most expensive and most perishable of all blood products. If a unit of platelets is not used within 5 days of the time it is produced from whole blood, it must be destroyed. The demand for platelets is highly variable as they are needed in only a limited number of medical contexts. When they are used, however, multiple units are often needed. The hospitals supplied by the blood bank collectively owned the blood bank and set prices. As a result they could return a unit of platelets up to the time it outdated and not be charged for it. Thus, there was little incentive to manage inventories in an efficient manner. Many of the larger hospitals ordered almost twice the number of platelet units that they used each year resulting in the need to destroy thousands of units of this expensive blood product. Other hospitals ordered almost all of their needed platelets on a STAT or emergency basis. The blood bank often had to ship the units to these hospitals using a taxi or express courier at significant expense to the system. Clearly an improved system was needed. We proposed a revision to their pricing policies along with a system in which selected hospitals would maintain an inventory of platelets for use in neighboring hospitals. This would allow the system to take advantage of the risk-pooling effect. We selected the hospitals at which inventory would be maintained using a *P*-median model (Hakimi, 1963, 1964; Daskin, 1995). The model did not account directly for the working inventory or safety stock (risk-pooling) effects, for the transport costs from the blood bank to the selected hospitals or the fixed costs of establishing the facilities. Clearly, this was only a first-cut approach. The research outlined below is aimed, in part, at developing a more comprehensive and more accurate model of such a situation.

The remainder of this paper is organized as follows. In section 2 we review relevant related literature. The model we propose is formulated in section 3. The formulation we obtain is a mixed integer non-linear programming problem which can be viewed as an extension of the traditional uncapacitated fixed charge facility location problem. In addition to the standard facility location and local distribution costs, the model includes cost components representing working and safety stock inventories at the distribution centers as well as transport costs from the supplier(s) to the distribution centers. These inventory and supplier-to-DC transport costs introduce significant non-linearities into the model. Section 4 outlines a number of key properties of the model we propose and introduces an additional assumption on the demand distributions that we use to develop a solution algorithm for the problem. Section 5 outlines our solution procedure for the problem. Computational results are presented in section 6. In section 7, we present our conclusions and outline directions for future work.

2. Literature Review

Inventory theory literature tends to focus on finding optimal inventory replenishment strategies at the DCs and the retailer outlets. This work usually assumes that the number and locations of the DCs are given. See, for example, Graves et al. (1993), Nahmias (1997) and Zipkin (1997). On the other hand, location theory tends to focus on developing models for determining the number of DCs and their locations, as well as the DC-retailer assignments. This work usually includes fixed facility setup costs and transportation costs, but the operational inventory and shortage costs are typically ignored. Daskin and Owen (1999) provide an overview of facility location modeling as do the recent texts by Daskin (1995) and Drezner (1995).

Eppen (1979) studied the so-called "risk pooling effects," namely the effects of inventory-cost savings achieved by grouping retailers. Assume customer demands are normally distributed with a mean m_i and a standard deviation s_i for customer *i*. Then the expected total inventory cost under the decentralized mode for n retailers is $K\sum_{i=1}^{n} s_i$. If the demands of the n retailers are independent, the optimal cost under a centralized mode can be expressed by $K\sqrt{\sum_{i=1}^{n} s_i^2}$, which is less than $K\sum_{i=1}^{n} s_i$, where K is a constant depending on the holding and penalty costs and the standard normal loss function.

Recently, there are several new studies that combine inventory management and routing decisions. For example, Federgruen and Zipkin (1984), Federgruen and Simchi-Levi (1995), Viswanathan and Mathur (1997), Chan, Federgruen, and Simchi-Levi(1998) and Kleywegt, Nori, and Savelsbergh (2000).

Also, several models combine location and routing decisions; for instance, Laporte and Dejax (1989), Berman, Jaillet and Simchi-Levi (1995), and Berger, Coullard and Daskin (1998).

Shen, Coullard and Daskin (2000) studied model presented below, but they use set partitioning approach. We compared our computational results with theirs. Teo, Ou and Goh (2000) studied the impact on inventory costs with consolidation of distribution centers. They also propose an algorithm that solves for a distribution system with the total fixed facility location cost and inventory costs within $\sqrt{2}$ of the optimal. But they ignore the costs to ship from the supplier to the DCs and from the DCs to the retailers in their model. Finally, Erlebacher and Meller (2000) formulate a highly non-linear integer location/inventory model. They attack the problem by using a continuous approximation as well as a number of construction and bounding heuristics. For problems with 16 customers, they obtained solutions that were between 3.78% and nearly 36% of a lower bound. An exchange heuristic improved the solution considerably. Computation times on a 600 node problem using the exchange heuristic averaged 117 hours on a Sun Ultra Sparcstation.

3. Model Formulation

We consider a three-tiered system consisting of one or more suppliers, distribution centers and retailers. We assume that the locations of the suppliers and the retailers are known and that the suppliers have infinite capacity at least from the perspective of the system being modeled. The problem is to determine the optimal number of distribution center, their location, the retailers assigned to each distribution center, and the optimal ordering policy at the distribution centers. We do not explicitly model the inventory maintained by the retailers themselves. A key problem is that the demand that is seen by each distribution center is a function of the demands at the retailers assigned to the distribution center. Thus, the inventory policy – the reorder interval, reorder size, and safety stock – at the distribution center is a function of the assignment of retailers to the distribution center. Since these assignments are not known *a priori*, the inventory policy must also be endogenously determined.

To begin modeling the problem, let us assume for the moment that we know which customers are to be assigned to a specific distribution. center. Assume that the demand at each retailer is Normally distributed with a daily mean of m_i and a daily variance of s_i^2 and let S be the set of customers assigned to the distribution center. Let L be the lead time in days for deliveries from the supplier to the distribution center.

Assuming that the daily demands at each retailer are uncorrelated over time and across retailers, the lead time demand at the distribution center is Normally distributed with a mean of $L \sum_{i \in \boldsymbol{S}} m_i$ and a variance of $L \sum_{i \in \boldsymbol{S}} \boldsymbol{s}_i^2$. The safety stock required to ensure that stockouts occur with a probability of $\boldsymbol{a}$ or less is

$$z_{\boldsymbol{a}} \sqrt{L \sum_{i \in \boldsymbol{S}} \boldsymbol{s}_i^2} \tag{1}$$

where $z_{\boldsymbol{a}}$ is a standard Normal deviate such that $P(z \le z_{\boldsymbol{a}}) = \boldsymbol{a}$.

For the moment, let D be the expected annual demand (i.e., $D = \boldsymbol{c} \sum_{i \in \boldsymbol{S}} m_i$), where $\boldsymbol{c}$ is a constant used to convert daily demand into annual demand (e.g., 365 if demands occur every day of the year), let h be the holding cost per item per year and let F be the fixed cost of placing an order from the distribution center to the supplier. Then the annual cost of ordering inventory from the supplier at the distribution center is given by

$$Fn + \boldsymbol{b} v\left(\frac{D}{n}\right) n + \boldsymbol{q} \frac{hD}{2n} \tag{2}$$

where n is the (unknown) number of orders per year, $v(x)$ is the cost of shipping an order of size x from the supplier, and $\boldsymbol{b}$ and $\boldsymbol{q}$ are weights that we assign to transportation and inventory costs respectively so that we can later test the effects of varying the importance of these costs relative to the fixed facility costs.

The first term of (2) represents the total fixed cost of placing n orders per year. The second term represents the shipment cost $v\left(\frac{D}{n}\right)$ per shipment multiplied by the number of shipments per year and the weight, $\boldsymbol{b}$, associated with transport. D/n is the expected shipment size per shipment. The third term is cost of the average working inventory. On average, there will be $\frac{D}{2n}$ items of working inventory on hand at a cost of h per item per year. Taking the derivative of this expression with respect to n, the number of orders per year, we obtain

$$F + \boldsymbol{b} v\left(\frac{D}{n}\right) - \boldsymbol{b} n v'\left(\frac{D}{n}\right)\frac{D}{n^2} - \boldsymbol{q}\frac{hD}{2n^2} = F + \boldsymbol{b} v\left(\frac{D}{n}\right) - \boldsymbol{b} v'\left(\frac{D}{n}\right)\frac{D}{n} - \boldsymbol{q}\frac{hD}{2n^2} = 0 \tag{3}$$

If $v(x)$ is linear in x (e.g., if $v(x) = g + ax$) then $v'(x)$ is a constant (e.g., a) and the expression above becomes

$$F + \boldsymbol{b}g + \boldsymbol{b}a\frac{D}{n} - \boldsymbol{b}a\frac{D}{n} - \boldsymbol{q}\frac{hD}{2n^2} = F + \boldsymbol{b}g - \boldsymbol{q}\frac{hD}{2n^2} = 0 \tag{4}$$

Solving for n, we obtain $n = \sqrt{\dfrac{\boldsymbol{q}hD}{2(F + \boldsymbol{b}g)}}$. Substituting this into the cost function (2) we obtain a working inventory cost of

$$\begin{aligned} &F\sqrt{\frac{\boldsymbol{q}hD}{2(F + \boldsymbol{b}g)}} + \boldsymbol{b}g\sqrt{\frac{\boldsymbol{q}hD}{2(F + \boldsymbol{b}g)}} + \boldsymbol{b}aD + \boldsymbol{q}\frac{hD}{2}\sqrt{\frac{2(F + \boldsymbol{b}g)}{\boldsymbol{q}hD}} = \\ &\sqrt{\frac{\boldsymbol{q}hD(F + \boldsymbol{b}g)}{2}} + \boldsymbol{b}aD + \sqrt{\frac{\boldsymbol{q}hD(F + \boldsymbol{b}g)}{2}} = \sqrt{2\boldsymbol{q}hD(F + \boldsymbol{b}g)} + \boldsymbol{b}aD \end{aligned} \tag{5}$$

Recall that this cost includes the costs of placing orders at the distribution center, transporting goods from the supplier to the distribution center and holding the working inventory at the DC.

Unfortunately, the derivation of the safety stock (1) and the working inventory cost (5) assume that we know the assignment of retailers to the distribution center as denoted by the set ***S***. The identity of this set is *not* known *a priori* and must be determined endogenously.

To simultaneously determine the locations of the DCs, assignments of the retailers to the DCs, and working and safety stock inventory costs, let us define the following additional inputs and sets:

I	set of retailers indexed by i
J	set of candidate DC sites indexed by j
f_j	fixed (annual) cost of locating a distribution center at candidate site j, for each $j \in \boldsymbol{J}$
d_{ij}	cost per unit to ship between retailer i and candidate DC site j, for each $i \in \boldsymbol{I}$ and $j \in \boldsymbol{J}$
$\boldsymbol{c}$	days per year (used to convert daily demand and variance values to annual values).

In addition, we define the following decision variables:

$$X_j = \begin{cases} 1 & \text{if we locate at candidate site } j \\ 0 & \text{if not} \end{cases}$$

$$Y_{ij} \quad = \quad \begin{cases} 1 & \text{if demands at retailer } i \text{ are assigned to a DC at candidate site } j \\ 0 & \text{if not} \end{cases}$$

With this notation, we can formulate the problem as follows:

$$\text{Minimize} \quad \sum_{j \in \boldsymbol{J}} f_j X_j + \left(\boldsymbol{b} \sum_{j \in \boldsymbol{J}} \sum_{i \in \boldsymbol{I}} \boldsymbol{c}\, d_{ij}\, m_i Y_{ij} \right) +$$

$$\left(\sum_{j \in \boldsymbol{J}} \sqrt{2qh \left(F_j + \boldsymbol{b}\, g_j \right) \sum_{i \in \boldsymbol{I}} \boldsymbol{c}\, m_i Y_{ij}} + \boldsymbol{b} \sum_{j \in \boldsymbol{J}} a_j \sum_{i \in \boldsymbol{I}} \boldsymbol{c}\, m_i Y_{ij} \right) + \tag{6}$$

$$qh\, z_{\boldsymbol{a}} \sum_{j \in \boldsymbol{J}} \sqrt{\sum_{i \in \boldsymbol{I}} L\, \boldsymbol{s}_i^2 Y_{ij}}$$

$$\text{Subject to} \quad \sum_{j \in \boldsymbol{J}} Y_{ij} = 1 \qquad \forall i \in \boldsymbol{I} \tag{7}$$

$$Y_{ij} \le X_j \qquad \forall i \in \boldsymbol{I}; \forall j \in \boldsymbol{J} \tag{8}$$

$$X_j \in \{0,1\} \qquad \forall j \in \boldsymbol{J} \tag{9}$$

$$Y_{ij} \in \{0,1\} \qquad \forall i \in \boldsymbol{I}; \forall j \in \boldsymbol{J} \tag{10}$$

The first term of the objective function (6) is the fixed cost of locating facilities. The second term represents the local delivery cost. Note that $\sum_{i \in \boldsymbol{I}} \boldsymbol{c}\, m_i Y_{ij}$ represents the total annual demand assigned to distribution center j. Thus, the third term represents the total working inventory cost (5) where we have added the obvious subscripts j to the fixed and unit shipping costs g and a respectively obtaining g_j and a_j. In addition, the fixed cost of placing an order, F, has been made DC-specific, obtaining F_j. The fourth term represents the safety stock inventory cost. Constraint (7) states that each demand node must be assigned to a DC. Constraint (8) stipulates that the assignments can only be made to open DCs. Finally, constraints (9) and (10) are standard integrality constraints, with (10) representing single-sourcing constraints, meaning that all of the demand at a retailer must be assigned to the same DC.

The objective function can be rearranged as follows:

$$
\text{Minimize} \quad \sum_{j \in \boldsymbol{J}} f_j X_j + \left(\beta \sum_{j \in \boldsymbol{J}} \sum_{i \in \boldsymbol{I}} \chi \mu_i \left(d_{ij} + a_j \right) Y_{ij} \right) +
$$

$$
\left(\sum_{j \in \boldsymbol{J}} \sqrt{2\theta h \left(F_j + \beta g_j \right) \sum_{i \in \boldsymbol{I}} \chi \mu_i Y_{ij}} \right) + \theta h z_{\alpha} \sum_{j \in \boldsymbol{J}} \sqrt{\sum_{i \in \boldsymbol{I}} L \sigma_i^2 Y_{ij}} = \tag{11}
$$

$$
\sum_{j \in \boldsymbol{J}} \left\{ f_j X_j + \sum_{i \in \boldsymbol{I}} \hat{d}_{ij} Y_{ij} + K_j \sqrt{\sum_{i \in \boldsymbol{I}} \mu_i Y_{ij}} + \Theta \sqrt{\sum_{i \in \boldsymbol{I}} \hat{\sigma}_i^2 Y_{ij}} \right\}
$$

where

$$
\begin{aligned}
\hat{d}_{ij} &= \beta \chi \mu_i \left(d_{ij} + a_j \right) \\
K_j &= \sqrt{2\theta h \chi \left(F_j + \beta g_j \right)} \\
\Theta &= \theta h z_{\alpha} \\
\hat{\sigma}_i^2 &= L \sigma_i^2
\end{aligned}
$$

In (11) we have grouped the linear term representing the marginal transport costs from the supplier to the distribution center (represented by terms in a_j) with the local delivery costs (represented by terms in d_{ij}). Thus, $\hat{d}_{ij}$ captures local delivery costs from the DCs to the retailers as well as the marginal cost of shipping a unit from a supplier to a DC. K_j captures the working inventory effects due to the fixed ordering costs at the DC as well as the fixed transport costs from a supplier to a DC. Finally, Θ captures the safety stock costs at the distribution centers.

The constraints (7)-(10) are identical to those of the traditional uncapacitated fixed charge facility location problem (Balinski, 1965; Erlenkotter, 1978; Korkel, 1989). Thus, the solution approach that we propose below mirrors some of the solution approaches used for that problem. However, these approaches must be modified significantly to account for the final two non-linear terms in the objective function (11).

4. Model properties

Before outlining the solution approach, it is worth noting a number of properties of the model. First, unlike the traditional uncapacitated fixed charge location model in which it is always optimal to assign demands to the facility that can serve the demands at least cost (i.e., the facility j with the smallest value of d_{ij} for retailer i), in this problem it may be optimal to assign retailers to a more remote distribution center. Doing so may reduce the working inventory, safety stock inventory and transport costs from the supplier to the DC sufficiently to offset the increased local delivery costs. Figure 1 illustrates a small example in which this occurs. Table 1 provides the input data and parameters for the problem while Table 2 compares the total costs for different DC locations and different customer assignments for the problem. The first column gives the DC locations; the next three give the retailer assignments to DCs; the next three give the facility, local delivery and inventory cost terms; and the last two columns give the total cost and the difference between the optimal total cost and the cost of the solution indicated in that row. In this example, it is optimal to assign demands at retailer B to a facility at A with a cost per unit shipped of 101 despite the fact that there is a facility at C with a smaller unit shipping cost for the B to C channel.

Not only does this phenomenon occur in small (contrived) cases, but we have found that it occurs in our computational results using realistic national demand data, particularly when the inventory-related costs are large relative to the other costs (i.e., when q is large relative to b). Typically we find this behavior associated with small retail nodes that almost equidistant to two different DCs. We also can construct examples in which it is optimal to *locate* a DC at a particular node, but for demands from that node to be assigned to a different DC. In other words, it is conceivable that the optimal solution would locate a facility in Chicago, for example, but would assign demands from Chicago to a facility in Minneapolis. Again, the intuitive reason for this is that the reduction in inventory and supplier-to-DC transport costs (all of which entail risk-pooling terms) more than offset the increased local delivery costs from the DC to the retailers. The reader is referred to Shen (2000) or Shen, Coullard and Daskin (2000) for a simple example in which this occurs. In practice, it is unlikely that a supply chain manager would organize her distribution system in this way even if doing so would result in small cost savings. Fortunately, we will be able to prove that this does not occur once we make one additional assumption.

To simplify the model somewhat further we assume that the variance-to-mean ratio at each retailer is identical for all retailers. In other words, we assume that $\frac{s_i^2}{m_i} = g$, $\forall i \in \boldsymbol{I}$. While this may seem like a restrictive assumption, if demands arise from a Poisson process, then this assumption is exact and we are merely approximating a Poisson demand process by Normally distributed demands, which is a good approximation for sufficiently large demand values (Montgomery, Runger and Hubele, 1998). Also, in many other contexts (e.g., transportation planning with random travel times) it is common to assume a single variance-to-mean ratio (Sheffi, 1985). With this assumption, the objective function can be rewritten as

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j \in \boldsymbol{J}} \left\{ f_j X_j + \sum_{i \in \boldsymbol{I}} \hat{d}_{ij} Y_{ij} + K_j \sqrt{\sum_{i \in \boldsymbol{I}} m_i Y_{ij}} + \Theta \sqrt{\sum_{i \in \boldsymbol{I}} \hat{s}_i^2 Y_{ij}} \right\} = \\
& \sum_{j \in \boldsymbol{J}} \left\{ f_j X_j + \sum_{i \in \boldsymbol{I}} \hat{d}_{ij} Y_{ij} + K_j \sqrt{\sum_{i \in \boldsymbol{I}} m_i Y_{ij}} + \Theta \sqrt{\sum_{i \in \boldsymbol{I}} Lg\, m_i Y_{ij}} \right\} = \qquad (12) \\
& \sum_{j \in \boldsymbol{J}} \left\{ f_j X_j + \sum_{i \in \boldsymbol{I}} \hat{d}_{ij} Y_{ij} + \hat{K}_j \sqrt{\sum_{i \in \boldsymbol{I}} m_i Y_{ij}} \right\}
\end{aligned}$$

where $\hat{K}_j = K_j + \Theta\sqrt{Lg}$.

This assumption does two things. First, it reduces the number of non-linear terms from two to one. This will facilitate the solution approach outlined below in section 5. Second, as shown in the following theorem, under this assumption it is never optimal to open a DC at a node and to then serve the demands from that node from another DC.

Theorem: If α>0.5, $q > 0$ and h>0 and all inter-nodal distances are positive and $s_i^2 / m_i = g$ $\forall i \in \boldsymbol{I}$, then if there is a DC at node j then demands at node j are served by that DC.

Proof: Assume that there is a DC at node j, but that the demands at node j are served by a DC at some other node, node k in an optimal solution. Since there is a DC at node j, there must be other demand nodes with a total positive mean demand that are assigned to the DC at j. Let this total mean demand be H_j. Similarly, let H_k be the total mean demand assigned to the DC at node k *excluding* the demand originating at node j whose mean value is m_j. (Note that while H_k is defined to include the demand originating at node k, H_j does ***not*** include the demand originating at node j.) Let d_{kj} be the cost of shipping one unit of demand from k to j and, as before, $\Theta = \theta h z_\alpha$. This situation is illustrated in Figure 2 below.

Since serving the demand that originates at j from the facility at k is (assumed to be) optimal, we have

$$d_{kj} m_j + \Theta\sqrt{H_k + m_j} + \Theta\sqrt{H_j} \le \Theta\sqrt{H_j + m_j} + \Theta\sqrt{H_k} \tag{13}$$

or the cost of serving the demand originating at node j from k is less than or equal to the cost of serving the demand originating at node j from the facility at j. Also, we have

$$d_{kj} m_j + \Theta\sqrt{H_k + m_j} + \Theta\sqrt{H_j} \le d_{kj} m_j + \tilde{d}_{kj} H_j + \Theta\sqrt{H_k + H_j + m_j} \tag{14}$$

or the cost of serving the demand originating at node j from k and serving other demands (H_j) at node j is less than the cost of serving the demand at node j from node k **and** serving the demand assigned to node j from the facility at k. Doing the latter incurs an additional transport cost per unit of $\tilde{d}_{kj} \le d_{kj}$ since the demands assigned to node j can always be routed through j when served from k. Note that the term $\tilde{d}_{kj} H_j$ does not represent the *total* local delivery cost for the demands that are being reassigned from j to k; rather it represents the *average additional* cost of assigning these demands from j to k. Thus, we can rewrite (14) as

$$\begin{aligned} d_{kj} m_j + \Theta\sqrt{H_k + m_j} + \Theta\sqrt{H_j} &\le d_{kj} m_j + \tilde{d}_{kj} H_j + \Theta\sqrt{H_k + H_j + m_j} \\ &\le d_{kj}\left(m_j + H_j\right) + \Theta\sqrt{H_k + H_j + m_j} \end{aligned} \tag{15}$$

Rearranging (13) we have

$$d_{kj} \le \Theta \frac{\sqrt{H_j + m_j} - \sqrt{H_j}}{m_j} - \Theta \frac{\sqrt{H_k + m_j} - \sqrt{H_k}}{m_j} \tag{16}$$

and rearranging (15) we have

$$
\begin{aligned}
d_{kj} &\geq \Theta \frac{\sqrt{H_k + m_j} - \sqrt{H_k + H_j + m_j}}{H_j} + \Theta \frac{\sqrt{H_j}}{H_j} \\
&= \Theta \frac{\sqrt{H_k + m_j} - \sqrt{H_k + H_j + m_j}}{H_j} + \Theta \frac{\sqrt{H_j} - \sqrt{0}}{H_j}
\end{aligned}
\tag{17}
$$

Combining (16) and (17) and dividing by the positive coefficient Θ, we have

$$
\frac{\sqrt{H_k + m_j} - \sqrt{H_k + H_j + m_j}}{H_j} + \frac{\sqrt{H_j} - \sqrt{0}}{H_j} \leq \frac{\sqrt{H_j + m_j} - \sqrt{H_j}}{m_j} - \frac{\sqrt{H_k + m_j} - \sqrt{H_k}}{m_j} \tag{18}
$$

or

$$
\frac{\sqrt{H_j} - \sqrt{0}}{H_j} - \frac{\sqrt{H_k + H_j + m_j} - \sqrt{H_k + m_j}}{H_j} \leq \frac{\sqrt{H_j + m_j} - \sqrt{H_j}}{m_j} - \frac{\sqrt{H_k + m_j} - \sqrt{H_k}}{m_j} \tag{19}
$$

But,

$$
\frac{\sqrt{H_j} - \sqrt{0}}{H_j} > \frac{\sqrt{H_j + m_j} - \sqrt{H_j}}{m_j} \tag{20}
$$

by the concavity of the square root operator since the left hand side is the slope of the square root function between 0 and H_j, while the right hand side is the slope of the function between H_j and $H_j + m_j$. Similarly, we have

$$
\frac{\sqrt{H_k + H_j + m_j} - \sqrt{H_k + m_j}}{H_j} < \frac{\sqrt{H_k + m_j} - \sqrt{H_k}}{m_j} \tag{21}
$$

since the left hand side is the slope of the square root function between $H_k + m_j$ and $H_k + H_j + m_j$ while the right hand side is the slope of the square root function between H_k and $H_k + m_j$. Thus, combining (20) and (21), we obtain

$$
\frac{\sqrt{H_j} - \sqrt{0}}{H_j} - \frac{\sqrt{H_k + H_j + m_j} - \sqrt{H_k + m_j}}{H_j} > \frac{\sqrt{H_j + m_j} - \sqrt{H_j}}{m_j} - \frac{\sqrt{H_k + m_j} - \sqrt{H_k}}{m_j} \tag{22}
$$

which contradicts (18). Thus, we cannot have demands originating at node j served by a DC at some other facility k while some demands are served from the DC at j. In other words, if there is a DC at j, it must serve the demands that originate at j.

5. Solution approach

5.1 Finding a lower bound

The model formulated in section 3 which accounts for the fixed DC location costs, local distribution costs from the DCs to the retailers, working and safety stock inventory at the DCs and transport costs from the supplier(s) to the DCs is a variant of the uncapaciated fixed charge location problem. To solve this problem, we will use Lagragnian relaxation (Fisher, 1981, 1985) embedded in branch and bound. In particular, we will relax constraint (7) to obtain the following Lagrangian problem.

$$\underset{\lambda}{Max}\ \underset{X,Y}{Min} \quad \sum_{j\in J}\left\{f_j X_j + \sum_{i\in I}\hat{d}_{ij}Y_{ij} + \hat{K}_j\sqrt{\sum_{i\in I}\mu_i Y_{ij}}\right\} + \sum_{i\in I}\lambda_i\left(1-\sum_{j\in J}Y_{ij}\right) =$$

$$\sum_{j\in J}\left\{f_j X_j + \sum_{i\in I}\left(\hat{d}_{ij}-\lambda_i\right)Y_{ij} + \hat{K}_j\sqrt{\sum_{i\in I}\mu_i Y_{ij}}\right\} + \sum_{i\in I}\lambda_i \tag{23}$$

Subject to

$$Y_{ij} \le X_j \qquad \forall i\in I;\forall j\in J \tag{8}$$

$$X_j \in \{0,1\} \qquad \forall j\in J \tag{9}$$

$$Y_{ij} \in \{0,1\} \qquad \forall i\in I;\forall j\in J \tag{10}$$

For fixed values of the Lagrange multipliers, λ_i, we want to minimize (23) over the location variables, X_j, and the assignment variables, Y_{ij}. In the absence of the non-linear term in the assignment variables, solving the problem is simply a matter of computing, $V_j = f_j + \sum_{i\in I}\min\left(0,\hat{d}_{ij}-\lambda_i\right)$ for each candidate site $j\in J$, and setting $X_j=1$ for those candidate sites for which $V_j \le 0$. (If no V_j value is non-positive, we identify the smallest positive V_j and set the corresponding $X_j = 1$.) The assignment variables are then easy to determine. One simply sets $Y_{ij}=1$ if $\hat{d}_{ij}-\lambda_i \le 0$ and $X_j=1$ and $Y_{ij}=0$ otherwise. However, the presence of the non-linear term makes finding an appropriate value of V_j, the value of including candidate site j in the solution, more difficult. To do so, we need to be able to solve a subproblem of the following form for each candidate DC:

$$\textbf{SP(j)} \qquad \min \qquad \tilde{V}_j = \sum_{i \in \boldsymbol{I}} b_i Z_i + \sqrt{\sum_{i \in \boldsymbol{I}} c_i Z_i} \tag{24}$$

$$\text{subject to} \qquad Z_i \in \{0,1\} \qquad \forall i \in \boldsymbol{I} \tag{25}$$

where $b_i = \hat{d}_{ij} - \boldsymbol{l}_i$ and $c_i = \hat{K}_j^2 m_i \geq 0$. In (24)-(25) we have replace the assignment variables Y_{ij} by Z_i to simplify the notation since **SP(j)** is specific to DC j.

In Shen, Coullard and Daskin (2000), we show that this subproblem can be solved by the following procedure.

Step 1: Partitioning the set $\boldsymbol{I}$ into three sets: $\boldsymbol{I}^+ = \{i : b_i \geq 0\}$, $\boldsymbol{I}^0 = \{i : b_i < 0 \text{ and } c_i = 0\}$ and $\boldsymbol{I}^- = \{i : b_i < 0 \text{ and } c_i > 0\}$. (We note that under our assumption of the variance being proportional to the mean, the set $\boldsymbol{I}^0$ will generally be empty. We include it below for completeness.)

Step 2: Sort the elements of $\boldsymbol{I}^-$ so that $\frac{b_1}{c_1} \leq \frac{b_2}{c_2} \leq \bullet\bullet\bullet \leq \frac{b_n}{c_n}$ where $n = \left|\boldsymbol{I}^-\right|$.

Step 3: Compute the partial sums $S_m = \sum_{i \in \boldsymbol{I}^0} b_i Z_i + \sqrt{\sum_{i \in \boldsymbol{I}^0} c_i Z_i} + \sum_{\substack{i=1 \\ i \in \boldsymbol{I}^-}}^{m} b_i Z_i + \sqrt{\sum_{\substack{i=1 \\ i \in \boldsymbol{I}^-}}^{m} c_i Z_i}$.

Step 4: Selecting the value of m that results in the minimum value of S_m.

Since the major step in this algorithm is the sorting of the elements of $\boldsymbol{I}^-$ in step 2, the algorithm has complexity $O(|\boldsymbol{I}|\log|\boldsymbol{I}|)$. This problem must be solved for each $j \in \boldsymbol{J}$, so solving (23) for given values of the Lagrange multipliers has complexity $O(|\boldsymbol{J}||\boldsymbol{I}|\log|\boldsymbol{I}|)$. The proof of the optimality of this procedure relies on the concavity of the square root function. The interested reader is referred to Shen, Coullard and Daskin (2000) or Shen (2000).

Solving subproblem SP(j) for each j enables us to solve the Lagrangian problem (23) and (8)-(10) for fixed values of the Lagrange multipliers in a manner identical to that used for the uncapacitated fixed charge problem. We add f_j to the optimal objective function value of SP(j) to obtain the value of using a DC at candidate site j in the Lagrangian solution. We then select all DCs with negative values (or the one with the smallest positive value if all have non-negative values). For each opened DC, the corresponding values of the assignment variables are identical to the Z_i values in subproblem SP(j); for unopened DCs (those for which $X_j = 0$ in the Lagrangian solution), $Y_{ij} = 0 \ \forall i \in \boldsymbol{I}$.

Having solved the Lagrangian problem, we need to find the optimal Lagrange multipliers. We do so using a standard subgradient optimization procedure (Fisher 1981, 1985). The optimal value of (23) is a lower bound on the objective function (12).

5.2 Finding an upper bound

At each iteration of the Lagrangian procedure, we find an upper bound as follows. We initially fix the DC locations at those sites for which $X_j = 1$ in the current Lagrangian solution. Then we assign retailers to DCs in a two-phased process. First, for each retailer for which $\sum_{j \in \boldsymbol{J}} Y_{ij} \geq 1$ (each retailer assigned to at least one open DC in the Lagrangian solution), we assign the retailer to the DC for which $Y_{ij} = 1$ and that increases the cost the least based on the assignments made so far. In the test cases reported below, retailers are processed in order of non-increasing mean demand. In the second phase, we process retailers for which $\sum_{j \in \boldsymbol{J}} Y_{ij} = 0$ (retailers that were not assigned to any open DC in the Lagrangian procedure). We assign each such retailer to the open DC which increases the total cost the least based on the assignments made so far. Note that the key difference between phases 1 and 2 is that in phase 1 we limit the possible assignments of a retailer to those to which the retailer was tentatively assigned in the Lagrangian procedure thereby utilizing information provided by the Lagrangian solution. In phase 2, however, we are dealing with retailers that were unassigned in the Lagrangian solution. Hence, for these retailers, we consider all possible assignments to open DCs.

5.3 Retailer reassignments

If the value of the upper bound that we compute using the procedure outlined above is better than the best known upper bound, we try to improve the bound further by considering all possible single retailer moves from the DC to which a retailer is currently assigned to another DC. The *value* of the reassignment is generally equal to the sum of the changes in the second and third terms of (12). However, if reasigning retailer i from a DC at site j to another DC will remove all of the assigned demand at site j, then the value of the move is augmented by the fixed cost, f_j, of that DC (since we can remove the site). If the DC is forced into the solution by an additional constraint imposed by the branch and bound algorithm in which the Lagrangian procedure is embedded, the DC cannot be removed from the solution and we do not add the fixed cost into the value of the proposed retailer reassignment. We do not, however, actually remove an open DC from consideration until no improving reassignments can be found. Thus, the value of reassigning retailer i to an open DC with no currently assigned demand equals the sum of the changes in the second and third terms of (12) *minus* the fixed cost for the DC since we would need to re-open the site.

For each retailer, the best possible reassignment is found and performed before finding reassignments for the next retailer. We continue looping through all retailers until we complete one entire loop without finding an improving reassignment. At that point, an open DC with no assigned demand is removed and its fixed cost is actually subtracted from the upper bound on the solution.

5.4 DC exchange algorithm improvements

After the termination of the Lagrangian procedure, we applied a variant of the exchange algorithm proposed by Teitz and Bart (1968) for the P-median problem. For each DC in the current solution, we find the best substitute DC that is not in the current solution. For each such potential exchange, retailers are assigned in a greedy manner to the DC which increases the cost the least based on the assignments made so far. Thus, this process is like the second phase assignment process in obtaining the upper bound at each iteration of the Lagrangian procedure in that a retailer can be assigned to *any* open DC. If a DC exchange is found that improves the solution, we make the exchange; otherwise we proceed to the next open DC to try to find an improving exchange involving that DC. If any improving exchanges are found, we try single retailer reassignments (as discussed in section 5.3) to the best DC configuration we have found and then restart the search for improving exchanges. If a pass through all possible exchanges is made without finding an improving exchange, the exchange algorithm terminates.

5.5 Variable fixing

In addition to the DC exchange heuristic outlined above, at the end of the Lagrangian procedure at the root node, we employ a variable fixing technique. In essence, this can be thought of as performing branch and bound on all of the DC locations, without revising any of the Lagrange multipliers. It uses the following rules:

Node exclusion rule: If candidate DC j is not currently part of the best-known solution and if $LB + f_j + \tilde{V}_j > UB$ then candidate DC j cannot be part of the optimal solution, where LB and UB are the current lower and upper bounds on the solution, respectively.

Node inclusion rule: If candidate DC j is part of the best-known solution and if $LB - \left(f_j + \tilde{V}_j\right) > UB$ then candidate DC j must be part of the optimal solution. Note that when node j is part of the best-known solution, $f_j + \tilde{V}_j$ will generally be negative so that the left hand side of the inequality above will generally be greater than LB.

5.6 Branch and bound

After the exchange algorithm and the variable forcing routine, either the lower bound equals the upper bound or there are *no* unforced DC location variables. In that case, the solution corresponding to the upper bound is optimal, so the lower bound is set to the upper bound and the algorithm terminates. In our experience this often occurs as indicated below. If, however, the lower bound is less than the upper bound and some candidate DC locations are not forced in or out of the solution, we employ branch and bound. We branch on the free (non-forced) DC location with the largest assigned demand. If all DC locations are forced into the solution then we branch on the first un-forced candidate location in the list of candidate locations. In all cases, we first exclude the node on which we branch and then include the node. Branching is done in a depth-first manner.

6. Computational results

In this section, we outline computational results from two experiments. The first was designed to test the algorithm's computational capabilities and to compare the algorithm with a set partitioning approach to the same problem proposed by Shen (2000) and Shen, Coullard, and Daskin (2000). For that experiment we employed two datasets: one had 88 retail locations and the other had 150 retail locations. The second experiment was based on a modification of the 150-node data set and was designed to assess the sensitivity of the results with respect to changes in the relative importance of the transport and inventory terms. We also used this dataset to assess the impacts of significant reductions in the fixed order costs as might result from the use of e-commerce technologies. In all cases, each retail location was also a candidate DC location. Also, in all cases, we set d_{ij}, the unit cost of shipping from candidate DC j to retailer i, to the great circle distance between these locations.

For all cases, the parameters for the Lagrangian procedure are shown in Table 3.

Two datasets were used for the initial set of tests. They are minor modifications of the 88 and 150-node datasets given in Daskin (1995). For the 88-node dataset – representing the 50 largest cities in the 1990 U.S. census along with the 48 capitals of the continental U.S. – the mean demand was obtained by dividing the population data by 1000 and rounding the result to the nearest integer. Fixed facility location costs were obtained by dividing the facility location costs in Daskin (1995) by 100. For the 150-node dataset – representing the 150 largest cities in the continental U.S. for the 1990 census – the mean demand was obtained in the same manner. The fixed facility costs were all set to 100, one-one-thousandth of the value in the dataset given by Daskin. These changes were made to allow us to deal with smaller numbers.

Table 4 presents the coefficients used in all the runs for both datasets.

Table 5 presents the results for the two datasets. As the transport costs increase (as b goes up), the number of DCs goes up. Conversely, as inventory costs increase (as q goes up), the number of DCs goes down. Also, when inventory considerations dominate (q is very large relative to b), it is sometimes optimal to assign one or more retailers to a DC that is other than the least cost DC in terms of local delivery cost (d_{ij}). For example, for the final case (150 nodes, $b = 0.001$ and $q = 1$), three retailers were assigned to DCs other than the one with the smallest local delivery cost. Reassigning each to the least local delivery cost DC would increase the cost by $3.37 or 0.027%. The local delivery costs decrease by almost $11 out of roughly $4,870 (0.22%), while the safety stock, fixed order costs, and working inventory costs each increase by about $4.7 or 0.245%. Thus, while it is optimal to assign these three retailers to DCs other than the ones that could provide the least cost local delivery service, reassigning them to the least-cost DCs would not increase the total cost very much.

Table 5 also compares the computation times for the algorithm presented above with those obtained in Shen, Coullard, and Daskin (2000) using a set partitioning approach. Times obtained for our model are on a Dell Latitude CPx computer running at 650 MHz using Windows 98. The program was written in Delphi 5. The Shen, Coullard, and Daskin times are on a Sun Spark Station running the SunOS 4.1.3u5 operating system. Our computation times are consistently lower than those obtained using the column generation approach. However, our times tend to grow with b while the column generation times decrease with b. Thus, for very large values of b relative to q, it might be better to use the column generation approach. In many cases, we did not need to use branch and bound at all; when we did very few nodes deeded to be evaluated. Finally, the variable forcing rules were exceptionally effective, forcing almost all nodes in or out of the solution at the root node. This is in part due to the very tight bounds obtained at the root node.

Finally, we consider a different variant of the 150-node dataset given in Daskin (1995). In this case, we divided the demands by 250 and truncated the result to obtain the mean daily demand. Table 6 gives the model coefficients for this set of runs, which were designed to better replicate real-world conditions. Two supplier locations were considered – Chicago and Phoenix – and the fixed shipping cost from the supplier to a candidate DC was taken to be function of the distance from the closer supplier to the candidate DC site as shown in Table 9.

Figure 3 shows the optimal solution for a base case of $b = 0.00025$ and $q = 0.1$. The supplier in Chicago serves 6 of the 10 DCs and about 73% of the total demand. Figure 4 breaks down the total cost of approximately \$4.49 million into its constituent parts. Note that in this case, the transport costs from the suppliers to the DCs as well as the safety stock costs – all buried in the "other" category – are negligible. Figures 5 and 6 show the sensitivity of the cost and number of facilities to changes in the transport and inventory cost weights. As the relative importance of inventory costs goes up, the number of DCs located goes down and as the importance of transport costs goes up, the number of DCs increases. The figures also compare this model with the traditional uncapacitated fixed charge (UFC) location model. Not surprisingly, the total cost for the UFC model is always below that of the model presented above which includes additional cost components. Also, due to the concavity of the additional cost terms, the UFC consistently locates more DCs than does the model above.

Finally, we consider how the solution might change if the fixed cost of placing an order decreased significantly from \$1000 per order to \$10 per order as might be the case with e-commerce technologies. The total cost decreases from \$4.49 million to \$2.9 million. Interestingly, it is now optimal to open 14 DCs instead of the 10 found in the base case. This is reassuring since the demand-weighted average local delivery distance to a retailer from a DC will decrease from 127.1 to 88.8 miles. It is reassuring that the average distance decreases in this case, since e-commerce is often associated with expectations of improved customer service. Figure 7 shows the new solution. The supplier in Chicago serves 9 of the DCs and about 71% of the demand. Compared to the cost breakdown in the base case (Figure 4), in this case, 48% of the total cost is in facility costs and another 44.3% is in local delivery costs. Order costs and working inventory costs each constitute 3.3% of the total costs while safety stock costs and shipment costs from the supplier to the DCs are again negligible.

If we could not relocate or add to the set of open DCs used in the base case, but could simply reoptimize the retailer assignments and inventory management policies, the costs would be 4.1% higher than those for the solution shown in Figure 7. If we were constrained to using the 10 DCs in the base case, but could add DCs, it would be optimal to use 3 additional DCs located in Cleveland, OH, Denver, CO, and Richomond, VA. The total cost in this case is only 1.1% greater than that of the optimal solution in Figure 7.

7. Conclusions and directions for future work

We have presented a new facility location model that incorporates working and safety stock inventory costs at the distribution centers as well as the economies of scale that exist in the transport costs from suppliers to DCs. A Lagrangian solution algorithm was presented for the case in which the ratio of the variance of demand at the retailers to the mean demand is the same for all retailers. A number of improvement heuristics were outlined for the problem. Also, two variable forcing rules that are applied at the root node of the branch and bound tree were discussed.

The algorithm was tested on two datasets consisting of 88 and 150 nodes respectively. The computational results compare favorably with those of the set partitioning approach proposed by Shen (2000) and Shen, Coullard and Daskin (2000). In many cases, branch and bound was not needed because the bounds at the root node proved the optimality of the solution and/or because we could force all of the candidate DCs into or out of the solution. In a final set of tests, we included explicit supplier locations and modified the inputs to better reflect actual conditions. When fixed order costs are significantly reduced, the number of facilities located increases. Despite our best efforts to reflect realistic conditions, it is important to test the data with actual corporate data to confirm that the model is accurately capturing the relevant costs.

A number of extensions should be considered. First, we need to develop ways of solving the problem when the ratio of the variance to the mean is not identical for all retailers. Second, we hope to consider the cases with multiple items as well as a constraint on the maximum allowable inventory at a DC and a constraint on the maximum demand that can be served by a supplier. Finally, it might be possible to incorporate local delivery cost estimates that better reflect less-than-truckload routing since such approximations often involve the square root of demand (Daganzo, 1991). We are working in all of these areas.

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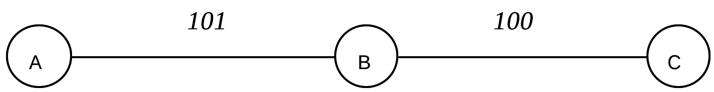

Figure 1: Sample figure for non-closest assignment

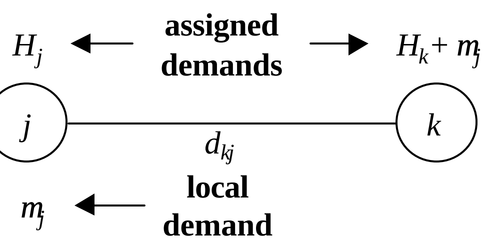

Figure 2 – Small network for proof of Theorem 1

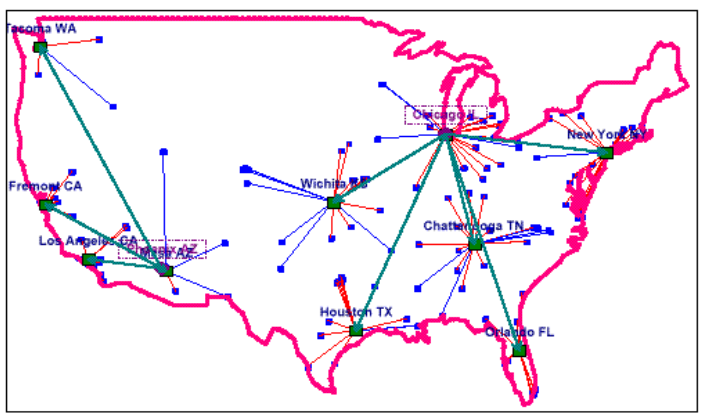

Figure 3: Solution for base case

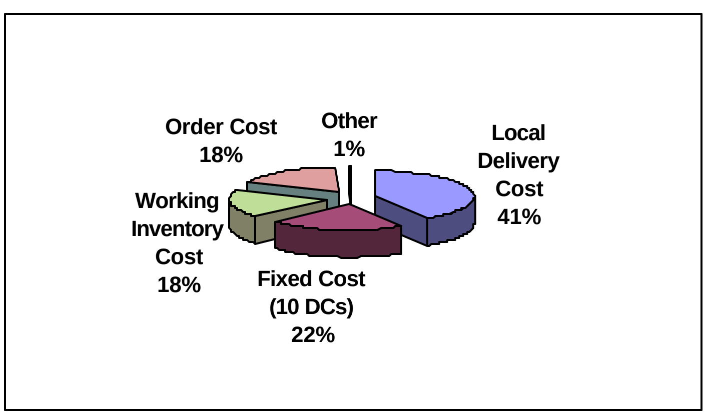

Figure 4: Distribution of costs for base case

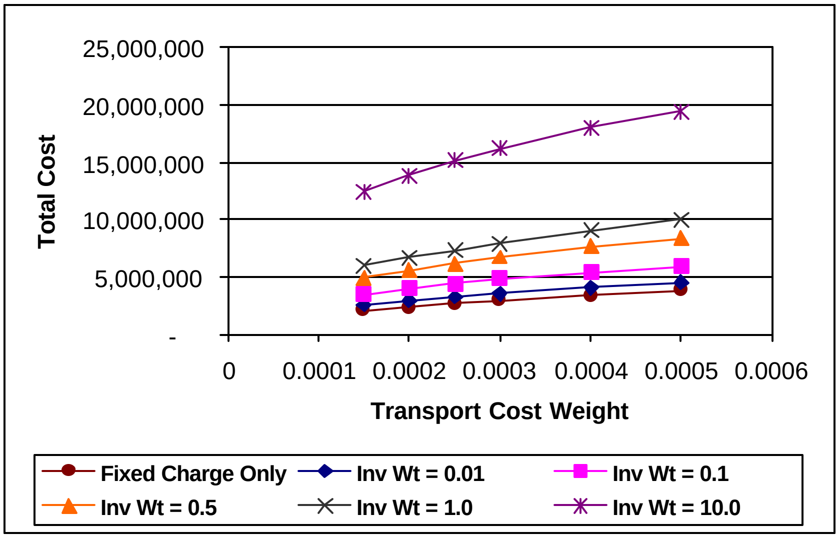

Figure 5: Sensitivity of cost to changes in transport and inventory weights

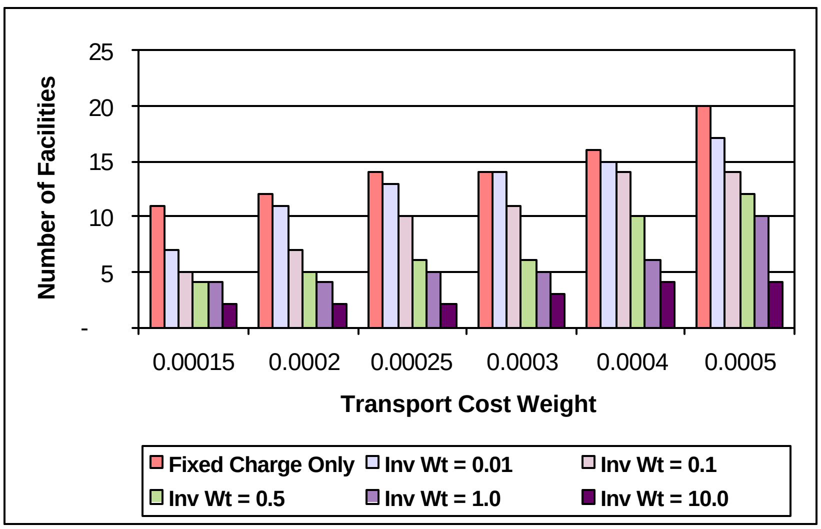

Figure 6: Sensitivity of number of DCs to changes in transport and inventory weights

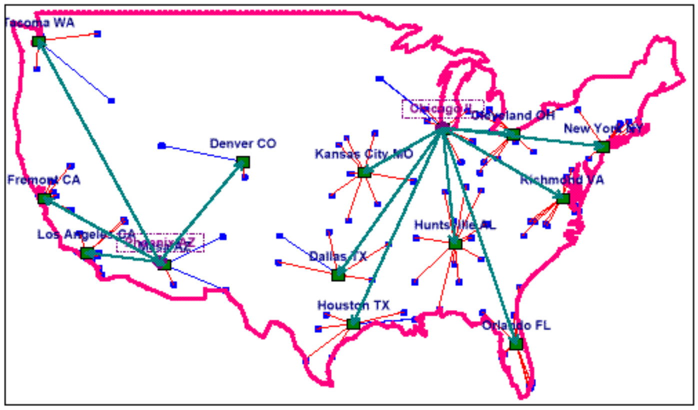

Figure 7: Solution with reduced fixed ordering costs

	NODE		
	A	**B**	**C**
Fixed Cost	10	1000	10
Mean	100	1	1

Table 1a: Data for network in Figure 1

F	Fixed Order cost	10	b	Transport weight	1	L	Lead time	1
$g_j \ \forall j \in \boldsymbol{J}$	Fixed shipping cost from supplier to DC	1	z_{α}	Service level parameter	1.96	h	Unit holding cost	2
$a_j \ \forall j \in \boldsymbol{J}$	Unit shipping cost from supplier to DC	1	q	Inventory weight	1	c	Days per year	1

Table 1b: Parameters for network in Figure 1

	Retailer Assignments							
Facilities	**A**	**B**	**C**	**Facility cost**	**Local Delivery Cost**	**Inventory Cost**	**Total Cost**	**Cost - min cost**
A	A	A	A	10	304	106.58	420.58	181.97
B	B	B	B	1,000	10,301	106.58	11,407.58	11,168.97
C	C	C	C	10	20,301	106.58	20,417.58	20,178.97
A,C	A	C	C	20	101	120.46	241.46	2.84
A,C	A	A	C	20	102	116.61	238.61	0.00
A,B	A	B	B	1,010	101	120.46	1,231.46	992.84
A,B,C	A	B	C	1,020	0	126.64	1,146.64	908.03

Table 2: Costs of different DC locations and retailer assignments

Parameter	**Value**
Maximum number of iterations	400
Minimum alpha multiplier	0.00000001
Maximum number of iterations before halving alpha	12
Crowder's damping factor	0.3
Initial Lagrange multiplier value	$10\overline{m} + 10 f_j$

Table 3: Parameters for Lagrangian relaxation runs

F	Fixed Order cost	10	g	Variance to mean ratio	1.00	L	Lead time	1
$g_j \ \forall j \in \boldsymbol{J}$	Fixed shipping cost from supplier to DC	10	$z_{\boldsymbol{a}}$	Service level parameter	1.96	h	Unit holding cost	1
$a_j \ \forall j \in \boldsymbol{J}$	Unit shipping cost from supplier to DC	5				$\boldsymbol{c}$	Days per year	1

Table 4: Model coefficients for 88 and 150-node test problems

F	Fixed Order cost	1000	g	Variance to mean ratio	1.00	L	Lead time (days)	7
$g_j \ \forall j \in \boldsymbol{J}$	Fixed shipping cost from supplier to DC	40 +0.1(distance)	$z_{\boldsymbol{a}}$	Service level parameter	1.96	h	Unit holding cost	25
$a_j \ \forall j \in \boldsymbol{J}$	Unit shipping cost from supplier to DC	0.5				$\boldsymbol{c}$	Days per year	250

Table 6: Model coefficients for final set of test problems

No. Nodes	Beta	Theta	Objective Function	# of DCs	# Retailers assigned to non-closest DC	Iter.	B&B Nodes	Time (sec.)	# DCs forced IN at root node	# DCs forced OUT at root node	Shen et al. time (sec.)	Shen et al. columns
88	0.001	0.1	13,229.55	9	0	215	1	2.15	9	79	626	12,246
88	0.002	0.1	19,975.37	11	0	372	1	3.74	10	77	65	3,417
88	0.003	0.1	25,306.68	15	0	221	1	2.75	15	73	22	2,352
88	0.004	0.1	28,752.64	21	0	486	3	5.93	13	62	15	1,409
88	0.005	0.1	31,390.69	23	0	321	1	3.29	23	65	8	846
88	0.001	0.1	13,229.55	9	0	215	1	2.15	9	79	626	12,246
88	0.002	0.2	20,491.17	10	0	291	1	2.91	10	77	89	4,185
88	0.005	0.5	33,794.94	22	0	400	1	4.83	22	66	7	906
88	0.005	0.1	31,390.69	23	0	321	1	3.29	23	65	8	846
88	0.005	0.5	33,794.94	22	0	400	1	4.83	22	66	7	906
88	0.005	1	35,876.10	21	0	120	1	1.21	21	67	12	1,222
88	0.005	5	47,348.38	17	1	129	1	1.37	17	71	60	4,579
88	0.005	10	57,959.54	12	2	183	1	1.86	12	76	145	9,158
88	0.005	20	74,760.97	9	2	400	1	4.18	9	79	869	20,114
150	0.0004	0.01	3,977.27	15	0	779	3	22.03	11	134	252	9,624
150	0.0006	0.01	4,867.76	21	0	422	3	14.28	21	128	180	8,075
150	0.0008	0.01	5,580.71	26	0	403	3	15.22	26	123	55	3,351
150	0.001	0.01	6,162.99	28	0	405	3	15.98	28	121	29	2,318
150	0.0005	0.01	4,459.32	18	0	423	3	13.73	15	130	277	10,907
150	0.001	0.02	6,410.09	28	1	400	1	15.66	28	122	45	2,730
150	0.002	0.04	8,988.62	41	0	171	1	4.78	41	109	22	1,286
150	0.001	0.01	6,162.99	28	0	405	3	15.98	28	121	29	2,318
150	0.001	0.1	7,508.49	26	1	301	1	12.36	26	124	102	4,233
150	0.001	0.5	10,175.71	21	2	400	1	13.73	21	129	261	11,795
150	0.001	1	12,380.63	15	3	466	3	13.95	14	133	730	21,944

Table 5: Results for 88 and 150 node datasets