Feedback vertex sets and disjoint cycles in planar (di)graphs

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Abstract. We present new fixed parameter algorithms for the FEEDBACK VERTEX SET problem on planar graphs. We show that if a planar graph has a feedback vertex set of cardinality at most k then its treewidth is bounded by $\sqrt{k}$. An approximate tree decomposition can be obtained in linear time, and this is used to find an algorithm computing a feedback vertex set in time $(p^{\sqrt{k}}n)$ time for some constant p. An easy argument shows that the problem is in linear time reducible to a problem kernel of at most $O(k^2)$ vertices, and this kernel can be used to obtain an algorithm that runs in time $O(q^{\sqrt{k}} + n)$ for some other constant q.
We also show that finding k vertex disjoint cycles in a planar graph can be done in $O(c^{\sqrt{k}}n)$ time for some constant c. To this extend we show that that every planar graph with a maximum number of k vertex disjoint cycles has treewidth $O(\sqrt{k})$.

Le premier pas

Definition 1. *A* feedback vertex set, (FVS)*, of a graph G is a set U of vertices such that every cycle of G passes through at least one vertex of U.*

Instance: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Does G have a feedback vertex set of at most k vertices?

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It is easy to see that computing a minimum cardinality FVS is an NP-complete problem by a reduction from VERTEX COVER [23]. Simply add an extra vertex for every edge in the graph and make it adjacent to both endpoints of the edge. This reduction preserves planarity and vertex cover remains NP-complete for planar graphs. The problem remains also NP-complete for digraphs with no in– or out–degree exceeding 2 and for planar graphs with no in– or outdegree exceeding 3. There are quite a lot of graph classes for which the FVS problem turns out to be polynomial solvable. To name a few, this is the case for 3-regular graphs [39, 29], chordal graphs and interval graphs [14, 45, 31], permutation graphs and trapezoid graphs (in time $O(nm)$ [12]), and cocomparability graphs ($O(n^2m)$ [12]).

Definition 2. *Consider an algorithm for a parameterised problem (I, k) where I is the problem instance and k the parameter. The algorithm is called* uniformly polynomial *if it runs in time $O(f(k)|I|^c)$ where $|I|$ is the size of I, $f(k)$ is an arbitrary function, and c a constant. A parameterised problem is* fixed parameter tractable, (FPT)*, if it admits a uniformly polynomial algorithm.*

In this paper we pitch our attention towards undirected & unweighted planar graphs. We present new FPT results for the FVS problem when restricted to planar graphs.
Notice that "being acyclic within k vertices" is a minor closed property, (since being acyclic is minor closed [21]). Hence there is a linear time recognition algorithm for every fixed k for graphs in general. Hence FVS is fixed parameter tractable [17]. A concrete algorithm to solve the k-FEEDBACK VERTEX SET problem in $O((2k+1)^k n^2)$ time is described in [17] (page 32). It is remarked that the problem can be solved alternatively in time $O((17k^4)!(n+e))$. For descriptions of algorithms with these timebounds see [9, 16]. For a randomised algorithm which completes in $O(c4^k kn)$ steps with probability at least $1-(1-\frac{1}{4^k})^{c4^k}$ see [7].

The FVS problem is a typical node-deletion problem for a hereditary property (being acyclic in this case). It was shown that the node deletion problem for every non-trivial hereditary property is MAX SNP-hard [34]. Hence it cannot have a polynomial time approximation scheme unless P=NP [2]. It was shown recently [11] that for all MAX SNP-hard problems finding exact solutions in subexponential time (i.e., $O(2^{o(k)}p(n))$ parameterised algorithms) is not possible unless P=W[1]. For graphs in general there are by now quite a few approximation algorithms known with a constant (larger than 2) approximation ratio for the FEEDBACK VERTEX SET problem [3, 6, 20]. We show that for planar graphs the optique is much better. Although, as far as we know, there is no algorithm for the FVS for planar graphs in the vicinity significantly closer than the general 2-approximation [3].

Remark 1. As far as approximations are concerned, both the vertex and arc case are each reducible to one another [22]. The (directed) FEEDBACK *arc* SET problem is polynomial for planar digraphs [25]. It is known that FVS in planar digraphs can be approximated with performance guarantee bounded by the maximum degree or by the number of cyclic faces minus 1 [40]. Apparently the FPT–(non)ness of FVS for unweighted digraphs is open.

Remark 2. Every *rich* planar graph (i.e., a planar graph in which every vertex has degree at least 3) has a face of length at most 5. (This follows immediately from Euler's formula. See Lemma 4 or, e.g., [6].) Hence there is a *bounded search tree* algorithm for unweighted planar FVS running in time $O(5^k n)$: Reduce the graph until every vertex has degree at least 3, and repeatedly find faces of length at most 5, building a bounded search tree [17]. Combining this with the kernel reduction as described in Section 2 we obtain an algorithm that runs in time $O(c^k + n)$ on planar graphs. In this paper we describe how to improve this to $O(q^{\sqrt{k}} + n)$.

Definition 3. *A* cycle packing *in a graph G is a set of vertex disjoint cycles.*

Instance:	A graph $G = (V, E)$.
Parameter:	An integer k .
Question:	Does G have a cycle packing with k elements?

Definition 4. *For a graph G we let $cp(G)$ stand for the maximum cardinality of a cycle packing.*

Definition 5. *For a graph G let $fvs(G)$ be the minimum cardinality of a feedback vertex set in G. Let $fvs(k) = \max\{fvs(G) \mid cp(G) \leq k\}$.*

It was shown by Erdös and Pósa that $fvs(k) = \Theta(k \log k)$. Bollobás proved $fvs(1) = 3$ [10] (see [43]). Other unpublished proofs were given by Pósa and by Sachs (see [18, 43]). Voss shows that $fvs(2) = 6$ and $9 \leq fvs(3) \leq 12$ [43, 44].
Notice that the k-DISJOINT CYCLES problem is NP-complete (also for planar graphs) because it contains PARTITION INTO TRIANGLES as a special case [23]. The k-DISJOINT CYCLES problem was shown to be FPT in [16, 9]. We can show that for *planar graphs* $fvs(G) = O(cp(G))$. In Sections 3 and 4 we give an algorithm that solves the k-DISJOINT CYCLES problem for planar graphs in time $O(c^{\sqrt{k}} n)$.

1 Treewidth vs. FVS

Definition 6. *A* dominating set *D in a graph G is a set of vertices such that every vertex not in D has at least one neighbour in D.*

Let $\gamma(G)$ denote the *domination number* of a graph G, i.e., the minimum cardinality of a dominating set in G. We use the result of [1]. (For general information on minors and treewidth we refer to [15, 27].)

Theorem 1. *If a planar graph G has a dominating set of size at most k then the treewidth of G is at most $\delta\sqrt{k}$ for some constant δ. There is a linear time algorithm that finds a tree decomposition of this width.*

Remark 3. The best upperbound known to us at present is $\delta < 6\sqrt{34}$ [1].

Theorem 2. *If G is a planar graph with a FVS of cardinality k, then the treewidth of G is $O(\sqrt{k})$.*

Proof. Assume that G has a FVS F of cardinality k. We show that there is a planar supergraph H of G such that F is a dominating set in H.
Consider a plane embedding of G. Since every face is a cycle, it must contain at least one vertex of F. For every face, fix one such vertex in F and add edges between every other vertex of the face and this vertex of F. Call this new graph H.
Observe that the set F is a dominating set in the planar graph H. By Theorem 2 H has treewidth $O(\sqrt{k})$, and since G is a subgraph of H, also G has treewidth $O(\sqrt{k})$. □

In the remainder of this section we describe an algorithm computing a FVS using a tree decomposition of a planar graph $G = (V, E)$. Let $\ell - 1$ be the *width* of this tree decomposition (i.e., every bag of this decomposition has at most ℓ vertices). Let the pair $(T, \mathcal{S})$ designate the tree decomposition of G where T is a rooted tree T and $\mathcal{S} = \{S_i \mid i \in V(T)\}$ a collection of subsets of $V(G)$ called *bags*, each corresponding uniquely with a node in T. By definition the following three conditions are satisfied:

1. Every vertex of G is contained in at least one bag $S_i \in \mathcal{S}$,
2. both endpoints of every edge of G are contained in at least one common bag S_i, and
3. for every vertex x of G, if x appears in bags S_i and S_j then it appears in all bags corresponding with nodes that appear on the path in T between the points i and j.

As customary in treewidth algorithms, we work our way from the leaves other than the root in T up to the root. Let G_i be the subgraph of G induced by the union of bags corresponding with nodes in the subtree of $i \in V(T)$. For each i we define $\Gamma_i \in MG_i$ as the minor of G_i obtained from G_i by contracting all edges in G_i incident with at most one vertex in S_i. Clearly $\Gamma_i \in MG$ is a minor of G with vertex set S_i.
Call a pair $\{x, y\}$ in Γ_i an *insignia* if there is an x, y-path in G_i with internal vertices in $V(G_i) - S_i$. Notice that the edges of S_i are insignia as well as all other edges of Γ_i.

Definition 7. *A set F_i is called a* partial FVS *for G_i if every cycle of G_i without vertices in S_i is hit by F_i and every cycle contained in S_i is hit by F_i.*

We *classify* partial FVSs F_i with $|F_i| \leq k$ for G_i by the following characteristic:

1. the intersection $F_i \cap S_i$,
2. for every insignia $\{x, y\}$ in Γ_i an indicator taking the values 0,1 or ∞ according whether there are 0, 1 or more different (as vertex sets) paths P from x to y with $P \cap S_p = \{x, y\}$ and $F_i \cap P = \emptyset$, and

3. for every vertex $x \in S_i$ an indicator taking the value TRUE if there is a cycle C with $C \cap F_i = \emptyset$ and $C \cap S_i = \{x\}$, and
4. the minimum cardinality of a partial FVS with the aforementioned intersection and indicators.

Remark 4. Notice that if there are at least 2 different paths between x and y without vertices of a partial FVS then either these paths are internally vertex disjoint or the indicator for x or for y must be true.

Proposition 1. *Let r be the root of T. There exists a feedback vertex set with cardinality at most k if and only if there is a partial FVS F for G_r with cardinality at most k such that there are no insignia $\{x, y\}$ with value ∞ and such that for every $x \in S_r$ the indicator for x is* FALSE.

Lemma 1. *The number of different classifications of partial FVSs is at most c^ℓ for some constant c.*

Proof. The graph Γ_i is a planar graph on ℓ vertices since planarity is preserved under taking of minors. Hence its number of edges is bounded by $3\ell - 6$. Each insignia of Γ_i takes a value 0, 1 or ∞. Each vertex indicator takes a value TRUE or FALSE and the number of different intersections of FVSs with S_i is at most 2^ℓ. Hence the total number of different classifications is at most $3^{3\ell-6}2^{2\ell}$. $\square$

A tree decomposition $(T, \mathcal{S})$ is in *standard form* if each node of T is one of 4 different types. A node p is a START NODE if p is a leaf of the tree different from the root. It is an INTRODUCE NODE if it has exactly one child q and the bag at node p has one more vertex x than the bag at node q, i.e., $S_p = S_q + x$. Node p is a FORGET NODE if S_p contains one vertex x *less* than S_q, i.e., $S_p = S_q - x$. Finally, a node p is a JOIN NODE if it has two children q and r and $S_p = S_q = S_r$.

Correctness of the following statement is readily checked (see [27]):

Proposition 2. *Given a tree decomposition (with a linear number of nodes) for a graph G then it can be turned into a tree decomposition in standard form with the same width and with at most $4n$ nodes in linear time.*

Henceforth assume that a tree decomposition of width ℓ in standard form with at most $O(n)$ nodes is given. We protract by describing the computation of the classification characteristics of partial FVSs for each of the 4 different types of nodes. In order to shorten the description we pose that the minors can be computed easily and we abstract the precise formulation of this procedure:

Proposition 3. *In each of the four cases the minor $\Gamma_p \in MG_p$ for a node p can be obtained from the minor(s) of its child(ren) in $O(\ell)$ time.*

START NODES. Let p be a start node. In this case the only insignia are edges in $G[S_p]$. All vertex indicators are FALSE since $\Gamma_p G[S_p]$. Compute by exhaustive search all partial FVSs (i.e., the collection of all subsets of vertices such that every cycle in S_p has at least one vertex in the subset). For the insignia set the counter to 1 if none of the endpoints is in the FVS.

JOIN NODES. Let p be a join node with children q and r. We compare partial FVSs of S_q and S_r which have the same intersection in S_p. A pair x, y of vertices in S_p is an insignia if it is an insignia in at least one of S_q and S_r. Its counter has a value which is the sum of the values in S_q and S_r (or ∞ if this is at least 2 and minus 1 if $\{x, y\} \in E$). Each vertex indicator is exactly TRUE if it is TRUE in at least one of S_q and S_r. The minimum cardinality of a partial FVS at p is given by the minimum sum over the pairs of FVSs in S_q and S_r giving rise to the required intersection and indicators.

FORGET NODES. Let p be a forget node with child q and let $x \in S_q - S_p$. Let F be a partial FVS at node q and consider its intersection and indicators at node q. If the indicator for x was TRUE then F is no longer a partial FVS for p. Assume the indicator for x at node q is FALSE. Then $F \cap S_p = (F \cap S_q) - x$. A pair $a, b \in S_p$ is an insignia if either it was an insignia in S_q or if there were insignia between a and x and b and x. The number of paths without internal vertices in S_p from a to b goes up in the latter case if $x \notin Fi \cap S_q$. If $y \in S_p$, $x, y \notin F$, and if the indicator for $\{x, y\}$ was ∞, then the indicator for y is now TRUE.

INTRODUCE NODES. Let p be an introduce node with child q and let $x \in S_p - S_q$. Let F be a partial FVS at node q and consider its intersection and indicators at node q. The indicator for x at node p is FALSE and the only insignia incident with x are edges in S_p. We consider two possibilities: either x is added to F or not. By definition, the new set F can be a partial FVS at node p only if there is no cycle in S_p without vertices in F. It is clear how to compute the classification for node p from the one at q.

Correctness of the procedure described above is evident, and we obtain:

Theorem 3. *There exists an algorithm running in time $O(c^{\sqrt{k}}n)$ which checks if a planar graph G has a FVS of cardinality at most k.*

We end this section with an observation for the directed case. Consider the problem of finding a minimum set of vertices in a planar digraph which hits all directed cycles. It is easy to see that the algorithm can be adapted to solve this situation.

2 Reduction to a problem kernel

Lemma 2. *Let G be a planar graph. Any vertex incident with at least $2k + 2$ faces must be in any FVS of size at most k.*

Proof. If a vertex x would be incident with so many faces, then there would be at least $k + 1$ faces with only the vertex x in common. If x were not in the FVS we could only get a FVS of cardinality more than k. □

We may assume that the graph is biconnected, otherwise we can perform dynamic programming on the biconnected components. Hence, we may assume that every vertex has degree at most $2k+1$. We may also assume that every vertex has degree at least 3 (as long as the graph is no triangle). We can now apply a lemma by H.J. Voss [43].

Lemma 3. *If G is a graph with no vertex of degree less than 3, then for any feedback vertex set F, $n \leq |F|(\Delta+1)-2$, where Δ is the maximal degree in G.*

Corollary 1. *There is a linear time reduction to an $O(k^2)$ kernel.*

Using the result of Section 1 we obtain our main result:

Theorem 4. *There exists an $O(c^{\sqrt{k}}+n)$ algorithm for the k-FEEDBACK VERTEX SET problem on planar graphs for some constant c.*

3 Treewidth vs. disjoint cycles

Let G be a planar graph with a maximum number of k disjoint cycles. We show that $fvs(G) \leq 5k$. It follows that $tw(G) = O(\sqrt{k})$. Using a tree-decomposition with this width we show in the next section that we can compute a maximum number of disjoint cycles in linear time. Recall that a graph is called *rich* if every vertex has degree at least 3. The following lemma is well known.

Lemma 4. *Every rich planar graph has a face of length at most 5.*

Proof. Assume that every face has length at least 6. Let ϕ be the number of faces. Counting the pairs of faces and incident edges in two ways we obtain $2e \geq 6\phi$ since every edge is in at most two faces. By Euler's formula $e-n+2=\phi \leq \frac{e}{3}$. Hence $2e \leq 3n-6$. But the degree of every vertex is at least 3, so $2e \geq 3n$. □

Theorem 5. *Let G be a planar graph. Then $fvs(G) \leq 5\ cp(G)$.*

Proof. Start with an empty set F and repeatedly take the following steps until not a one is apt.

FIRSTLY, repeatedly, remove pendant vertices from G until G has no more vertices of degree ≤ 1.
SECONDLY, repeatedly, take out vertices of degree 2 that have non-adjacent neighbours and connect its two neighbours by an edge.
THIRDLY, assume G has a vertex of degree 2 and the two neighbours are connected by an edge. Then put the triangle in F and rip it out of G.
FOURTHLY, and finally, assume that G has no more vertices of degree ≤ 2. Then G has a face with length at most 5 by Lemma 4. Choose such a face $\mathbb{F}$ and put all its vertices in F. Rip $\mathbb{F}$ out of G.

Let C be a chordless cycle of G. As long as C has length at least 4 and contains degree 2 vertices these are removed in the second step. If C ends up with 3 vertices and one of them has degree 2, then the third step ensures that C contains a vertex which is put in the feedback vertex set. If C has only vertices of degree

at least 3 then none of its vertices is removed from the graph without it being put in the feedback vertex set (in the third or fourth phase). This shows that F is a feedback vertex set in G.
Notice that whenever vertices are put in the set F all the vertices of a cycle of length at most 5 are removed from the graph. For each cycle that is removed at most three vertices are put in F. This shows that F contains at most 5 times the maximum number of vertex disjoint cycles of G. □

The next theorem is an immediate consequence of Theorem 5 and Theorem 1.

Theorem 6. *Let G be a planar graph with $cp(G) \leq k$. Then $tw(G) = O(\sqrt{k})$.*

Clearly, for any graph G, $fvs(G) \geq cp(G)$. In general it was shown in [18,43] that there exist family of graphs G with $fvs(G) = \Theta(cp(G) \log cp(G))$. We feel fairly confident that there is a much tighter link for planar graphs:

Conjecture 1. [a] For planar graphs G, $fvs(G) \leq 2\ cp(G)$.

[a] Jones' conjecture.

Notice that the factor 2 would be optimal as shown by the 3-sun. For directed graphs a similar conjecture is known as Younger's conjecture (or Gallai–Younger conjecture). A cycle packing in a digraph is a set of vertex disjoint directed cycles. Younger conjectured that for every k there is a $g(k)$ such that any digraph D without a cycle packing of k elements contains a feedback vertex set of at most $g(k)$ vertices [8,46]. It is the directed analogue of the earlier Erdös–Pósa Theorem. Younger's conjecture for general digraphs was proved in [37]. The bound given for general graphs is "worse" than exponential but it shows the FPT–ness of k-DISJOINT CYCLES in digraphs.
For planar digraphs it has been shown that if D does not contain a cycle packing with k elements then D contains a set of at most $O(k \log(k) \log\log(k))$ vertices whose removal leaves an acyclic graph [36]. Combining results of [24] and [36] shows that $g(k)$ can be chosen as Ck for some constant C in the planar case [1].
For planar digraphs D with maximum in–degree and out–degree 2 it has been shown that $fvs(D) < 4\ cp(D)$ [32].

4 A treewidth algorithm for disjoint cycles

In the previous section we showed that if G is a planar graph with $cp(G) = k$ then $tw(G) \leq C\sqrt{k}$ for some constant C. There exists a linear time algorithm to find a tree decomposition of this width [1]. In this section we describe a treewidth algorithm computing a maximum set of vertex disjoint cycles in a planar graph G.

[1] See the footnote in [36]

Let a tree decomposition $(T, \mathcal{S})$ with the set of bags $\mathcal{S} = \{S_i \mid i \in V(T)\}$ and with width $\ell - 1$ in *standard form* be given. Let G_i be the subgraph of G induced by the union of bags corresponding with nodes in the subtree of $i \in V(T)$. For each i we define $\Gamma_i \in MG_i$ as the minor of G_i obtained from G_i by contracting all edges incident with at most one vertex of S_i. Hence Γ_i is a minor of G_i with vertex set S_i.

Definition 8. *A pair $PC = (\mathcal{C}_i, \mathcal{P}_i)$ of paths and cycles in G_i is a* partial cycle collection *for G_i if*

1. *each cycle of $\mathcal{C}_i$ is vertex disjoint with all other cycles and paths of $\mathcal{P}_i$,*
2. *the paths and cycles of PC share no vertices outside S_i and if a path is just an edge of S_i it is vertex disjoint with all other paths and cycles,*
3. *each path of $\mathcal{P}_i$ has its endpoints in S_i and its internal vertices in $V(G_i) - S_i$,*
4. *there are at most two paths in $\mathcal{P}_i$ sharing the same pair of endpoints in S_i, and*
5. *each cycle of $\mathcal{C}_i$ has all vertices either in $V(G_i) - S_i$ or all vertices in S_i or exactly one vertex in S_i.*

We assume some plane embedding of the graph G is given. We classify partial cycle collections $PC = (\mathcal{C}_i, \mathcal{P}_i)$ for G_i by the following characteristic:

1. the number c_i of cycles of $\mathcal{C}_i$ without vertices of S_i (or ∞ if this number is more than k),
2. for each path $P \in \mathcal{P}_i$ the two end-vertices of $P \cap S_i$,
3. for each insignia in S_i the values 0, 1 or 2 according whether there are zero, one, or two paths in $\mathcal{P}_i$ with these endpoints, and
4. for every vertex $x \in S_i$ an indicator taking the value TRUE if there is a cycle $C \in \mathcal{C}_i$ with $C \cap S_i = \{x\}$.

Proposition 4. *Let r be the root of T. Consider a partial cycle collection $PC = (\mathcal{C}_r, \mathcal{P}_r)$. Let s be the number of vertices with indicator* TRUE*. Add one or two edges in $G[S_p]$ between pairs of end-vertices of paths of $\mathcal{P}_p$ (this could create double edges) and remove the vertices of which the indicator has the value* TRUE*. Now compute the maximum number r_p of vertex disjoint cycles in this graph. The graph G has a collection with at least k vertex disjoint cycles if and only if there is a partial cycle collection at p with $s + r_p + c_p \geq k$.*

Lemma 5. *The number of different characteristics of partial cycle collections is at most $(k+1)2^{7\ell}$.*

Proof. Notice that the number of edges in Γ_i is at most $3\ell - 6$ since Γ_i is planar. Each edge of Γ_i can be an edge of one or two paths of $\mathcal{P}_i$, hence there are $O(2^{2(3\ell-6)})$ choices for the path collection $\mathcal{P}_i$. The number of choices for subsets of vertices with indicator TRUE is clearly bounded by 2^{ℓ}. □

We describe the computation of the different characteristics of partial cycle collections in each of the four cases.

START NODES. Let p be a start node. The path collection is *any* subset of vertex disjoint edges of S_i in this case. The indicator for every vertex must be FALSE and $c_p = 0$.

JOIN NODES. Let p be a join node with children q and r. Consider a pair of characteristics for q and r of which the intersections with S_p coincide. We take the union of path collections at q and r and if two paths have the same endpoints we set the counter to the maximum of the sum and 2. Let $c_p = c_q + c_r$. A vertex' indicator gets the value TRUE when it is TRUE in at least one of the partial cycle collections at q and r.

FORGET NODES. Let p be a forget node with child q and let $x \in S_q - S_p$. Consider a partial cycle collection $PC = (\mathcal{P}_q, \mathcal{C}_q)$ at node q. If the indicator for x was TRUE then this cycle has now no more vertices in S_p, hence $c_p = c_q + 1$.
If the indicator for x is FALSE and if x is the endpoint of a path then we update the partial cycle collection for node p as follows. Assume that x is incident with paths of $\mathcal{P}_q$. Choose arbitrary two endpoints y_1 and y_2 of these paths. Each choice gives a new partial cycle collection. If $y_1 = y_2$ then set the indicator for y_1 to TRUE. If $y_1 \neq y_2$ then create one new path between y_1 and y_2.

INTRODUCE NODES. Let p be an introduce node with child q and let $x \in S_p - S_q$. consider a partial cycle collection $PC = (\mathcal{P}_q, \mathcal{C}_q)$ at node q. We can add paths to this collection for edges $\{x, y\}$ for vertices y not incident with any path or cycle of PC.

Correctness of the procedure described above should be clear, and we obtain:

Theorem 7. *There exists an algorithm running in time $O(c^{\sqrt{k}}n)$ which checks if a planar graph G has k vertex disjoint cycles.*

Consider the problem of finding the maximum number of vertex disjoint (directed) cycles in a digraph. Clearly the algorithm described in this section can be adapted to fit this problem.

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By Kostochka's result [28,41], there exists a constant $c \in \mathbb{R}$ such that for every $r \in \mathbb{N}$, every graph G of *average* degree $d(G) \geq cr\sqrt{\log r}$ has a K_r minor. Hence, if $d(G) \geq c(k+3)\sqrt{\log(k+3)}$, then every FVS must have cardinality at least k since a K_{k+3} has no FVS with less than k vertices. Combining this with a result by Thomassen [42] this implies that that there exists a constant $c \in \mathbb{R}$ such that for every graph G with *girth* $g(G) \geq cr\sqrt{\log r}$ and $\delta \geq 3$ has a K_r minor. This would give an alternative search tree algorithm for graphs in general (Remark 2). See also the result of Monien and Schulz [33]. For *weighted* undirected graphs, the algorithm MiniWCycle given in [6] translates to a kernel with a factor $\log n$ for the FVS problem.

It would be interesting to know some optimal bound for the minimum cardinality of a feedback vertex set compared to the number of independent cycles in a graph when restricted to planar graphs [18].

It is unclear if a non-trivial kernelization of the DISJOINT CYCLES problem exists for planar graphs. In this case we are interested in the *existence* of a set of disjoint cycles only. (Since the time needed to list them could well be $\Theta(n)$.)

Concerning digraphs, we would like to draw some attention to the LOOP CUTSET problem for digraphs stated in [6]. The factor 4 approximation described in this paper is a direct application of their algorithm SubG-2-3. It seems that a reduction to a problem kernel for this problem like we described it for the FVS problem is somewhat problematic.

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