Facets of The Cardinality Constrained Circuit Polytope

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Abstract

The Cardinality Constrained Circuit Problem (CCCP) is the problem of finding a minimum cost circuit in a graph where the circuit is constrained to have at most k edges. The CCCP is NP-Hard. We present classes of facet-inducing inequalities for the convex hull of feasible circuits.

1 Introduction

In the knapsack constrained circuit problem (KCCP), we are given an undirected graph $G = (V, E)$, a cost c_e for each edge $e \in E$, a weight $w_v \geq 0$ for each vertex $v \in V$, and an integer k. The objective is to find a minimum cost circuit (i.e., a simple cycle) subject to the constraint that the sum of the weights on the vertices in the circuit is at most k. The KCCP is easily seen to be NP-hard, because when we subtract a sufficiently large constant from the cost of each edge and set $k = \sum_{v \in V} w_v$, we obtain a traveling salesman problem.

Although the KCCP is an interesting optimization problem, its importance to us stems from the fact that it can be used to model the pricing problem in branch-and-price algorithms for the vehicle routing problem. For a comprehensive discussion of branch-and-price algorithms for vehicle routing problems the reader is referred to [DSD84].

In branch-and-price algorithms for vehicle routing problems, the pricing problem is usually solved by dynamic programming, i.e., multi-label shortest path algorithms. Solving the pricing problem by a branch-and-cut algorithm, rather than a dynamic programming algorithm, may have several computational advantages. First, good feasible solutions, corresponding to columns with a negative reduced cost, may be found quicker because multi-label shortest path algorithms find feasible solutions only when the sink node is labeled, which may take a long time if the underlying network is large. Secondly, using a branch-and-cut algorithm, it is not necessary to solve the problem to optimality to show that no negative reduced column exists. As soon as the global lower bound becomes nonnegative, we know the optimal solution will be nonnegative as well. Finally, if the state space cannot be pruned by restrictions such as time windows, a dynamic programming approach begins to resemble exhaustive search.

We are interested in investigating the advantages and disadvantages of using a branch-and-cut algorithm, rather than using dynamic programming, to solve the pricing problem within a branch-and-price algorithm for vehicle routing problems.

Developing a branch-and-cut algorithm for the KCCP is also interesting from another perspective. The polytope defining the set of feasible solutions to the KCCP is the intersection of two other polyhedra, namely the knapsack polytope and the circuit polytope. We know a lot about the structure of both these polyhedra, and it is interesting to learn more about the value of this knowledge when it comes to developing a branch-and-cut algorithm for the KCCP.

To facilitate our investigation, we have decided to start with the special case of unit weights, i.e., the cardinality constrained circuit problem (CCCP). The CCCP models the pricing problem that arises in branch-and-price algorithms for the vehicle routing problem with unit demands.

Before we discuss problems that are related to the KCCP and the CCCP, we first present a transformation that converts node related information to edge related information. This transformation allows us to look at the problems from different perspectives. The weight w_v for $v \in V$ can be placed on the edges by introducing an edge weight $w_e = 0.5(w_u + w_v)$ for all $e = \{u, v\} \in E$ and requiring that the sum of the weights on the edges in the circuit be at most k.

In the capacitated prize collecting traveling salesman problem (CPCTSP) [BCSL96], we are given an undirected graph $G = (V, E)$, a travel cost t_e for $e \in E$, a reward p_v and weight w_v for $v \in V$, a depot node $v_0 \in V$, and a capacity $W \in \mathbb{Z}_+$. The objective is to find a route, or set of edges, R starting and ending in v_0 that maximizes the collected rewards minus the incurred travel costs, i.e., $\sum_{v \in V(R)} p_v - \sum_{e \in R} t_e$, subject to the constraint that the total weight on the route, i.e., $\sum_{v \in V(R)} w_v$, does not exceed the capacity W.

In the orienteering problem (OP), we are given a graph $G = (V, E)$, rewards p_v for $v \in V$, a depot node $v_0 \in V$, travel costs t_e for $e \in E$, and an upper bound $Q \in \mathbb{Z}_+$. The objective is to find a route R starting and ending at v_0 that maximizes the total collected reward ($\sum_{v \in V(R)} p_v$) subject to the constraint that the total travel cost ($\sum_{e \in R} t_e$) is less than Q. Heuristics for solving the OP are given by Golden *et al.* [GWL88] and Ramesh *et al.* [RYK92]; and polyhedral approaches for OP are given by Fischetti *et al.* [FGT98] and Leifer and Rosenwein [LR94].

Applying the transformation presented above, we see that the CPCTSP and OP are equivalent problems. Furthermore, if we remove the requirement that the route goes through the depot node, then the CPCTSP and OP are also equivalent to the KCCP.

In the prize collecting traveling salesman problem (PCTSP) [Bal89], there is a reward for visiting a node and penalty for not visiting a node. The objective is to minimize the sum of the penalties and the travel costs subject to the constraint that the total collected reward should be greater than or equal to a given minimum. A heuristic for the PCTSP in which the prize requirement is not considered is given by Bienstock et al. [BGSLW93]. Polyhedral based approaches to the PCTSP are presented by Balas [Bal89] and Pillai [Pil92].

If we remove the remove the knapsack constraint from the KCCP, we are left with the weighted girth problem (WGP) or circuit problem (CP), where we are trying to find a minimum cost circuit in a graph. Bauer [Bau94, Bau97] studies the WGP problem in great detail, presents facet defining inequalities for its underlying polyhedron, and provides a branch-and-cut approach for its solution. Wang [Wan95] examines both the CP and the closely related Eulerian subtour problem.

The remainder of the paper is organized as follows. In Section 2, we give two integer programming formulations for the CCCP and present basic results on the facial structure of the polyhedra associated with the convex hulls of feasible solutions for both formulations. In Section 3, we show that many facet inducing inequalities for the WGP polyhedron are also facet inducing for the CCCP polyhedron. In Section 4, we derive new classes of facets for the CCCP polyhedron.

2 Integer Programming Formulations of the CCCP

In order to ease the exposition, we will first introduce a few definitions. For $V' \subseteq V$, define

$$
\begin{aligned}
E(V') &\equiv \{(i,j) \in E : i \in V', j \in V'\}, \text{ and} \\
\delta(V') &\equiv \{(i,j) \in E : i \in V', j \notin V'\}.
\end{aligned}
$$

In addition, we will write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$. For a given subset of edges $E' \subseteq E$, we use the notation

$$V(E') \equiv \{v \in V : E' \cap \delta(v) \neq \emptyset\}$$

to define the set of nodes spanned by E'.

To formulate the CCCP, we use decision variables $x_e,\ e \in E$, and $y_v,\ v \in V$, to describe a circuit C with the following meanings:

$$x_e = \begin{cases} 1, & \text{if } e \in C, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$y_v = \begin{cases} 1 & \text{if } v \in V(C), \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience, we often write $x(E')$ for $\sum_{e \in E'} x_e$ for a set of edges $E' \subseteq E$, and $y(V')$ for $\sum_{v \in V'} y_v$ for a set of vertices $V' \subseteq V$. Also, for two subsets $S, T \subseteq E$, $S \cap T = \emptyset$, we let

$$x(S:T) = \sum_{s \in S, t \in T} x_{s,t}$$

An integer programming formulation of the CCCP can be given as follows:

Minimize

$$\sum_{e \in E} c_e x_e \tag{2.1}$$

subject to

$$\begin{aligned}
x(\delta(v)) &= 2y_v && \forall v \in V, && (2.2)\\
x(\delta(S)) &\geq 2(y_u + y_v - 1) && \forall S \subset V,\ 3 \leq |S| \leq n-3, && \\
& && u \in S,\ v \in V \setminus S, && (2.3)\\
x(E) &\geq 3, && && (2.4)\\
x(E) &\leq k, && && (2.5)\\
x_e &\in \{0,1\} && \forall e \in E, && (2.6)\\
y_v &\in \{0,1\} && \forall v \in V. && (2.7)
\end{aligned}$$

In this formulation, the *degree equations* (2.2) ensure that a feasible solution goes exactly once through each visited node, and the *disjoint circuit elimination constraints* (2.3) make sure that our solution is a connected circuit. Constraint (2.4) eliminates the null circuit, constraint (2.5) is the *cardinality constraint*, and constraints (2.6) and (2.7) give the integrality conditions on our variables.

Let $\mathcal{C}_n$ be the set of circuits of K_n, the complete graph on n nodes, and let χ^C be the incidence vector of a circuit C. We are interested in studying the *cardinality constrained circuit polytope*

$$\begin{aligned}
P_C^{n,k} &= \operatorname{conv}\{(\chi^C, \chi^{V(C)})^T \in \mathbb{R}^{|E|+|V|} | C \in \mathcal{C}_n, |C| \leq k\}\\
&= \operatorname{conv}\{(x,y)^T \in \mathbb{R}^{|E|+|V|} | (x,y) \text{ satisfies } (2.2)-(2.7)\}.
\end{aligned}$$

$P_C^{n,k}$ is precisely the intersection of the two polytopes:

$$\begin{aligned}
P_C^n &= \operatorname{conv}\{(x,y) \in \mathbb{R}^{|E|+|V|} | (x,y) \text{ satisfies } (2.2)-(2.4), (2.6)-(2.7)\} \text{ and}\\
P_k^n &= \operatorname{conv}\{x \in \mathbb{R}^{|E|} | x \text{ satisfies } (2.5) \text{ and } (2.6)\}.
\end{aligned}$$

Hence, any valid inequality for P_C^n or P_k^n is also valid for $P_C^{n,k}$. We will show that in many cases facet defining inequalities for P_C^n and P_k^n are also facet defining for $P_C^{n,k}$.

By substituting out the node variables y_v $(v \in V)$ using (2.2), we obtain a formulation of the CCCP that uses only edge variables x_e $(e \in E)$.
Minimize

$$\sum_{e \in E} c_e x_e$$

subject to

$$\begin{aligned}
x(\delta(v)) &\leq 2 && \forall v \in V, && (2.8)\\
x(\delta(v) \setminus e) - x_e &\geq 0 && \forall v \in V, e \in \delta(v), && (2.9)\\
x_e + x((u:T)) + x((v:S)) & && && \\
-x((S:T)) &\leq 2 && \forall e = (u,v) \in E \text{ such that} && \\
& && S,T \text{ is a partition of} && \\
& && V \setminus \{u,v\}, |S|, |T| \geq 2, && (2.10)\\
x(E) &\geq 3, && && (2.11)\\
x(E) &\leq k, && && (2.12)\\
x_e &\in \{0,1\} \ \forall e \in E. && && (2.13)
\end{aligned}$$

The *degree constraints* (2.8) and the *parity constraints* (2.9) ensure that every vertex has degree zero or two. The *disjoint circuit elimination constraints* (2.10) ensure our circuit is connected. Since there are no node variables in this formulation, the associated cardinality constrained circuit polytope and circuit polytope have different definitions:

$$\begin{aligned}
\tilde{P}_C^{n,k} &= \text{conv}\{\chi^C \in \mathbb{R}^{|E|} \mid C \in \mathcal{C}_n, |C| \leq k\}\\
&= \{x \in \mathbb{R}^{|E|} | x \text{ satisfies } (2.8) - (2.13)\}
\end{aligned}$$

and

$$\tilde{P}_C^n = \text{conv}\{\chi^C \in \mathbb{R}^E \mid C \in \mathcal{C}_n, \}.$$

Bauer [Bau94] [Bau97] and Wang [Wan95] have studied the facial structure of $\tilde{P}_C^n$ and we will frequently use their results.

Our next goal is to show some properties of the polytopes introduced above and to establish relations between them. Similar proofs appear in [BCSL96].

Theorem 2.1 *For* $4 \leq k \leq n$,
$dim(\tilde{P}_C^n) = dim(P_C^n) = dim(\tilde{P}_C^{n,k}) = dim(P_C^{n,k}) = |E| = n(n-1)/2.$

Proof. Bauer [Bau94] and Wang [Wan95] establish that $\dim(\tilde{P}_C^n) = |E|$. Their proofs use circuits of at most length four, so $\dim(\tilde{P}_C^{n,k}) = |E|$. The rank of the set of equalities $(x(\delta(v)) = 2y_v \ \forall v \in V)$ is $|V|$; thus, $\dim(P_C^n) \leq |E|$ and $\dim(P_C^{n,k}) \leq |E|$. The proofs of Bauer and Wang show that there are $|E| + 1$ circuits (of length at most 4) whose incidence vectors $x \in \mathbb{R}^{|E|}$ are affinely independent. For these same circuits, the incidence vectors in terms of edge and node variables $(x,y)^T \in \mathbb{R}^{|E|+|V|}$ are also affinely independent, so $\dim(P_C^n) \geq |E|$ and $\dim(P_C^{n,k}) \geq |E|$. $\square$

Since the two formulations of the CCCP describe the same set of feasible circuits, we would also suspect that their polyhedra have the same facets.

Theorem 2.2 *If the inequality $a^T x \leq a_0$ is facet defining for $\tilde{P}_C^{n,k}$, it is also facet defining for $P_C^{n,k}$. If $b^T x + d^T y \leq b_0$ is facet defining for $P_C^{n,k}$, then $h^T x \leq b_0$, where $h = (h)_{ij} \equiv b_{ij} + \frac{1}{2}(d_i + d_j)$ is facet defining for $\tilde{P}_C^{n,k}$.*

Proof. If $a^T x \leq a_0$ is facet defining for $\tilde{P}_C^{n,k}$, there are $|E|$ affinely independent circuits that satisfy $a^T x = a_0$. These same $|E|$ circuits written in terms of edge and node variables are also affinely independent and satisfy the equality $a^T x = a_0$. If $b^T x + d^T y \leq b_0$ is facet defining for $P_C^{n,k}$, then there are $|E|$ affinely independent circuits $(x^1, y^1)^T, (x^2, y^2)^T, \ldots, (x^{|E|}, y^{|E|})^T \in \mathbb{R}^{|E|+|V|}$ such that $b^T x^j + d^T y^j = b_0 \ \forall j = 1, 2, \ldots |E|$. As $y_v = x(\delta(v))/2 \ \forall v \in V$, we find by substitution that these circuits also satisfy $h^T x = b_0$. Further, the incidence vectors of circuits $x^1, x^2, \ldots x^{|E|}$ are affinely independent, for if not, we would have

$$|E| > \text{rank} \begin{bmatrix} x^1 & x^2 & \ldots & x^{|E|} \\ -1 & -1 & \ldots & -1 \end{bmatrix} = \text{rank} \begin{bmatrix} x^1 & x^2 & \ldots & x^{|E|} \\ y^1 & y^2 & \ldots & y^{|E|} \\ -1 & -1 & \ldots & -1 \end{bmatrix} \tag{2.14}$$

which would imply that the original circuits $(x^1, y^1)^T, (x^2, y^2)^T, \ldots, (x^{|E|}, y^{|E|})^T \in \mathbb{R}^{|E|+|V|}$ were not affinely independent. $\square$

Since the polyhedra are the same, we use only the notation $P_C^{n,k}$ when referencing either polyhedron for the remainder of the paper.

Let us introduce the following two polytopes which are closely related to P_C^n and $P_C^{n,k}$. For a node set $K \subseteq V$, we let

$$P_C^K = \text{conv}\{(\chi^C, \chi^{V(C)})^T \in \mathbb{R}^{|E(K)|+|K|} \mid C \text{ is a circuit in } G(K) = (K, E(K))\}$$

and

$$\begin{aligned} P_C^{n,K} &= \text{conv}\{(\chi^C, \chi^{V(C)})^T \in \mathbb{R}^{|E|+|V|} \mid C \in \mathcal{C}_n, \ C \text{ covers only nodes in } G(K)\} \\ &= \text{conv}\{(\chi^C, \chi^{V(C)})^T \in \mathbb{R}^{|E|+|V|} \mid C \in \mathcal{C}_n, \ x_e = 0 \text{ for all } e \notin E(K), \\ &\qquad\qquad y_v = 0 \text{ for all } v \notin K\}. \end{aligned}$$

The following lemma will be useful in helping us characterize when facet defining inequalities for P_C^n are also facet defining for $P_C^{n,k}$.

Lemma 2.3 *Let $4 \leq k < n$ and let $ax + fy \leq a_0$ be facet defining for P_C^n. Suppose there is a set $K \subseteq V$ with $|K| \leq k$ such that the restriction $\tilde{a}x + \tilde{f}y \leq a_0$ of $ax + fy \leq a_0$ to $G(K) = (K, E(K))$ is facet defining for P_C^K. Moreover, assume that for any $e \in E \setminus E(K)$ there is a circuit $C \in \mathcal{C}_n$ with $e \in C$, $|C| \leq k$ and $a\chi^C + f\chi^{V(C)} = a_0$. Then $ax + fy \leq a_0$ is facet defining for $P_C^{n,k}$.*

Proof. Since $\tilde{a}x + \tilde{f}y \le a_0$ defines a facet of P_C^K, we can conclude that $ax + fy \le a_0$ defines a facet of $P_C^{n,K}$. With $P_C^{n,K} = P_C^{n,k} \cap \{(x,y)^T \in \mathbb{R}^{E+V} \mid x_e = 0 \text{ for all e } \in E \setminus E(K), y_v = 0 \text{ for all } v \in K\}$, the claim follows from the existence of a circuit $C \in \mathcal{C}_n$ with $e \in C$, $|C| \le k$, and $a\chi^C + f\chi^{V(C)} = a_0$ $\forall e \in E \setminus E(K)$. □

Bauer [Bau94] has characterized when the basic inequalities defining P_C^n are facet inducing. Because of Theorem 2.2 and the fact the proofs require only circuits of lengths 3 and 4, we have

Theorem 2.4

(i) *The* **trivial inequalities** $x_e \ge 0$, $e \in E$, *define facets of* $P_C^{n,k}$ *for* $n \ge 5$ *and* $k \ge 4$.

(ii) *The* **degree constraints** $x(\delta(v)) \le 2$, $v \in V$, *define facets of* $P_C^{n,k}$ *for* $n \ge 5$ *and* $k \ge 4$.

(iii) *The* **parity constraints** $x(\delta(v) \setminus x_e) - x_e \ge 0$, $v \in V, e \in \delta(v)$, *are facet inducing for* $P_C^{n,k}$, $n \ge 5$ *and* $k \ge 4$.

(iv) *Let* $e = (u,v) \in E$, *and let* S, T *be a partition of* $V \setminus \{u,v\}$ *with* $S, T \ge 2$. *Then the* **disjoint circuit elimination constraint**

$$x_e + x((u:T)) + x((v:S)) - x((S:T)) \le 2$$

defines a facet of $P_C^{n,k}$, $n \ge 6$ *and* $k \ge 4$.

(v) *The inequality* $x(E) \ge 3$ *is facet defining for* $P_C^{n,k}$, $n \ge 5$ *and* $k \ge 4$.

Next, we show that the cardinality constraint, which is facet inducing for P_k^n is also a facet inducing for $P_C^{n,k}$.

Theorem 2.5 *Let* $4 \le k < n$. *Then the* **cardinality constraint**

$$x(E) \le k$$

is facet defining for $P_C^{n,k}$.

Proof. Assume that there is an inequality $bx \le b_0$, $b \in \mathbb{R}^{|E|}$, $b \ne \mathbf{0}$, which is valid for $P_C^{n,k}$ and satisfies $\{x \in P_C^{n,k} \mid x(E) = k\} \subseteq \{x \in P_C^{n,k} \mid bx = b_0\}$.

Let $f = (u,v)$ and $g = (w,z)$ be any two nonadjacent edges in E, define $h = (v,w)$, $l = (u,z)$, and let C be a circuit of cardinality k containing the edges f and h, but not the node z. With $C' = C \setminus \{f,h\} \cup \{g,l\}$, we have $b\chi^C = b\chi^{C'} = b_0$ and thus

$$b_f + b_h = b_g + b_l.$$

Analogously, we can derive

$$b_g + b_h = b_l + b_f$$

and get $b_g = b_f$. Since for any two adjacent edges, we can find an edge which is not adjacent to either of them, we get the same coefficient for all edges $e \in E$. This immediately yields that $bx \le b_0$ is a positive multiple of $x(E) \le k$. □

In the next sections, we introduce valid inequalities and facets for the polyhderon $P_C^{n,k}$ that do not explicitly appear as inequalities in the integer programming formulation. In Section 3, we give conditions under which some facets of P_C^n are also facets of $P_C^{n,k}$. In Section 4, we derive valid inequalities and facets for $P_C^{n,k}$ which are not valid inequalities for P_C^n. Some of the these are obtained from lifting facets of P_C^n that define only lower dimensional faces of $P_C^{n,k}$, and others are obtained independently.

3 Inequalities from the Circuit Polytope

In this section, we show that two classes of valid inequalities for P_C^n are also valid for $P_C^{n,k}$. The *cut inequalities* are facet inducing for both P_C^n and $P_C^{n,k}$ for $n \geq 7$. The *forest inequalities* are facet defining for P_C^n for $n \geq 7$, and facet inducing for $P_C^{n,k}$ if an easy to check condition is satisfied.

3.1 The Cut Inequalities

The *cut inequalities*, introduced by Seymour [Sey79], generalize the parity constraints (2.9). They are shown to be facet defining for P_C^n, $n \geq 2$, by Bauer [Bau94]. Her proof involves only circuits of length 3 and 4 and thus immediately yields the following theorem.

Theorem 3.1 *Let $n \geq 7$ and $k \geq 4$. For $S \subseteq V$, $3 \leq |S| \leq n-3$, and $e \in \delta(S)$, the* **cut inequality**

$$x(\delta(S) \setminus e) - x_e \geq 0$$

is facet defining for $P_C^{n,k}$.

3.2 The Forest Inequalities

Bauer [Bau94, Bau97] gives several classes of facet defining inequalities for P_C^n that are derived using the fact that the traveling salesman polytope is a face of the circuit polytope. Among them are the *forest inequalities*, which are obtained from the facet inducing clique tree inequalities of Grötschel and Pulleyblank [GP86]. It turns out that, if an easy to check condition on the "size" of the inequality is satisfied, the inequality is also a facet for $P_C^{n,k}$.

A *clique tree* is a connected graph composed of cliques which satisfy the following properties (in the following a clique tree is always considered a subgraph of K_n):

(1) The cliques are partitioned into two sets, the set of *handles* and the set of *teeth.*

(2) No two teeth intersect.

(3) No two handles intersect.

(4) Each tooth contains at least two nodes, at most $n-2$ nodes, and at least one node not belonging to any handle.

(5) For each handle, the number of teeth intersecting it is odd and at least three.

(6) If a tooth and a handle intersect, then their intersection is an articulation set of the clique tree.

Suppose we are given a clique tree with handles $H_1, \ldots, H_r$ and teeth $T_1, \ldots, T_s$. For every tooth T_j, we denote by t_j the number of handles which intersect T_j. Choose from every tooth T_j, $1 \leq j \leq t$, a node $r(j)$ not belonging to any handle, referred to as the *root* of T_j. Choose from every nonempty intersection $H_i \cap T_j$, $1 \leq i \leq r$, $1 \leq j \leq s$, of a tooth and a handle a node $u(i,j)$, which we call the *link* of H_i and T_j. Define R to be the set of all roots $r(j)$, $1 \leq j \leq s$, and U to be the set of all links $u(i,j)$, where $1 \leq i \leq r$, $1 \leq j \leq s$, and $H_i \cap T_j \neq \emptyset$. Using the result of Bauer [Bau94] and performing the substitution $x(\delta(v)) = 2y_v \; \forall v \in V$, the following can be shown to be true.

Theorem 3.2 *Let a clique tree of K_n, $n \geq 6$, be given by a set of handles $H_1, \ldots, H_r$, $r \geq 1$, and a set of teeth $T_1, \ldots, T_s$. Let R be a set of roots and U a set of links. Then the* **forest inequality**

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} y(H_i) + \sum_{j=1}^{s} (y(T_i) - t_i) - y(U) - y(R) + \frac{3}{2}(s-1)$$

is facet defining for P_C^n.

Using Lemma 2.3, we can find a sufficient condition for a forest inequality to be facet defining for $P_C^{n,k}$.

Theorem 3.3 *Let $n \geq 9$, $4 \leq k < n$, and let $ax + fy \leq a_0$ be a forest inequality with set of roots R and set of links U. Then $ax + fy \leq a_0$ is facet defining for $P_C^{n,k}$ whenever $|R| + |U| + 2 \leq k$.*

Proof. We let $K = U \cup R$ and apply Lemma 2.3. Let $e = (u,v)$ be any edge not contained in $E(K)$. The inequality $\tilde{a}x + \tilde{f}y \leq a_0$, where $(\tilde{a}, \tilde{f})$ is the restriction of (a,f) to $G_K^{uv} = (K \cup \{u,v\}, E(K \cup \{u,v\}))$ is also a forest inequality and hence there is a circuit C of length at most $|K \cup \{u,v\}| \leq k$ containing e and satisfying $\tilde{a}\chi^C + \tilde{f}\chi^{V(C)} = a\chi^C + f\chi^{V(C)} = a_0$. □

If $r = 1$ and $|T_j \cap H_1| = 1 \; \forall j = 1, \ldots, s$, we call the inequality a *simple forest inequality.* Simple forest inequalities correspond to the *simple comb inequalities* of the TSP polytope. If, in addition, $|T_j| = 2 \; \forall j = 1, \ldots, s$, we call the resulting inequality a *2-forest inequality.* 2-forest inequalities correspond to *2-matching inequalities* of the TSP polytope.

4 Inequalities Specific to the Cardinality Constrained Circuit Polytope

In this section, we derive valid inequalities for $P_C^{n,k}$ that are not valid for P_C^n. The first two, the *cardinality-path inequalities* and the *k-partition inequalities* are derived from first principles. They are also shown to be facet inducing for $P_C^{n,k}$. The second two, the *cardinality-tree inequalities* and the *maximal set inequalities* are obtained by strengthening inequalities known to be valid and facet inducing for P_C^n.

4.1 The Cardinality-Path Inequalities

If P is a path with k edges and C is a circuit of cardinality at most k, then the cardinality-path inequality corresponding to P says that C never uses more edges of P than inner nodes of P.

Theorem 4.1 *Let $4 \leq k < n$, P be a path in K_n consisting of k edges, and $\dot{P}$ denote the set of inner nodes of P. Then the* **cardinality-path** *inequality*

$$x(P) \leq y(\dot{P})$$

defines a facet of $P_C^{n,k}$.

Proof. First, we prove that the cardinality-path inequalities are valid for $P_C^{n,k}$. Suppose for a contradiction that C is a feasible circuit satisfying $x(P) \geq y(\dot{P}) + 1$ for some path P consisting of k edges. By definition $y(\dot{P}) = k - 1$, so $x(P) \geq k$. Since C is a circuit, $x(C \setminus P) \geq 1$, but then $x(C) = x(C \setminus P) + x(P) > k$, so C was not feasible.

We will use the "indirect-method" to show that the cardinality-path inequality is facet defining. Let us assume that we have an inequality $bx \leq dy + b_0$, $b \in \mathbb{R}^{|E|}$, $d \in \mathbb{R}^{|V|}$, $(b, d) \neq \mathbf{0}$, which is valid for $P_C^{n,k}$ and satisfies $\{(x,y)^T \in P_C^{n,k} \mid x(P) = y(\dot{P})\} \subseteq \{(x,y)^T \in P_C^{n,k} \mid bx = dy + b_0\}$. If we can show that there is a $\lambda \geq 0$ and a vector $\mu \in \mathbb{R}^{|V|}$, such that $bx - dy = \lambda(x(P) - y(\dot{P})) + \sum_{v \in V} \mu_v(x(\delta(v)) - 2y_v)$ and $b_0 = 0$, then $x(P) \leq y(\dot{P})$ is facet defining for $P_C^{n,k}$.

The columns in the coefficient matrix corresponding to the node variables y_v $v \in V$ are linearly independent. Therefore, with an appropriate choice of multipliers μ_v, we can fix the values of d_v $v \in V$ to be

$$d_v = \begin{cases} 1, & \text{if } v \in \dot{P}, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $V = \{1, \ldots, n\}$ and $P = \{(1,2), (2,3), \ldots, (k, k+1)\}$. Define the following circuits:

$$\begin{aligned}
C_0 &= \{(1,2), (2, k+1), (k, k+1), (1, k)\}, & \\
C_{1,j} &= \{(1,2), (2,3), \ldots (j-1, j), (1, j)\}, & 3 \leq j \leq k, \\
\tilde{C}_{1,j} &= \{(1,2), (2,3), \ldots (j-1, j), & \\
&\qquad (1, k+1), (j, k+1)\}, & 2 \leq j \leq k-1, \\
C_{j,k+1} &= \{(j, j+1), (j+1, j+2), & \\
&\qquad \ldots, (k, k+1), (j, k+1)\}, & 2 \leq j \leq k-1, \\
\tilde{C}_{j,k+1} &= \{(j, j+1), (j+1, j+2), & \\
&\qquad \ldots, (k, k+1), (1, j), (1, k+1)\}, & 3 \leq j \leq k, \\
C_{i,j} &= \{(1,2), \ldots, (i-1, i), (i, j), (j, j+1), & \\
&\qquad \ldots, (k, k+1), (1, k+1)\}, & 2 \leq i < j - 1 \leq k.
\end{aligned}$$

The incidence vectors of all circuits defined above satisfy $x(P) = y(\dot{P})$ and hence $bx = dy + b_0$. We will first show that $b_0 = 0$. Observe that for any $j \in \{3, \dots, k-1\}$ we have

$$\chi^{\tilde{C}_{1,2}} + \chi^{\tilde{C}_{k,k+1}} + \chi^{C_{1,j}} + \chi^{C_{j,k+1}} - \chi^{C_0} - \chi^{\tilde{C}_{1,j}} - \chi^{\tilde{C}_{j,k+1}} = 0$$

and thus $b\mathbf{0} = d\mathbf{0} + b_0$ which yields $b_0 = 0$.

We start determining the coefficients of b by looking at the circuits $C_{1,j}$ and $\tilde{C}_{1,j}$ for some $j \in \{3, \dots, k-1\}$. We get

$$\begin{array}{l} b_{1,2} + \dots + b_{j-1,j} + b_{1,j} = j-1 \\ b_{1,2} + \dots + b_{j-1,j} + b_{1,k+1} + b_{j,k+1} = j-1 \end{array}$$

and thus $b_{1,j} = b_{1,k+1} + b_{j,k+1}$. Analogously, from $C_{j,k+1}$ and $\tilde{C}_{j,k+1}$, we get $b_{j,k+1} = b_{1,j} + b_{1,k+1}$ and hence $b_{j,k+1} = b_{1,j}$ and $b_{1,k+1} = 0$.

Now, consider the circuits $C_{1,j}$ and $C_{1,j+1}$, $3 \le j \le k-1$. We have

$$\begin{array}{l} b_{1,2} + \dots + b_{j-1,j} + b_{1,j} = j-1 \\ b_{1,2} + \dots + b_{j-1,j} + b_{j,j+1} + b_{1,j+1} = j \end{array}$$

and thus

$$b_{j,j+1} + b_{1,j+1} - b_{1,j} = 1. \tag{4.15}$$

The circuits $C_{j,k+1}$ and $C_{j+1,k+1}$ give us, for $2 \le j \le k-2$,

$$\begin{array}{l} b_{j,j+1} + b_{j+1,j+2} + \dots + b_{k,k+1} + b_{j,k+1} = k-j \\ b_{j+1,j+2} + \dots + b_{k,k+1} + b_{j+1,k+1} = k-j-1 \end{array}$$

and hence

$$b_{j,j+1} + b_{j,k+1} - b_{j+1,k+1} = 1. \tag{4.16}$$

Since for $3 \le j \le k-1$, we have $b_{1,j} = b_{j,k+1}$ and for $2 \le j \le k-2$ we have $b_{1,j+1} = b_{j+1,k+1}$, we get, with the equations (4.15) and (4.16) $b_{j,j+1} = 1$ and $b_{1,j} = b_{1,j+1}$ for $3 \le j \le k-2$. Thus, we know that $b_{j,j+1} = 1$ for $3 \le j \le k-2$ and $b_{1,j} = b_{j,k+1} = \alpha$ for $3 \le j \le k-1$.

To calculate the coefficients $b_{2,3}$ and $b_{k-1,k}$, we look at $C_{1,3}$ and $C_{k-1,k+1}$. We get

$$\begin{array}{l} b_{1,2} + b_{2,3} + \alpha = 1 \\ b_{k-1,k} + b_{k,k+1} + \alpha = 1 \end{array}$$

and thus

$$b_{1,2} + b_{2,3} - b_{k-1,k} - b_{k,k+1} = 0. \tag{4.17}$$

From the circuit $C_{2,k+1}$ we get

$$b_{2,3} + k - 4 + b_{k-1,k} + b_{k,k+1} + b_{2,k+1} = k-1$$

or equivalently

$$b_{2,3} + b_{k-1,k} + b_{k,k+1} + b_{2,k+1} = 3.$$

Adding this equation to (4.17), we obtain $b_{1,2}+2b_{2,3}+b_{2,k+1} = 3$. From $\tilde{C}_{1,2}$, we get $b_{1,2}+b_{2,k+1} = 1$, and thus, we have $b_{2,3} = 1$. Analogously, we can derive $b_{k-1,k} = 1$. Equation (4.15) for $j = k-1$ gives us $b_{1,k} = \alpha$ and equation (4.16) for $j = 2$ gives us $b_{2,k+1} = \alpha$.

From the circuits $\tilde{C}_{1,2}$ and $\tilde{C}_{k,k+1}$, we get $b_{1,2} + \alpha = 1$ and $b_{k,k+1} + \alpha = 1$ and thus $b_{1,2} = b_{k,k+1} = 1 - \alpha$.

Now, let $i \in \{2, \ldots, k-2\}$, $j \in \{4, \ldots, k\}$ and $j > i+1$. The circuit $C_{i,j}$ gives us $2(1-\alpha) + (i-2) + (k-j) + b_{i,j} = (i-1) + (k+1-j)$ and thus $2(1-\alpha) + b_{i,j} = 2$. Hence, we have $b_{i,j} = 2\alpha$ for all $2 \leq i < j-1 \leq k-1$.

So far, we considered all edges with both endnodes in $V(P)$. Now we will determine the coefficients for edges with one node in $V(P)$. Assume now that we have a node $v \notin V(P)$. Observe that, if C is a circuit satisfying $\chi^C(P) = \chi^{V(C)}(\dot{P})$, every circuit $\tilde{C}$ obtained from C by replacing an edge $(i,j) \in C \setminus P$ by the edges (i,v) and (v,j) also satisfies $\chi^{\tilde{C}}(P) = \chi^{V(\tilde{C})}(\dot{P})$. Thus, we know that for all $(i,j) \in E(V(P)) \setminus P$, we have $b_{i,v} + b_{v,j} = b_{i,j}$. This yields

$$\begin{aligned} b_{1,v} \qquad\quad + b_{k+1,v} &= 0 \\ b_{1,v} + b_{j,v} \qquad\qquad &= \alpha \\ b_{l,v} + b_{k+1,v} &= \alpha \end{aligned}$$

where $j \in \{3, \ldots, k\}$ and $l \in \{2, \ldots, k-1\}$. We get $b_{1,v} = b_{k+1,v} = 0$ and $b_{j,v} = \alpha$ for $j \in \{2, \ldots, k\}$.

Finally, for two nodes $v, w \in V \setminus V(P)$, we get $b_{v,w} = 0$ by looking at the circuit $C = \{(1,2), (1,v), (v,w), (2,w)\}$.

Now, let $\lambda = 1 - 2\alpha$ and consider the vector $b - \lambda\chi^P$. We have

$$(b - \lambda\chi^P)_e = \begin{cases} 1-\alpha-(1-2\alpha) = \alpha & \text{for } e = (1,2) \text{ or } e = (k,k+1), \\ \alpha & \text{for } e = (1,j),\ 3 \leq j \leq k \text{ or} \\ & \quad e = (j,k+1),\ 2 \leq j \leq k-1, \\ 1-(1-2\alpha) = 2\alpha & \text{for } e = (j,j+1),\ 2 \leq j \leq k-1, \\ 2\alpha & \text{for } e = (i,j),\ 2 \leq i < j-1 \leq k-1, \\ 0 & \text{for } e = (1,k+1), \\ \alpha & \text{for } e = (j,v),\ v \notin V(P),\ j \in \{2,\ldots,k\}, \\ 0 & \text{for } e = (1,v),\ e = (v,k+1) \text{ or} \\ & \quad e = (v,w),\ v,w \notin V(P). \end{cases}$$

Thus $bx - \lambda x(P) = \alpha \sum_{v \in \dot{P}} x(\delta(v)) = 2\alpha y(\dot{P})$. With $2\alpha = 1 - \lambda$, we have $bx - \lambda x(P) = (1-\lambda) y(\dot{P})$ which is equivalent to

$$bx - y(\dot{P}) = \lambda x(P) - \lambda y(\dot{P}).$$

Since e.g. the circuit $C = \{(1,3), (3,k+1), (1,k+1)\}$ satisfies $b\chi^C \leq \chi^{V(C)}(\dot{P})$, we have $2\alpha \leq 1$, which is equivalent to $\lambda \geq 0$. We assumed that $b, d \neq \mathbf{0}$ and thus we have shown that the inequality $bx \leq dy + b_0$ is a positive multiple of $x(P) \leq y(\dot{P})$. $\square$

4.2 The k-Partition Inequalities

The k-partition inequalities ensure that we use enough edges across a partition of V into sets of size $k-1$.

Theorem 4.2 *Let $4 \le k < n$, $s = \lceil \frac{n}{k-1} \rceil$ and $V = \bigcup_{i=1}^{s} V_i$ be a partition of V with $|V_i| = k-1$ for $1 \le i \le s-1$ and $|V_s| \le k-1$. Then*

$$2\sum_{i=1}^{s} x(E(V_i)) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \le 2(k-1)$$

is facet defining for $P_C^{n,k}$.

Proof. To see that the k-partition inequalities are valid, note that all circuits of length $k-1$ satisfy the inequality. Further, a circuit of length k must use at least two edges in $\cup_{i=1}^{s-1} \cup_{j=i+1}^{s} (V_i : V_j)$, so the inequality is satisfied in this case as well.

We now show that the k-partition inequalities are facet defining for $P_C^{n,k}$. Let the inequality $2\sum_{i=1}^{s} x(E(V_i)) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \le 2(k-1)$ be denoted by $ax \le a_0$. Suppose that we have an inequality $bx \le b_0$, $b \in \mathbb{R}^{|E|}$, $b \ne \mathbf{0}$, which is valid for $P_C^{n,k}$ and satisfies $\{x \in P_C^{n,k} | ax = a_0\} \subseteq \{x \in P_C^{n,k} | bx = b_0\}$.

Let $b^i x \le b_0$ be the restriction of the inequality $bx \le b_0$ to $E(V_i)$, for $1 \le i \le s$. We first show that we may write $b^i = \tilde{b}^i + \mu^i A^i$ where $\mu^i \in \mathbb{R}^{k-1}$, A^i is the node edge incidence matrix corresponding to $G_i = (E(V_i), V_i)$ and where $k-1$ components of $\tilde{b}^i$, which correspond to a linearly independent subset of the columns of A^i, are fixed to any values.

Since each Hamiltonian circuit H of G_i satisfies $a\chi^H = a_0$, it is also true that $b^i\chi^H = b_0$. Therefore the inequality $b^i x \le b_0$ must be a linear combination of the constraints $x(\delta^i(v)) \le 2$, where $\delta^i(v)$ is the set of all edges in $E(V_i)$ which are incident to v. Therefore, $b^i = \sigma^i A^i$.

Without loss of generality, assume that the columns $1, 2, \ldots, k-1$ of A^i are linearly independent. The submatrix $\bar{A}^i$ generated by these columns is nonsingular, so we may select any values of $\tilde{b}_1^i, \tilde{b}_2^i, \ldots, \tilde{b}_{k-1}^i$ and find multipliers $\tau^i = (\tau_i^i, \tau_2^i \ldots, \tau_{k-1}^i)$ such that $(\tilde{b}_1^i, \tilde{b}_2^i, \ldots, \tilde{b}_{k-1}^i) = \tau^i \bar{A}^i$. Let $\tilde{b}^i = \tau^i A^i$. Then

$$b_i = \tilde{b}^i + \sigma A^i - \tau^i A^i = \tilde{b}^i + \mu^i A^i,$$

with $\mu^i \equiv \sigma^i - \tau^i$, and $k-1$ components of $\tilde{b}^i$ are fixed to values of our choosing.

Now let $i = 1$ and $V_1 = \{1, \ldots, k-1\}$. We fix $\tilde{b}_{1,t}^1 = 2$ for all $2 \le t \le k-1$ and $\tilde{b}_{2,3}^1 = 2$. If $k = 4$, we have $\tilde{b}_{r,t}^1 = 2$, for all $r, t \in \{1, 2, 3\}$, $r \ne t$.

Else, if $k \ge 6$, let $r \in \{4, \ldots, k-2\}$ and consider the circuits

$$C = \{(1,r), (r, r-1), \ldots, (4,2), (2,3), (3, r+1), \ldots, (k-2, k-1), (1, k-1)\}$$

and

$$\tilde{C} = \{(1,2), (2,4), \ldots, (r-1, r), (r, 3), (3, r+1), \ldots, (k-2, k-1), (k-1, 1)\}.$$

For $k = 5$ and $r = 4$ or $k \ge 6$ and $r = k-1$, define

$$C = \{(1,r), (r, r-1), \ldots, (4,2), (2,3), (3,1)\}$$

and

$$\tilde{C} = \{(1,2),(2,4),\ldots,(r-1,r),(r,3),(3,1)\}.$$

From C and $\tilde{C}$, we get

$$\tilde{b}^1_{1,r} + \tilde{b}^1_{2,3} = \tilde{b}^1_{1,2} + \tilde{b}^1_{3,r}$$

and hence $\tilde{b}^1_{3,r} = 2$. By iterating this argument, we get $\tilde{b}^1_{q,r} = 2$ for all $q, r \in \{1,\ldots,k-1\}$, $q \neq r$.

Thus, we have $b^1 = 2\mathbf{I} + \mu^1 A^1$, where $\mathbf{I}$ is the vector of 1's. Similarly, we can derive $b^i = 2\mathbf{I} + \mu^i A^i$, $1 \leq i \leq s-1$.

Now, let $i \in \{2,\ldots,s\}$ with $|V_i| \geq 2$. Let $v \in V_1 \setminus \{1,2\}$ and let P_v be a path in $E(V_1)$ having endnodes 1 and 2 and containing all nodes of V_1 except node v. Let $e = (u,w) \in E(V_i)$ and consider the circuit

$$C_e = P_v \cup \{(u,w),(1,u),(2,w)\}.$$

We have

$$b^1\chi^{C_e} = 2(k-3) + 2\sum_{j=1,j\neq v}^{k-1} \mu^1_j - \mu^1_1 - \mu^1_2 + b_{u,w} + b_{1,u} + b_{2,w} = b_0$$

and since the last equation holds for all $v \in V_1 \setminus \{1,2\}$, we get $\mu^1_v =: \lambda^1$ for all $v \in V_1 \setminus \{1,2\}$. Repeating this argument, we get $\mu^1_v =: \lambda^1$ also for $v \in \{1,2\}$ and hence $b^1 = 2\mathbf{I} + (\lambda^1\mathbf{I})A^1$. Analogously, we get $b^i = 2\mathbf{I} + (\lambda^i\mathbf{I})A^i$, for all $1 \leq i \leq s$ where $|V_i| \geq 2$.

Moreover, any Hamiltonian circuit in V_i, $1 \leq i \leq s-1$, gives us $b_0 = 2(k-1) + 2\lambda^i(k-1)$ and thus, we conclude $\lambda_i =: \lambda$ for all $1 \leq i \leq s-1$ and $b^i = 2\mathbf{I} + (\lambda\mathbf{I})A^i$.

Now, choose an arbitrary node $z \notin V_1$. Let

$$C_{1,2} = P_{1,2} \cup \{(1,z),(2,z)\}$$

and

$$C_{2,3} = P_{2,3} \cup \{(2,z),(3,z)\}$$

where $P_{r,t}$, $r,t \in V_1$, $r \neq t$, denotes a Hamiltonian path in V_1 with endnodes r and t. We have

$$\begin{aligned} b\chi^{C_{1,2}} &= 2(k-2) + 2(k-2)\lambda + b_{1,z} + b_{2,z} &&= b_0 \\ b\chi^{C_{2,3}} &= 2(k-2) + 2(k-2)\lambda + \phantom{b_{1,z}} + b_{2,z} + b_{3,z} &&= b_0 \end{aligned}$$

and thus $b_{1,z} = b_{3,z}$. Similarly, we get $b_{i,z} = b_{j,z}$ for all $i,j \in \{1,\ldots,k-1\}$. Moreover, we have $b_0 = 2(k-1) + 2\lambda(k-1)$ and thus $b_{i,z} = 1 + \lambda$ for all $i \in \{1,\ldots,k-1\}$. With analogous arguments, we get also $b_{r,z} = 1 + \lambda$ for all $r \in V_i$, $z \in V \setminus V_i$, $1 \leq i \leq s-1$.

Thus, the coefficients $b_{u,w}$ where $u,w \in V_s$ remain to be calculated. For $e = (u,w)$, $u,w \in V_s$, consider again

$$C_e = P_v \cup \{(u,w),(1,u),(2,w)\}.$$

We have

$$b_0 = 2(k-3) + 2\lambda(k-3) + b_{u,w} + (1+\lambda) + (1+\lambda)$$

and hence, with $b_0 = 2(k-1) + 2\lambda(k-1)$, we conclude $b_{u,w} = 4 + 4\lambda - 2(1+\lambda) = 2 + 2\lambda$.

Altogether, we have

$$b_e = \begin{cases} 2(1+\lambda) & \text{if } e \in E(V_i),\ i \in \{1,\ldots,s\}, \\ 1+\lambda & \text{if } e \in (V_i : V_j),\ i,j \in \{1,\ldots,s\},\ i \neq j \end{cases}$$

and thus $b_e = (1+\lambda)a_e$ for all $e \in E$. Since e.g. the circuit $C = \{(1,2),(2,n),(1,n)\}$ satisfies $b\chi^C \leq 2(k-1) + 2\lambda(k-1)$, we get $\lambda \geq -1$ and thus $(1+\lambda) \geq 0$, which completes our proof. □

4.3 Maximal Set Inequalities

Wang [Wan95] introduces a class of inequalities he calls the *multipartition inequalities* and shows them to be facet defining for P_C^n. His theorem is stated below.

Theorem 4.3 *Let $K_n = (V,E)$, $4 \leq k < n$ and a partition of V be given by $V = \bigcup_{i=1}^s V_i$ where $|V_i| \geq 2$ for all $1 \leq i \leq s$. Moreover, let $T_i \subseteq E(V_i)$, $1 \leq i \leq s$, be a spanning tree of the complete graph induced by V_i and $\overline{T}_i$ be its complement with respect to $E(V_i)$. Then a facet of P_C^n is induced by the inequality*

$$2\sum_{i=1}^{s} x(\overline{T}_i) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \geq 2.$$

These inequalities do not in general induce facets of $P_C^{n,k}$ but can be strengthened by replacing the sets $\overline{T}_i$, which are the complements of maximal sets in $E(V_i)$ not containing any circuit, by complements of maximal sets in $E(V_i)$ not containing any circuit of cardinality less than or equal to k.

Theorem 4.4 *Let $K_n = (V,E)$, $4 \leq k < n$, and let a partition of V be given by $V = \bigcup_{i=1}^s V_i$ where $|V_i| \geq 2$ for all $1 \leq i \leq s$. Moreover, let $M_i \subseteq E(V_i)$, $1 \leq i \leq s$, be a maximal edge set with respect to $E(V_i)$ not containing any circuit of cardinality less than or equal to k. Let $\overline{M}_i = E(V_i) \setminus M_i$ be its complement with respect to $E(V_i)$. Then a facet of $P_C^{n,k}$ is induced by the* **maximal set inequality**

$$2\sum_{i=1}^{s} x(\overline{M}_i) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \geq 2.$$

Proof. The inequality is valid, since every feasible circuit either uses at least one edge in $\cup_{i=1}^s \overline{M}_i$ or two edges in $\cup_{i=1}^{s-1} \cup_{j=i+1}^{s} (V_i : V_j)$. Let us denote the inequality $2\sum_{i=1}^s x(\overline{M}_i) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \geq 2$ by $ax \geq 2$. Assume that we have an inequality $bx \geq b_0$, $b \in \mathbb{R}^{|E|}$, which is valid for $P_C^{n,k}$ and satisfies $\{x \in P_C^{n,k} \mid ax = 2\} \subseteq \{x \in P_C^{n,k} \mid bx = b_0\}$. We show that $bx \geq b_0$ is simply a scalar multiple of $ax \geq 2$.

We first show that $b_e = \frac{b_0}{2}$ for all $e \in (V_i, V_j)$ where $1 \leq i < j \leq s$ and $b_f = 0$ for all $f \in M_i$, $1 \leq i \leq s$. Let, without loss of generality, $i = 1$ and $j = 2$. Let $x, y \in V_1$ and $v, w \in V_2$ be such that $m_1 = (x, y) \in M_1$ and $m_2 = (v, w) \in M_2$ and denote the edge (x, v) with e. Observe that e is substitutional for all edges in $(V_1 : V_2)$, since we can always find edges in M_1 and M_2 adjacent to a given edge in (V_1, V_2). Let the other edges induced by x, y, v, w be denoted as in Figure 4.1.

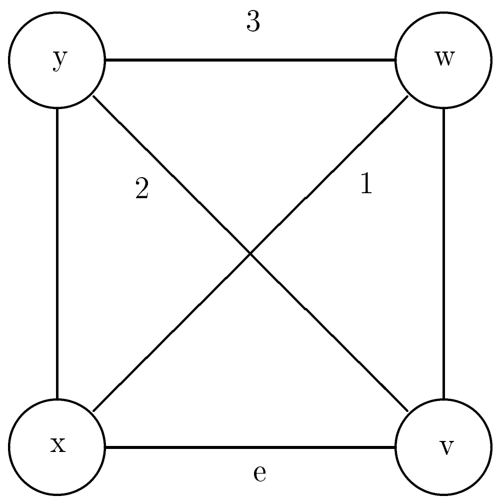

Figure 4.1: Description of Edges

From the circuits $\{e, 1, m_2\}$, $\{e, 2, m_1\}$, $\{2, 3, m_2\}$ and $\{1, 3, m_1\}$, we derive the equations

$$\begin{array}{rcl} b_e + b_1 + b_{m_2} &=& b_0 \\ b_e + b_2 + b_{m_1} &=& b_0 \\ b_2 + b_3 + b_{m_2} &=& b_0 \\ b_1 + b_3 + b_{m_1} &=& b_0. \end{array}$$

From these equations, we may conclude that $b_1 = b_2$, $b_{m_1} = b_{m_2}$, and $b_3 = b_e$.

Adding the first two equations and subtracting the equation

$$b_1 + b_2 + b_{m_1} + b_{m_2} = b_0,$$

which we get from the circuit $\{1, 2, m_1, m_2\}$, yields $b_e = \frac{b_0}{2}$. Hence, also $b_3 = \frac{b_0}{2}$ and thus with the equation

$$b_e + b_3 + b_{m_1} + b_{m_2} = b_0,$$

we conclude $b_{m_1} = b_{m_2} = 0$. Also, $b_1 = b_2 = \frac{b_0}{2}$.

We now show that $b_g = b_0$ for all edges in $\overline{M}_i$, $1 \le i \le s$. Let w.l.o.g. $i = 1$ and $g \in \overline{M}_1$. Since $g \notin M_1$, there must be a circuit $C \in M_1 \cup g$ with $|C| \le k$. This circuit satisfies $b_0 = b\chi^C = b_g$ and thus we conclude $b_g = b_0$.

Since $bx \ge b_0$ has to be valid also for the circuit $\{e, 1, 2, 3\}$, we know that $b_0 > 0$, which completes the proof. □

4.4 The Cardinality-Tree Inequalities

Wang [Wan95] introduces the following generalization of the degree inequalities.

Theorem 4.5 *Let T be a spanning tree of K_n. For each $e = (u, v) \in E \setminus T$, define l_e^T as the length of the unique (u, v) path in T. The* **tree inequality**

$$\sum_{e \in T} x_e + \sum_{e \notin T} (2 - l_e^T) x_e \leq 2$$

is a valid inequality for P_C^n. If T is such that every edge $e \in T$ is in a star $K_{1,3} \subseteq T$, then the inequality is also facet defining for P_C^n.

Using the fact that all circuits must be of length at most k, this inequality can be strengthened.

Theorem 4.6 *Let T be a spanning tree of K_n. For each $e = (u, v) \in E \setminus T$, define l_e^T as the length of the unique (u, v) path in T. Define*

$$w_{l_e}^T = \begin{cases} 2 - l_e^T & \text{if } 2 \leq l_e^T \leq k - 1, \\ 4 - 2k + l_e^T & \text{if } k \leq l_e^T \leq \lceil 3k/2 \rceil - 2, \\ 2 - k/2 & \text{if } k \text{ is even and } l_e^T \geq 3k/2 - 1, \\ (3 - k)/2 & \text{if } k \text{ is odd and } l_e^T = \lceil 3k/2 \rceil - 1 + 2i \text{ for some } i \in \mathbb{Z}_+, \\ (5 - k)/2 & \text{if } k \text{ is odd and } l_e^T = \lceil 3k/2 \rceil + 2i \text{ for some } i \in \mathbb{Z}_+. \end{cases}$$

The **cardinality-tree inequality**

$$\sum_{e \in T} x_e + \sum_{e \in E \setminus T} w_{l_e}^T x_e \leq 2$$

is a valid inequality for $P_C^{n,k}$.

Proof. The proof relies on the concept of coefficient improvement. We will improve the coefficients of the variables corresponding to edges $e \in E \setminus T$. These coefficients w_e start out at their initial values $2 - l_e^T$, and it is our goal to show that they can be increased to at least $w_{l_e}^T$, thereby strengthening the inequality. Note first that the coefficients of edges $f \in E \setminus T$ with $l_f^T \leq k - 1$ are not changed, so the "new" coefficients $w_{l_f}^T$ are certainly valid in this case. For an edge $f \in E \setminus T$, define the coefficient improvement problem (CIP) for f as

$$\begin{aligned} \max_x \quad & z_f \equiv \sum_{e \in T} x_e + \sum_{e \in E \setminus T} w_e x_e \\ \text{subject to:} \quad & x \text{ is the incidence vector of a circuit } C, \\ & |C| \leq k, \\ & x_f = 1. \end{aligned}$$

With the above definitions, we can state the following lemma.

Lemma 4.7 *The coefficient $w_f^T = 2 - l_f^T$ can be increased by $2 - z_f$.*

The following lemma will also come in handy:

Lemma 4.8 *Define* P_e^T *as the unique path in* T *between the endpoints of* e. *If* $f \in C \cap (E \setminus T)$ *and* $(C \setminus f) \cap (E \setminus T) \neq \emptyset$, *then* $P_f^T \setminus C \subseteq \bigcup_{e \in (C \setminus f) \cap (E \setminus T)} P_e^T$.

Proof. Consider the graph G_C with edge set $\cup_{e \in C \cap (E \setminus T)} (P_e^T \bigcup C)$. G_C is planar, so draw G_C with C as its outer face, as in Figure 4.2. In this embedding, edges on interior faces are in $\cup_{e \in C \cap (E \setminus T)} P_e^T$. Let $g \in P_f^T \setminus C$. g is on the boundary of two interior faces, so g is also on some P_e^T for $e \in (C \setminus f) \cap (E \setminus T)$. $\square$

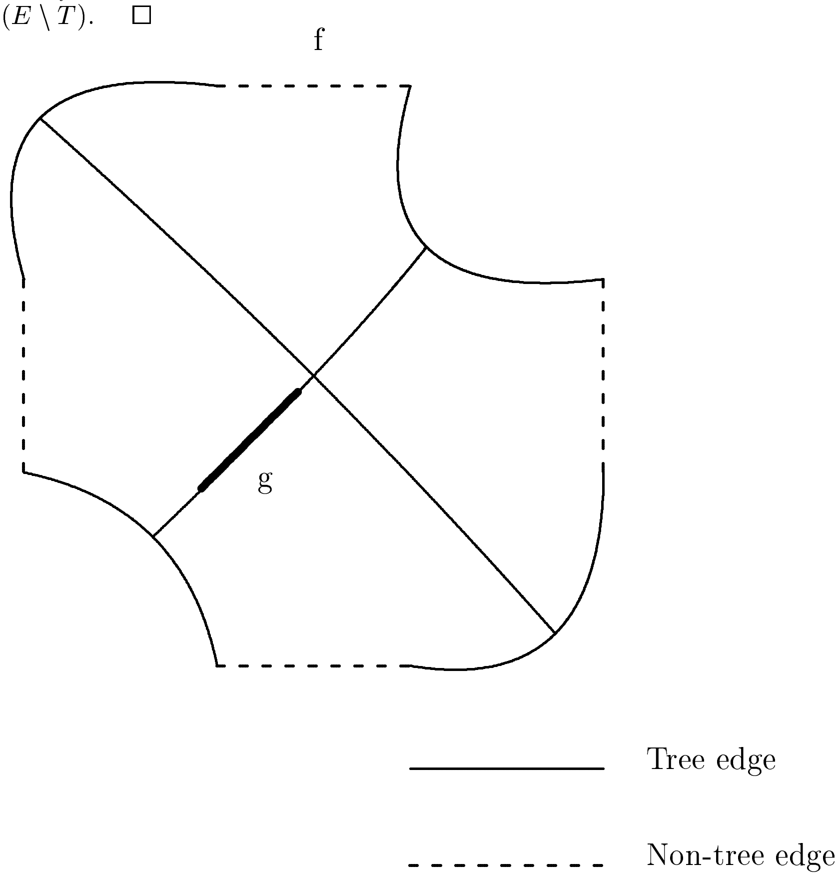

Figure 4.2: An embedding of C and P_e^T

Now, suppose for a contradiction that the coefficient for edge $f \in E \setminus T$ cannot be increased to $w_{l_f}^T$, this implies by Lemma 4.7 that

$$2 - l_f^T + 2 - z_f \leq w_{l_f}^T - 1. \tag{4.18}$$

Since edge f must be in an optimal solution x to problem (CIP), we have

$$z_f = \sum_{e \in T} x_e + 2 - l_f^T + \sum_{e \in (E \setminus T) \setminus f} w_e x_e. \tag{4.19}$$

Combining (4.18) and (4.19) gives that

$$w_{l_f}^T + \sum_{e \in (E \setminus T) \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e. \tag{4.20}$$

Let C be the optimal circuit defined by incidence vector x. Define m as $m \equiv |C \cap (E \setminus T)|$. We now proceed to prove the validity of the coefficient improvements on a case by case basis.
Case I. $k \leq l_f^T \leq \lceil 3k/2 \rceil - 2$, so that $w_{l_f}^T = 4 - 2k + l_f^T$.

Subcase I.a.1. k is even and $\exists g \in (C \setminus f) \cap (E \setminus T)$ such that $l_g^T \geq k$.

In this subcase, it can be shown by definition of $w_{l_e}^T$ that $w_{l_f}^T \leq 2 - k/2$, and $w_g \leq 2 - k/2$. Also by definition of $w_{l_e}^T$, we know that $w_e \leq 0 \; \forall e \in E \setminus T$. Using these facts and (4.20), we may write

$$2 - k/2 + 2 - k/2 \geq w_{l_f}^T + w_g \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e,$$

which yields

$$\sum_{e \in T} x_e \geq k - 1.$$

Since $f, g \in C \cap (E \setminus T)$, the circuit C with incidence vector x has $|C| \geq k+1$. But this contradicts our cardinality constraint.

Subcase I.a.2. k is odd and $\exists g \in (C \setminus f) \cap (E \setminus T)$ such that $l_g^T \geq k$.

Suppose first that one of the following occurs:

- $k \leq l_f^T < \lceil 3k/2 \rceil - 2$,
- $k \leq l_g^T < \lceil 3k/2 \rceil - 2$, or
- $l_g^T = \lceil 3k/2 \rceil - 1 + 2i$ for some $i \in \mathbb{Z}_+$.

By definitions of the weights $w_{l_e}^T$, it can be shown that either $w_{l_f}^T \leq (3-k)/2$ and $w_g \leq (5-k)/2$, or $w_{l_f}^T \leq (5-k)/2$ and $w_g \leq (3-k)/2$. Using (4.20), we can now write

$$(3-k)/2 + (5-k)/2 \geq w_{l_f}^T + w_g \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e,$$

which yields

$$\sum_{e \in T} x_e \geq k - 1.$$

As in subcase I.a.1, we have a contradiction since the circuit C is too long.

Now suppose that $l_f^T = \lceil 3k/2 \rceil - 2$ and $l_g^T = \lceil 3k/2 \rceil - 2$ or $l_g^T = \lceil 3k/2 \rceil + 2i$ for $i \in \mathbb{Z}_+$. We can write $l_g^T = \lceil 3k/2 \rceil + 2(j-1)$ for some $j \in \mathbb{Z}_+$. By definition of the weights, we know that $w_{l_f}^T = w_g = (5-k)/2$. Using (4.20), we have

$$(5-k)/2 + (5-k)/2 \geq w_{l_f}^T + w_g \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e,$$

which yields

$$\sum_{e \in T} x_e \geq k - 2.$$

It must be that $\sum_{e \in T} x_e = k - 2$, or else we violated our cardinality constraint.

Suppose f, g, C, and T look as in Figure 4.3. Let l_{xy} denote the length of the path in T from x to y. Then we know that

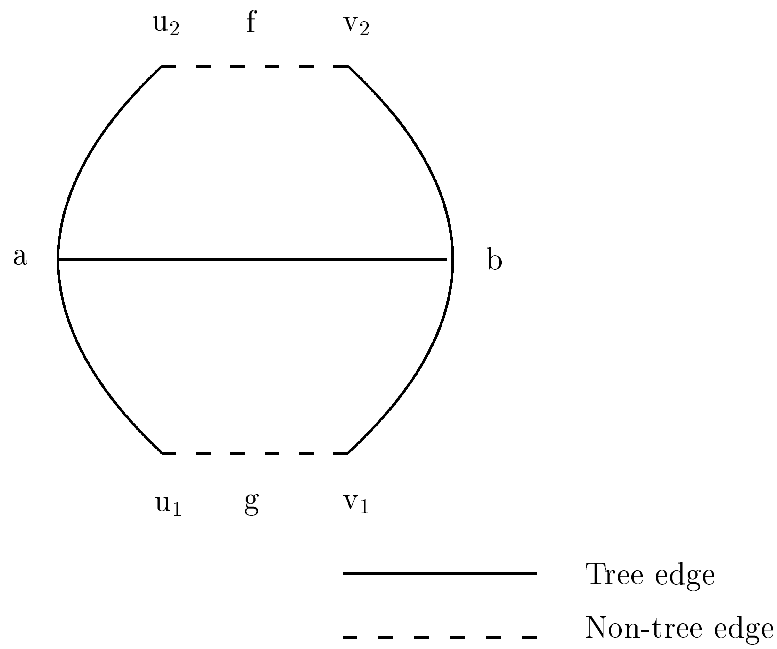

Figure 4.3: The circuit, tree, and edges in $C \cap (E \setminus T)$

$$l_{au_1} + l_{ab} + l_{bv_1} = (3k+1)/2, \tag{4.21}$$

$$l_{au2} + l_{ab} + l_{bv_2} = (3k+1)/2 + 2(j-1) \text{ for some } j \in \mathbb{Z}_+, \tag{4.22}$$

$$l_{au_1} + l_{au_2} + l_{bv_1} + l_{bv_2} = k - 2. \tag{4.23}$$

Subtracting (4.21) from (4.22) and adding it to (4.23) yields:

$$2(l_{au_2} + l_{bv_2}) = k - 2 + 2(j-1).$$

The left hand side of this equation is even and the right hand side is odd, a contradiction.

Subcase I.b $\nexists g \in C \cap (E \setminus T \setminus f)$ such that $l_g^T \geq k$.

In this subcase we know that $w_{l_f}^T = 4 - 2k + l_f^T$ and $w_e = (2 - l_e^T)\ \forall e \in C \cap (E \setminus T \setminus f)$. Recall that $m \equiv |C \cap (E \setminus T)|$. Beginning with inequality (4.20) we deduce the following:

$$\begin{aligned} 4 - 2k + l_f^T + \sum_{e \in (C \setminus f) \cap (E \setminus T)} (2 - l_e^T) &\geq 3 - \sum_{e \in T} x_e \\ 4 - 2k + l_f^T + 2(m-1) - \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T &\geq 3 + m - k \\ l_f^T - \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T &\geq k + 1 - m. \end{aligned}$$

Using Lemma 4.8, this is equivalent to

$$|P_f^T \cap C| - \sum_{e \in (C \setminus f) \cap (E \setminus T)} |P_e^T \setminus P_f^T| \geq k + 1 - m. \tag{4.24}$$

It is clear that

$$|C| \geq |P_f^T \cap C| + m. \tag{4.25}$$

Combining (4.24) and (4.25) gives

$$|C| \geq k + 1 + \sum_{e \in E \setminus T \setminus f} |P_e^T \setminus P_f^T| \geq k + 1,$$

a contradiction, since C is too long.

Case II. $l_f^T \geq \lceil 3k/2 \rceil - 1$.
Subcase II.a.1. k is even, and $\exists g \in (C \setminus f) \cap (E \setminus T)$ such that $l_g^T \geq k$.

In this subcase we know by definition of the weights $w_{l_e}^T$ that $w_{l_f}^T = 2 - k/2$, and $w_g \leq 2 - k/2$. A contradiction is derived in a manner similar to subcase I.a.1.

Subcase II.a.2. k is even, and $\nexists g \in (C \setminus f) \cap (E \setminus T)$ such that $l_g^T \geq k$.

In this subcase, we know that $w_{l_f}^T = 2 - k/2$ and $w_e = 2 - l_e^T\ \forall e \in (C \setminus f) \cap (E \setminus T)$. Using (4.20), we can get the following inequalities:

$$\begin{aligned} 2 - k/2 + \sum_{e \in (C \setminus f) \cap (E \setminus T)} (2 - l_e^T) &\geq 3 - \sum_{e \in T} x_e \\ 2 - k/2 + 2(m-1) - \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T &\geq 3 + m - k \\ - \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T &\geq 3 - m - k/2. \end{aligned} \tag{4.26}$$

Adding the inequality $l_f^T \geq 3k/2 - 1$ to (4.26), we get

$$l_f^T - \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T \geq k - m + 2.$$

Using Lemma 4.8 we find that this is equivalent to

$$|P_f^T \cap C| - \sum_{e \in (C \setminus f) \cap (E \setminus T)} |P_e^T \setminus P_f^T| \geq k - m + 2.$$

This leads us to

$$|C| \geq |P_f^T \cap C| + m \geq k + 2 + \sum_{e \in E \setminus T \setminus f} |P_e^T \setminus P_f^T| \geq k + 2,$$

which is a contradiction, since C is too long.

Subcase II.b.1.i. k is odd, $l_f^T = \lceil 3k/2 \rceil - 1 + 2i$ for some $i \in \mathbb{Z}_+$, and $\exists g \in C \cap (E \setminus T \setminus f)$ such that $l_g^T \geq k$.

In this subcase, we know that $w_{l_f}^T = (3-k)/2$. Further, we know that $w_g \leq (5-k)/2$. Using (4.20), we have

$$(3-k)/2 + (5-k)/2 \geq w_{l_f}^T + w_g \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e,$$

which yields

$$\sum_{e \in T} x_e \geq k - 1,$$

a contradiction.

Subcase II.b.1.ii. k is odd, $l_f^T = \lceil 3k/2 \rceil - 1 + 2i$ for some $i \in \mathbb{Z}_+$, and $\nexists g \in C \cap (E \setminus T \setminus f)$ such that $l_g^T \geq k$.

We know that $w_{l_f}^T = (3-k)/2$ and $w_e = 2 - l_e^T \ \forall e \in (C \setminus f) \cap (E \setminus T)$. (4.20) implies that

$$(3-k)/2 + \sum_{e \in (C \setminus f) \cap (E \setminus T)} (2 - l_e^T) \geq 3 - \sum_{e \in T} x_e.$$

Using the fact that $\sum_{e \in T} x_e \leq k - m$, this inequality can be manipuated to

$$- \sum_{e \in (C \setminus f) \cap (E \setminus T)} l_e^T \geq (7-k)/2 - m \tag{4.27}$$

Furthermore, we know that

$$l_f^T \geq \lceil 3k/2 \rceil - 1 = (3k-1)/2 \tag{4.28}$$

Adding inequalities (4.27) and (4.28) and applying Lemma 4.8, we get that

$$|P_f^T \cap C| - \sum_{e \in (C \setminus f) \cap (E \setminus T)} |P_e^T \setminus P_f^T| \geq k - m + 3.$$

Therefore,

$$|C| \geq |P_f^T \cap C| + m \geq k + 3 + \sum_{e \in E \setminus T \setminus f} |P_e^T \setminus P_f^T| \geq k + 3,$$

which gives the contradiction $|C| > k$.

Subcase II.b.2.i. k is odd, $l_f^T = \lceil 3k/2 \rceil + 2i$ for some $i \in \mathbb{Z}_+$, and $\exists g \in C \cap (E \setminus T \setminus f)$ such that $l_g^T \geq k$.

Suppose first that $k \leq l_g^T < \lceil 3k/2 \rceil - 2$ or $l_g^T = \lceil 3k/2 \rceil - 1 + 2i$ for some $i \in \mathbb{Z}_+$. In this case $w_g \leq (3-k)/2$. Using (4.20) we can say

$$(5-k)/2 + (3-k)/2 \geq w_{l_f}^T + w_g \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e,$$

which yields

$$\sum_{e \in T} x_e \geq k - 1,$$

a contradiction.

Now suppose that $l_g^T = \lceil 3k/2 \rceil - 2$ or $l_g^T = \lceil 3k/2 \rceil + 2i$ for some $i \in \mathbb{Z}_+$, which can equivalently be written as $l_g^T = \lceil 3k/2 \rceil + 2(j-1)$ for some $j \in \mathbb{Z}_+$. From (4.20), we get

$$w_{l_f}^T + w_g = (5-k)/2 + (5-k)/2 \geq w_{l_f}^T + \sum_{e \in E \setminus T \setminus f} w_e x_e \geq 3 - \sum_{e \in T} x_e.$$

This inequality implies that $\sum_{e \in T} x_e \geq k-2$, but since edges f and g are also in the circuit, it must be that $\sum_{e \in T} x_e = k-2$, or else the cardinality constraint would be violated. Referring to Figure 4.3, we know that

$$l_{au_1} + l_{ab} + l_{bv_1} = (3k+1)/2 + 2i, \text{ for some } i \in \mathbb{Z}_+ \tag{4.29}$$

$$l_{au2} + l_{ab} + l_{bv_2} = (3k+1)/2 + 2(j-1), \text{ for some } j \in \mathbb{Z}_+ \tag{4.30}$$

$$l_{au_1} + l_{au_2} + l_{bv_1} + l_{bv_2} = k - 2. \tag{4.31}$$

Subtracting (4.29) from (4.30) and adding it to (4.31) yields:

$$2(l_{au_2} + l_{bv_2}) = k - 2 + 2(i - j + 1).$$

The left hand side of this equation is even and the right hand side is odd, a contradiction.

Subcase II.b.2.ii. k is odd, $l_f^T = \lceil 3k/2 \rceil + 2i$ for some $i \in \mathbb{Z}_+$, and $\nexists g \in C \cap (E \setminus T \setminus f)$ such that $l_g^T \geq k$.

In this subcase, we know that $w^T_{l_f} = (5-k)/2$ and $w_e = 2 - l^T_e \quad \forall e \in (C \setminus f) \cap (E \setminus T)$. Therefore, (4.20) implies that

$$(5-k)/2 + \sum_{e \in (C \setminus f) \cap (E \setminus T)} (2 - l^T_e) \geq 3 - \sum_{e \in T} x_e.$$

Using the fact that $\sum_{e \in T} x_e \leq k - m$, this inequality can be manipulated to give

$$- \sum_{e \in (C \setminus f) \cap (E \setminus T)} l^T_e \geq (5-k)/2 - m \tag{4.32}$$

Furthermore, we know that

$$l^T_f \geq \lceil 3k/2 \rceil = (3k-1)/2 \tag{4.33}$$

Adding inequalities (4.32) and (4.33) and applying Lemma 4.8, we get that

$$|P^T_f \cap C| - \sum_{e \in (C \setminus f) \cap (E \setminus T)} |P^T_e \setminus P^T_f| \geq k - m + 3.$$

Therefore,

$$|C| \geq |P^T_f \cap C| + m \geq k + 3 + \sum_{e \in E \setminus T \setminus f} |P^T_e \setminus P^T_f| \geq k + 3,$$

which gives the contradiction $|C| > k$.

We have shown that in all possible cases, if the coefficient of an edge $e \in E \setminus T$ in the original tree inequality cannot be improved to $w^T_{l_e}$, then there is a contradiction. □

Theorem 4.9 *Let $n \geq 5$, $k \geq 4$. Let a spanning tree T of K_n have the following properties:*

- *If v is a leaf node of T with adjacent node u, then $|\delta(u)| \geq 3$.*
- *If $e \in E \setminus T$ is such that $l^T_e \geq \lceil 3k/2 \rceil - 1$, then there exists and edge $f \in E \setminus T$ with $l^T_f = l^T_e - 1$, and a circuit C, $|C| = k$, consisting of e, f and edges of T.*

Then the inequality

$$\sum_{e \in T} x_e + \sum_{e \in E \setminus T} w^T_{l_e} x_e \leq 2$$

is facet inducing for $P^{n,k}_C$.

Proof. Let us denote the inequality $\sum_{e \in T} x_e + \sum_{e \in E \setminus T} w^T_{l_e} x_e \leq 2$ by $ax \leq 2$. Assume that we have an inequality $bx \leq b_0$, $b \in \mathbb{R}^{|E|}$, which is valid for $P^{n,k}_C$ that satisfies $\{x \in P^n_C \mid ax = 2\} \subseteq \{x \in P^n_C \mid bx = b_0\}$. We show that $bx \leq b_0$ is a scalar multiple of $ax \leq 2$, which implies that it is facet defining for $P^{n,k}_C$.

We first determine the coefficients for edges $e \in T$ incident to a node adjacent to a leaf of T. We have the situation shown in Figure 4.4.

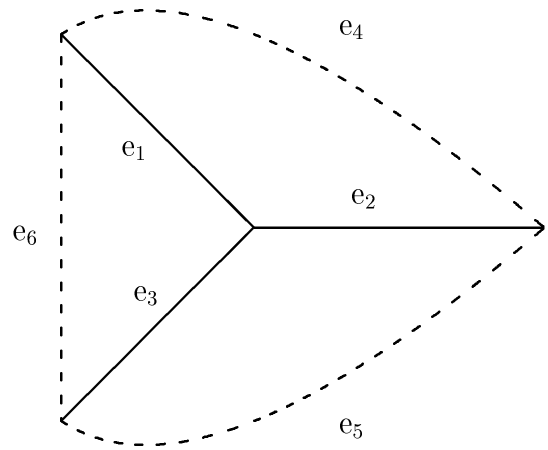

Figure 4.4: Description of Edges

The circuits $\{e_1, e_2, e_4\}, \{e_2, e_3, e_5\}, \{e_1, e_3, e_6\}, \{e_1, e_2, e_5, e_6\}, \{e_2, e_3, e_4, e_6\}$, and $\{e_1, e_3, e_4, e_5\}$ are on the face $ax \leq 2$, so

$$\begin{array}{rcl} b_{e_1} + b_{e_2} + b_{e_4} &=& b_0 \\ b_{e_2} + b_{e_3} + b_{e_5} &=& b_0 \\ b_{e_1} + b_{e_3} + b_{e_6} &=& b_0 \\ b_{e_1} + b_{e_2} + b_{e_5} + b_{e_6} &=& b_0 \\ b_{e_2} + b_{e_3} + b_{e_4} + b_{e_6} &=& b_0 \\ b_{e_1} + b_{e_3} + b_{e_4} + b_{e_5} &=& b_0. \end{array}$$

From these equations, we can deduce that $b_{e_1} = b_{e_2} = b_{e_3} = b_0/2$.

We next determine the coefficients for the remaining edges $e \in T$. If $e \in T$ and e is not incident to a node adjacent to a leaf, then we have a situation as shown in Figure 4.5.

The circuits $\{e, e_1, e_3\}, \{e, e_2, e_4\}$, and $\{e_1, e_2, e_3, e_4\}$ lie on the face $ax \leq 2$, so

$$\begin{array}{rcl} b_e + b_{e_1} + b_{e_3} &=& b_0 \\ b_e + b_{e_2} + b_{e_4} &=& b_0 \\ b_{e_1} + b_{e_2} + b_{e_3} + b_{e_4} &=& b_0 \end{array}$$

Adding the first two equations and subtracting the third yields that $b_e = b_0/2$.

We next determine the coefficients for edges $e \in E \setminus T$ with $l_e^T \leq k-1$. The fundamental circuit of e with respect to T lies on the face $ax \leq 2$, so

$$\begin{aligned} l_e^T(b_0/2) + b_e &= b_0 \Rightarrow \\ b_e &= (b_0/2)(2 - l_e^T) = (b_0/2)w_{l_e^T}. \end{aligned} \tag{4.34}$$

For edges $e \in E \setminus T$ such that $k \leq l_e^T \leq \lceil 3k/2 \rceil - 2$ consider the circuit shown in Figure 4.6.

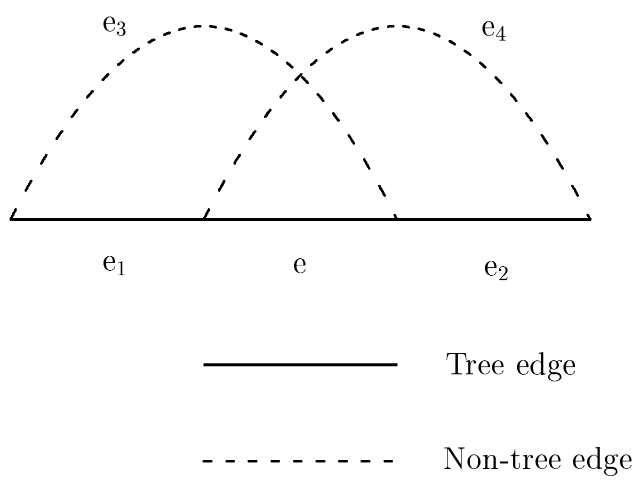

Figure 4.5: Description of Edges

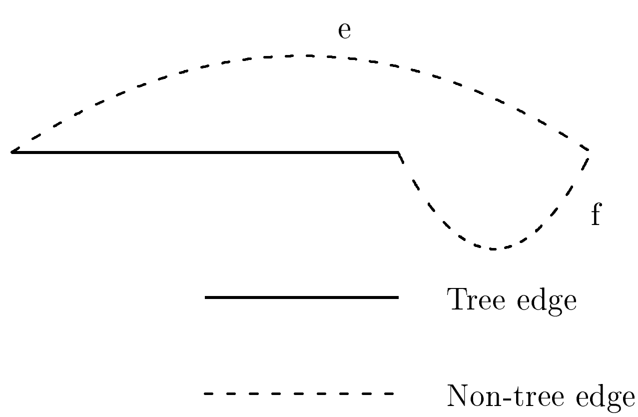

Figure 4.6: A circuit

This circuit consists of the edge e, $k-2$ edges of the P_e, and a "shortcut" edge f. Edge f has $l_f^T = l_e^T - (k-2) \leq k-1$. This circuit lies on the face $ax \leq 2$, since

$$\begin{aligned} w_{l_e}^T + w_{l_f}^T + k - 2 &= 4 - 2k + l_e^T + 2 - l_f^T + k - 2 \\ &= 4 - 2k + l_e^T + 2 - (l_e^T - (k-2)) + k - 2 \\ &= 2. \end{aligned}$$

Therefore we know that

$$b_e + b_f + (b_0/2)(k-2) = b_0. \tag{4.35}$$

Since $l_f^T \leq k-1$, we know from (4.34) that $b_f = (b_0/2)(2 - l_f^T)$. Substituting this into (4.35) gives

$$b_e = (b_0/2)(4 - 2k + l_e^T). \tag{4.36}$$

Finally, consider an edge $f_1 \in E \setminus T$ with $l_{f_1}^T \geq \lceil 3k/2 \rceil - 1$. By assumption, there is a circuit C_1, $|C_1| = k$, consisting of f_1, g_1 and edges of T, such that $l_{g_1}^T = l_{f_1}^T - 1$. We first show that this circuit lies on the face $ax \leq 2$. There are four cases to consider.

If k is odd and $l_{f_1}^T = \lceil 3k/2 \rceil - 1$, then $l_{g_1}^T = \lceil 3k/2 \rceil - 2$ and $w_{l_{g_1}}^T = 4 - 2k + l_{g_1}^T = 4 - 2k + (3k+1)/2 - 2 = (5-k)/2$. Therefore,

$$w_{l_{f_1}}^T + w_{l_{g_1}}^T + k - 2 = (3-k)/2 + (5-k)/2 + k - 2 = 2.$$

If k is odd and $l_{f_1}^T \geq \lceil 3k/2 \rceil$, then since $l_{g_1}^T = l_{f_1}^T - 1$,

$$w_{l_{f_1}}^T + w_{l_{g_1}}^T + k - 2 = (3-k)/2 + (5-k)/2 + k - 2 = 2.$$

If k is even and $l_{f_1}^T = \lceil 3k/2 \rceil - 1$, then $l_{g_1}^T = \lceil 3k/2 \rceil - 2$, and $w_{l_{g_1}}^T = 4 - 2k + l_{g_1}^T = 4 - 2k + 3k/2 - 2 = 2 - k/2$. So,

$$w_{l_{f_1}}^T + w_{l_{g_1}}^T + k - 2 = 2 - k/2 + 2 - k/2 + k - 2 = 2.$$

If k is even and $l_{f_1}^T \geq \lceil 3k/2 \rceil$, then

$$w_{l_{f_1}}^T + w_{l_{g_1}}^T + k - 2 = 2 - k/2 + 2 - k/2 + k - 2 = 2.$$

If $l_{g_1}^T \geq \lceil 3k/2 \rceil - 1$, then by assumption there exists another circuit C_2, consisting of $k-2$ edges of T, edge g_1, and an edge $g_2 \in E \setminus T$, with $l_{g_2}^T = l_{g_1}^T - 1$. C_2 is also on the face $ax \leq 2$ by the same argument that showed that C_1 was on the face $ax \leq 2$.

More generally, there is a series of circuits $\{C_1, C_2, \ldots, C_{t-1}, C_t\}$, such that the circuit C_j consists of $k-2$ edges of T and two edges $f_j, g_j \in E \setminus T$. Further, $l_{g_j}^T = l_{f_j}^T - 1$ for $j = 1 \ldots, t$, $g_j = f_{j+1}$ for $j = 1, \ldots, t-1$, and $l_{g_t} = \lceil 3k/2 \rceil - 2$. The incidence vector of C_j lies on the face $ax \leq 2$.

Since each C_j is on the face $ax \leq 2$, it must also be on the face $bx \leq b_0$, which leads us to the following series of equalities:

$$\begin{aligned} b_{f_t} + b_{g_t} + (b_0/2)(k-2) &= b_0 \\ b_{f_{t-1}} + b_{g_{t-1}} + (b_0/2)(k-2) &= b_0 \\ &\vdots \\ b_{f_1} + b_{g_1} + (b_0/2)(k-2) &= b_0. \end{aligned}$$

Substituting for the g_j, we have

$$b_{f_t} + b_{g_t} + (b_0/2)(k-2) = b_0 \quad (4.37)$$

$$b_{f_{t-1}} + b_{f_t} + (b_0/2)(k-2) = b_0 \quad (4.38)$$

$$\vdots$$

$$b_{f_1} + b_{f_2} + (b_0/2)(k-2) = b_0. \quad (4.39)$$

If k is even, then we have already determined that $b_{g_t} = (b_0/2)(4-2k+3k/2-2) = (b_0/2)(2-k/2)$. Substituting into (4.38), we get that

$$b_{f_t} = b_0 - (b_0/2)(k-2+2-k/2) = (b_0/2)(2-k/2).$$

By substituting into the equations up to (4.39) in turn, we can determine that $b_{f_1} = (b_0/2)(2-k/2)$.

If k is odd, then we know that $b_{g_t} = (b_0/2)(4-2k+(3k+1)/2-2) = (b_0/2)((5-k)/2)$. Substituting into (4.38), we get that

$$b_{f_t} = b_0 - (b_0/2)(k-2+(5-k)/2) = (b_0/2)((3-k)/2).$$

Substitution of this expression for b_{f_t} into (4.39) gives

$$b_{f_{t-1}} = b_0 - (b_0/2)(k-2+(3-k)/2) = (b_0/2)((5-k)/2).$$

Repeating this process for the remaning equations up to (4.39), we get that

$$b_{f_1} = \begin{cases} (3-k)/2 & \text{if } t \text{ is even} \\ (5-k)/2 & \text{if } t \text{ is odd.} \end{cases}$$

This is equivalent to

$$b_{f_1} = \begin{cases} (b_0/2)(3-k)/2 & \text{if } l^T_{f_1} = \lceil 3k/2 \rceil - 1 + 2i \text{ for some } i \in \mathbb{Z}_+, \\ (b_0/2)(5-k)/2 & \text{if } l^T_{f_1} = \lceil 3k/2 \rceil + 2i \text{ for some } i \in \mathbb{Z}_+. \end{cases}$$

At this point, we have shown that

$$b_e = \begin{cases} (b_0/2)(2-l^T_e) & 2 \le l^T_e \le k-1, \\ (b_0/2)(4-2k+l^T_e) & k \le l^T_e \lceil 3k/2 \rceil - 2, \\ (b_0/2)(2-k/2) & k \text{ even and } l^T_e \ge 3k/2-1, \\ (b_0/2)((3-k)/2) & k \text{ odd and } l^T_e = \lceil 3k/2 \rceil - 1 + 2i \text{ for some } i \in \mathbb{Z}_+, \\ (b_0/2)((5-k)/2) & k \text{ odd and } l^T_e = \lceil 3k/2 \rceil + 2i \text{ for some } i \in \mathbb{Z}_+, \end{cases}$$

or $b_e = (b_0/2)a_e$ for all $e \in E$. In Figure 4.4, the circuit C consisting of edges e_4, e_5, and e_6 must satisfy $bx \le b_0$, so $b_0 \ge 0$. We have therefore shown that the inequality $bx \le b_0$ is a positive multiple of $ax \le 2$, and the proof is complete. □

Using techniques found in this proof, we can show that the condition that if v is a leaf node of T with adjacent node u, then $|\delta(u)| \ge 3$ is both necessary and sufficient for the tree inequalities to be facet defining for P^n_C. This strengthens Theorem 4.5 of Wang [Wan95].

5 Conclusions

We have presented many classes of facet inducing inequalities for the polyhedron arising from an integer programming formulation of the cardinality constrained circuit problem. A branch and cut algorithm based on these inequalities is presented in [BLS01].

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