A Remarkable Property of the Dynamic Optimization Extremals*

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Abstract

We give conditions under which a function $F(t, x, u, \psi_0, \psi)$ satisfies the relation $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x}$ along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of an optimal control problem, where H is the corresponding Hamiltonian. The relation generalizes the well known fact that the equality $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}$ holds along the extremals of the problem, and that in the autonomous case $H \equiv$ *constant*. As applications of the new relation, methods for obtaining conserved quantities along the Pontryagin extremals and for characterizing problems possessing given constants of the motion are obtained.

Keywords: dynamic optimization, optimal control, Pontryagin extremals, constants of the motion.

1 Introduction

A dynamic optimization continuous problem poses the question of what is the optimal magnitude of the choice variables, at each point of time, in a given interval. To tackle such problems, three major approaches are available: dynamic programming; the calculus of variations; and the powerful

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optimal control approach. All these techniques are well known in the literature of operations research (see e.g. [3, 4, 30]), systems theory (see e.g. [13]), economics (see e.g. [8, 19] and [21, Capítulo 14]) and management sciences (see e.g [12])). Here we are concerned with the methods and procedures of optimal control. This approach allows the effective study of many optimization problems arising in such fields as engineering, astronautics, mathematics, physics, economics, business management and operations research, due to its ability to deal with restrictions on the variables and nonsmooth functions (see e.g. [12, 17, 26]).

At the core of optimal control theory is the Pontryagin maximum principle – the celebrated first order necessary optimality condition – whose solutions are called (Pontryagin) extremals and which are obtained through a function H called Hamiltonian, akin to the Lagrangian function used in ordinary calculus optimization problems (see e.g. [20, 26])). For autonomous problems of optimal control, i.e. when the Hamiltonian H does not depend explicitly on time t, a basic property of the Pontryagin extremals is the remarkable feature that the corresponding Hamiltonian is constant along the extremals (see e.g. [22, 16]). In classical mechanics this property corresponds to energy conservation (see e.g. [18, 23]), while in the calculus of variations it corresponds to the second Erdmann necessary optimality condition (see e.g. [9]). For problems of optimal control that depend upon time t explicitly (non-autonomous problems), the property amounts to the fact that the total derivative with respect to time of the corresponding Hamiltonian equals the partial derivative of the Hamiltonian with respect to time:

$$\frac{\mathrm{d}H}{\mathrm{d}t}\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \frac{\partial H}{\partial t}\left(t, x(t), u(t), \psi_0, \psi(t)\right) \tag{1}$$

(see e.g. [22, 2, 14]). This corresponds to the DuBois-Reymond necessary condition of the calculus of variations (see e.g. [7]). Recent applications, in many different contexts of the calculus of variations and optimal control, show the fundamental nature of the property (1). It has been used in [11, 1, 24] to establish Lipschitzian regularity of minimizers; in [10] to establish some existence results; and in [28, 29] to prove some generalizations of first Noether's theorem. The techniques used in the proof of the relation are also very useful, and have been applied in contexts far away from dynamic optimization (see e.g. [15]).

In this note we give conditions under which a function $F(t, x, u, \psi_0, \psi)$ satisfies the equality

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} \tag{2}$$

along the Pontryagin extremals. For $F = H$ equality (2) reduces to (1). As a corollary, we obtain a necessary and sufficient condition for $F(t, x, u, \psi_0, \psi)$ to be a constant of the motion. From it one is able to find constants of the motion that depend on the control and that are not momentum maps. The condition provides also a method for the characterization of optimal control problems with given constants of the motion. All these possibilities are illustrated with examples.

2 Preliminaries

Without loss of generality (see e.g. [2]), we will be considering the optimal control problems in Lagrange form with fixed initial time a and fixed terminal time b $(a < b)$.

2.1 Formulation of the Optimal Control Problem

The problem consists of minimize a cost functional of the form

$$J[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t))\,\mathrm{d}t\,, \tag{3}$$

called the performance index, among all the solutions of the vector differential equation

$$\dot{x}(t) = \varphi(t, x(t), u(t))\ . \tag{4}$$

The *state trajectory* $x(\cdot)$ is a n-vector absolutely continuous function

$$x(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n)\ ;$$

and the *control* $u(\cdot)$ is a r-vector measurable and bounded function satisfying the control constraint $u(t) \in \Omega$,

$$u(\cdot) \in L_\infty([a, b]; \Omega)\ .$$

The set $\Omega \subseteq \mathbb{R}^r$ is called the *control set.* In general, the problem may include some boundary conditions and state constrains, but they are not relevant for the present study: the results obtained are independent of those restrictions. We assume the functions $L : [a, b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $\varphi : [a, b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ to be continuous on $[a, b] \times \mathbb{R}^n \times \Omega$ and to have continuous derivatives with respect to t and x.

2.2 The Pontryagin Maximum Principle

We shall now formulate the celebrated Pontryagin maximum principle [22], which is a first-order necessary optimality condition. The maximum principle provides a generalization of the classical calculus of variations first-order necessary optimality conditions and can treat problems in which upper and lower bounds are imposed on the control variables – a possibility of considerable interest in operations research (see [12]).

Theorem 1 (Pontryagin maximum principle). *Let $(x(\cdot), u(\cdot))$ be a minimizer of the optimal control problem. Then there exists a nonzero pair $(\psi_0, \psi(\cdot))$, where $\psi_0 \leq 0$ is a constant and $\psi(\cdot)$ a n-vector absolutely continuous function with domain $[a, b]$, such that the following hold for almost all t on the interval $[a, b]$:*

(i) *the Hamiltonian system*

$$\begin{cases} \dot{x}(t) & = & \dfrac{\partial H\left(t, x(t), u(t), \psi_0, \psi(t)\right)}{\partial \psi}\,, \\ \dot{\psi}(t) & = & -\dfrac{\partial H\left(t, x(t), u(t), \psi_0, \psi(t)\right)}{\partial x}\,; \end{cases}$$

(ii) *the maximality condition*

$$H\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \max_{v \in \Omega} H\left(t, x(t), v, \psi_0, \psi(t)\right)\,;$$

with the Hamiltonian $H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u)$.

Definition 1. A quadruple $(x(\cdot),\ u(\cdot),\ \psi_0,\ \psi(\cdot))$ satisfying the Hamiltonian system and the maximality condition is called a (Pontryagin) extremal.

Remark 1. The maximality condition is a static optimization problem. The method of solving the optimal control problem (3)–(4) via the maximum principle consists of finding the solutions of the Hamiltonian system by the elimination of the control with the aid of the maximality condition. The required optimal solutions are found among these extremals.

The proof of the following theorem can be found, for example, in [22, 2].

Theorem 2. *If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal, then the function $H\left(t, x(t), u(t), \psi_0, \psi(t)\right)$ is an absolutely continuous function of t and satisfies the equality (1), where on the left-hand side we have the total derivative with respect to t, and on the right-hand side the partial derivative of the Hamiltonian with respect to t.*

As a particular case of Theorem 2, when the Hamiltonian does not depend explicitly on t, that is when the optimal control problem is autonomous – functions L and φ do not depend on t – then the value of the Hamiltonian evaluated along an arbitrary Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem turns out to be constant:

$$H(x(t), u(t), \psi_0, \psi(t)) \equiv const\,, \quad t \in [a, b]\,.$$

We remark that Theorem 2 is a consequence of the Pontryagin maximum principle. We shall generalize Theorem 2 in Section 3. Before, we review some facts from functional analysis needed in the proof of our result.

2.3 Facts from Functional Analysis

First we introduce the concept of an absolutely continuous function in t uniformly with respect to s.

Definition 2. Let $\phi(s, t)$ be a real valued function defined on $[a, b] \times [a, b]$. The function $\phi(s, t)$ is said to be an absolutely continuous function in t uniformly with respect to s if, given $\varepsilon > 0$, there exists $\delta > 0$, independent of s, such that for every finite collection of disjoint intervals $(a_j, b_j) \subseteq [a, b]$

$$\sum_j (b_j - a_j) \le \delta \Rightarrow \sum_j |\phi(s, b_j) - \phi(s, a_j)| \le \varepsilon \quad (s \in [a, b])\ .$$

The proof of the following two propositions can be found in [14, p. 74].

Proposition 3. *Let $F(t, x, u, \psi_0, \psi)$, $F : [a, b] \times \mathbb{R}^n \times \Omega \times \mathbb{R}_0^- \times \mathbb{R}^n \to \mathbb{R}$, be continuously differentiable with respect to t, x, ψ for u fixed, and assume that there exists a function $G(\cdot) \in L_1([a, b]; \mathbb{R})$ such that*

$$\left\|\nabla_{(t,x,\psi)} F(t, x(t), u(s), \psi_0, \psi(t))\right\| \le G(t) \quad (s, t \in [a, b])\ .$$

Then $\phi(s, t) = F(t, x(t), u(s), \psi_0, \psi(t))$ is absolutely continuous in t uniformly with respect to s on $[a, b]$.

Proposition 4. *Let $\phi(s, t)$, $\phi : [a, b] \times [a, b] \to \mathbb{R}$, be an absolutely continuous function in t uniformly with respect to s satisfying*

$$\phi(t, t) = \max_{s \in [a,b]} \phi(s, t)$$

in a set dense in $[a, b]$. Then the function $\phi(t, t)$ can be uniquely extended to a function $m(t)$ absolutely continuous on $[a, b]$.

3 Main Result

Our result is a generalization of the Theorem 2.

Theorem 5. *If $F(t,x,u,\psi_0,\psi)$ is a real valued function as in Proposition 3 and besides satisfies*

$$F(t,x(t),u(t),\psi_0,\psi(t)) = \max_{v\in\Omega} F(t,x(t),v,\psi_0,\psi(t)) \tag{5}$$

a.e. in $t\in[a,b]$ along any Pontryagin extremal $(x(\cdot),u(\cdot),\psi_0,\psi(\cdot))$ of the optimal control problem, then $t\to F(t,x(t),u(t),\psi_0,\psi(t))$ is absolutely continuous and the equality

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\cdot\frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi}\cdot\frac{\partial H}{\partial x} \tag{6}$$

holds along the extremals.

Proof. Our proof is an extension of the standard proof of Theorem 2. Let $(x(\cdot),u(\cdot),\psi_0,\psi(\cdot))$ be a Pontryagin extremal of the problem. Setting $v = u(s)$ in (5) we obtain that $\phi(s,t) = F(t,x(t),u(s),\psi_0,\psi(t))$ satisfies

$$\phi(t,t) \geq \phi(s,t)\,, \quad s\in[a,b]\,, \tag{7}$$

for t in a set of full measure on $[a,b]$. Proposition 4 then implies that $m(t) = \phi(t,t) = F(t,x(t),u(t),\psi_0,\psi(t))$ is an absolutely continuous function on $[a,b]$. It remains to prove that

$$\dot{m}(t) = \frac{\partial F}{\partial t}(\pi(t)) + \frac{\partial F}{\partial x}(\pi(t))\cdot\frac{\partial H}{\partial \psi}(\pi(t)) - \frac{\partial F}{\partial \psi}(\pi(t))\cdot\frac{\partial H}{\partial x}(\pi(t))\,,$$

where $\pi(t) = (t,x(t),u(t),\psi_0,\psi(t))$. Since

$$\frac{m(t+h)-m(t)}{h} = \frac{\phi(t+h,t+h)-\phi(t,t+h)}{h} + \frac{\phi(t,t+h)-\phi(t,t)}{h}$$

and by the hypotheses the left-hand side and the second addend on the right-hand side have a limit as $h\to 0$, one concludes that the first term on the right must have a limit as well. From (7) $\phi(t+h,t+h)\geq\phi(t,t+h)$ and it follows that $\frac{\phi(t+h,t+h)-\phi(t,t+h)}{h}$ is nonnegative when $h>0$ and nonpositive when $h<0$; thus, its limit must be zero when $h\to 0$. We obtain in this way that

$$\begin{aligned}\dot{m}(t) &= \lim_{h\to 0}\frac{F(t+h,x(t+h),u(t),\psi_0,\psi(t+h)) - F(t,x(t),u(t),\psi_0,\psi(t))}{h}\\ &= \frac{\partial F}{\partial t}(\pi(t)) + \frac{\partial F}{\partial x}(\pi(t))\cdot\dot{x}(t) + \frac{\partial F}{\partial \psi}(\pi(t))\cdot\dot{\psi}(t)\,,\end{aligned}$$

and the conclusion follows from the Hamiltonian system. ∎

Corollary 6. *Let* $F(t, x, u, \psi_0, \psi)$, $F : [a, b] \times \mathbb{R}^n \times \Omega \times \mathbb{R}_0^- \times \mathbb{R}^n \to \mathbb{R}$*, be continuously differentiable with respect to* t*,* x*,* ψ *for* u *fixed; and* $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ *be an extremal. If*

(i) $F(t, x(t), u(t), \psi_0, \psi(t))$ *is absolutely continuous in* t*;*

(ii) $F(t, x(t), u(t), \psi_0, \psi(t)) = \max_{v \in \Omega} F(t, x(t), v, \psi_0, \psi(t))$*;*

then the equality (6) *holds along the extremal.*

Possible applications of Theorem 5 follow in the next section.

4 Applications of the Main Result

Solving the Hamiltonian system by the elimination of the control with the aid of the maximality condition is typically a difficult task. Therefore it is worthwhile to look for circumstances which make the solution easier. This is the case when the extremals don't change the value of a given function. Indeed, the existence of such a function, called constant of the motion, may be used for reducing the dimension of the Hamiltonian system (see e.g. [27, Módulo 5]). In extreme cases, with a sufficiently large number of (independent) constants of the motion, one can solve the problem completely.

4.1 Constants of the Motion

From Theorem 5 one immediately obtains a necessary and sufficient condition for a function to be a constant of the motion.

Definition 3. A quantity $F(t, x, u, \psi_0, \psi)$ which is constant along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem, is called a *constant of the motion.*

Corollary 7. *Under the conditions of Theorem 5,* $F(t, x, u, \psi_0, \psi)$ *is a constant of the motion if and only if*

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} = 0 \tag{8}$$

holds along the Pontryagin extremals of the optimal control problem.

Example 1. ($n = 4$, $r = 2$, $\Omega = \mathbb{R}^2$) Let us consider the problem

$$\int_a^b \left((u_1(t))^2 + (u_2(t))^2 \right) \, dt \longrightarrow \min \,,$$

$$\begin{cases} \dot{x_1}(t) = x_3(t) \\ \dot{x_2}(t) = x_4(t) \\ \dot{x_3}(t) = -x_1(t) \left((x_1(t))^2 + (x_2(t))^2 \right) + u_1(t) \\ \dot{x_4}(t) = -x_2(t) \left((x_1(t))^2 + (x_2(t))^2 \right) + u_2(t) \,. \end{cases}$$

The corresponding Hamiltonian function is

$$H\left(x_1, x_2, x_3, x_4, u_1, u_2, \psi_0, \psi_1, \psi_2, \psi_3, \psi_4\right) = \psi_0 \left(u_1^2 + u_2^2\right) + \psi_1 x_3 \\ + \psi_2 x_4 - \psi_3 x_1 \left(x_1^2 + x_2^2\right) + \psi_3 u_1 - \psi_4 x_2 \left(x_1^2 + x_2^2\right) + \psi_4 u_2 \,.$$

We claim that

$$F = -\psi_1 x_2 + \psi_2 x_1 - \psi_3 x_4 + \psi_4 x_3 \tag{9}$$

is a constant of the motion for the problem. Direct calculations show that

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{4} \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial \psi_i} - \sum_{i=1}^{4} \frac{\partial F}{\partial \psi_i} \frac{\partial H}{\partial x_i} = \psi_4 u_1 - \psi_3 u_2 \,. \tag{10}$$

From the maximality condition it follows that $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$, that is, $2\psi_0 u_1 + \psi_3 = 0$ and $2\psi_0 u_2 + \psi_4 = 0$. Using these last two identities in (10) one concludes from Corollary 7 that (9) is a constant of the motion.

4.2 Characterization of Optimal Control Problems

We shall endeavor here to find a method to synthesize optimal control problems with given constants of the motion. If a function F is fixed *a priori*, we can regard equality (8) as a partial differential equation in the unknown Hamiltonian H. Obviously, if this differential equation admits a solution, then an optimal control problem can be constructed with the constant of the motion F. We shall illustrate the general idea in special situations.

Example 2. The Hamiltonian H is a constant of the motion if and only if $\frac{\partial H}{\partial t} = 0$. Condition is trivially satisfied for autonomous problems.

Example 3. Function $\psi x + Ht$ is a constant of the motion if and only if $H = \frac{\partial H}{\partial x} x - \frac{\partial H}{\partial \psi} \psi - \frac{\partial H}{\partial t} t$. Condition is satisfied, for example, for problems of the form $(0 < a < b)$

$$\int_a^b \frac{L\left(tx(t), u(t)\right)}{t} dt \longrightarrow \min ,$$
$$\dot{x}(t) = \frac{\varphi\left(tx(t), u(t)\right)}{t^2} .$$

Example 4. We conclude from Corollary 7 that a necessary and sufficient condition for $H\psi x$ to be a constant of the motion is

$$\psi x \frac{\partial H}{\partial t} + \psi H \frac{\partial H}{\partial \psi} - H x \frac{\partial H}{\partial x} = 0 .$$

A simple problem with constant of the motion $H\psi x$ is therefore

$$\int_a^b L\left(u(t)\right) dt \longrightarrow \min ,$$
$$\dot{x}(t) = \varphi\left(u(t)\right) x(t) .$$

Example 5. The following optimization problem is important in the study of cubic polynomials on Riemannian manifolds (see [6, p. 39] and [25]). Here we consider the particular case when one has 2-dimensional state and n controls:

$$\int_0^T \left(\left(u_1(t)\right)^2 + \cdots + \left(u_n(t)\right)^2 \right) dt \longrightarrow \min , \tag{11}$$
$$\begin{cases} \dot{x_1}(t) = x_2(t) , \\ \dot{x_2}(t) = X_1\left(x_1(t)\right) u_1(t) + \cdots + X_n\left(x_1(t)\right) u_n(t) . \end{cases}$$

Functions $X_i(\cdot)$, $i = 1, \ldots, n$, are assumed smooth. The Hamiltonian for the problem is

$$H = \psi_0 \left(u_1^2 + \cdots + u_n^2\right) + \psi_1 x_2 + \psi_2 \left(X_1(x_1) u_1 + \cdots + X_n(x_1) u_n\right) .$$

As far as the problem is autonomous, the Hamiltonian is a constant of the motion. We are interested in finding a new constant of the motion for the problem. We will look for one of the form

$$F = k_1 \psi_1 x_1 + k_2 \psi_2 x_2 ,$$

where k_1 and k_2 are constants. This is a typical constant of the motion, known in the literature by *momentum map* (see [5]). First we note that

$$\frac{\partial F}{\partial t} = 0 , \frac{\partial F}{\partial x_1} = k_1 \psi_1 , \frac{\partial F}{\partial x_2} = k_2 \psi_2 , \frac{\partial F}{\partial \psi_1} = k_1 x_1 , \frac{\partial F}{\partial \psi_2} = k_2 x_2 ,$$

and

$$\frac{\partial H}{\partial x_1} = \psi_2 \left(X_1'(x_1)u_1 + \cdots + X_n'(x_1)u_n \right) , \frac{\partial H}{\partial x_2} = \psi_1 ,$$
$$\frac{\partial H}{\partial \psi_1} = x_2 , \frac{\partial H}{\partial \psi_2} = X_1(x_1)u_1 + \cdots + X_n(x_1)u_n .$$

Substituting these quantities into (8) we obtain that

$$k_1\psi_1 x_2 + k_2\psi_2 \left(X_1(x_1)u_1 + \cdots + X_n(x_1)u_n\right) \\ - k_1 x_1 \psi_2 \left(X_1'(x_1)u_1 + \cdots + X_n'(x_1)u_n \right) - k_2 x_2 \psi_1 = 0 .$$

The equality is trivially satisfied if $k_1 = k_2$ and $X_i'(x_1)x_1 = X_i(x_1)$, $i = 1, \ldots, n$. We have just proved the following proposition.

Proposition 8. *If the homogeneity condition* $X_i(\lambda x_1) = \lambda X_i(x_1)$ *(*$i = 1, \ldots, n$*),* $\forall \lambda > 0$*, holds, then* $\psi_1(t)x_1(t)+\psi_2(t)x_2(t)$ *is constant in* $t \in [0, T]$ *along the extremals of the problem* (11)*.*

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