Global and Local Convergence of Line Search Filter Methods for Nonlinear Programming

Andreas Wächter* and Lorenz T. Biegler†

CAPD Technical Report B-01-09
Department of Chemical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213
(August 23, 2001; revised May 29, 2002)

Abstract

Line search methods for nonlinear programming using Fletcher and Leyffer's filter method, which replaces the traditional merit function, are proposed and their global and local convergence properties are analyzed. Previous theoretical work on filter methods has considered trust region algorithms and only the question of global convergence. The presented framework is applied to barrier interior point and active set SQP algorithms. Under mild assumptions it is shown that every limit point of the sequence of iterates generated by the algorithm is feasible, and that there exists at least one limit point that is a stationary point for the problem under consideration. Furthermore, it is shown that the proposed methods do not suffer from the Maratos effect if the search directions are improved by second order corrections, so that fast local convergence to strict local solutions is achieved. A new alternative filter approach employing the Lagrangian function instead of the objective function with identical global convergence properties is briefly discussed.

Keywords: nonlinear programming – nonconvex constrained optimization – filter method – line search – SQP – interior point – barrier method – global convergence – local convergence – Maratos effect – second order correction

1 Introduction

Recently, Fletcher and Leyffer [12] have proposed filter methods, offering an alternative to merit functions, as a tool to guarantee global convergence in algorithms for nonlinear programming (NLP). The underlying concept is that trial points are accepted if they improve the objective function *or* improve the constraint violation instead of a combination of those two measures defined by a merit function. The practical results reported for the filter trust region sequential quadratic programming (SQP) method in [12] are encouraging, and soon global convergence results for related algorithms were established [10, 13]. Other researchers have also proposed global convergence results for different trust region based filter methods, such as for an interior point (IP) approach [26], a bundle method for non-smooth optimization [11], and a pattern search algorithm for derivative-free optimization [1].

*IBM T.J. Watson Research Center, Yorktown Heights, NY; E-mail: `andreasw@watson.ibm.com`
†Carnegie Mellon University, Pittsburgh, PA; E-mail: `lb01@andrew.cmu.edu`

In this paper we propose and analyze a filter method framework based on line search which can be applied to barrier IP methods as well as active set SQP methods. The motivation given by Fletcher and Leyffer for the development of the filter method [12] is to avoid the necessity to determine a suitable value of the penalty parameter in the merit function. In addition, assuming that Newton directions are usually "good" directions (in particular if exact second derivative information is used) filter methods have the potential to be more efficient than algorithms based on merit functions, as they generally accept larger steps. However, in the context of a line search method, the filter approach offers another important advantage regarding robustness. It has been known for some time that line search methods can converge to "spurious solutions", infeasible points that are not even critical points for a measure of infeasibility, if the gradients of the constraints become linearly dependent at non-feasible points. In [23], Powell gives an example for this behavior. More recently, the authors demonstrated another global convergence problem for many line search IP methods on a simple well-posed example [29]. Here, the affected methods generate search directions that point outside of the region $\mathcal{I}$ defined by the inequality constraints because they are forced to satisfy the linearization of the equality constraints. Consequently, an increasingly smaller fraction of the proposed step can be taken, and the iterates eventually converge to an infeasible point at the boundary of $\mathcal{I}$, which once again is not even a stationary point for any measure of infeasibility (see also [19] for a detailed discussion of "feasibility control"). Using a filter approach within a line search algorithm helps to overcome these problems. If the trial step size becomes too small in order to guarantee sufficient progress towards a solution of the problem, the proposed filter method reverts to a feasibility restoration phase, whose goal is to deliver a new iterate that is at least sufficiently less infeasible. As a consequence, the global convergence problems described above cannot occur.

This paper is organized as follows. For easy comprehension of the derivation and analysis of the proposed line search filter methods, the main part of the paper will consider the particular case of a barrier IP method. At the very end of the paper we will show how the presented techniques can be applied to active set SQP methods. The presented barrier method can be of the primal or primal-dual type, and differs from the IP filter algorithm proposed by M. Ulbrich, S. Ulbrich, and Vicente [26] in that the barrier parameter is kept constant for several iterations. This enables us to base the acceptance of trial steps directly on the (barrier) objective function instead of only on the norm of the optimality conditions. Therefore the presented method can be expected to be less likely to converge to saddle points or maxima than the algorithm proposed in [26]. Recently, Benson, Shanno, and Vanderbei [2] proposed several heuristics based on the idea of filter methods, for which improved efficiency compared to their previous merit function approach are reported. Their approach is different from the one proposed here in many aspects, and no global convergence analysis is given.

In Section 2 we will motivate and state the algorithm for the solution of the barrier problem with a fixed value of the barrier parameter. The method is motivated by the trust region SQP method proposed by Fletcher et. al. [10]. An important difference, however, lies in the condition that determines when to switch between certain sufficient decrease criteria; this modification allows us to show fast local convergence of the proposed line search filter method.

We will then show in Section 3 that every limit point of the sequence of iterates generated by the algorithm is feasible, and that there is at least one limit point that satisfies the first order optimality conditions for the barrier problem. The assumptions made are less restrictive than those made for previously proposed line search IP methods for NLP (e.g. [9, 30, 25]).

In Section 4 the local convergence properties of the algorithm will be discussed. As Fletcher and Leyffer pointed out in [12], filter methods can also suffer from the so-called Maratos effect [18], which leads to short step sizes arbitrarily close to a solution of the problem, and hence to a poor local convergence behavior, in particular in an SQP framework. We will show that full steps for search directions, possibly improved by a second order correction, will eventually be accepted in the neighborhood of a strict local solution of the problem satisfying the usual second order optimality conditions. As a consequence, fast local convergence can be established for the solution of the barrier problem with a fixed value of the barrier parameter.

In Section 5.1 we propose an alternative measure for the filter acceptance criteria. Here, a trial point is accepted if it reduces the infeasibility or the value of the Lagrangian function (instead of the objective function). The global convergence results still hold for this modification.

Having presented the line search filter framework on the example of a barrier method we will finally show in Section 5.2 how it can be applied to SQP methods preserving the same global and local convergence properties. In Section 5.3 we briefly point out that our local convergence analysis can also be applied to a slightly modified version of the trust region filter SQP method proposed by Fletcher et. al. [10].

Notation. We will denote the i-th component of a vector $v \in \mathbb{R}^n$ by $v^{(i)}$. Norms $\|\cdot\|$ will denote a fixed vector norm and its compatible matrix norm unless otherwise noted. For brevity, we will use the convention $(x, \lambda) = (x^T, \lambda^T)^T$ for vectors x, λ. For a matrix A, we will denote by $\sigma_{\min}(A)$ the smallest singular value of A, and for a symmetric, positive definite matrix A we call the smallest eigenvalue $\lambda_{\min}(A)$. Given two vectors $v, w \in \mathbb{R}^n$, we define the convex segment $[v, w] := \{v + t(w - v) : t \in [0, 1]\}$. Finally, we will denote by $O(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta\, t_k$ for some constant $\beta > 0$ independent of k, and by $o(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta_k t_k$ for some positive sequence $\{\beta_k\}$ with $\lim_k \beta_k = 0$.

2 A Line Search Filter Approach for a Barrier Method

The algorithm that will be discussed in Sections 2, 3, and 4 is a *barrier method*. Here, we assume that the optimization problem (NLP) is stated as

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{1a}$$

$$\text{s.t.} \quad c(x) = 0 \tag{1b}$$

$$x \geq 0, \tag{1c}$$

where the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the equality constraints $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $m < n$ are sufficiently smooth. The algorithm can be changed in an obvious way if only some or none of the variables have bounds.

A barrier method solves a sequence of *barrier problems*

$$\min_{x \in \mathbb{R}^n} \quad \varphi_\mu(x) := f(x) - \mu \sum_{i=1}^{n} \ln(x^{(i)}) \tag{2a}$$

$$\text{s.t.} \quad c(x) = 0 \tag{2b}$$

for a decreasing sequence μ_l of *barrier parameters* with $\lim_l \mu_l = 0$. Local convergence of barrier methods as $\mu \to 0$ has been discussed in detail by other authors, in particular by Nash and Sofer [20] for primal methods, and by Gould, Orban, Sartenaer, and Toint [15, 16] for primal-dual methods. In those approaches, the barrier problem (2) is solved to a certain tolerance $\epsilon > 0$ for a fixed value of the barrier parameter μ. The parameter μ is then decreased and the tolerance ϵ is tightened for the next barrier problem. It is shown that if the parameters μ and ϵ are updated in a particular fashion, the new starting point, enhanced by an extrapolation step with the cost of one regular iteration, will eventually solve the next barrier problem well enough in order to satisfy the new tolerance. Then the barrier parameter μ will be decreased again immediately (without taking an additional step), leading to a superlinear convergence rate of the overall interior point algorithm for solving the original NLP (1).

Consequently, the step acceptance criterion in the solution procedure for a fixed barrier parameter μ becomes irrelevant as soon as the (extrapolated) starting points are immediately accepted. Until then, we can consider the (approximate) solution of the individual barrier problems as independent procedures (similar to the approach taken in [4] and [5]). The focus of this paper are the properties of the line search filter approach, and we will therefore only address the convergence properties of an algorithm for solving the barrier problem (2) for a *fixed* value of the barrier parameter μ, and only give some additional comments on the overall IP method in Remark 6 at the end of Section 3.

The first order optimality conditions, or Karush-Kuhn-Tucker (KKT) conditions, (see e.g. [22]) for the barrier problem with a value of the barrier parameter μ (from now on fixed) are

$$\nabla f(x) - \mu X^{-1}e + A(x)\lambda = 0 \tag{3a}$$
$$c(x) = 0 \tag{3b}$$

where $X := \mathrm{diag}(x)$, and the symbol e denotes the vector of all ones of appropriate dimension. We denote with $A(x) := \nabla c(x)$ the transpose of the Jacobian of the constraints c. The vector λ corresponds for the Lagrange multipliers for the equality constraints (2b). Since at a local solution x_*^μ of (2) we have $x_*^\mu > 0$, this inequality is enforced for all iterates, i.e.

$$x_k > 0 \tag{4}$$

for all k.

Given an initial estimate x_0 with $x_0 > 0$, the line search algorithm proposed in this paper generates a sequence of improved estimates x_k of the solution for the barrier problem (2). For this purpose in each iteration k a search direction d_k is computed from the linearization of the KKT conditions (3),

$$\begin{bmatrix} H_k & A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} d_k \\ \lambda_k^+ \end{pmatrix} = - \begin{pmatrix} \nabla f(x_k) - \mu X_k^{-1} e \\ c(x_k) \end{pmatrix}. \tag{5}$$

Here, $A_k := A(x_k)$, and H_k denotes the Hessian $\nabla^2_{xx}\mathcal{L}_\mu(x_k, \lambda_k)$ of the Lagrangian

$$\mathcal{L}_\mu(x, \lambda) := \varphi_\mu(x) + c(x)^T\lambda \tag{6}$$

of the barrier problem (2), or an approximation to it, where λ_k is some estimate of the optimal multipliers corresponding to the equality constraints (2b). λ_k^+ in (5) can be used to determine a new estimate λ_{k+1} for the next iteration. Note that also primal-dual barrier methods (see e.g. [30, 14, 26]) fall into this class. In this case we have $H_k = W_k + X_k^{-1}V_k$, with W_k being the Hessian of the Lagrangian of the *original* NLP (1) and $V_k := \mathrm{diag}(v_k)$ an approximation for some dual variables $v_k \approx \mu X_k^{-1}e$. As is common for most line search methods, we will assume that the projection of the Hessian approximation H_k onto the null space of the constraint Jacobian is sufficiently positive definite.

After a search direction d_k has been computed, a step size $\alpha_k \in (0, 1]$ is determined in order to obtain the next iterate

$$x_{k+1} := x_k + \alpha_k d_k. \tag{7}$$

The step size α_k has to be chosen so that also the next iterate satisfies the positivity requirement (4). For this purpose we determine the largest step size $\alpha_k^{\max} \in (0,1]$ that satisfies the *fraction-to-the-boundary rule*, that is

$$\alpha_k^{\max} := \max\{\alpha \in (0,1] : x_k + \alpha d_k \geq (1-\tau)x_k\} \tag{8}$$

for a fixed parameter $\tau \in (0,1)$, usually chosen close to 1.

Furthermore, we want to guarantee that ideally the sequence $\{x_k\}$ of iterates converges to a solution of the barrier problem (2). In this paper we consider a backtracking line search procedure, where a decreasing sequence of step sizes $\alpha_{k,l} \in (0, \alpha_k^{\max}]$ $(l = 0, 1, 2, \ldots)$ is tried until some acceptance criterion is satisfied. Traditionally, a trial step size $\alpha_{k,l}$ is accepted if the corresponding trial point

$$x_k(\alpha_{k,l}) := x_k + \alpha_{k,l} d_k \tag{9}$$

provides sufficient reduction of a *merit function*, such as the exact penalty function [17]

$$\phi_\rho(x) = \varphi_\mu(x) + \rho\, \theta(x) \tag{10}$$

where we define the infeasibility measure $\theta(x)$ by

$$\theta(x) = \|c(x)\|\,. \tag{11}$$

Under certain regularity assumptions it can be shown that a strict local minimum of the exact penalty function coincides with a local solution of the barrier (2) if the value of the *penalty parameter* $\rho > 0$ is chosen sufficiently large [17].

In order to avoid the determination of an appropriate value of the penalty parameter ρ, Fletcher and Leyffer [12] proposed the concept of a *filter method* in the context of a trust region SQP algorithm. In the remainder of this section we will describe how this concept can be applied to the line search barrier framework outlined above.

The underlying idea is to interpret the barrier problem (2) as a bi-objective optimization problem with two goals: minimizing the constraint violation $\theta(x)$ and minimizing the barrier function $\varphi_\mu(x)$. A certain emphasis is placed on the first measure, since a point has to be feasible in order to be an optimal solution of the barrier problem. Here, we do not require that a trial point $x_k(\alpha_{k,l})$ provides progress in a merit function such as (10), which combines these two goals as a linear combination into one single measure. Instead, the trial point $x_k(\alpha_{k,l})$ is accepted if it improves feasibility, i.e. if $\theta(x_k(\alpha_{k,l})) < \theta(x_k)$, *or* if it improves the barrier function, i.e. if $\varphi_\mu(x_k(\alpha_{k,l})) < \varphi_\mu(x_k)$. Note, that this criterion is less demanding than the enforcement of decrease in the penalty function (10) and will in general allow larger steps.

Of course, this simple concept is not sufficient to guarantee global convergence. Several precautions have to be added as we will outline in the following; these are closely related to those proposed in [10]. (The overall line search filter algorithm is formally stated on page 9.)

1. *Sufficient Reduction.* Line search methods that use a merit function ensure *sufficient* progress towards the solution. For example, they may do so by enforcing an Armijo condition for the exact penalty function (10) (see e.g. [22]). Here, we borrow the idea from [10, 13] and replace this condition by requiring that the next iterate provides at least as much progress in one of the measures θ or φ_μ that corresponds to a small fraction of the current constraint violation, $\theta(x_k)$. More precisely, for fixed constants $\gamma_\theta, \gamma_\varphi \in (0,1)$, we say that a trial step size $\alpha_{k,l}$ provides sufficient reduction with respect to the current iterate x_k, if

$$\theta(x_k(\alpha_{k,l})) \leq (1-\gamma_\theta)\theta(x_k) \tag{12a}$$

$$\text{or} \quad \varphi_\mu(x_k(\alpha_{k,l})) \leq \varphi_\mu(x_k) - \gamma_\varphi \theta(x_k). \tag{12b}$$

In a practical implementation, the constants $\gamma_\theta, \gamma_\varphi$ typically are chosen to be small. However, relying solely on this criterion would allow the acceptance of a sequence $\{x_k\}$ that always provides sufficient reduction with respect to the constraint violation (12a) and converges to a feasible, but non-optimal point. In order to prevent this, we change to a different sufficient reduction criterion whenever for the current trial step size $\alpha_{k,l}$ the *switching condition*

$$m_k(\alpha_{k,l}) < 0 \qquad \text{and} \qquad [-m_k(\alpha_{k,l})]^{s_\varphi}[\alpha_{k,l}]^{1-s_\varphi} > \delta\,[\theta(x_k)]^{s_\theta} \tag{13}$$

holds with fixed constants $\delta > 0, s_\theta > 1, s_\varphi > 2s_\theta$, where

$$m_k(\alpha) := \alpha \nabla\varphi_\mu(x_k)^T d_k \tag{14}$$

is the linear model of the barrier function φ_μ into direction d_k. We choose to formulate the switching condition (13) in terms of a general model $m_k(\alpha)$ as it will allow us later, in Section 5.1, to define the algorithm for an alternative measure that replaces "$\varphi_\mu(x)$".

If the switching condition (13) holds, instead of insisting on (12), we require that an Armijo-type condition for the barrier function,

$$\varphi_\mu(x_k(\alpha_{k,l})) \leq \varphi_\mu(x_k) + \eta_\varphi m_k(\alpha_{k,l}), \tag{15}$$

is satisfied (see [10]). Here, $\eta_\varphi \in (0, \frac{1}{2})$ is a fixed constant. It is possible that for several trial step sizes $\alpha_{k,l}$ with $l = 1, \ldots, \tilde{l}$ condition (13), but not (15) is satisfied. In this case we note that for smaller step sizes the switching condition (13) may no longer be valid, so that the method reverts to the acceptance criterion (12).

The switching condition (13) deserves some discussion. On the one hand, for global convergence we need to ensure that close to a feasible but non-optimal point $\bar{x}$ a new iterate indeed leads to progress in the objective function (and not only the infeasibility measure). Lemma 2 below will show that $m_k(\alpha) \leq -\alpha\epsilon$ for some $\epsilon > 0$ and all $\alpha \in (0, 1]$ for iterates x_k in a neighborhood of $\bar{x}$. Therefore, the switching condition is satisfied, if $\alpha_{k,l} > (\delta/\epsilon^{s_\varphi})[\theta(x_k)]^{s_\theta}$. The fact that the right hand side is $o(\theta(x_k))$ allows us to show in Lemma 10 that sufficient decrease in the objective function (15) is indeed obtained by the new iterate close to $\bar{x}$. On the other hand, in order to show that full steps are taken in the neighborhood of a strict local solution x_*^μ we need to ensure that then the Armijo condition (15) is only enforced (i.e. the switching condition is only true) if the progress predicted by the linear model m_k is large enough so that the full step, possibly improved by a second order correction step, is accepted. This is shown in Lemma 14 below, and it is crucial for its proof that the switching condition with $\alpha_{k,0} = 1$ implies $\theta(x_k) = O(\|d_k\|^{\frac{s_\varphi}{s_\theta}}) = o(\|d_k\|^2)$. Note that the switching conditions used in [10, 13] do not imply this latter relationship (see also Section 5.3).

2. *Filter as taboo-region.* It is also necessary to avoid cycling. For example, this may occur between two points that alternatingly improve one of the measures, θ and φ_μ, and worsen the other one. For this purpose, Fletcher and Leyffer [12] propose to define a "taboo region" in the half-plane $\{(\theta, \varphi_\mu) \in \mathbb{R}^2 : \theta \geq 0\}$. They maintain a list of $(\theta(x_p), \varphi_\mu(x_p))$-pairs (called *filter*) corresponding to (some of) the previous iterates x_p and require that a point, in order to be accepted, has to improve at least one of the two measures compared to those previous iterates. In other words, a trial step $x_k(\alpha_{k,l})$ can only be accepted, if

$$\begin{aligned} \theta(x_k(\alpha_{k,l})) &< \theta(x_p) \\ \text{or} \qquad \varphi_\mu(x_k(\alpha_{k,l})) &< \varphi_\mu(x_p) \end{aligned}$$

for all $(\theta(x_p), \varphi_\mu(x_p))$ in the current filter.

In contrast to the notation in [12, 10], for the sake of a simplified notation we will define the filter in this paper not as a list but as a *set* $\mathcal{F}_k \subseteq [0,\infty) \times \mathbb{R}$ containing *all* (θ, φ_μ)-pairs that are "prohibited" in iteration k. We will say, that a trial point $x_k(\alpha_{k,l})$ is *acceptable to the filter* if its (θ, φ_μ)-pair does not lie in the taboo-region, i.e. if

$$\Big(\theta(x_k(\alpha_{k,l})), \varphi_\mu(x_k(\alpha_{k,l}))\Big) \notin \mathcal{F}_k. \tag{16}$$

During the optimization we will make sure that the current iterate x_k is always acceptable to the current filter $\mathcal{F}_k$.

At the beginning of the optimization, the filter is initialized to be empty, $\mathcal{F}_0 := \emptyset$, or — if one wants to impose an explicit upper bound on the constraint violation — as $\mathcal{F}_0 := \{(\theta, \varphi) \in \mathbb{R}^2 : \theta \geq \theta_{\max}\}$ for some $\theta_{\max} > \theta(x_0)$. Throughout the optimization the filter is then augmented in some iterations after the new iterate x_{k+1} has been accepted. For this, the updating formula

$$\mathcal{F}_{k+1} := \mathcal{F}_k \cup \Big\{(\theta, \varphi) \in \mathbb{R}^2 : \theta \geq (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad \varphi \geq \varphi_\mu(x_k) - \gamma_\varphi \theta(x_k)\Big\} \tag{17}$$

is used (see also [10]). If the filter is not augmented, it remains unchanged, i.e. $\mathcal{F}_{k+1} := \mathcal{F}_k$. Note, that then $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all k. This ensures that *all* later iterates will have to provide sufficient reduction with respect to x_k as defined by criterion (12), if the filter has been augmented in iteration k. Note, that for a practical implementation it is sufficient to store the "corner entries"

$$\Big((1-\gamma_\theta)\theta(x_k),\ \varphi_\mu(x_k) - \gamma_\varphi \theta(x_k)\Big) \tag{18}$$

(see [10]).

It remains to decide which iterations should augment the filter. Since one motivation of the filter method is to make it generally less conservative than a penalty function approach, we do not want to augment the filter in every iteration. In addition, as we will see in the discussion of the next safeguard below, it is important for the proposed method that we never include feasible points in the filter. The following rule from [10] is motivated by these considerations.

We will always augment the filter if for the accepted trial step size $\alpha_{k,l}$ the switching condition (13) or the Armijo condition (15) do not hold. Otherwise, if the filter is not augmented, the value of the barrier objective function is strictly decreased (see Eq. (37) below). To see that this indeed prevents cycling let us assume for a moment that the algorithms generates a cycle of length l

$$x_K, x_{K+1}, \ldots, x_{K+l-1}, x_{K+l} = x_K, x_{K+l+1} = x_{K+1}, \ldots \tag{19}$$

Since a point x_k can never be reached again if the filter is augmented in iteration k, the existence of a cycle would imply that the filter is not augmented for all $k \geq K$. However, this would imply that $\varphi_\mu(x_k)$ is a strictly decreasing sequence for $k \geq K$, giving a contradiction, so that (19) cannot be a cycle.

3. *Feasibility restoration phase.* If the linear system (5) is consistent, d_k satisfies the linearization of the constraints and we have $\theta(x_k(\alpha_{k,l})) < \theta(x_k)$ whenever $\alpha_{k,l} > 0$ is sufficiently small. It is not guaranteed, however, that there exists a trial step size $\alpha_{k,l} > 0$ that indeed provides *sufficient* reduction as defined by criterion (12). Furthermore, if the search direction d_k points outside of the non-negative orthant $\{x \in \mathbb{R} : x \geq 0\}$ and x_k is close to the boundary of this region, it is possible (e.g. in the example problem in [29]) that the first trial step size $\alpha_{k,0} = \alpha_k^{\max}$ with $\alpha_k^{\max}$ from (8) is already too small to allow sufficient decrease in θ and φ_μ.

In this situation, where no admissible step size can be found, the method switches to a *feasibility restoration phase*, whose purpose is to find a new iterate x_{k+1} merely by decreasing the constraint violation θ, so that x_{k+1} satisfies (12) and is also acceptable to the current filter. In this paper, we do not specify the particular procedure for this feasibility restoration phase. It could be any iterative algorithm for decreasing θ, possibly ignoring the objective function, and different methods could even be used at different stages of the optimization procedure.

Since we will make sure that a feasible iterate is never included in the filter, the algorithm for the feasibility restoration phase usually should be able to find a new acceptable iterate $x_{k+1} > 0$ unless it converges to a stationary point of θ. The latter case may be important information for the user, as it indicates that the problem seems (at least locally) infeasible. If the feasibility restoration phase terminates successfully by delivering a new admissible iterate $x_{k+1} > 0$, the filter is augmented according to (17) to avoid cycling back to the problematic point x_k.

In order to detect the situation where no admissible step size can be found and the restoration phase has to be invoked, we propose the following rule. Consider the case when the current trial step size $\alpha_{k,l}$ is still large enough so that the switching condition (13) holds for some $\alpha \leq \alpha_{k,l}$. In this case, we will not switch to the feasibility restoration phase, since there is still the chance that a shorter step length might be accepted by the Armijo condition (15). Therefore, we can see from the switching condition (13) and the definition of m_k (14) that we do not want to revert to the feasibility restoration phase if $\nabla\varphi_\mu(x_k)^T d_k < 0$ and

$$\alpha_{k,l} > \frac{\delta[\theta(x_k)]^{s_\theta}}{[-\nabla\varphi_\mu(x_k)^T d_k]^{s_\varphi}}.$$

However, if the switching condition (13) is not satisfied for the current trial step size $\alpha_{k,l}$ and all shorter trial step sizes, then the decision whether to switch to the feasibility restoration phase is based on the linear approximations

$$\theta(x_k + \alpha d_k) = \theta(x_k) - \alpha\theta(x_k) + O(\alpha^2) \quad (\text{since } A_k^T d_k + c(x_k) = 0) \tag{20a}$$

$$\varphi_\mu(x_k + \alpha d_k) = \varphi_\mu(x_k) + \alpha\nabla\varphi_\mu(x_k)^T d_k + O(\alpha^2) \tag{20b}$$

of the two measures. This predicts that the sufficient decrease condition for the infeasibility measure (12a) may not be satisfied for step sizes satisfying $\alpha_{k,l} \leq \gamma_\theta$. Similarly, in case $\nabla\varphi_\mu(x_k)^T d_k < 0$, the sufficient decrease criterion for the barrier function (12b) may not be satisfied for step sizes satisfying

$$\alpha_{k,l} \leq \frac{\gamma_\varphi\theta(x_k)}{-\nabla\varphi_\mu(x_k)^T d_k}.$$

We can summarize this in the following formula for a minimal trial step size

$$\alpha_k^{\min} := \gamma_\alpha \cdot \begin{cases} \min\left\{\gamma_\theta, \frac{\gamma_\varphi\theta(x_k)}{-\nabla\varphi_\mu(x_k)^T d_k}, \frac{\delta[\theta(x_k)]^{s_\theta}}{[-\nabla\varphi_\mu(x_k)^T d_k]^{s_\varphi}}\right\} & \\ \qquad\qquad \text{if } \nabla\varphi_\mu(x_k)^T d_k < 0 & \\ \gamma_\theta \qquad \text{otherwise} & \end{cases} \tag{21}$$

and switch to the feasibility restoration phase when $\alpha_{k,l}$ becomes smaller than $\alpha_k^{\min}$. Here, $\gamma_\alpha \in (0, 1]$ is a safety-factor that might be useful in a practical implementation in order to compensate for the neglected higher order terms in the linearization (20) and to avoid invoking the feasibility restoration phase unnecessarily.

It is possible, however, to employ more sophisticated rules to decide when to switch to the feasibility restoration phase while still maintaining the convergence properties. These rules could, for example, be based on higher order approximations of θ and/or φ_μ. We only need to ensure that the algorithm does not switch to the feasibility restoration phase as long as (13) holds for a step size $\alpha \leq \alpha_{k,l}$ where $\alpha_{k,l}$ is the current trial step size, and that the backtracking line search procedure is finite, i.e. it eventually either delivers a new iterate x_{k+1} or reverts to the feasibility restoration phase.

The proposed method also allows to switch to the feasibility restoration phase in any iteration, in which the infeasibility $\theta(x_k)$ is not too small. For example, this might be necessary, when the Jacobian of the constraints A_k^T is (nearly) rank-deficient, so that the linear system (5) is (nearly) singular and no search direction can be computed.

We are now ready to formally state the overall algorithm for solving the barrier problem (2) (for a fixed value of the barrier parameter μ).

Algorithm I

Given: Starting point $x_0 > 0$; constants $\theta_{\max} \in (\theta(x_0), \infty]$; $\gamma_\theta, \gamma_\varphi \in (0, 1)$; $\delta > 0$; $\gamma_\alpha \in (0, 1]$; $s_\theta > 1$; $s_\varphi > 2s_\theta$; $0 < \tau_1 \leq \tau_2 < 1$.

1. *Initialize.*

   Initialize the filter $\mathcal{F}_0 := \{(\theta, \varphi) \in \mathbb{R}^2 : \theta \geq \theta_{\max}\}$ and the iteration counter $k \leftarrow 0$.

2. *Check convergence.*

   Stop, if x_k is a local solution (or at least stationary point) of the barrier problem (2), i.e. if it satisfies the KKT conditions (3) for some $\lambda \in \mathbb{R}^m$.

3. *Compute search direction.*

   Compute the search direction d_k from the linear system (5). If this system is (almost) singular, go to feasibility restoration phase in Step 9.

4. *Apply fraction-to-the-boundary rule.*

   Compute the maximal step size $\alpha_k^{\max}$ from (8).

5. *Backtracking line search.*

   5.1. *Initialize line search.*
   Set $\alpha_{k,0} = \alpha_k^{\max}$ and $l \leftarrow 0$.

   5.2. *Compute new trial point.*
   If the trial step size becomes too small, i.e. $\alpha_{k,l} < \alpha_k^{\min}$ with $\alpha_k^{\min}$ defined by (21), go to the feasibility restoration phase in Step 9.
   Otherwise, compute the new trial point $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l} d_k$.

   5.3. *Check acceptability to the filter.*
   If $x_k(\alpha_{k,l}) \in \mathcal{F}_k$, reject the trial step size and go to Step 5.5.

   5.4. *Check sufficient decrease with respect to current iterate.*

   5.4.1. *Case I. The switching condition (13) holds:*
   If the Armijo condition for the barrier function (15) holds, accept the trial step and go to Step 6.
   Otherwise, go to Step 5.5.

5.4.2. *Case II. The switching condition (13) is not satisfied:*
If (12) holds, accept the trial step and go to Step 6.
Otherwise, go to Step 5.5.

5.5. *Choose new trial step size.*
Choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l \leftarrow l+1$, and go back to Step 5.2.

6. *Accept trial point.*

   Set $\alpha_k := \alpha_{k,l}$ and $x_{k+1} := x_k(\alpha_k)$.

7. *Augment filter if necessary.*

   If one of the conditions (13) or (15) does not hold, augment the filter according to (17); otherwise leave the filter unchanged, i.e. set $\mathcal{F}_{k+1} := \mathcal{F}_k$.

   (Note, that Step 5.3 and Step 5.4.2 ensure, that $(\theta(x_{k+1}), \varphi_\mu(x_{k+1})) \notin \mathcal{F}_{k+1}$.)

8. *Continue with next iteration.*

   Increase the iteration counter $k \leftarrow k+1$ and go back to Step 2.

9. *Feasibility restoration phase.*

   Compute a new iterate $x_{k+1} > 0$ by decreasing the infeasibility measure θ, so that x_{k+1} satisfies the sufficient decrease conditions (12) and is acceptable to the filter, i.e. $(\theta(x_{k+1}), \varphi_\mu(x_{k+1})) \notin \mathcal{F}_k$. Augment the filter according to (17) (for x_k) and continue with the regular barrier iteration in Step 8.

Remark 1 *From Step 5.5 it is clear that* $\lim_l \alpha_{k,l} = 0$*. In the case that* $\theta(x_k) > 0$ *it can be seen from (21) that* $\alpha_k^{\min} > 0$*. Therefore, the algorithm will either accept a new iterate in Step 5.4, or switch to the feasibility restoration phase. If on the other hand* $\theta(x_k) = 0$ *and the algorithm does not stop in Step 2 at a KKT point, then the positive definiteness of* H_k *on the null space of* A_k *implies that* $\nabla\varphi_\mu(x_k)^T d_k < 0$ *(see e.g. Lemma 4). Therefore,* $\alpha_k^{\min} = 0$*, and the Armijo condition (15) is satisfied for a sufficiently small step size* $\alpha_{k,l}$*, i.e. a new iterate will be accepted in Step 5.4.1. Overall, we see that the inner loop in Step 5 will always terminate after a finite number of trial steps, and the algorithm is well-defined.*

Remark 2 *The mechanisms of the filter ensure that* $(\theta(x_k), \varphi_\mu(x_k)) \notin \mathcal{F}_k$ *for all* k*. Furthermore, the initialization of the filter in Step 1 and the update rule (17) imply that for all* k *the filter has the following property.*

$$(\bar{\theta}, \bar{\varphi}) \notin \mathcal{F}_k \quad \Longrightarrow \quad (\theta, \varphi) \notin \mathcal{F}_k \text{ if } \theta \leq \bar{\theta} \text{ and } \varphi \leq \bar{\varphi}. \tag{22}$$

Remark 3 *For practical purposes, it might not be efficient to restrict the step size by enforcing an Armijo-type decrease (15) in the objective function, if the current constraint violation is not small. It is possible to change the switching rule (i.e. Step 5.4) so that (15) only has to be satisfied whenever* $\theta(x_k) \leq \theta_{\text{sml}}$ *for some* $\theta_{\text{sml}} > 0$ *without affecting the convergence properties of the method [28].*

Remark 4 *The proposed method has many similarities with the trust region filter SQP method proposed and analyzed in [10]. As pointed out above, we chose a modified switching rule (13) in order to be able to show fast local convergence in Section 4. Further differences result from the fact, that the proposed method follows a line search approach, so that in contrast to [10] the* *actual step taken does not necessarily satisfy the linearization of the constraints, i.e. we might have $A_k^T(x_k - x_{k+1}) \neq c(x_k)$ in some iterations. As a related consequence, the condition when to switch to the feasibility restoration phase in Step 5.2 could not be chosen to be the detection of infeasibility of the trust region QP, but had to be defined by means of a minimal step size (21). Finally, since we are considering a barrier method, the inequality constraints (1c), that are only implicitly present in the barrier problem (2) and are handled by means of the fraction-to-the-boundary rule (8), allow certain quantities, such as $\nabla\varphi_\mu(x_k)$, to potentially become unbounded. Due to these differences, the global convergence analysis presented in [10] does not apply to the proposed line search filter method.*

Notation. In the remainder of this paper we will denote the set of indices of those iterations, in which the filter has been augmented according to (17), by $\mathcal{A} \subseteq \mathbb{N}$; i.e.

$$\mathcal{F}_k \subsetneqq \mathcal{F}_{k+1} \qquad \Longleftrightarrow \qquad k \in \mathcal{A}.$$

The set $\mathcal{R} \subseteq \mathbb{N}$ will be defined as the set of all iteration indices in which the feasibility restoration phase is invoked. Since Step 9 makes sure that the filter is augmented in every iteration in which the restoration phase is invoked, we have $\mathcal{R} \subseteq \mathcal{A}$. We will denote with $\mathcal{R}_{\text{inc}} \subseteq \mathcal{R}$ the set of those iteration counters, in which the linear system (5) is too ill-conditioned or singular, so that the restoration phase is invoked from Step 3.

3 Global Convergence

3.1 Assumptions and Preliminary Results

Let us first state the assumptions necessary for the global convergence analysis of Algorithm I. Since the barrier objective function (2a) and its derivatives become unbounded as x_k approaches the boundary of the non-negative orthant $\{x \in \mathbb{R}^n : x \geq 0\}$, it is more convenient to scale the first rows and columns of the linear system (5) by X_k to obtain

$$\begin{bmatrix} \tilde{H}_k & \tilde{A}_k \\ \tilde{A}_k^T & 0 \end{bmatrix} \begin{pmatrix} \tilde{d}_k \\ \lambda_k^+ \end{pmatrix} = -\begin{pmatrix} X_k \nabla f(x_k) - \mu e \\ c(x_k) \end{pmatrix}, \tag{23}$$

where $\tilde{A}_k := \tilde{A}(x_k)$ with $\tilde{A}(x) := XA(x)$, $\tilde{d}_k := X_k^{-1} d_k$, and $\tilde{H}_k := X_k H_k X_k$.

We first state the assumptions in technical terms, and will discuss afterwards their practical relevance.

Assumptions G. *Let $\{x_k\}$ be the sequence generated by Algorithm I, where we assume that the feasibility restoration phase in Step 9 always terminates successfully and that the algorithm does not stop in Step 2 at a first-order optimal point.*

(G1) *There exists an open set $\mathcal{C} \subseteq \mathbb{R}^n$ with $[x_k, x_k + \alpha_k^{\max} d_k] \subseteq \mathcal{C}$ for all $k \notin \mathcal{R}_{\text{inc}}$, so that f and c are differentiable on $\mathcal{C}$, and their function values, as well as their first derivatives, are bounded and Lipschitz-continuous over $\mathcal{C}$.*

(G2) *The iterates are bounded, i.e. there exists $M_x > 0$ with $\|x_k\| \leq M_x$ for all k.*

(G3) *The matrices H_k approximating the Hessian of the Lagrangian in (5) are uniformly bounded for all $k \notin \mathcal{R}_{\text{inc}}$.*

(G4) *There exists a constant* $\theta_{\rm inc}$*, so that* $k \notin \mathcal{R}_{\rm inc}$ *whenever* $\theta(x_k) \leq \theta_{\rm inc}$*, i.e. the linear system (5) is "sufficiently consistent" and the restoration phase is not invoked from Step 3 close to feasible points.*

(G5) *There exists a constant* $M_A > 0$*, so that for all* $k \notin \mathcal{R}_{\rm inc}$ *we have*

$$\sigma_{\min}(\tilde{A}_k) \geq M_A.$$

(G6) *The scaled Hessian approximations* $\tilde{H}_k$ *are uniformly positive definite on the null space of the scaled Jacobian* $\tilde{A}_k^T$*. In other words, there exists a constant* $M_H > 0$*, so that for all* $k \notin \mathcal{R}_{\rm inc}$

$$\lambda_{\min}\left(\tilde{Z}_k^T \tilde{H}_k \tilde{Z}_k\right) \geq M_H, \tag{24}$$

where the columns of $\tilde{Z}_k \in \mathbb{R}^{n\times(n-m)}$ *form an orthonormal basis matrix of the null space of* $\tilde{A}_k^T$*.*

Assumptions (G1) and (G3) merely establish smoothness and boundedness of the problem data. Assumption (G2) may be considered rather strong since it explicitly excludes divergence of the iterates. In particular in an interior point framework this might constitute a problematic issue. However, it is necessary in our analysis to make this assumption as it guarantees that the barrier objective function (2a) is bounded below. Note, that this assumption is also made in [9, 26, 30, 31].

As we will see later in Lemma 2, Assumption (G6) ensures a certain descent property and it is similar to common assumptions on the reduced Hessian in SQP line search methods (see e.g. [22]). To guarantee this requirement in a practical implementation, one could compute a QR-factorization of $\tilde{A}_k$ to obtain matrices $\tilde{Z}_k \in \mathbb{R}^{n\times(n-m)}$ and $\tilde{Y}_k \in \mathbb{R}^{n\times m}$ so that the columns of $[\tilde{Z}_k\ \tilde{Y}_k]$ form an orthonormal basis of $\mathbb{R}^n$, and the columns of $\tilde{Z}_k$ are a basis of the null space of $\tilde{A}_k^T$ (see e.g. [14]). Then, the overall scaled search direction can be decomposed into two orthogonal components,

$$\tilde{d}_k = q_k + p_k, \quad \text{where} \tag{25a}$$

$$q_k := \tilde{Y}_k \bar{q}_k \quad \text{and} \quad p_k := \tilde{Z}_k \bar{p}_k, \tag{25b}$$

with

$$\bar{q}_k := -\left[\tilde{A}_k^T \tilde{Y}_k\right]^{-1} c(x_k) \tag{26a}$$

$$\bar{p}_k := -\left[\tilde{Z}_k^T \tilde{H}_k \tilde{Z}_k\right]^{-1} \tilde{Z}_k^T \left(X_k \nabla f(x_k) - \mu e + \tilde{H}_k q_k\right) \tag{26b}$$

(see e.g. [22]). The eigenvalues for the reduced scaled Hessian in (26b) (the term in square brackets) could be monitored and modified if necessary. However, this procedure is prohibitive for large-scale problems, and in those cases one instead might employ heuristics to ensure at least positive definiteness of the reduced Hessian, for example, by monitoring and possibly modifying the inertia of the iteration matrix in (5) or (23) (see e.g. [27]). Note, on the other hand, that (24) holds in the neighborhood of a local solution x_*^μ satisfying the sufficient second order optimality conditions (see Assumption (L2) in Section 4), if H_k approaches the exact Hessian of the Lagrangian of the barrier problem (2). Then, close to x_*^μ, no eigenvalue correction will be necessary and fast local convergence can be expected, assuming that full steps will be taken close to x_*^μ. The question, how the eigenvalue corrections (and the constant M_H in (24)) have to be chosen as μ is driven to zero in the overall interior point method, is beyond the scope of this paper.

The regularity requirement (G5) ensures that, whenever the scaled gradients of the constraints become (nearly) linearly dependent, the method has to switch to the feasibility restoration phase in Step 3. In practice one could monitor the singular values of $\tilde{Y}_k^T \tilde{A}_k$ in (26a), which are identical to the singular values of $\tilde{A}$, as a criterion when to switch to the restoration phase in Step 3.

Note that for $x \geq 0$, rank-deficiency of the scaled Jacobian $XA(x)$, i.e. $\sigma_{\min}(XA(x)) = 0$, is equivalent to the statement that the gradients of the equality constraints and of the bound constraints active at x,

$$\nabla c_1(x), \ldots, \nabla c_m(x), \qquad \text{and} \qquad e_i \text{ for } i \in \{j : x^{(j)} = 0\}, \tag{27}$$

are linearly dependent. With this in mind we can replace Assumptions (G4) and (G5) by the following assumption.

(G5*) *At all* feasible *points x the gradients of the active constraints (27) are linearly independent.*

If (G5*) holds, there exists constants $b_1, b_2 > 0$, so that

$$\theta(x_k) \leq b_1 \qquad \Longrightarrow \qquad \sigma_{\min}(\tilde{A}_k) \geq b_2$$

due to the continuity of $\sigma_{\min}(XA(x))$ as a function of x and the boundedness of the iterates. If we now decide to invoke the feasibility restoration phase in Step 3 whenever $\sigma_{\min}(\tilde{A}_k) \leq b_3\theta(x_k)$ for some fixed constant $b_3 > 0$, then Assumptions (G4) and (G5) hold.

In contrast to most previously analyzed interior point methods for general nonlinear programming (with the exception of [4]), this allows the treatment of degenerate constraints at non-feasible points. Assumption (G5*) is considerably less restrictive than those made in the analysis of [9, 26, 30, 31], where it is essentially required that the gradients of all equality constraints and active inequality constraints (27) are linearly independent at *all* points, and not only at all *feasible* points. The assumptions made in [25] are weaker than this, but still require at all points linear independence of the gradients of all *active* equality and inequality constraints, also at infeasible points. Also note that Assumption (G5*) is satisfied in the problematic example presented by the authors in [29].

Similar to the analysis in [10], we will make use of a *first order criticality measure* $\chi(x_k) \in [0, \infty]$ with the property that, if a subsequence $\{x_{k_i}\}$ of iterates with $\chi(x_{k_i}) \to 0$ converges to a feasible limit point x_*^μ, then x_*^μ corresponds to a first order optimal solution (assuming that certain constraint qualifications such as linear independence of the constraint gradients hold at x_*^μ; see Assumption (G5*)). In the case of the barrier method Algorithm I this means that there exist λ_*^μ, so that the KKT conditions (3) are satisfied for (x_*^μ, λ_*^μ).

For the convergence analysis of the barrier method we will define the criticality measure for iterations $k \notin \mathcal{R}_{\text{inc}}$ as

$$\chi(x_k) := \|\bar{p}_k\|_2\,, \tag{28}$$

with $\bar{p}_k$ from (26b). Note that this definition is unique, since p_k in (25a) is unique due to the orthogonality of $\tilde{Y}_k$ and $\tilde{Z}_k$, and since $\|\bar{p}_k\|_2 = \|p_k\|_2$ due to the orthonormality of $\tilde{Z}_k$. For completeness, we may define $\chi(x_k) := \infty$ for $k \in \mathcal{R}_{\text{inc}}$.

In order to see that $\chi(x_k)$ defined in this way is indeed a criticality measure under Assumptions G, let us consider a subsequence of iterates $\{x_{k_i}\}$ with $\lim_i \chi(x_{k_i}) = 0$ and $\lim_i x_{k_i} = x_*^\mu$ for some feasible limit point $x_*^\mu \geq 0$. From Assumption (G4) we then have $k_i \notin \mathcal{R}_{\text{inc}}$ for i sufficiently large. Furthermore, from Assumption (G5) and (26a) we have $\lim_i \bar{q}_{k_i} = 0$, and then from $\lim_i \chi(x_{k_i}) = 0$, (28), (26b), and Assumption (G6) we have that

$$\lim_{i\to\infty} \|\tilde{Z}_{k_i}^T \left(X_{k_i} \nabla f(x_{k_i}) - \mu e\right)\| = \lim_{i\to\infty} \|\tilde{Z}_{k_i}^T X_{k_i} \nabla \varphi_\mu(x_{k_i})\| = 0. \tag{29}$$

$Z_{k_i} := X_{k_i}^{-1}\tilde{Z}_{k_i}$ is a null space matrix of the unscaled Jacobian $A_{k_i}^T$. If $x_*^\mu > 0$, then $X_{k_i}^{-1}$ is uniformly bounded, and from (29) we have $\lim_i \|Z_{k_i}^T \nabla\varphi_\mu(x_{k_i})\| = 0$, which is a well-known optimality measure (see e.g. [22]).

However, we also need to consider the possibility that the l-th component $(x_*^\mu)^{(l)}$ of the limit point x_*^μ is zero. Since $\tilde{Y}_{k_i}$ and $\tilde{Z}_{k_i}$ in (25b) have been chosen to be orthogonal and $\tilde{Z}_{k_i}$ is an orthonormal basis of the null space of $\tilde{A}_{k_i}^T$, premultiplying a vector by $\tilde{Z}_{k_i}^T$ gives the orthogonal projection of this vector onto the null space of the scaled Jacobian $\tilde{A}_k^T$ in the scaled space. Therefore, we can write (29) equivalently as

$$\lim_{i\to\infty} \left\| \left((I - \tilde{A}_{k_i} \left[\tilde{A}_{k_i}^T \tilde{A}_{k_i}\right]^{-1} \tilde{A}_{k_i}^T \right) (X_{k_i}\nabla f(x_{k_i}) - \mu e) \right\| = 0.$$

Rearranging terms we then obtain

$$\lim_{i\to\infty} X_{k_i} \left(\nabla f(x_{k_i}) - A_{k_i} \left[\tilde{A}_{k_i}^T \tilde{A}_{k_i}\right]^{-1} \tilde{A}_{k_i}^T \left(X_{k_i}\nabla f(x_{k_i}) - \mu e\right) \right) = \mu e. \qquad (30)$$

Since $\sigma_{\min}(\tilde{A}_{k_i})$ is uniformly bounded away from zero due to Assumption (G5), the expression in the large round brackets on the left hand side of (30) is bounded, so that the l-th component of the left hand side expression would converge to zero, whereas μ is nonzero. This contradiction shows that a limit point of a subsequence with $\chi(x_{k_i}) \to 0$ actually satisfies $x_*^\mu > 0$.

Before we begin the global convergence analysis, let us state some preliminary results.

Lemma 1 *Suppose Assumptions G hold. Then there exist constants $M_{\tilde{d}}$, M_d, M_λ, $M_m > 0$, such that*

$$\|\tilde{d}_k\| \le M_{\tilde{d}}, \qquad \|d_k\| \le M_d, \qquad \|\lambda_k^+\| \le M_\lambda, \qquad |m_k(\alpha)| \le M_m\alpha \qquad (31)$$

for all $k \notin \mathcal{R}_{\rm inc}$ and $\alpha \in (0,1]$. Furthermore, there exists a constant $\bar{\alpha}^{\max} > 0$, so that for all $k \notin \mathcal{R}_{\rm inc}$ we have $\alpha_k^{\max} \ge \bar{\alpha}^{\max} > 0$.

Proof. From (G1) and (G2) it is clear that the right hand side of (23) is uniformly bounded. Additionally, Assumptions (G3), (G5), and (G6) guarantee that the inverse of the matrix in (23) exists and is uniformly bounded for all $k \notin \mathcal{R}_{\rm inc}$. Consequently, the solution of (23), $(\tilde{d}_k, \lambda_k^+)$, as well as $d_k = X_k\tilde{d}_k$ are uniformly bounded. It then also follows that

$$m_k(\alpha)/\alpha = \nabla\varphi_\mu(x_k)^T d_k = \left(X_k\nabla f(x_k) - \mu e\right)^T \tilde{d}_k$$

is uniformly bounded.

The fraction-to-the-boundary rule (8) can be reformulated as

$$\alpha_k^{\max}\tilde{d}_k \ge -\tau e.$$

Hence, since $\tilde{d}_k$ is uniformly bounded for all $k \notin \mathcal{R}_{\rm inc}$, $\alpha_k^{\max}$ is uniformly bounded away from zero for all $k \notin \mathcal{R}_{\rm inc}$. □

The following result shows that the search direction is a direction of sufficient descent for the barrier objective function at points that are sufficiently close to feasible and non-optimal.

Lemma 2 *Suppose Assumptions G hold. Then the following statement is true:*

> *If $\{x_{k_i}\}$ is a subset of iterates for which $\chi(x_{k_i}) \geq \epsilon$ with a constant $\epsilon > 0$ independent of i then there exist constants $\epsilon_1, \epsilon_2 > 0$, such that*
>
> $$\theta(x_{k_i}) \leq \epsilon_1 \qquad \Longrightarrow \qquad m_{k_i}(\alpha) \leq -\epsilon_2 \alpha.$$
>
> *for all i and $\alpha \in (0, 1]$.*

Proof. Consider a subset $\{x_{k_i}\}$ of iterates with $\chi(x_{k_i}) = \|\bar{p}_{k_i}\|_2 \geq \epsilon$. Then, by Assumption (G4), for all x_{k_i} with $\theta(x_{k_i}) \leq \theta_{\text{inc}}$ we have $k_i \notin \mathcal{R}_{\text{inc}}$. Furthermore, with $q_{k_i} = O(\|c(x_{k_i})\|)$ (from (26a) and Assumption (G5)) it follows that for $k_i \notin \mathcal{R}_{\text{inc}}$

$$\begin{aligned}
m_{k_i}(\alpha)/\alpha &= \nabla\varphi_\mu(x_{k_i})^T d_{k_i} \\
&= \nabla\varphi_\mu(x_{k_i})^T X_{k_i} \tilde{d}_{k_i} \\
&\overset{(25)}{=} \nabla\varphi_\mu(x_{k_i})^T X_{k_i} \tilde{Z}_{k_i} \bar{p}_{k_i} + \nabla\varphi_\mu(x_{k_i})^T X_{k_i} q_{k_i} \\
&\overset{(26b)}{=} -\bar{p}_{k_i}^T \left[\tilde{Z}_{k_i}^T \tilde{H}_{k_i} \tilde{Z}_{k_i}\right] \bar{p}_{k_i} - \bar{p}_{k_i}^T \tilde{Z}_{k_i}^T \tilde{H}_{k_i} q_{k_i} \\
&\qquad + \left(X_{k_i} \nabla f(x_{k_i}) - \mu e\right)^T q_{k_i} \qquad\qquad (32) \\
&\overset{(G3),(G6)}{\leq} -c_1 \|\bar{p}_{k_i}\|_2^2 + c_2 \|\bar{p}_{k_i}\|_2 \|c(x_{k_i})\| + c_3 \|c(x_{k_i})\| \\
&\leq \chi(x_{k_i}) \left(-\epsilon\, c_1 + c_2 \theta(x_{k_i}) + \frac{c_3}{\epsilon} \theta(x_{k_i})\right)
\end{aligned}$$

for some constants $c_1, c_2, c_3 > 0$, where we used $\chi(x_{k_i}) \geq \epsilon$ in the last inequality. If we now define

$$\epsilon_1 := \min\left\{\theta_{\text{inc}}, \frac{\epsilon^2\, c_1}{2(c_2\, \epsilon + c_3)}\right\},$$

it follows for all x_{k_i} with $\theta(x_{k_i}) \leq \epsilon_1$ that

$$m_{k_i}(\alpha) \leq -\alpha \frac{\epsilon\, c_1}{2} \chi(x_{k_i}) \leq -\alpha \frac{\epsilon^2\, c_1}{2} =: -\alpha \epsilon_2.$$

□

Lemma 3 *Suppose Assumptions (G1) and (G2) hold. Then there exist constants $C_\theta, C_\varphi > 0$, so that for all $k \notin \mathcal{R}_{\text{inc}}$ and $\alpha \leq \alpha_k^{\max}$*

$$|\theta(x_k + \alpha d_k) - (1-\alpha)\theta(x_k)| \leq C_\theta \alpha^2 \|d_k\|^2 \qquad (33a)$$

$$|\varphi_\mu(x_k + \alpha d_k) - \varphi_\mu(x_k) - m_k(\alpha)| \leq C_\varphi \alpha^2 \|\tilde{d}_k\|^2. \qquad (33b)$$

Since the proof of this lemma is similar to the proof of Lemma 4 in [4], we omit it for the sake of brevity.

Finally, we show that Step 9 (feasibility restoration phase) of Algorithm I is well-defined. Unless the feasibility restoration phase terminates at a stationary point of the constraint violation it is essential that reducing the infeasibility measure $\theta(x)$ eventually leads to a point that is acceptable to the filter. This is guaranteed by the following lemma which shows that no (θ, φ)-pair corresponding to a feasible point is ever included in the filter.

Lemma 4 *Suppose Assumptions G hold. Then*

$$\theta(x_k) = 0 \quad \Longrightarrow \quad m_k(\alpha) < 0 \qquad \textit{and} \tag{34}$$

$$\Theta_k := \min\{\theta : (\theta, \varphi) \in \mathcal{F}_k\} > 0 \tag{35}$$

for all k and $\alpha \in (0, 1]$.

Proof. If $\theta(x_k) = 0$, we have from Assumption (G4) that $k \notin \mathcal{R}_{\text{inc}}$. In addition, it then follows $\chi(x_k) > 0$ because Algorithm I would have terminated otherwise in Step 2, in contrast to Assumptions G. Considering the decomposition (25), it follows as in (32) that

$$\frac{m_k(\alpha)}{\alpha} = \nabla\varphi_\mu(x_k)^T d_k \leq -c_1\chi(x_k)^2 < 0,$$

i.e. (34) holds.

The proof of (35) is by induction. It is clear from Step 1 of Algorithm I, that the claim is valid for $k = 0$ since $\theta_{\max} > 0$. Suppose the claim is true for k. Then, if $\theta(x_k) > 0$ and the filter is augmented in iteration k, it is clear from the update rule (17), that $\Theta_{k+1} > 0$, since $\gamma_\theta \in (0, 1)$. If on the other hand $\theta(x_k) = 0$, Lemma 2 applied to the singleton $\{x_k\}$ implies that $m_k(\alpha) < 0$ for all $\alpha \in (0, \alpha^{\max}]$, so that the switching condition (13) is true for all trial step sizes. Therefore, Step 5.4 considers always "Case I", and the reason for α_k having been accepted must have been that α_k satisfies (15). Consequently, the filter is not augmented in Step 7. Hence, $\Theta_{k+1} = \Theta_k > 0$. □

3.2 Feasibility

In this section we will show that under Assumptions G the sequence $\theta(x_k)$ converges to zero, i.e. all limit points of $\{x_k\}$ are feasible.

Lemma 5 *Suppose that Assumptions G hold, and that the filter is augmented only a finite number of times, i.e. $|\mathcal{A}| < \infty$. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0. \tag{36}$$

Proof. Choose K, so that for all iterations $k \geq K$ the filter is not augmented in iteration k; in particular, $k \notin \mathcal{R}_{\text{inc}} \subseteq \mathcal{A}$ for $k \geq K$. From Step 7 in Algorithm I we then have, that for all $k \geq K$ both conditions (13) and (15) are satisfied for α_k. From (13) it follows with M_m from Lemma 1 that

$$\delta[\theta(x_k)]^{s_\theta} < [-m_k(\alpha_k)]^{s_\varphi}[\alpha_k]^{1-s_\varphi} \leq M_m^{s_\varphi}\alpha_k$$

and hence (since $1 - 1/s_\varphi > 0$)

$$c_4[\theta(x_k)]^{s_\theta - \frac{s_\theta}{s_\varphi}} < [\alpha_k]^{1-\frac{1}{s_\varphi}} \qquad \text{with} \qquad c_4 := \left(\frac{\delta}{M_m^{s_\varphi}}\right)^{1-\frac{1}{s_\varphi}},$$

which implies

$$\begin{aligned} \varphi_\mu(x_{k+1}) - \varphi_\mu(x_k) &\overset{(15)}{\leq} \eta_\varphi m_k(\alpha_k) \\ &\overset{(13)}{<} -\eta_\varphi\delta^{\frac{1}{s_\varphi}}[\alpha_k]^{1-\frac{1}{s_\varphi}}[\theta(x_k)]^{\frac{s_\theta}{s_\varphi}} \\ &< -\eta_\varphi\delta^{\frac{1}{s_\varphi}}c_4[\theta(x_k)]^{s_\theta}. \end{aligned} \tag{37}$$

Hence, for all $i = 1, 2, \ldots$,

$$\begin{aligned}\varphi_\mu(x_{K+i}) &= \varphi_\mu(x_K) + \sum_{k=K}^{K+i-1} (\varphi_\mu(x_{k+1}) - \varphi_\mu(x_k)) \\ &< \varphi_\mu(x_K) - \eta_\varphi \delta^{\frac{1}{s_\varphi}} c_4 \sum_{k=K}^{K+i-1} [\theta(x_k)]^{s_\theta}.\end{aligned}$$

Since $\varphi_\mu(x_{K+i})$ is bounded below (from Assumptions (G1) and (G2)), the series on the right hand side in the last line is bounded, which in turn implies (36). □

The following lemma considers a subsequence $\{x_{k_i}\}$ with $k_i \in \mathcal{A}$ for all i. Its proof can be found in [10, Lemma 3.3].

Lemma 6 *Let $\{x_{k_i}\}$ be a subsequence of iterates generated by Algorithm I, so that the filter is augmented in iteration k_i, i.e. $k_i \in \mathcal{A}$ for all i. Furthermore assume that there exist constants $c_\varphi \in \mathbb{R}$ and $C_\theta > 0$, so that*

$$\varphi_\mu(x_{k_i}) \geq c_\varphi \qquad \text{and} \qquad \theta(x_{k_i}) \leq C_\theta$$

for all i (for example, if Assumptions (G1) and (G2) hold). It then follows that

$$\lim_{i\to\infty} \theta(x_{k_i}) = 0.$$

The previous two lemmas prepare the proof of the following theorem.

Theorem 1 *Suppose Assumptions G hold. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0.$$

Proof. In the case, that the filter is augmented only a finite number of times, Lemma 5 implies the claim. If in the other extreme there exists some $K \in \mathbb{N}$, so that the filter is updated by (17) in *all* iterations $k \geq K$, then the claim follows from Lemma 6. It remains to consider the case, where for all $K \in \mathbb{N}$ there exist $k_1, k_2 \geq K$ with $k_1 \in \mathcal{A}$ and $k_2 \notin \mathcal{A}$.

The proof is by contradiction. Suppose, $\limsup_k \theta(x_k) = M > 0$. Now construct two subsequences $\{x_{k_i}\}$ and $\{x_{l_i}\}$ of $\{x_k\}$ in the following way.

1. Set $i \leftarrow 0$ and $k_{-1} = -1$.

2. Pick $k_i > k_{i-1}$ with
$$\theta(x_{k_i}) \geq M/2 \tag{38}$$
and $k_i \notin \mathcal{A}$. (Note that Lemma 6 ensures the existence of $k_i \notin \mathcal{A}$ since otherwise $\theta(x_{k_i}) \to 0$.)

3. Choose $l_i := \min\{l \in \mathcal{A} : l > k_i\}$, i.e. l_i is the first iteration after k_i in which the filter is augmented.

4. Set $i \leftarrow i + 1$ and go back to Step 2.

Thus, every x_{k_i} satisfies (38), and for each x_{k_i} the iterate x_{l_i} is the first iterate after x_{k_i} for which $(\theta(x_{l_i}), \varphi_\mu(x_{l_i}))$ is included in the filter.

Since (37) holds for all $k = k_i, \ldots, l_i - 1 \notin \mathcal{A}$, we obtain for all i

$$\varphi_\mu(x_{l_i}) \leq \varphi_\mu(x_{(k_i+1)}) < \varphi_\mu(x_{k_i}) - \eta_\varphi \delta^{\frac{1}{s_\varphi}} c_4 [M/2]^{s_\theta}. \tag{39}$$

This ensures that for all $K \in \mathbb{N}$ there exists some $i \geq K$ with $\varphi_\mu(x_{k_{(i+1)}}) \geq \varphi_\mu(x_{l_i})$ because otherwise (39) would imply

$$\varphi_\mu(x_{k_{(i+1)}}) < \varphi_\mu(x_{l_i}) < \varphi_\mu(x_{k_i}) - \eta_\varphi \delta^{\frac{1}{s_\varphi}} c_4 [M/2]^{s_\theta}$$

for all i and consequently $\lim_i \varphi_\mu(x_{k_i}) = -\infty$ in contradiction to the fact that $\{\varphi_\mu(x_k)\}$ is bounded below (from Assumptions (G1) and (G2)). Thus, there exists a subsequence $\{i_j\}$ of $\{i\}$ so that

$$\varphi_\mu(x_{k_{(i_j+1)}}) \geq \varphi_\mu(x_{l_{i_j}}). \tag{40}$$

Since $x_{k_{(i_j+1)}} \notin \mathcal{F}_{k_{(i_j+1)}} \supseteq \mathcal{F}_{l_{i_j}}$ and $l_{i_j} \in \mathcal{A}$, it follows from (40) and the filter update rule (17), that

$$\theta(x_{k_{(i_j+1)}}) \leq (1 - \gamma_\theta)\theta(x_{l_{i_j}}). \tag{41}$$

Since $l_{i_j} \in \mathcal{A}$ for all j, Lemma 6 yields $\lim_j \theta(x_{l_{i_j}}) = 0$, so that from (41) we obtain $\lim_j \theta(x_{k_{i_j}}) = 0$ in contradiction to (38). □

3.3 Optimality

In this section we will show that Assumptions G guarantee that at least one limit point of $\{x_k\}$ is a first order optimal point for the barrier problem (2).

The first lemma shows conditions under which it can be guaranteed that there exists a step length bounded away from zero so that the Armijo condition (15) for the barrier function is satisfied.

Lemma 7 *Suppose Assumptions G hold. Let $\{x_{k_i}\}$ be a subsequence with $k_i \notin \mathcal{R}_{\text{inc}}$ and $m_{k_i}(\alpha) \leq -\alpha\epsilon_2$ for a constant $\epsilon_2 > 0$ independent of k_i and for all $\alpha \in (0, 1]$. Then there exists some constant $\bar{\alpha} > 0$, so that for all k_i and $\alpha \leq \bar{\alpha}$*

$$\varphi_\mu(x_{k_i} + \alpha d_{k_i}) - \varphi_\mu(x_{k_i}) \leq \eta_\varphi m_{k_i}(\alpha). \tag{42}$$

Proof. Let $M_{\tilde{d}}, \bar{\alpha}^{\max}$ and C_φ be the constants from Lemma 1 and Lemma 3. It then follows for all $\alpha \leq \bar{\alpha}$ with

$$\bar{\alpha} := \min\left\{\bar{\alpha}^{\max}, \frac{(1-\eta_\varphi)\epsilon_2}{C_\varphi M_{\tilde{d}}^2}\right\}$$

that

$$\begin{aligned} & \varphi_\mu(x_{k_i} + \alpha d_{k_i}) - \varphi_\mu(x_{k_i}) - m_{k_i}(\alpha) \\ \stackrel{(33b)}{\leq} \; & C_\varphi \alpha^2 \|\tilde{d}_{k_i}\|^2 \leq \alpha(1-\eta_\varphi)\epsilon_2 \\ \leq \; & -(1-\eta_\varphi) m_{k_i}(\alpha), \end{aligned}$$

which implies (42). □

Let us again first consider the "easy" case, in which the filter is augmented only a finite number of times.

Lemma 8 *Suppose that Assumptions G hold and that the filter is augmented only a finite number of times, i.e.* $|\mathcal{A}| < \infty$*. Then*

$$\lim_{k\to\infty} \chi(x_k) = 0.$$

Proof. Since $|\mathcal{A}| < \infty$, there exists $K \in \mathbb{N}$ so that $k \notin \mathcal{A}$ for all $k \geq K$. Suppose, the claim is not true, i.e. there exists a subsequence $\{x_{k_i}\}$ and a constant $\epsilon > 0$, so that $\chi(x_{k_i}) \geq \epsilon$ for all i. From (36) and Lemma 2 there exist $\epsilon_1, \epsilon_2 > 0$ and $\tilde{K} \geq K$, so that for all $k_i \geq \tilde{K}$ we have $\theta(x_{k_i}) \leq \epsilon_1$ and

$$m_{k_i}(\alpha) \leq -\alpha\epsilon_2 \quad \text{for all} \quad \alpha \in (0, 1]. \tag{43}$$

It then follows from (15) that for $k_i \geq \tilde{K}$

$$\varphi_\mu(x_{k_i+1}) - \varphi_\mu(x_{k_i}) \leq \eta_\varphi m_{k_i}(\alpha_{k_i}) \leq -\alpha_{k_i}\eta_\varphi\epsilon_2.$$

Reasoning similarly as in proof of Lemma 5, one can conclude that $\lim_i \alpha_{k_i} = 0$, since $\varphi_\mu(x_{k_i})$ is bounded below and since $\varphi_\mu(x_k)$ is monotonically decreasing (from (37)) for all $k \geq \tilde{K}$. We can now assume without loss of generality that $\tilde{K}$ is sufficiently large, so that $\alpha_{k_i} < \bar{\alpha}^{\max}$ with $\bar{\alpha}^{\max}$ from Lemma 1. This means that for $k_i \geq \tilde{K}$ the first trial step $\alpha_{k,0} = \alpha_k^{\max}$ has not been accepted. The last rejected trial step size $\alpha_{k_i,l_i} \in [\alpha_{k_i}/\tau_2, \alpha_{k_i}/\tau_1]$ during the backtracking line search procedure then satisfies (13) since $k_i \notin \mathcal{A}$ and $\alpha_{k_i,l_i} > \alpha_{k_i}$. Thus, it must have been rejected because it violates (15), i.e. it satisfies

$$\varphi_\mu(x_{k_i} + \alpha_{k_i,l_i} d_{k_i}) - \varphi_\mu(x_{k_i}) > \eta_\varphi m_{k_i}(\alpha_{k_i,l_i}), \tag{44}$$

or it has been rejected because it is not acceptable to the current filter, i.e.

$$(\theta(x_{k_i} + \alpha_{k_i,l_i} d_{k_i}), \varphi_\mu(x_{k_i} + \alpha_{k_i,l_i} d_{k_i})) \in \mathcal{F}_{k_i} = \mathcal{F}_K. \tag{45}$$

We will conclude the proof by showing that neither (44) nor (45) can be true for sufficiently large k_i.

To (44): Since $\lim_i \alpha_{k_i} = 0$, we also have $\lim_i \alpha_{k_i,l_i} = 0$. In particular, for sufficiently large k_i we have $\alpha_{k_i,l_i} \leq \bar{\alpha}$ with $\bar{\alpha}$ from Lemma 7, i.e. (44) cannot be satisfied for those k_i.

To (45): Let $\Theta_K := \min\{\theta : (\theta, \varphi) \in \mathcal{F}_K\}$. From Lemma 4 we have $\Theta_K > 0$. Using Lemma 1 and Lemma 3, we then see that

$$\theta(x_{k_i} + \alpha_{k_i,l_i} d_{k_i}) \leq (1 - \alpha_{k_i,l_i})\theta(x_{k_i}) + C_\theta M_d^2 [\alpha_{k_i,l_i}]^2.$$

Since $\lim_i \alpha_{k_i,l_i} = 0$ and from Theorem 1 also $\lim_i \theta(x_{k_i}) = 0$, it follows that for k_i sufficiently large we have $\theta(x_{k_i} + \alpha_{k_i,l_i} d_{k_i}) < \Theta_K$ which contradicts (45). □

The next lemma establishes conditions under which a step size can be found that is acceptable to the current filter (see (16)).

Lemma 9 *Suppose Assumptions G hold. Let* $\{x_{k_i}\}$ *be a subsequence with* $k_i \notin \mathcal{R}_{\text{inc}}$ *and* $m_{k_i}(\alpha) \leq -\alpha\epsilon_2$ *for a constant* $\epsilon_2 > 0$ *independent of* k_i *and for all* $\alpha \in (0, 1]$*. Then there exist constants* $c_5, c_6 > 0$ *so that*

$$(\theta(x_{k_i} + \alpha d_{k_i}), \varphi_\mu(x_{k_i} + \alpha d_{k_i})) \notin \mathcal{F}_{k_i}$$

for all k_i *and* $\alpha \leq \min\{c_5, c_6\theta(x_{k_i})\}$*.*

Proof. Let $M_d, M_{\tilde{d}}, \bar{\alpha}^{\max}, C_\theta, C_\varphi$ be the constants from Lemma 1 and Lemma 3. Define $c_5 := \min\{\bar{\alpha}^{\max}, \epsilon_2/(M_{\tilde{d}}^2\, C_\varphi)\}$ and $c_6 := 1/(M_d^2\, C_\theta)$.

Now choose an iterate x_{k_i}. The mechanisms of Algorithm I ensure (see comment in Step 7), that

$$(\theta(x_{k_i}), \varphi_\mu(x_{k_i})) \notin \mathcal{F}_{k_i}. \tag{46}$$

For $\alpha \leq c_5$ we have $\alpha^2 \leq \frac{\alpha\epsilon_2}{M_{\tilde{d}}^2\, C_\varphi} \leq \frac{-m_{k_i}(\alpha)}{C_\varphi \|\tilde{d}_{k_i}\|^2}$, or equivalently

$$m_{k_i}(\alpha) + C_\varphi \alpha^2 \|\tilde{d}_{k_i}\|^2 \leq 0,$$

and it follows with (33b) that

$$\varphi_\mu(x_{k_i} + \alpha d_{k_i}) \leq \varphi_\mu(x_{k_i}), \tag{47}$$

since $\alpha \leq c_5 \leq \bar{\alpha}^{\max} \leq \alpha_k^{\max}$. Similarly, for $\alpha \leq c_6\theta(x_{k_i}) \leq \frac{\theta(x_{k_i})}{\|d_{k_i}\|^2\, C_\theta}$, we have $-\alpha\theta(x_{k_i}) + C_\theta \alpha^2 \|d_{k_i}\|^2 \leq 0$ and thus from (33a)

$$\theta(x_{k_i} + \alpha d_{k_i}) \leq \theta(x_{k_i}). \tag{48}$$

The claim then follows from (46), (47) and (48) using (22). □

The last lemma in this section shows that in iterations corresponding to a subsequence with only non-optimal limit points the filter is eventually not augmented. This result will be used in the proof of the main global convergence theorem to yield a contradiction.

Lemma 10 *Suppose Assumptions G hold. Let $\{x_{k_i}\}$ be a subsequence with $\chi(x_{k_i}) \geq \epsilon$ for a constant $\epsilon > 0$ independent of k_i. Then there exists $K \in \mathbb{N}$, so that for all $k_i \geq K$ the filter is not augmented in iteration k_i, i.e. $k_i \notin \mathcal{A}$.*

Proof. Since by Theorem 1 we have $\lim_i \theta(x_{k_i}) = 0$, it follows from Lemma 2 that there exist constants $\epsilon_1, \epsilon_2 > 0$, so that

$$\theta(x_{k_i}) \leq \epsilon_1 \quad \text{and} \quad m_{k_i}(\alpha) \leq -\alpha\epsilon_2 \tag{49}$$

for k_i sufficiently large and $\alpha \in (0, 1]$; without loss of generality we can assume that (49) is valid for all k_i. We can now apply Lemma 7 and Lemma 9 to obtain the constants $\bar{\alpha}, c_5, c_6 > 0$. Choose $K \in \mathbb{N}$, so that for all $k_i \geq K$

$$\theta(x_{k_i}) < \min\left\{\theta_{\text{inc}}, \frac{\bar{\alpha}}{c_6}, \frac{c_5}{c_6}, \left[\frac{\tau_1 c_6 \epsilon_2^{s_\varphi}}{\delta}\right]^{\frac{1}{s_\theta - 1}}\right\} \tag{50}$$

with τ_1 from Step 5.5. For all $k_i \geq K$ with $\theta(x_{k_i}) = 0$ we can argue as in the proof of Lemma 4 that both (13) and (15) hold in iteration k_i, so that $k_i \notin \mathcal{A}$.

For the remaining iterations $k_i \geq K$ with $\theta(x_{k_i}) > 0$ we note that this implies that $k_i \notin \mathcal{R}_{\text{inc}}$,

$$\frac{\delta\, [\theta(x_{k_i})]^{s_\theta}}{\epsilon_2^{s_\varphi}} < \tau_1 c_6 \theta(x_{k_i}) \tag{51}$$

(since $s_\theta > 1$), as well as

$$c_6\theta(x_{k_i}) < \min\{\bar{\alpha}, c_5\}. \tag{52}$$

Now choose an arbitrary $k_i \geq K$ with $\theta(x_{k_i}) > 0$ and define

$$\beta_{k_i} := c_6\theta(x_{k_i}) \stackrel{(52)}{=} \min\{\bar{\alpha}, c_5, c_6\theta(x_{k_i})\}. \tag{53}$$

Lemma 7 and Lemma 9 then imply, that a trial step size $\alpha_{k_i,l} \leq \beta_{k_i}$ will satisfy both

$$\varphi_\mu(x_{k_i}(\alpha_{k_i,l})) \leq \varphi_\mu(x_{k_i}) + \eta_\varphi m_{k_i}(\alpha_{k_i,l}) \tag{54}$$

and

$$\Big(\theta(x_{k_i}(\alpha_{k_i,l})), \varphi_\mu(x_{k_i}(\alpha_{k_i,l}))\Big) \notin \mathcal{F}_{k_i}. \tag{55}$$

If we now denote with $\alpha_{k_i,L}$ the first trial step size satisfying both (54) and (55), the backtracking line search procedure in Step 5.5 then implies that for $\alpha \geq \alpha_{k_i,L}$

$$\alpha \geq \tau_1\beta_{k_i} \stackrel{(53)}{=} \tau_1 c_6\theta(x_{k_i}) \stackrel{(51)}{>} \frac{\delta[\theta(x_{k_i})]^{s_\theta}}{\epsilon_2^{s_\varphi}}$$

and therefore for $\alpha \geq \alpha_{k_i,L}$

$$\delta[\theta(x_{k_i})]^{s_\theta} < \alpha\epsilon_2^{s_\varphi} = [\alpha]^{1-s_\varphi}(\alpha\epsilon_2)^{s_\varphi} \stackrel{(49)}{\leq} [\alpha]^{1-s_\varphi}[-m_{k_i}(\alpha)]^{s_\varphi}.$$

This means, the switching condition (13) is satisfied for $\alpha_{k_i,L}$ and all previous trial step sizes. Consequently, for all trial step sizes $\alpha_{k_i,l} \geq \alpha_{k_i,L}$, Case I is considered in Step 5.4. We also have $\alpha_{k_i,l} \geq \alpha_{k_i}^{\min}$, i.e. the method does not switch to the feasibility restoration phase in Step 5.2 for those trial step sizes. Consequently, $\alpha_{k_i,L}$ is indeed the accepted step size α_{k_i}. Since it satisfies both (13) and (54), the filter is not augmented in iteration k_i. □

We are now ready to prove the main global convergence result.

Theorem 2 *Suppose Assumptions G hold. Then*

$$\lim_{k\to\infty} \theta(x_k) = 0 \tag{56a}$$

$$\textit{and} \quad \liminf_{k\to\infty} \chi(x_k) = 0. \tag{56b}$$

In other words, all limit points are feasible, and there exists a limit point $x_*^\mu > 0$ of $\{x_k\}$ which is a first order optimal point for the barrier problem (2).

Proof. (56a) follows from Theorem 1. In order to show (56b), we have to consider two cases:

i) The filter is augmented only a finite number of times. Then Lemma 8 proves the claim.

ii) There exists a subsequence $\{x_{k_i}\}$, so that $k_i \in \mathcal{A}$ for all i. Now suppose, that $\limsup_i \chi(x_{k_i}) > 0$. Then there exists a subsequence $\{x_{k_{i_j}}\}$ of $\{x_{k_i}\}$ and a constant $\epsilon > 0$, so that $\lim_j \theta(x_{k_{i_j}}) = 0$ and $\chi(x_{k_{i_j}}) > \epsilon$ for all k_{i_j}. Applying Lemma 10 to $\{x_{k_{i_j}}\}$, we see that there is an iteration k_{i_j}, in which the filter is not augmented, i.e. $k_{i_j} \notin \mathcal{A}$. This contradicts the choice of $\{x_{k_i}\}$, so that $\lim_i \chi(x_{k_i}) = 0$, which proves (56b).

That a limit point x_*^μ with $\theta(x_*^\mu) = \chi(x_*^\mu) = 0$ lies indeed in the interior of the non-negative orthant, i.e. $x_*^\mu > 0$, has been argued in the paragraph before the statement of Lemma 1. □

Remark 5 *It is not possible to obtain a stronger result in Theorem 2, such as "*$\lim_k \chi(x_k) = 0$*". The reason for this is that even arbitrarily close to a strict local solution the restoration phase might be invoked even though the search direction is very good. This can happen if the current filter contains "old historic information" corresponding to previous iterates that lie in a different region of* $\mathbb{R}^n$ *but had values for* θ *and* φ_μ *similar to those for the current iterate. For example, if for the current iterate* $(\theta(x_k), \varphi_\mu(x_k))$ *is very close to the current filter (e.g. there exists filter pairs* $(\bar\theta, \bar\varphi) \in \mathcal{F}_k$ *with* $\bar\theta < \theta(x_k)$ *and* $\bar\varphi \approx \varphi_\mu(x_k)$*) and the barrier function* φ_μ *has to be increased in order to approach the optimal solution, the trial step sizes can be repeatedly rejected in Step 5.3 so that finally* $\alpha_{k,l}$ *becomes smaller than* $\alpha_k^{\min}$ *and the restoration phase is triggered. Without making additional assumptions on the restoration phase we only know that the next iterate* x_{k+1} *returned from the restoration phase is less infeasible, but possibly far away from any KKT point.*

In order to avoid that x_{k+1} *diverts from a strict local solution* x_*^μ *(satisfying the usual second order Assumptions L below), we propose the following procedure. If the restoration phase is invoked at points where the KKT error (the norm of the left hand side of (3)) is small, continue to take steps into the usual search directions* d_k *from (5) (now within the restoration phase), as long as the KKT error is decreased by a fixed fraction. If this is not possible, we have to revert to a different algorithm for the feasibility restoration phase. If* x_k *is sufficiently close to* x_*^μ*, Assumptions L ensure that* x_*^μ *is a point of attraction for Newton's method, so that this procedure will be able to eventually deliver a new iterate* x_{k+1} *which is sufficiently close to feasibility in order to be accepted by the current filter and at the same time approaches* x_*^μ*, so that overall* $\lim_k x_k = x_*^\mu$ *is guaranteed.*

Remark 6 *For the overall barrier method as the barrier parameter* μ *is driven to zero, we may simply re-start Algorithm I by deleting the current filter whenever the barrier parameter changes. Alternatively, we may choose to store the values of the two terms* $f(x_l)$ *and* $\sum_i \ln(x_l^{(i)})$ *in the barrier function* $\varphi_\mu(x_l)$ *separately for each corner entry (18) in the filter, which would allow one to initialize the filter for the new barrier problem under consideration of already known information. Details on such a procedure are beyond the scope of this paper.*

4 Local Convergence

In this section we will discuss the local convergence properties of Algorithm I. As mentioned by Fletcher and Leyffer [12], the filter approach can still suffer from the so-called Maratos effect [18], even though it is usually less restrictive in terms of accepting steps than a penalty function approach. The Maratos effect occurs if, even arbitrarily close to a strict local solution of the barrier problem, a full step d_k increases *both* the barrier function φ_μ and the constraint violation θ, leads to insufficient progress with respect to the current iterate and is rejected. This can result in poor local convergence behavior. As a remedy, Fletcher and Leyffer propose to improve the step d_k, if it has been rejected, by means of a second order correction which aims to further reduce infeasibility.

In the following we will show that second order correction steps are indeed able to prevent the Maratos effect. For clarity, the analysis is still done for the barrier approach described in Section 2 for a *fixed* value of the barrier parameter μ, even though strictly speaking the barrier problem for a given value of the barrier parameter is only solved approximately in the overall barrier method, and its overall local performance depends on its behavior as μ converges to zero (see e.g. [15, 16, 20]). More relevant is the avoidance of the Maratos effect in an active set SQP method. Later in Section 5.2 we will show that the following local convergence results can also be applied to those methods.

4.1 Second Order Correction Steps

Let us first outline the procedure for the second order correction.

If in iteration k

i) $\alpha_k^{\max} = 1$ with $\alpha_k^{\max}$ defined in (8),

ii) the first trial step size $\alpha_{k,0} = 1$ has been rejected in Step 5.3 or Step 5.4 of Algorithm I, and

iii) $\theta(x_k) \le \theta_{\text{soc}}$ for some fixed constant $\theta_{\text{soc}} \in (0, \infty]$,

then, instead of immediately continuing with the selection of a shorter trial step size $\alpha_{k,1}$ in Step 5.5, we first compute a second order correction step and accept it if it satisfies our usual acceptance criteria, as outlined next.

Algorithm SOC

5.1*. *Compute second order correction step.*

Solve the linear system

$$\begin{bmatrix} H_k^{\text{soc}} & A_k^{\text{soc}} \\ (A_k^{\text{soc}})^T & 0 \end{bmatrix} \begin{pmatrix} d_k^{\text{soc}} \\ \lambda_k^{\text{soc}} \end{pmatrix} = - \begin{pmatrix} g_k^{\text{soc}} \\ c(x_k + d_k) + c_k^{\text{soc}} \end{pmatrix}, \tag{57}$$

(particular admissible choices of $H_k^{\text{soc}}, A_k^{\text{soc}}, g_k^{\text{soc}}, c_k^{\text{soc}}$ are discussed below) to obtain the second order correction step d_k^{soc} and define

$$\bar{x}_{k+1} := x_k + d_k + d_k^{\text{soc}}.$$

5.2*. *Check fraction-to-the-boundary rule.*

If

$$x_k + d_k + d_k^{\text{soc}} \ge (1 - \tau) x_k \tag{58}$$

is *not* satisfied, reject the second order correction step and continue with Step 5.5 (of Algorithm I).

5.3*. *Check acceptability to the filter.*

If $\bar{x}_{k+1} \in \mathcal{F}_k$, reject the second order correction step and go to Step 5.5.

5.4*. *Check sufficient decrease with respect to current iterate.*

5.4.1*. *Case I. The switching condition*

$$m_k(1) < 0 \qquad \textit{and} \qquad [-m_k(1)]^{s_\varphi} > \delta \, [\theta(x_k)]^{s_\theta} \tag{59}$$

holds:

If the Armijo condition for barrier function

$$\varphi_\mu(\bar{x}_{k+1}) \le \varphi_\mu(x_k) + \eta_\varphi m_k(1) \tag{60}$$

holds, accept $\bar{x}_{k+1}$ and go to Step 6.

Otherwise, go to Step 5.5.

5.4.2*. *Case II. The switching condition (59) is not satisfied:* If

$$\begin{aligned} \theta(\bar{x}_{k+1}) &\leq (1-\gamma_\theta)\theta(x_k) && \text{(61a)} \\ \text{or} \quad \varphi_\mu(\bar{x}_{k+1}) &\leq \varphi_\mu(x_k) - \gamma_\varphi\theta(x_k) && \text{(61b)} \end{aligned}$$

hold, accept $\bar{x}_{k+1}$ and go to Step 6.
Otherwise, go to Step 5.5.

If $\bar{x}_{k+1}$ has been accepted by Algorithm SOC as the next iterate, we also replace Step 7 of Algorithm I by

7*. If one of the conditions (59) or (60) does not hold, augment the filter according to (17); otherwise leave the filter unchanged, i.e. set $\mathcal{F}_{k+1} := \mathcal{F}_k$.

It can be verified easily that this modification of Algorithm I does not affect its global convergence properties.

Second order correction steps of the form (57) are discussed by Conn, Gould, and Toint in [7, Section 15.3.2.3]. Here we assume that H_k^{soc} is uniformly positive definite on the null space of $(A_k^{\text{soc}})^T$, and that in a neighborhood of a strict local solution we have

$$g_k^{\text{soc}} = o(\|d_k\|), \qquad A_k - A_k^{\text{soc}} = O(\|d_k\|), \qquad c_k^{\text{soc}} = o(\|d_k\|^2). \tag{62}$$

In [7], the analysis is made for the particular choices $c_k^{\text{soc}} = 0$, $A_k^{\text{soc}} = A(x_k + p_k)$ for some $p_k = O(\|d_k\|)$, and $H_k = \nabla^2_{xx}\mathcal{L}_\mu(x_k, \lambda_k)$ in (5) for multiplier estimates λ_k. However, the careful reader will be able to verify that the results that we will use from [7] still hold as long as

$$(W_k^\mu - H_k)d_k = o(\|d_k\|), \tag{63}$$

if x_k converges to a strict local solution x_*^μ of the barrier problem with corresponding multipliers λ_*^μ, where

$$W_k^\mu = \nabla^2_{xx}\mathcal{L}_\mu(x_k, \lambda_*^\mu) \overset{(6)}{=} \nabla^2\varphi_\mu(x_k) + \sum_{i=1}^m (\lambda_*^\mu)^{(i)}\nabla^2 c^{(i)}(x_k). \tag{64}$$

Popular choices for the quantities in the computation of the second order correction step in (57) that satisfy (62) are for example the following.

(a) $H_k^{\text{soc}} = I$, $g_k^{\text{soc}} = 0$, $c_k^{\text{soc}} = 0$, and $A_k^{\text{soc}} = A_k$ or $A_k^{\text{soc}} = A(x_k + d_k)$, which corresponds to a least-square step for the constraints.

(b) $H_k^{\text{soc}} = X_k^2$, $g_k^{\text{soc}} = 0$, $c_k^{\text{soc}} = 0$, and $A_k^{\text{soc}} = A_k$ or $A_k^{\text{soc}} = A(x_k + d_k)$, which corresponds to a least-square step for the constraints in a different norm which takes the proximity to the boundary into account.

(c) $H_k^{\text{soc}} = H_k$, $g_k^{\text{soc}} = 0$, $c_k^{\text{soc}} = 0$, and $A_k^{\text{soc}} = A_k$, which is very inexpensive since this choice allows to reuse the factorization of the linear system (5).

(d) H_k^{soc} being the Hessian approximation corresponding to $x_k + d_k$, $g_k^{\text{soc}} = \nabla\varphi_\mu(x_k + d_k) + A(x_k + d_k)^T\lambda_k^+$, $c_k^{\text{soc}} = 0$, and $A_k^{\text{soc}} = A(x_k + d_k)$ which corresponds to the step in the next iteration, supposing that $x_k + d_k$ has been accepted. This choice has the flavor of the watchdog technique [6].

(e) If d_k^{soc} is a second order correction step, and $\bar{d}_k^{\text{soc}}$ is an additional second order correction step (i.e. with "$c(x_k + d_k)$" replaced by "$c(x_k + d_k + d_k^{\text{soc}})$" in (57)), then $d_k^{\text{soc}} + \bar{d}_k^{\text{soc}}$ can be understood as a single second order correction step for d_k (in that case with $c_k^{\text{soc}} \neq 0$). Similarly, several consecutive correction steps can be considered as a single one.

4.2 Local Convergence Analysis with Second Order Correction Steps

We start the analysis by stating the necessary assumptions.

Assumptions L. *Assume that $\{x_k\}$ converges to a local solution $x_*^\mu > 0$ of the barrier problem (2) and that the following holds.*

(L1) *The functions f and c are twice continuously differentiable in a neighborhood of x_*^μ.*

(L2) *x_*^μ satisfies the following sufficient second order optimality conditions.*

- *x_*^μ is feasible, i.e. $\theta(x_*^\mu) = 0$,*
- *there exists $\lambda_*^\mu \in \mathbb{R}^m$ so that the KKT conditions (3) are satisfied for (x_*^μ, λ_*^μ),*
- *the constraint Jacobian $A(x_*^\mu)$ has full rank, and*
- *the Hessian of the Lagrangian $W_*^\mu = \nabla_{xx}^2 \mathcal{L}_\mu(x_*^\mu, \lambda_*^\mu)$ is positive definite on the null space of $A(x_*^\mu)^T$.*

(L3) *In (57), $H_k^{\rm soc}$ is uniformly positive definite on the null space of $(A_k^{\rm soc})^T$, and (62) holds.*

(L4) *The Hessian approximations H_k in (5) satisfy (63).*

The assumption $x_k \to x_*^\mu$ has been discussed in Remark 5 on page 22. Assumption (L4) is reminiscent of the Dennis-Moré characterization of superlinear convergence [8]. However, this assumption is stronger than necessary for superlinear convergence [3] which requires only that $Z_k^T(W_k^\mu - H_k)d_k = o(\|d_k\|)$, where Z_k is a null space matrix for A_k^T.

First we summarize some preliminary results.

Lemma 11 *Suppose Assumptions G and L hold. Then there exists a neighborhood U_1 of x_*^μ, so that for all $x_k \in U_1$ we have*

$$
\begin{aligned}
d_k^{\rm soc} &= o(\|d_k\|) && \text{(65a)}\\
\alpha_k^{\max} &= 1 && \text{(65b)}\\
x_k + d_k + d_k^{\rm soc} &\geq (1-\tau)x_k && \text{(65c)}\\
m_k(1) &= O(\|d_k\|) && \text{(65d)}\\
c(x_k + d_k + d_k^{\rm soc}) &= o(\|d_k\|^2) && \text{(65e)}
\end{aligned}
$$

Proof. Since from Assumption (L3) the matrix in (57) has a uniformly bounded inverse and the right hand side is $o(\|d_k\|)$, claim (65a) follows. Furthermore, since $x_*^\mu > 0$ and $d_k, d_k^{\rm soc} \to 0$ as $x_k \to x_*^\mu$, we have (65b) and (65c). (65d) follows from the boundedness of $\nabla\varphi_\mu(x_k)$ and (14). Finally, from

$$
\begin{aligned}
c(x_k + d_k + d_k^{\rm soc}) &= c(x_k + d_k) + A(x_k + d_k)^T d_k^{\rm soc} + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(57)}{=} -c_k^{\rm soc} - (A_k^{\rm soc})^T d_k^{\rm soc} + (A(x_k) + O(\|d_k\|))^T d_k^{\rm soc}\\
&\qquad + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(62)}{=} o(\|d_k\|^2) + O(\|d_k\|\|d_k^{\rm soc}\|) + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(65a)}{=} o(\|d_k\|^2)
\end{aligned}
$$

for x_k close to x_*^μ the last claim (65e) follows. $\square$

In order to prove our local convergence result we will make use of two results established in [7] regarding the effect of second order correction steps on the exact penalty function (10). Note, that we will employ the exact penalty function only as a technical device, but the algorithm never refers to it. We will also use the following model of the penalty function

$$q_\rho(x_k, d) = \varphi_\mu(x_k) + \nabla\varphi_\mu(x_k)^T d + \frac{1}{2} d^T H_k d + \rho \left\| A_k^T d + c(x_k) \right\|. \tag{66}$$

The first result follows from Theorem 15.3.7 in [7].

Lemma 12 *Suppose Assumptions G and L hold. Let ϕ_ρ be the exact penalty function (10) and q_ρ defined by (66) with $\rho > \|\lambda_*\|_D$, where $\|\cdot\|_D$ is the dual norm to $\|\cdot\|$. Then,*

$$\lim_{k\to\infty} \frac{\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\rm soc})}{q_\rho(x_k, 0) - q_\rho(x_k, d_k)} = 1. \tag{67}$$

The next result follows from Theorem 15.3.2 in [7].

Lemma 13 *Suppose Assumptions G hold. Let (d_k, λ_k^+) be a solution of the linear system (5), and let $\rho > \|\lambda_k^+\|_D$. Then*

$$q_\rho(x_k, 0) - q_\rho(x_k, d_k) \geq 0. \tag{68}$$

The next lemma shows that in a neighborhood of x_*^μ Step 5.4.1* of Algorithm SOC will be successful if the switching condition (59) holds.

Lemma 14 *Suppose Assumptions G and L hold. Then there exists a neighborhood $U_2 \subseteq U_1$ of x_*^μ so that whenever the switching condition (59) holds, the Armijo condition (60) is satisfied for the step $d_k + d_k^{\rm soc}$.*

Proof. Choose U_1 to be the neighborhood from Lemma 11. It then follows that for $x_k \in U_1$ satisfying (59) that (58) holds and that

$$\theta(x_k) < \delta^{-\frac{1}{s_\theta}} [-m_k(1)]^{\frac{s_\varphi}{s_\theta}} \overset{(65d)}{=} O(\|d_k\|^{\frac{s_\varphi}{s_\theta}}) = o(\|d_k\|^2), \tag{69}$$

since $\frac{s_\varphi}{s_\theta} > 2$.

Since $\eta_\varphi < \frac{1}{2}$, Lemma 12 and (68) imply that there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have for some constant $\rho > 0$ with $\rho > \|\lambda_k^+\|_D$ independent of k that

$$\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\rm soc}) \geq \left(\frac{1}{2} + \eta_\varphi\right) (q_\rho(x_k, 0) - q_\rho(x_k, d_k)). \tag{70}$$

We then have

$$\begin{aligned} & \varphi_\mu(x_k) - \varphi_\mu(x_k + d_k + d_k^{\rm soc}) \\ \overset{(10)}{=} \; & \phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\rm soc}) - \rho\left(\theta(x_k) - \theta(x_k + d_k + d_k^{\rm soc})\right) \\ \overset{(70),(65e),(69)}{\geq} \; & \left(\frac{1}{2} + \eta_\varphi\right) (q_\rho(x_k, 0) - q_\rho(x_k, d_k)) + o(\|d_k\|^2) \\ \overset{(66),(69)}{=} \; & -\left(\frac{1}{2} + \eta_\varphi\right) \left(\nabla\varphi_\mu(x_k)^T d_k + \frac{1}{2} d_k^T H_k d_k\right) + o(\|d_k\|^2). \end{aligned}$$

Since $m_k(1) = \nabla\varphi_\mu(x_k)^T d_k$, this implies with the boundedness of λ_k^+, $\bar{p}_k$, and q_k (from (25)) that

$$
\begin{aligned}
& \varphi_\mu(x_k) + \eta_\varphi m_k(1) - \varphi_\mu(x_k + d_k + d_k^{\rm soc}) \\
\geq\ & -\frac{1}{2}\nabla\varphi_\mu(x_k)^T d_k - \left(\frac{1}{4} + \frac{\eta_\varphi}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(5)}{=}\ & \frac{1}{2}\left(d_k^T H_k d_k + d_k^T A_k \lambda_k^+\right) - \left(\frac{1}{4} + \frac{\eta_\varphi}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(5)}{=}\ & \left(\frac{1}{4} - \frac{\eta_\varphi}{2}\right) d_k^T H_k d_k - \frac{1}{2} c(x_k)^T \lambda_k^+ + o(\|d_k\|^2) \\
\overset{(69)}{=}\ & \left(\frac{1}{4} - \frac{\eta_\varphi}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(25)}{=}\ & \left(\frac{1}{4} - \frac{\eta_\varphi}{2}\right) \bar{p}_k^T \tilde{Z}_k^T \tilde{H}_k \tilde{Z}_k \bar{p}_k + O(\|q_k\|) + o(\|d_k\|^2).
\end{aligned} \tag{71}
$$

Finally, using repeatedly the orthonormality of $[\tilde{Z}_k\ \tilde{Y}_k]$ as well as the boundedness of $\{x_k\}$, we have

$$
\begin{aligned}
q_k\ &=\ O(\bar{q}_k) \overset{(26a),(G5)}{=} O(\theta(x_k)) \overset{(69)}{=} o(\|d_k\|^2) = o(\tilde{d}_k^T \tilde{d}_k) \\
&\overset{(25a)}{=}\ o(p_k^T p_k + q_k^T q_k) \overset{(25b)}{=} o(\|\bar{p}_k\|^2) + o(\|q_k\|^2)
\end{aligned}
$$

and therefore $q_k = o(\|\bar{p}_k\|^2)$, as well as

$$
d_k \overset{(25a)}{=} O(\|q_k\|) + O(\|p_k\|) \overset{(25b)}{=} o(\|\bar{p}_k\|) + O(\|\bar{p}_k\|) = O(\|\bar{p}_k\|).
$$

Hence, (60) is implied by (71), Assumption (G6) and $\eta_\varphi < \frac{1}{2}$, if x_k is sufficiently close to x_*^μ. □

It remains to show that also the filter and the sufficient reduction criterion (12) do not interfere with the acceptance of full steps close to x_*^μ. The following technical lemmas address this issue and prepare the proof of the main local convergence theorem.

Lemma 15 *Suppose Assumptions G and L hold. Then there exists a neighborhood $U_3 \subseteq U_2$ (with U_2 from Lemma 14) and constants $\rho_1, \rho_2, \rho_3 > 0$ with*

$$
\begin{aligned}
\rho_3\ &=\ (1-\gamma_\theta)\rho_2 - \gamma_\varphi && \text{(72a)} \\
2\gamma_\theta\rho_2\ &<\ (1+\gamma_\theta)(\rho_2 - \rho_1) - 2\gamma_\varphi && \text{(72b)} \\
2\rho_3\ &\geq\ (1+\gamma_\theta)\rho_1 + (1-\gamma_\theta)\rho_2, && \text{(72c)}
\end{aligned}
$$

so that for all $x_k \in U_3$ we have $\|\lambda_k^+\|_D < \rho_i$ for $i = 1, 2, 3$, and the second order correction step is always tried in Algorithm SOC if $x_k + d_k$ is rejected. Furthermore, for all $x_k \in U_3$ we have

$$
\phi_{\rho_i}(x_k) - \phi_{\rho_i}(x_k + d_k + \bar{d}_k^{\rm soc}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_k, 0) - q_{\rho_i}(x_k, d_k)\right) \overset{(68)}{\geq} 0 \tag{73}
$$

for $i = 2, 3$ and all choices

$$
\begin{aligned}
\bar{d}_k^{\rm soc}\ &=\ d_k^{\rm soc}, && \text{(74a)} \\
\bar{d}_k^{\rm soc}\ &=\ \sigma_k d_k^{\rm soc} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\rm soc}, && \text{(74b)} \\
\bar{d}_k^{\rm soc}\ &=\ \sigma_k d_k^{\rm soc} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\rm soc} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\rm soc}, && \text{(74c)} \\
\text{or}\quad \bar{d}_k^{\rm soc}\ &=\ \sigma_k d_k^{\rm soc} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\rm soc} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\rm soc} \\
&\qquad + d_{k+3} + \sigma_{k+3} d_{k+3}^{\rm soc}, && \text{(74d)}
\end{aligned}
$$

with $\sigma_k, \sigma_{k+1}, \sigma_{k+2}, \sigma_{k+3} \in \{0, 1\}$, as long as $x_{l+1} = x_l + d_l + \sigma_l d_k^{\rm soc}$ for $l \in \{k, \ldots, k+j\}$ with $j \in \{-1, 0, 1, 2\}$, respectively.

Proof. Let $U_3 \subseteq U_2$ be a neighborhood of x_*^μ, so that for all $x_k \in U_3$ we have $\theta(x_k) \leq \theta_{\text{soc}}$. Therefore, due to (65b) and (65c) the second order correction is always tried in Algorithm SOC if $x_k + d_k$ has been rejected. Since λ_k^+ is uniformly bounded for all k with $x_k \in U_3$, we can find $\rho_1 > \|\lambda_*^\mu\|$ with

$$\rho_1 > \|\lambda_k^+\|_D \tag{75}$$

for all k with $x_k \in U_3$. Defining now

$$\rho_2 := \frac{1+\gamma_\theta}{1-\gamma_\theta}\rho_1 + \frac{3\gamma_\varphi}{1-\gamma_\theta}$$

and ρ_3 by (72a), it is then easy to verify that $\rho_2, \rho_3 \geq \rho_1 > \|\lambda_k^+\|_D$ and that (72b) and (72c) hold. Since $(1+\gamma_\theta) < 2$, we have from Lemma 12 by possibly further reducing U_3 that (73) holds for $x_k \in U_3$, since according to (d) and (e) on page 24 all choices of $\bar{d}_k^{\text{soc}}$ in (74) can be understood as second order correction steps to d_k. □

Before proceeding with the next lemma, let us introduce some more notation.

Let U_3 and ρ_i be the neighborhood and constants from Lemma 15. Since $\lim_k x_k = x_*^\mu$, we can find $K_1 \in \mathbb{N}$ so that $x_k \in U_3$ for all $k \geq K_1$. Let us now define the level set

$$L := \{x \in U_3 : \phi_{\rho_3}(x) \leq \phi_{\rho_3}(x_*^\mu) + \kappa\}\,, \tag{76}$$

where $\kappa > 0$ is chosen so that for all $x \in L$ we have $(\theta(x), \varphi_\mu(x)) \notin \mathcal{F}_{K_1}$. This is possible, since $\Theta_{K_1} > 0$ from (35), and since $\max\{\theta(x) : x \in L\}$ converges to zero as $\kappa \to 0$, because x_*^μ is a strict local minimizer of ϕ_{ρ_3} [17]. Obviously, $x_*^\mu \in L$. For later reference let K_2 be the first iteration $K_2 \geq K_1$ with $x_{K_2} \in L$.

Furthermore, let us define for $k \in \mathbb{N}$

$$\mathcal{G}_k := \Big\{(\theta,\varphi) : \theta \geq (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad \varphi \geq \varphi_\mu(x_k) - \gamma_\varphi\theta(x_k)\Big\}$$

and $I_{k_1}^{k_2} := \{l = k_1, \ldots, k_2 - 1 : l \in \mathcal{A}\}$ for $k_1 \leq k_2$. Then it follows from the filter update rule (17) and the definition of $\mathcal{A}$ that for $k_1 \leq k_2$

$$\mathcal{F}_{k_2} = \mathcal{F}_{k_1} \cup \bigcup_{l \in I_{k_1}^{k_2}} \mathcal{G}_l. \tag{77}$$

Also note, that $l \in I_{k_1}^{k_2} \subseteq \mathcal{A}$ implies $\theta(x_l) > 0$. Otherwise, we would have from (34) that $m_k(\alpha_{k,l}) < 0$, so that (13) holds for all trial step sizes α, and the step must have been accepted in Step 5.4.1 or Step 5.4.1*, hence satisfying (15) or (60). This would contradict the filter update condition in Step 7 or 7*, respectively.

The last lemma will enable us to show in the main theorem of this section that, once the iterates have reached the level set L, the full step will always be acceptable to the current filter.

Lemma 16 *Suppose Assumptions G and L hold and let $x > 0$ and $l \geq K_1$ with $\theta(x_l) > 0$. Then the following statements hold.*

$$\left.\begin{array}{l}\textit{If } \phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l,0) - q_{\rho_2}(x_l,d_l)\right),\\ \textit{then } (\theta(x), \varphi_\mu(x)) \notin \mathcal{G}_l.\end{array}\right\} \tag{78}$$

$$\left.\begin{array}{l}\textit{If } x \in L \textit{ and } \phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_{K_2},0) - q_{\rho_2}(x_{K_2},d_{K_2})\right),\\ \textit{then } (\theta(x), \varphi_\mu(x)) \notin \mathcal{F}_{K_2}.\end{array}\right\} \tag{79}$$

Proof. To (78): Since $\rho_1 > \|\lambda_l^+\|_D$ we have from Lemma 13 that $q_{\rho_1}(x_l, 0) - q_{\rho_1}(x_l, d_l) \geq 0$, and hence using definition for q_ρ (66) and $A_l^T d_l + c(x_l) = 0$ (from (5)) that

$$\begin{aligned}
\phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) &\geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l, 0) - q_{\rho_2}(x_l, d_l)\right) \\
&= \frac{1+\gamma_\theta}{2}\left(q_{\rho_1}(x_l, 0) - q_{\rho_1}(x_l, d_l) + (\rho_2 - \rho_1)\theta(x_l)\right) \\
&\geq \frac{1+\gamma_\theta}{2}(\rho_2 - \rho_1)\theta(x_l). \qquad (80)
\end{aligned}$$

If $\varphi_\mu(x) < \varphi_\mu(x_l) - \gamma_\varphi\theta(x_l)$, the claim follows immediately. Otherwise, suppose that $\varphi_\mu(x) \geq \varphi_\mu(x_l) - \gamma_\varphi\theta(x_l)$. In that case, we have together with $\theta(x_l) > 0$ that

$$\begin{aligned}
\theta(x_l) - \theta(x) &\overset{(80),(10)}{\geq} \frac{1+\gamma_\theta}{2\rho_2}(\rho_2 - \rho_1)\theta(x_l) + \frac{1}{\rho_2}\left(\varphi_\mu(x) - \varphi_\mu(x_l)\right) \\
&\geq \frac{1+\gamma_\theta}{2\rho_2}(\rho_2 - \rho_1)\theta(x_l) - \frac{\gamma_\varphi}{\rho_2}\theta(x_l) \\
&\overset{(72b)}{>} -\gamma_\theta\theta(x_l),
\end{aligned}$$

so that $(\theta(x), \varphi_\mu(x)) \notin \mathcal{G}_l$.

To (79): Since $x \in L$, it follows by the choice of κ that $(\theta(x), \varphi_\mu(x)) \notin \mathcal{F}_{K_1}$. Thus, according to (77) it remains to show that for all $l \in I_{K_1}^{K_2}$ we have $(\theta(x), \varphi_\mu(x)) \notin \mathcal{G}_l$. Choose $l \in I_{K_1}^{K_2}$. As in (80) we can show that

$$\phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \geq \frac{1+\gamma_\theta}{2}(\rho_2 - \rho_1)\theta(x_{K_2}). \qquad (81)$$

Since $x \in L$ it follows from the definition of K_2 (as the first iterate after K_1 with $x_{K_2} \in L$) and the fact that $l < K_2$ that

$$\begin{aligned}
\phi_{\rho_3}(x_l) &\overset{(76)}{>} \phi_{\rho_3}(x_{K_2}) \overset{(10)}{=} \phi_{\rho_2}(x_{K_2}) + (\rho_3 - \rho_2)\theta(x_{K_2}) \\
&\overset{(81)}{\geq} \phi_{\rho_2}(x) + \left(\rho_3 - \frac{1+\gamma_\theta}{2}\rho_1 - \frac{1-\gamma_\theta}{2}\rho_2\right)\theta(x_{K_2}) \\
&\overset{(72c)}{\geq} \phi_{\rho_2}(x). \qquad (82)
\end{aligned}$$

If $\varphi_\mu(x) < \varphi_\mu(x_l) - \gamma_\varphi\theta(x_l)$, we immediately have $(\theta(x), \varphi_\mu(x)) \notin \mathcal{G}_l$. Otherwise we have $\varphi_\mu(x) \geq \varphi_\mu(x_l) - \gamma_\varphi\theta(x_l)$ which yields

$$\begin{aligned}
\theta(x) &\overset{(82),(10)}{<} \frac{1}{\rho_2}\left(\varphi_\mu(x_l) + \rho_3\theta(x_l) - \varphi_\mu(x)\right) \\
&\leq \frac{\rho_3 + \gamma_\varphi}{\rho_2}\theta(x_l) \\
&\overset{(72a)}{=} (1 - \gamma_\theta)\theta(x_l),
\end{aligned}$$

so that $(\theta(x), \varphi_\mu(x)) \notin \mathcal{G}_l$ which concludes the proof of (79). □

After these preparations we are finally able to show the main local convergence theorem.

Theorem 3 *Suppose Assumptions G and L hold. Then, for k sufficiently large full steps of the form* $x_{k+1} = x_k + d_k$ *or* $x_{k+1} = x_k + d_k + d_k^{\mathrm{soc}}$ *will be taken, and* x_k *converges to* x_*^μ *superlinearly.*

Proof. Recall that $K_2 \geq K_1$ is the first iteration after K_1 with $x_{K_2} \in L \subseteq U_3$. Hence, for all $k \geq K_2$ Lemma 11 and Lemma 15 imply that the second order correction step is always tried in Algorithm SOC if $x_k + d_k$ is rejected, and that $\alpha_k^{\max} = 1$ and (58) hold, i.e. the fraction-to-the-boundary rule is never active.

We now show by induction that the following statements are true for $k \geq K_2 + 2$:

$$\begin{aligned}
&(\mathrm{i}_k) \qquad \phi_{\rho_i}(x_l) - \phi_{\rho_i}(x_k) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in \{2,3\} \text{ and } K_2 \leq l \leq k-2 \\
&(\mathrm{ii}_k) \qquad x_k \in L \\
&(\mathrm{iii}_k) \qquad x_k = x_{k-1} + d_{k-1} + \sigma_{k-1} d_{k-1}^{\mathrm{soc}} \qquad \text{with } \sigma_{k-1} \in \{0,1\}.
\end{aligned}$$

We start by showing that these statements are true for $k = K_2 + 2$.

Suppose, the point $x_{K_2} + d_{K_2}$ is not accepted by the line search. In that case, define $\bar{x}_{K_2+1} := x_{K_2} + d_{K_2} + d_{K_2}^{\mathrm{soc}}$. Then, from (73) with $i = 3$, $k = K_2$, and (74a), we see from $x_{K_2} \in L$ and the definition of L that $\bar{x}_{K_2} \in L$. After applying (73) again with $i = 2$ it follows from (79) that $(\theta(\bar{x}_{K_2+1}), \varphi_\mu(\bar{x}_{K_2+1})) \notin \mathcal{F}_{K_2}$, i.e. $\bar{x}_{K_2+1}$ is not rejected in Step 5.3*. Furthermore, if the switching condition (59) holds, we see from Lemma 14 that the Armijo condition (60) with $k = K_2$ is satisfied for the point $\bar{x}_{K_2+1}$. In the other case, i.e. if (59) is violated (note that then (34) and (59) imply $\theta(x_{K_2}) > 0$), we see from (73) for $i = 2$, $k = K_2$, and (74a), together with (78) for $l = K_2$, that (61) holds. Hence, $\bar{x}_{K_2+1}$ is also not rejected in Step 5.4* and accepted as next iterate. Summarizing the discussion in this paragraph we can write $x_{K_2+1} = x_{K_2} + d_{K_2} + \sigma_{K_2} d_{K_2}^{\mathrm{soc}}$ with $\sigma_{K_2} \in \{0,1\}$.

Let us now consider iteration $K_2 + 1$. For $\sigma_{K_2+1} \in \{0,1\}$ we have from (73) for $k = K_2$ and (74b) that

$$\begin{aligned}
&\phi_{\rho_i}(x_{K_2}) - \phi_{\rho_i}(x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1} d_{K_2+1}^{\mathrm{soc}}) \\
\geq\ &\frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_{K_2}, 0) - q_{\rho_i}(x_{K_2}, d_{K_2})\right)
\end{aligned} \tag{83}$$

for $i = 2, 3$, which yields

$$x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1} d_{K_2+1}^{\mathrm{soc}} \in L. \tag{84}$$

If $x_{K_2+1} + d_{K_2+1}$ is accepted as next iterate x_{K_2+2}, we immediately obtain from (83) and (84) that (i_{K_2+2})–(iii_{K_2+2}) hold. Otherwise, we consider the case $\sigma_{K_2+1} = 1$. From (83), (84), and (79) we have for $\bar{x}_{K_2+2} := x_{K_2+1} + d_{K_2+1} + d_{K_2+1}^{\mathrm{soc}}$ that $(\theta(\bar{x}_{K_2+2}), \varphi_\mu(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2}$. If $K_2 \notin I_{K_2}^{K_2+1}$ it immediately follows from (77) that $(\theta(\bar{x}_{K_2+2}), \varphi_\mu(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$. Otherwise, we have $\theta(x_{K_2}) > 0$. Then, (83) for $i = 2$ together with (78) implies $(\theta(\bar{x}_{K_2+2}), \varphi_\mu(\bar{x}_{K_2+2})) \notin \mathcal{G}_{K_2}$, and hence with (77) we have $(\theta(\bar{x}_{K_2+2}), \varphi_\mu(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$, so that $\bar{x}_{K_2+2}$ is not rejected in Step 5.3*. Arguing similarly as in the previous paragraph we can conclude that $\bar{x}_{K_2+1}$ is also not rejected in Step 5.4*. Therefore, $x_{K_2+2} = \bar{x}_{K_2+2}$. Together with (83) and (84) this proves (i_{K_2+2})–(iii_{K_2+2}) for the case $\sigma_{K_2+1} = 1$.

Now suppose that (i_l)–(iii_l) are true for all $K_2 + 2 \leq l \leq k$ with some $k \geq K_2 + 2$. If $x_k + d_k$ is accepted by the line search, define $\sigma_k := 0$, otherwise $\sigma_k := 1$. Set $\bar{x}_{k+1} := x_k + d_k + \sigma_k d_k^{\mathrm{soc}}$. From (73) for (74c) we then have for $i = 2, 3$

$$\phi_{\rho_i}(x_{k-1}) - \phi_{\rho_i}(\bar{x}_{k+1}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_{k-1}, 0) - q_{\rho_i}(x_{k-1}, d_{k-1})\right) \geq 0. \tag{85}$$

Choose l with $K_2 \leq l < k - 1$ and consider two cases:

Case a): If $k = K_2 + 2$, then $l = K_2$, and it follows from (73) with (74d) that for $i = 2, 3$

$$\phi_{\rho_i}(x_l) - \phi_{\rho_i}(\bar{x}_{k+1}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l)\right) \geq 0. \tag{86}$$

Case b): If $k > K_2 + 2$, we have from (85) that $\phi_{\rho_i}(\bar{x}_{k+1}) \leq \phi_{\rho_i}(x_{k-1})$ and hence from (i_{k-1}) it follows that (86) also holds in this case.

In either case, (86) implies in particular that $\phi_{\rho_3}(\bar{x}_{k+1}) \leq \phi_{\rho_3}(x_{K_2})$, and since $x_{K_2} \in L$, we obtain

$$\bar{x}_{k+1} \in L. \tag{87}$$

If $x_k + d_k$ is accepted by the line search, (i_{k+1})–(iii_{k+1}) follow from (86), (85) and (87). If $x_k + d_k$ is rejected, we see from (87), (86) for $i = 2$ and $l = K_2$, and (79) that $(\theta(\bar{x}_{k+1}), \varphi_\mu(\bar{x}_{k+1})) \notin \mathcal{F}_{K_2}$. Furthermore, for $l \in I^k_{K_2}$ we have from (85) and (86) with (78) that $(\theta(\bar{x}_{k+1}), \varphi_\mu(\bar{x}_{k+1})) \notin \mathcal{G}_l$, and hence from (77) that $\bar{x}_{k+1}$ is not rejected in Step 5.3*. We can again show as before that $\bar{x}_{k+1}$ is not rejected in Step 5.4*, so that $x_{k+1} = \bar{x}_{k+1}$ which implies (i_{k+1})–(iii_{k+1}).

That $\{x_k\}$ converges to x^μ_* with a superlinear rate follows from (63) (see e.g. [21]). □

Remark 7 *As can be expected, the convergence rate of x_k towards x^μ_* is quadratic, if (63) is replaced by*

$$(W^\mu_k - H_k)d_k = O(\|d_k\|^2).$$

5 Alternative Algorithms

5.1 Measures based on the augmented Lagrangian Function

The two measures $\varphi_\mu(x)$ and $\theta(x)$ can be considered as the two components of the exact penalty function (10). Another popular choice for a merit function is the *augmented Lagrangian function* (see e.g. [22])

$$\ell_\rho(x, \lambda) := \varphi_\mu(x) + \lambda^T c(x) + \frac{\rho}{2} c(x)^T c(x), \tag{88}$$

where λ are multiplier estimates corresponding to the equality constraints (2b). If λ^μ_* are the multipliers corresponding to a strict local solution x^μ_* of the barrier problem, then there exists a penalty parameter $\rho > 0$, so that x^μ_* is a strict local minimizer of $\ell_\rho(x, \lambda^\mu_*)$.

In the line search filter method described in Section 2 we can alternatively follow an approach based on the augmented Lagrangian function rather than on the exact penalty function, by splitting the augmented Lagrangian function (88) into its two components $\mathcal{L}_\mu(x, \lambda)$ (defined in (6)) and $\theta(x)$. In Algorithm I we then replace all occurrences of the measure "$\varphi_\mu(x)$" by "$\mathcal{L}_\mu(x, \lambda)$". In addition to the iterates x_k we now also keep iterates λ_k as estimates of the equality constraint multipliers, and compute in each iteration k a search direction d^λ_k for those variables. This search direction can be obtained, for example, with no additional computational cost as $d^\lambda_k := \lambda^+_k - \lambda_k$ with λ^+_k from (5) or (23). Defining

$$\lambda_k(\alpha_{k,l}) := \lambda_k + \alpha_{k,l} d^\lambda_k,$$

the sufficient reduction criteria (12b) and (15) are then replaced by

$$\begin{aligned}
\mathcal{L}_\mu(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l})) &\leq \mathcal{L}_\mu(x_k, \lambda_k) - \gamma_\varphi \theta(x_k) \qquad \text{and} \\
\mathcal{L}_\mu(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l})) &\leq \mathcal{L}_\mu(x_k, \lambda_k) + \eta_\varphi m_k(\alpha_{k,l}),
\end{aligned}$$

respectively, where the model m_k for $\mathcal{L}_\mu$ is now defined as

$$
\begin{aligned}
m_k(\alpha) &:= \alpha \nabla\varphi_\mu(x)^T d_k - \alpha\lambda_k^T c(x_k) + \alpha(1-\alpha)c(x_k)^T d_k^\lambda && (89)\\
&= \mathcal{L}_\mu(x_k + \alpha d_k, \lambda_k + \alpha d_k^\lambda) - \mathcal{L}(x_k, \lambda_k) + O(\alpha^2) && (90)
\end{aligned}
$$

which is obtained by Taylor expansions of $\varphi_\mu(x)$ and $c(x)$ around x_k into direction d_k and the use of (5).

The switching condition (13) remains unchanged, but the definition of the minimum step size (21) has to be changed accordingly. The only requirements for this change are again that it is guaranteed that the method does not switch to the feasibility restoration phase in Step 5.2 as long as the switching condition (13) is satisfied for a trial step size $\alpha \leq \alpha_{k,l}$, and that the backtracking line search in Step 5 is finite.

One can verify that the global convergence analysis in Section 3 still holds with minor modifications [28]. Concerning local convergence, however, it is not clear to us at this point whether fast local convergence can also be achieved when the measure "$\mathcal{L}_\mu(x,\lambda)$" is used.

5.2 Line Search Filter SQP Methods

In this section we show how Algorithm I can be applied to line search SQP methods for the solution of nonlinear optimization problems of the form

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & f(x) && (91a)\\
\text{s.t.} \quad & c^{\mathcal{E}}(x) = 0 && (91b)\\
& c^{\mathcal{I}}(x) \geq 0, && (91c)
\end{aligned}
$$

where the functions f and $c := (c^{\mathcal{E}}, c^{\mathcal{I}})$ have the smoothness properties of f and c in Assumptions (G1) and (L1). A line search SQP method obtains search directions d_k as the solution of the quadratic program (QP)

$$
\begin{aligned}
\min_{d\in\mathbb{R}^n} \quad & g_k^T d + \frac{1}{2} d^T H_k d && (92a)\\
\text{s.t.} \quad & (A_k^{\mathcal{E}})^T d + c^{\mathcal{E}}(x_k) = 0 && (92b)\\
& (A_k^{\mathcal{I}})^T d + c^{\mathcal{I}}(x_k) \geq 0, && (92c)
\end{aligned}
$$

where $g_k := \nabla f(x_k)$, $A_k^{\mathcal{E}} := \nabla c^{\mathcal{E}}(x_k)$, $A_k^{\mathcal{I}} := \nabla c^{\mathcal{I}}(x_k)$, and H_k is (an approximation of the) Hessian of the Lagrangian

$$\mathcal{L}(x, \lambda, v) = f(x) + (c^{\mathcal{E}}(x))^T\lambda - (c^{\mathcal{I}}(x))^T v$$

of the NLP (91) with the Lagrange multipliers $v \geq 0$ corresponding to the inequality constraints (91c). We will denote the optimal QP multipliers corresponding to (92b) and (92c) with λ_k^+ and $v_k^+ \geq 0$, respectively.

We further define the infeasibility measure by

$$\theta(x) := \left\| \begin{pmatrix} c^{\mathcal{E}}(x) \\ c^{\mathcal{I}}(x)^{(-)} \end{pmatrix} \right\|,$$

where for a vector w the expression $w^{(-)}$ defines the vector with the components $\max\{0, -w^{(i)}\}$. Algorithm I can then be used with the following modifications.

1. All occurrences of "φ_μ" are replaced by "f".

2. The computation of the search direction in Step 3 is replaced by the solution of the QP (92). The restoration phase is invoked in this step, if the QP (92) is infeasible or not "sufficiently consistent" (see Assumption (G5**) below).

3. The fraction-to-the-boundary rule is no longer necessary, i.e. in Step 4 we always choose $\alpha_k^{\max} = 1$.

In order to state the assumptions necessary for a global convergence analysis let us again consider a decomposition of the search direction

$$d_k = q_k + p_k \tag{93}$$

where q_k is now defined as the solution of the QP

$$\begin{aligned} \min_{q \in \mathbb{R}^n} \quad & q^T q \\ \text{s.t.} \quad & (A_k^{\mathcal{E}})^T q + c^{\mathcal{E}}(x_k) = 0 \\ & (A_k^{\mathcal{I}})^T q + c^{\mathcal{I}}(x_k) \geq 0, \end{aligned}$$

i.e. q_k is the shortest vector satisfying the constraints in the QP (92), and p_k is simply defined as $d_k - q_k$. With these definitions we can now replace Assumptions (G5) and (G6) by

(G5**) *There exist constants $M_d, M_\lambda, M_v, M_q > 0$, so that for all $k \notin \mathcal{R}_{\text{inc}}$ we have*

$$\|d_k\| \leq M_d, \quad \|\lambda_k^+\| \leq M_\lambda, \quad \|v_k^+\| \leq M_v, \quad \|q_k\| \leq M_q \theta(x_k)$$

(G6**) *There exists a constant $M_H > 0$, so that for all $k \notin \mathcal{R}_{\text{inc}}$ we have*

$$d_k^T H_k d_k \geq M_H d_k^T d_k. \tag{94}$$

Assumption (G5**) is similar to the assumption expressed by Eq. (2.10) in [10]. Essentially, we assume that if the constraints of the QPs (92) become increasingly ill-conditioned, eventually the restoration phase will be triggered in Step 3. Together with Assumption (G4) this assumption also means that we suppose that the QP (92) is sufficiently consistent when feasible points are approached.

Assumption (G6**) again ensures descent in the objective function at sufficiently feasible points. This assumption has been made previously for global convergence proofs of SQP line search methods (see e.g. [24]). However, this assumption can be rather strong since even close to a strict local solution the exact Hessian might have to be modified in order to satisfy (94). In [28] an alternative and more natural assumption is considered for the NLP formulation (1) which only allows bound constraints as inequality constraints.

In order to see that the global convergence analysis in Section 3 still holds under the modified Assumptions G, let us first note that the objective function of the nonlinear problem solved by Algorithm I is now bounded since no "ln"-terms appear in the NLP (91) in contrast to the barrier problem (2). Therefore, the scaling (23) of the linear system (5) is no longer necessary. After defining the criticality measure again as $\chi(x_k) := \|p_k\|_2$ for $k \notin \mathcal{R}_{\text{inc}}$, the proofs are valid with minor straightforward modifications and with all occurrences of "φ_μ" and "$\tilde{d}_k$" replaced by "f"

and "d_k", respectively. Only the proof of Lemma 2 deserves special attention. From the optimality conditions for the QP (92) it follows in particular that

$$
\begin{aligned}
g_k + H_k d_k + A_k^{\mathcal{E}} \lambda_k^+ - A_k^{\mathcal{I}} v_k^+ &= 0 && \text{(95a)}\\
\left((A_k^{\mathcal{I}})^T d_k + c^{\mathcal{I}}(x_k)\right)^T v_k^+ &= 0 && \text{(95b)}\\
v_k^+ &\geq 0, && \text{(95c)}
\end{aligned}
$$

so that for all $k \notin \mathcal{R}_{\text{inc}}$

$$
\begin{aligned}
g_k^T d_k &\overset{(95a)}{=} -d_k^T H_k d_k - d_k^T A_k^{\mathcal{E}} \lambda_k^+ + d_k^T A_k^{\mathcal{I}} v_k^+\\
&\overset{(92b),(95b)}{=} -d_k^T H_k d_k + c^{\mathcal{E}}(x_k)^T \lambda_k^+ - c^{\mathcal{I}}(x_k)^T v_k^+\\
&\overset{(95c)}{\leq} -d_k^T H_k d_k + c^{\mathcal{E}}(x_k)^T \lambda_k^+ + \left(c^{\mathcal{I}}(x_k)^{(-)}\right)^T v_k^+\\
&\overset{(93)}{\leq} -M_H[\chi(x_k)]^2 + O\left(\chi(x_k)\theta(x_k)\right) + O(\theta(x_k))
\end{aligned}
$$

where we used Assumptions (G5**) and (G6**) in the last inequality. This corresponds to the second last line in (32), and we can conclude the proof of Lemma 2 as before.

Also the discussion of local convergence in Section 4 still applies if we assume that eventually the active set is not changed. To guarantee this, the computation of the second order correction step (57) and the feasibility restoration phase proposed in Remark 5 have to be adapted in order to take the active inequality constraints into account.

5.3 Fast Local Convergence of a Trust Region Filter SQP Method

As briefly mentioned in Section 2, the switching rule used in the trust region SQP-filter algorithm proposed by Fletcher et. al. [10] does not imply the relationship (69), and therefore the proof of Lemma 14 in our local convergence analysis does not hold for that method. However, it is easy to see that the global convergence analysis in [10] is still valid (in particular Lemma 3.7 and Lemma 3.10 in [10]), if the switching rule Eq. (2.19) in [10] is modified in analogy to (13) and replaced by

$$[m_k(d_k)]^{s_\varphi} \Delta_k^{1-s_\varphi} \geq \kappa_\theta \theta_k^\varphi,$$

where m_k is now the change of the objective function predicted by a quadratic model of the objective function, Δ_k the current trust region radius, $\kappa_\theta, \varphi > 0$ constants from [10] satisfying certain relationships, and the new constant $s_\varphi > 0$ satisfies $s_\varphi > 2\varphi$. Then the local convergence analysis in Section 4 is still valid (also for the quadratic model formulation), assuming that sufficiently close to a strict local solution the trust region is inactive, the trust region radius Δ_k is uniformly bounded away from zero, the (approximate) SQP steps s_k are computed sufficiently exact, and a second order correction as discussed in Section 4.1 is performed.

6 Conclusions

A framework for line search filter methods that can be applied to barrier methods and active set SQP methods has been presented. Global convergence has been shown under mild assumptions, which are, in particular, less restrictive than those made previously for some line search IP methods. The switching rule (13), differing from previously proposed rules, allows furthermore to establish that the Maratos effect can be avoided using second order corrections. We also proposed an alternative measure for the filter, using the Lagrangian function instead of the objective function, for which the global convergence properties still hold. However, it is subject to future research whether this new approach leads also to fast local convergence, possibly without the need of a second order correction step.

In a future paper we will present practical experience with the line search filter barrier method proposed in this paper. So far, our numerical results are very promising [28].

Acknowledgements.

Funding from the National Science Foundation (DMS-9706950) and by an unrestricted grant from the ExxonMobil Corporation is gratefully acknowledged. We are very indebted to Jorge Nocedal for intriguing discussions, in particular at the beginning of the work on this paper. We thank Caroline Sainvitu and Philippe Toint for their careful study of the manuscript and for catching several typos. We also thank an anonymous referee for his/her insightful comments which helped to improve the presentation of this paper.

References

[1] C. Audet and J. E. Dennis. A pattern search filter method for nonlinear programming without derivatives. Technical Report TR00-09, Department of Computational and Applied Mathematics, Rice University, Houston, TX, USA, March 2000.

[2] H. Y. Benson, D. F. Shanno, and R. J. Vanderbei. Interior-point methods for nonconvex nonlinear programming: Filter methods and merit functions. Technical report, Operations Research and Financial Engineering, Princeton University, Princeton, NJ, USA, December 2000.

[3] P. T. Boggs, J. W. Tolle, and P. Wang. On the local convergence of quasi-Newton methods for constrained optimization. *SIAM Journal on Control and Optimization*, 20:161–171, 1982.

[4] R. H. Byrd, J. Ch. Gilbert, and J. Nocedal. A trust region method based on interior point techniques for nonlinear programming. *Mathematical Programming*, 89:149–185, 2000.

[5] R. H. Byrd, G. Liu, and J. Nocedal. On the local behavior of an interior point method for nonlinear programming. In D. F. Griffiths and D. J. Higham, editors, *Numerical Analysis 1997*. Addison–Wesley Longman, Reading, MA, USA, 1997.

[6] R. M. Chamberlain, C. Lemarechal, H. C. Pedersen, and M. J. D. Powell. The watchdog technique for forcing convergence in algorithms for constrained optimization. *Mathematical Programming Study*, 16:1–17, 1982.

[7] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. SIAM, Philadelphia, PA, USA, 2000.

[8] J. E. Dennis and J. J. Moré. Quasi-Newton methods, motivation and theory. *SIAM Review*, 19(1):46–89, 1977.

[9] A. S. El-Bakry, R. A. Tapia, T. Tsuchiya, and Y. Zhang. On the formulation and theory of the Newton interior-point method for nonlinear programming. *Journal of Optimization Theory and Application*, 89(3):507–541, 1996.

[10] R. Fletcher, N. I. M. Gould, S. Leyffer, Ph. L. Toint, and A. Wächter. Global convergence of a trust-region SQP-filter algorithms for general nonlinear programming. Technical Report 99/03, Department of Mathematics, University of Namur, Belgium, May 1999. Revised October 2001. To appear in SIAM Journal on Optimization.

[11] R. Fletcher and S. Leyffer. A bundle filter method for nonsmooth nonlinear optimization. Technical Report NA/195, Department of Mathematics, University of Dundee, Scotland, December 1999.

[12] R. Fletcher and S. Leyffer. Nonlinear programming without a penalty function. *Mathematical Programming*, 91(2):239–269, 2002.

[13] R. Fletcher, S. Leyffer, and Ph. L. Toint. On the global convergence of a filter-SQP algorithm. *SIAM Journal on Optimization*, 13(1):44–59, 2002.

[14] D. M. Gay, M. L. Overton, and M. H. Wright. A primal-dual interior method for nonconvex nonlinear programming. In Y. Yuan, editor, *Advances in Nonlinear Programming*, pages 31–56. Kluwer Academic Publishers, Dordretch, 1998.

[15] N. I. M. Gould, D. Orban, A. Sartenaer, and P. L. Toint. Componentwise fast convergence in the solution of full-rank systems of nonlinear equations. Technical Report TR/PA/00/56, CERFACS, Toulouse, France, 1999. Revised May 2001.

[16] N. I. M. Gould, D. Orban, A. Sartenaer, and P. L. Toint. Superlinear convergence of primal-dual interior point algorithms for nonlinear programming. *SIAM Journal on Optimization*, 11(4):974–1002, 2001.

[17] S.-P. Han. A globally convergent method for nonlinear programming. *Journal of Optimization Theory and Application*, 22:297–309, 1977.

[18] N. Maratos. *Exact Penalty Function Algorithms for Finite Dimensional and Control Optimization Problems*. PhD thesis, University of London, London, UK, 1978.

[19] M. Marazzi and J. Nocedal. Feasibility control in nonlinear optimization. In A. DeVore, A. Iserles, and E. Suli, editors, *Foundations of Computational Mathematics*, volume 284 of *London Mathematical Society Lecture Note Series*, pages 125–154. Cambridge University Press, 2001.

[20] S. G. Nash and A. Sofer. Why extrapolation helps barrier methods. Technical report, Operations Research and Engineering Department, George Mason University, Fairfax, A 22030, USA, September 1998.

[21] J. Nocedal and M. Overton. Projected Hessian updating algorithms for nonlinearly constrained optimization. *SIAM Journal on Numerical Analysis*, 22:821–850, 1985.

[22] J. Nocedal and S. Wright. *Numerical Optimization*. Springer, New York, NY, USA, 1999.

[23] M. J. D. Powell. A hybrid method for nonlinear equations. In P. Rabinowitz, editor, *Numerical Methods for Nonlinear Algebraic Equations*, pages 87–114. Gordon and Breach Science Publishers, London, New York, Paris, 1970.

[24] K. Schittkowski. The nonlinear programming method of Wilson, Han and Powell with an augmented Lagrangian type line search function. *Numerische Mathematik*, 38, 1981.

[25] A.L. Tits, T.J. Urban, S. Bakhtiari, and C.T. Lawrence. A primal-dual interior-point method for nonlinear programming with strong global and local convergence properties. Technical Report TR 2001-3 R2, Department of Electrical and Computer Engineering and Institute for Systems Research, Univerity of Maryland, College Park, MD, July 2001.

[26] M. Ulbrich, S. Ulbrich, and L. N. Vicente. A globally convergent primal-dual interior-point filter method for nonconvex nonlinear programming. Technical Report TR00-12, Department of Computational and Applied Mathematics, Rice University, Houston, TX, USA, April 2000. Revised February 2002.

[27] R. J. Vanderbei and D. F. Shanno. An interior-point algorithm for nonconvex nonlinear programming. *Computational Optimization and Applications*, 13:231–252, 1999.

[28] A. Wächter. *An Interior Point Algorithm for Large-Scale Nonlinear Optimization with Applications in Process Engineering*. PhD thesis, Carnegie Mellon University, Pittsburgh, PA, USA, January 2002.

[29] A. Wächter and L. T. Biegler. Failure of global convergence for a class of interior point methods for nonlinear programming. *Mathematical Programming*, 88(2):565–574, 2000.

[30] H. Yamashita. A globally convergent primal-dual interior-point method for constrained optimization. *Optimization Methods and Software*, 10:443–469, 1998.

[31] H. Yamashita, H. Yabe, and T. Tanabe. A globally and superlinearly convergent primal-dual interior point trust region method for large scale constrained optimization. Technical report, Mathematical System Institute, Inc., Tokyo, Japan, July 1997. Revised July 1998.