Smoothing Method of Multipliers for Sum-Max Problems

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Abstract

We study nonsmooth unconstrained optimization problem, which includes sum of pairwise maxima of smooth functions. Minimum l_1-norm approximation is a particular case of this problem. Combining ideas Lagrange multipliers with smooth approximation of max-type function, we obtain a new kind of nonquadratic augmented Lagrangian. Our approach does not require artificial variables, and preserves sparse structure of Hessian in many practical cases. We present the corresponding method of multipliers, and its convergence analysis for a dual counterpart, resulting in a proximal point maximization algorithm. The practical efficiency of the algorithm is supported by computational results for large-scale problems, arising in structural optimization.

1 Introduction

We consider non-smooth convex unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} \left\{ F(x) = f(x) + \sum_{i=1}^{m} \max\left[\alpha_i h_i(x),\ \beta_i h_i(x)\right] \right\} \tag{1}$$

where $f(x)$ and $h_i(x),\ i=1,\dots,m$ are smooth convex functions, defined over entire space $\mathbb{R}^n$; $\alpha_i < \beta_i$ are certain constants. In order to guarantee convexity, we suppose, that for any particular index i, the values α_i and β_i are non-negative, if $h_i(x)$ is nonlinear. Problem in this setting arises for example in Truss Topology Design [1, 5]. This is a generalization of least l_1 norm approximation (regularization) problem, when the coefficients $\alpha_i = -1,\ \beta_i = 1$, and the functions $h_i(x)$ are affine:

$$F(x) = f(x) + \sum_{i=1}^{m} |h_i(x)| = f(x) + ||h(x)||_1\ , \qquad i = 1,\dots,m, \tag{2}$$

where $h(x) =: [h_1(x), h_2(x), .., h_m(x)]^T$. This kind of problems arises in many important areas of modern signal/image processing, in context of sparse representations, blind source separation, minimal total variation solutions, *etc.*(see for example [8, 21, 22]).

Further, for simplicity of notation, we consider the case when $\alpha_i \equiv \alpha$, $\beta_i \equiv \beta$ are independent of i; the extension onto the general case is quite straightforward.

Augmented Lagrangian approach Methods of multipliers, involving *non-quadratic augmented Lagrangians* [10, 6, 15, 3, 19, 20, 4, 7, 11] successfully compete with the interior-point and other methods in non-linear and semidefinite programming. They are especially efficient when a very high accuracy of solution is required. This success is partially explained by the fact, that due to iterative update of multipliers, the penalty parameter does not need to become extremely small in the neighborhood of solution. Additional improvement of numerical efficiency was obtained by using a "soft" *quadratic-logarithmic* penalty function, introduced in [4, 20]. This function has a bounded third derivative and hence is well matched to the Newton method.

In this article we combine ideas of method of multipliers with smooth approximation of a max-type function [2, 9]. This leads to a new type of augmented Lagrangian for sum-max problem, which does not require artificial variables. We show, that the powerful tool for analysis of augmented Lagrangian algorithms, based on their correspondence to the proximal point methods in dual space [18, 13, 14, 4], can be extended to our case. The practical efficiency of the algorithm is supported by computational results for large-scale problems, arising in structural optimization.

2 Smoothing of Max-function

Let us introduce a parameterized smooth approximation of the maximum function

$$r(t) = \max(\alpha t, \beta t) \ ,$$

shown at Fig.1 by solid line. The smoothing function

$$\varphi(t; \mu, c), \ \ \alpha < \mu < \beta, \ \ c > 0$$

(Fig.1, dashed line), has two parameters: μ and c. Parameter c defines accuracy of approximation of $r(\cdot)$: the approximation becomes perfect as $c \to \infty$. Parameter μ determines derivative of φ at $t = 0$ and will be used as an analog to Lagrange multiplier. The graph of the linear function μt (Fig.1, dot-dashed line) is tangent to the plot of $\varphi(\cdot; \mu, c)$ at the origin.

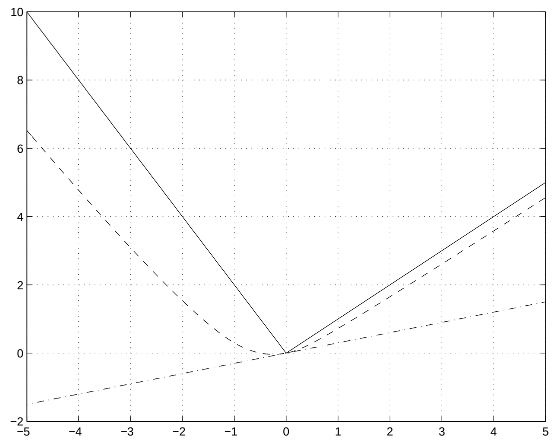

Figure 1: $\mathrm{Max}(\alpha t, \beta t)$ – solid line; $\varphi(t; \mu, c)$ – dashed line; μt - dot-dashed line

The function φ possesses the following properties:

(φ1) $\varphi(t; \mu, c)$ is convex in t;

(φ2) $\varphi(0; \mu, c) = 0$;

(φ3) $\varphi'_t(0; \mu, c) = \mu$;

(φ4) $\lim_{t \to -\infty} \varphi'_t(t; \mu, c) = \alpha$;

(φ5) $\lim_{t \to +\infty} \varphi'_t(t; \mu, c) = \beta$;

(φ6) $\lim_{c \to +\infty} \varphi(t; \mu, c) = r(t)$;

The particular form of function φ we introduce and prefer to use in our computations consists of three smoothly connected branches:

$$\varphi(t, \mu, c) = \begin{cases} \alpha t - p_1 \log \dfrac{t}{\tau_1} + s_1, & t < \tau_1 \leq 0 \\ \dfrac{ct^2}{2} + \mu t, & \tau_1 \leq t \leq \tau_2 \\ \beta t - p_2 \log \dfrac{t}{\tau_2} + s_2, & t > \tau_2 \geq 0 \ . \end{cases} \tag{3}$$

The coefficients $p_1, p_2, s_1, s_2, \tau_1, \tau_2$ are chosen to make the function φ continuous and twice differentiable at the joint points τ_1 and τ_2:

$$\begin{aligned} \tau_1 &= \frac{\alpha - \mu}{2c}; \quad \tau_2 = \frac{\beta - \mu}{2c}; \\ p_1 &= c\tau_1^2; \qquad p_2 = c\tau_2^2; \end{aligned}$$

$$s_1 = \frac{c}{2}\tau_1^2 + (\mu - \alpha)\tau_1;$$
$$s_2 = \frac{c}{2}\tau_2^2 + (\mu - \beta)\tau_2.$$

Remark: when $\tau_i = 0$, we put $p_i \log \frac{t}{\tau_i} = c\tau_i^2 \log \frac{t}{\tau_i} = 0$.

One can note, that when $\alpha \to 0$ and $\beta \to \infty$, $\varphi(\cdot)$ becomes a *quadratic-logarithmic* penalty for inequality constraint in nonquadratic augmented Lagrangian [4]. On the other hand, when $\alpha \to -\infty$ and $\beta \to \infty$, $\varphi(\cdot)$ becomes a quadratic penalty for equality constraint in a standard quadratic augmented Lagrangian [16]. In this way our approach generalizes known augmented Lagrangian techniques.

3 Lagrangian, Augmented Lagrangian and Duality of Sum-Max Problem

Now we introduce Lagrangian of sum-max problem (1)

$$L(x, u) = f(x) + \sum_i u_i h_i(x), \quad \alpha \leq u_i \leq \beta. \tag{4}$$

and augmented Lagrangian

$$M(x, u, c) = f(x) + \sum_{i=1}^{m} \varphi(h_i(x), u_i, c), \quad \alpha \leq u_i \leq \beta. \tag{5}$$

3.1 Duality of Sum-Max Problem

Here we consider a case when all the functions $h_i(.)$ are convex, $\beta > \alpha$ and $\alpha > 0$ when $h_i(.)$ is nonlinear .

Denote

$$I_n(x) = \{i : h_i(x) < 0\}; \tag{6}$$
$$I_0(x) = \{i : h_i(x) = 0\}; \tag{7}$$
$$I_p(x) = \{i : h_i(x) > 0\}; \tag{8}$$

Lemma 1 [necessary and sufficient optimality conditions]
A vector $x^* \in \mathbb{R}^n$ is a solution of the problem (1) iff there exists a vector $u^* \in \mathbb{R}^m$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} u_i^* \nabla h_i(x^*) = 0 \ , \tag{9}$$
$$u_i^* = \alpha, \qquad i \in I_n(x^*) \ , \tag{10}$$
$$u_i^* = \beta, \qquad i \in I_p(x^*) \ , \tag{11}$$
$$\alpha \leq u_i^* \leq \beta, \ i \in I_0(x^*) \ . \tag{12}$$

Proof. Subdifferential set of the convex function $F(x)$ in (1) can be represented as

$$\partial F(x) = \{\nabla f(x) + \sum_{i=1}^{m} u_i \nabla h_i(x)\} \ , \tag{13}$$

where

$$u_i = \alpha, \qquad i \in I_n(x), \tag{14}$$

$$u_i = \beta, \qquad i \in I_p(x), \tag{15}$$

$$\alpha \le u_i \le \beta, \ i \in I_0(x), \tag{16}$$

so (9) - (12) represent sufficient and necessary optimality condition

$$0 \in \partial F(x^*) \tag{17}$$

(see the subdifferential calculus developed in [16, 12]) □

Lemma 2 *For any given* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$, $\alpha \le u \le \beta$, *the following inequalities take place:*

$$F(x) \ge M(x, u, c) \ge L(x, u) \ . \tag{18}$$

Proof. Follows immediately from the inequality

$$\max(\alpha t, \beta t) \ge \varphi(t, \mu, c) \ge \mu t \ , \quad \alpha \le \mu \le \beta \tag{19}$$

illustrated by Fig.1. □

Theorem 1 [saddle point of Lagrangian]
A vector x^* *is a solution of the problem (1) iff there exists a vector* $u^* \in \mathbb{R}^m$, $\alpha \le u^* \le \beta$, *such that the pair* (x^*, u^*) *is a saddle point of the Lagrangian (4):*

$$L(x^*, u) \le L(x^*, u^*) \le L(x, u^*) \quad \forall x \in \mathbb{R}^n, \alpha \le u \le \beta \tag{20}$$

Proof. Suppose that x^* is a solution of the problem (1). Then by Lemma 1, conditions (9) – (12) hold. By (9)

$$\nabla_x L(x^*, u^*) = 0, \tag{21}$$

so by convexity of the Lagrangian with respect to x

$$x^* \in \arg\min_x L(x, u^*) \tag{22}$$

It is easy to check, that by (10) – (12)

$$u_i h_i(x) = \max(\alpha h_i(x), \beta h_i(x)),$$

hence $L(x^*, u^*) = F(x^*, u^*)$. Taking into account inequality (18), we obtain

$$L(x^*, u^*) \ge L(x^*, u) \ \ , \quad \alpha \le u \le \beta. \tag{23}$$

So by (22) and (23) the pair (x^*, u^*) is a saddle point.

Conversely, suppose that (x^*, u^*) is a saddle point of the Lagrangian. By definition (22) holds, hence (9) holds as well. By (20) and (4) we have

$$\sum_i u_i^* h_i(x^*) \geq \sum_i u_i h_i(x^*), \quad \alpha \leq u_i \leq \beta. \tag{24}$$

This implies that (10) and (11) must be satisfied, otherwise we could violate (24) putting corresponding u_i to α or β. Hence, conditions (9) – (12) hold, and by Lemma 1 the vector x^* is an optimal solution for the problem (1). □

Now we can introduce a dual function

$$G(u) = \min_x L(x, u). \tag{25}$$

By Theorem 1

$$F(x^*) = \min F(x) = \max_{\alpha \leq u \leq \beta} G(u) = G(u^*). \tag{26}$$

4 Smoothing Method of Multipliers

We introduce the following method of multipliers. The algorithm is iterative; at iteration k it performs the following steps:

1. Minimize augmented Lagrangian in x

$$x^{k+1} = \arg\min_x M(x, u^k, c_k); \tag{27}$$

2. Update the multipliers

$$u_i^{k+1} = \varphi'(h_i(x_{k+1}), u_i^k, c_k), \tag{28}$$

   where the derivative φ' is taken with respect to the first argument

3. Update the smoothing parameter (optional)

$$c_{k+1} = \gamma c_k, \quad \gamma > 1. \tag{29}$$

In practical implementation we sometimes restrict relative change of multipliers to some bounds in order to stabilize the method:

$$\gamma_1 < \frac{u_i^{k+1} - \alpha}{u_i^k - \alpha} < \gamma_2 \tag{30}$$

$$\gamma_1 < \frac{\beta - u_i^{k+1}}{\beta_i - u_i^k} < \gamma_2 \tag{31}$$

$$\alpha + \delta < u_i^{k+1} < \beta - \delta \tag{32}$$

We also restrict the smoothing parameter by some maximal value c_{max}. In numerical experiments presented in this work, our choice of the parameters was $\gamma = \gamma_1 = \frac{1}{\gamma_2} = 2$, $\delta = 10^{-6}$, $c_{max} = 10^3$. In general, the algorithm is rather insensitive to changes in the parameters in order of magnitude or more.

5 Proximal Point Algorithm as a Dual Interpretation of the Method

We show in this section that the algorithm (27 – 29) generates the same sequence $\{u^k\}$ as an appropriate (non-quadratic) proximal point algorithm applied to the maximization of the dual objective function G over box

$$\max_{\alpha \le u \le \beta} G(u) \ , \tag{33}$$

and present the proof of convergence of the algorithm.

Proposition 1 *After the update of multipliers (28) the vector x^{k+1} becomes a minimizer of the Lagrangian, which gives the value of the dual function*

$$L(x^{k+1}, u^{k+1}) = \min_x L(x, u^{k+1}) = G(u^{k+1}) \ . \tag{34}$$

Proof. By optimality condition for (27)

$$0 = f'(x^{k+1}) + \sum_{i=1}^{m} \varphi'[g_i(x^{k+1}), u_i^k, c_k] h_i'(x^{k+1}) \ . \tag{35}$$

Taking into account the updating formula (28), we obtain

$$0 = f'(x^{k+1}) + \sum_{i=1}^{m} u_i^{k+1} h_i'(x^{k+1}) \ , \tag{36}$$

which is exactly optimality condition for the Lagrangian $L(x^{k+1}, u^{k+1})$ with respect to the first argument. □

Proposition 2 *If $\bar{x}$ is a minimizer of the Lagrangian*

$$L(\bar{x}, \bar{u}) = \min_x L(x, \bar{u}) = G(\bar{u}) \ , \tag{37}$$

then

$$h(\bar{x}) \in \partial G(\bar{u}) \ , \tag{38}$$

where $\partial G(\bar{u})$ denotes subgdifferential set of the concave function G, and $h(x) =: (h_1(x), h_2(x), .., h_m(x))^T$

Proof. We will show that the vector h can be used as a gradient in Gradient Inequality for the concave function G(u)

$$\begin{aligned} G(u) &= \min_x \{f(x) + \sum u_i h_i(x)\} \le f(\bar{x}) + \sum u_i h_i(\bar{x}) \\ &= f(\bar{x}) + \sum \bar{u}_i h_i(\bar{x}) + \sum (u_i - \bar{u}_i) h_i(\bar{x}) \\ &= G(\bar{u}) + (u - \bar{u})^T h(\bar{x}) \end{aligned}$$

□

Lemma 3 [correspondence between method of multipliers and prox algorithm] *The algorithm (27),(28),(29) generates the same sequence of multipliers* $\{u_k\}$ *as a proximal point algorithm given by the following recurrent relation*

$$u^{k+1} = \arg\max_u \{G(u) - D_k(u, u^k)\}, \tag{39}$$

where the proximal term D_k *is given by*

$$D_k(u, u^k) = \sum_{i=1}^m \varphi^*(u_i, u_i^k, c_k) \ , \tag{40}$$

and φ^* *is the conjugate function of* φ *with respect to the first argument, given by Legendre transformation (see e.g. [16])*

$$\varphi^*(\lambda, \mu, c) = \sup_t \{\lambda t - \varphi(t, \mu, c)\}. \tag{41}$$

Proof. By the property of conjugate function

$$(\varphi^*)' = (\varphi')^{-1} \tag{42}$$

and updating formula (28), we obtain

$$(\varphi^*)'(u_i^{k+1}, u_i^k, c_k) = h_i(x_{k+1}). \tag{43}$$

So the gradient of the proximal term with respect to the first argument will be

$$\nabla_1 D_k(u^{k+1}, u^k) = h(x_{k+1}); \tag{44}$$

Taking into account (34) and (38), we obtain

$$\nabla_1 D_k(u^{k+1}, u^k) \in \partial G(u^{k+1}) \ , \tag{45}$$

which is precisely the necessary and sufficient condition for u^{k+1} to attain the maximum in (39). □

As example, the conjugate of the quadratic-logarithmic function (3) can be easily obtained by integrating the inverse of its derivative (see Figure 2)

$$\varphi^*(\lambda, \mu, c) = \begin{cases} -p_1 \log \dfrac{\lambda - \alpha}{l_1 - \alpha} + \dfrac{1}{2c}(l_1 - \mu)^2, & \alpha < \lambda < l_1 \\ \dfrac{1}{2c}(\lambda - \mu)^2, & l_1 \le \lambda \le l_2 \\ -p_2 \log \dfrac{\lambda - \beta}{l_2 - \beta} + \dfrac{1}{2c}(l_2 - \mu)^2, & l_2 < \lambda < \beta \end{cases} \tag{46}$$

where

$$l_1 = c\tau_1 + \mu; \quad l_2 = c\tau_2 + \mu;$$

To simplify notation we will omit the third argument in φ and denote

$$\psi(\lambda, \mu) \equiv \varphi^*(\lambda, \mu) \tag{47}$$

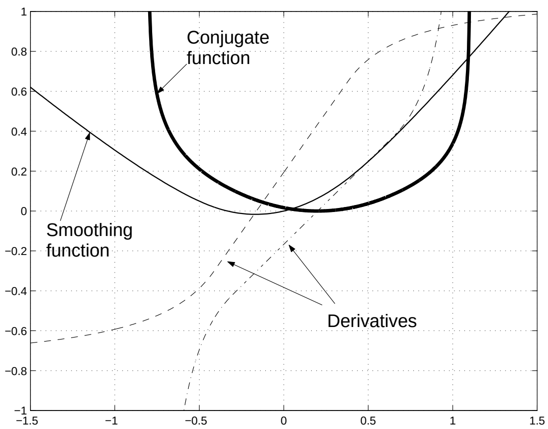

Figure 2: Smoothing function φ (solid line) and its conjugate ψ (bold solid line). Corresponding derivatives (dashed and dot-dashed lines) are mutually inverse functions, hence their plots are mutually symmetric with respect to the 45^o line.

Let $\alpha < \mu < \beta$. The conjugate function ψ has the following properties:

(ψ1.) ψ_1' increases monotonically
[since φ_1' increases monotonically and $\psi_1' = (\varphi_1')^{-1}$]

(ψ2.) $\psi(\cdot, \mu)$ is convex [by ψ1]

(ψ3.) $\psi_1'(\mu, \mu) = 0$ [since $\varphi_1'(0, \mu) = \mu$]

(ψ4.) $\psi(\lambda, \mu) = \infty$ for $\lambda < \alpha$ or $\lambda > \beta$ (barrier property)
$\lim_{\lambda \searrow \alpha} \psi_1'(\lambda, \mu) = -\infty, \quad \lim_{\lambda \nearrow \beta} \psi_1'(\lambda, \mu) = \infty$
[since $\lim_{t \to -\infty} \varphi'(t, \mu) = \alpha; \quad \lim_{t \to \infty} \varphi'(t, \mu) = \beta$]

(ψ5.) $\psi(\lambda, \lambda) = 0$
[$\psi(\lambda, \lambda) = \varphi^*(\lambda, \lambda) =: \sup(t\lambda - \varphi(t, \lambda)) = -\varphi(0, \lambda) = 0$ since $\varphi\prime(0, \lambda) = \lambda$]

(ψ6.) $\psi(\lambda, \mu) \geq 0$ [by ψ5, ψ3 and ψ2].

Denote

$$\text{Box}(\alpha, \beta) = \{u \in \mathbb{R}^m : \alpha \leq u \leq \beta\} . \tag{48}$$

Due to ($\psi 1 - \psi 6$), the proximal term (40) is a convex function with respect to the first argument and the following properties are satisfied:

(D1.) $D(u, v) \geq 0$, $u, v \in \text{Box}(\alpha, \beta)$

(D2.) $D(u, u) = 0$, $u \in \text{interior}(\text{Box}(\alpha, \beta))$

(D3.) $D(u, v) = \infty$ if $v \in \text{interior}(\text{Box}(\alpha, \beta))$, $u \notin \text{Box}(\alpha, \beta)$

Theorem 2 [monotonicity and convergence of the proximal point algorithm]
Let the proximal term (40) of the the algorithm (39) be based on the conjugate of the quadratic-logarithmic function (3). Then

(a) the sequence $\{u^k\}$ generated by the prox-algorithm (34) belongs to the $\mathrm{Box}(\alpha, \beta)$ and the sequence of function values $G(u^k)$ is nondecreasing;

(b) the set of accumulation points of the method (34) is nonempty and belongs to the solution set of the dual problem (33)

Proof.

(a) The sequence $\{u^k\}$ generated by the method (34) belongs to the $\mathrm{Box}(\alpha, \beta)$ (48) by barrier property ($D3$). Now:

$$\begin{aligned} G(u^{k+1}) - D(u^{k+1}, u^k) &\geq G(u^k) - D(u^k, u^k) \\ &= G(u^k)\ , \quad \text{by} \quad (D2)\ . \end{aligned}$$

Hence

$$G(u^{k+1}) - G(u^k) \geq D(u^{k+1}, u^k) \geq 0 \quad \text{by} \quad (D1)\ . \tag{49}$$

(b) $\mathrm{Box}(\alpha, \beta)$ is a compact set, hence the sequence $\{u^k\}$ has accumulation points. Arguing by contradiction, suppose that there is non-optimal accumulation point $\bar{u}$. Denote an "active" set of indices

$$I_a(\bar{u}) = \{i : \alpha < \bar{u}_i < \beta\}. \tag{50}$$

Consider a feasible neighborhood of the point $\bar{u}$, which is separated from the solution set U^* and from the boundaries of the feasible box in those coordinates, which are not at the boundary for $\bar{u}$:

$$N_\gamma(\bar{u}) = B_\gamma(\bar{u}) \bigcap \mathrm{Box}(\alpha, \beta)\ , \tag{51}$$

where $B_\gamma(\bar{u})$ is a ball centered in $\bar{u}$ with radius

$$\gamma = \frac{1}{2} \min \left\{ dist(\bar{u}, U^*),\ \min_{i \in I_a(\bar{u}_i)} \{\bar{u}_i - \alpha,\ \beta - \bar{u}_i\},\ \beta - \alpha \right\}\ . \tag{52}$$

For any point from $N_\gamma(\bar{u})$ the necessary optimality conditions are not satisfied, i.e. there exists a small positive number ϵ such that for any subgradient

$$g \in \bigcup_{u \in N_\gamma(\bar{u})} \partial G(u)\ , \tag{53}$$

at least for one value of the index i one of the following take place:

$$g_i < -\epsilon\ , \quad \bar{u}_i > \alpha \tag{54}$$

$$g_i > \epsilon\ , \quad \bar{u}_i < \beta \tag{55}$$

Let $u^k \in N_\gamma(\bar{u})$ be a point generated by the prox-algorithm. By (45)

$$\nabla_1 D_k(u^k, u^{k-1}) \in \partial G(u^k) \ , \tag{56}$$

hence taking into account (54), (55), (56) and (40), we conclude that there exists such index i that either

$$\psi_1'(u_i^k, u_i^{k-1}) > \epsilon \ , \quad \alpha \leq u_i^k \leq \beta - \gamma \tag{57}$$

or

$$\psi_1'(u_i^k, u_i^{k-1}) < -\epsilon \ , \quad \alpha + \gamma \leq u_i^k \leq \beta \tag{58}$$

It is shown in Proposition 3 in Appendix, that conditions (57) and (58) entail

$$\psi(u_i^k, u_i^{k-1}) > \delta > 0 \ , \tag{59}$$

where δ is a constant depending on ϵ, γ, and c. Recall that $\bar{u}$ is an accumulation point, so by (49) and (59) there exists an infinite set of indices $\{k_j\}$ such that

$$G(u^{k_j}) - G(u^{k_j - 1}) > \delta > 0 \ .$$

Thus, the infinite sequence $G(u^{k_j})$ increases each step at least by a positive constant, as shown above. Hence, it cannot be bounded, contrary to assumption $G^* < \infty$. □

6 Computational Results in Truss Topology Design

The problem is to find the stiffest truss which carries a given load and which consists of bars of a given total volume. The bars of the truss are a subset of the bars connecting all of a set of a priori chosen nodal points. The volume t_i of each bar is within prescribed upper and lower bounds U_i and L_i. The original formulation of the problem is the following (see [1] and [5] for details):

$$\min_{x,t} f^T x \tag{60}$$

subject to

$$\begin{aligned} A(t)x &= f && (61) \\ \sum_{i=1}^{m} t_i &= v && (62) \\ 0 \leq L_i &\leq t_i \leq U_i \ , \quad i = 1, \ldots, m && (63) \end{aligned}$$

where

# of variables	# of constraints	Upper bound on bar volume	Lower bound on bar volume	# of Newton steps	# of grad. evaluations
98	150	10	0	23	72
		0.1	0	19	80
		0.1	0.01	18	77
126	1234	10	0	46	147
		0.1	0	30	124
		0.1	0.001	30	91
		10	0.001	46	136
242	4492	10	0	44	136
342	8958	10	0	45	147
		0.1	0	36	124
		0.1	0.0005	32	121
450	15556	10	0	102	795
		0.01	0	160	1376
		0.001	0.0001	64	386

Table 1: Numerical Results for the Truss Topology Design Problems

N	–	number of nodes in the truss
$t = (t_i)$	–	m-dimensional vector of the bars' volumes ("design variables")
m	–	maximum number of bars ($m = \frac{1}{2}N(N-1)$)
$x = (x_j)$	–	n-dimensional vector of the displacements of the nodes ("analysis variables")
n	–	number of analysis variables $n = 2N$ (2D-trusses) or $n = 3N$ (3D-trusses)
f	–	n-dimensional vector of the loads on the nodes
v	–	given total volume of the truss
$L = (L_i)$	–	m-dimensional vector of lower bounds on the bars' volumes
$U = (U_i)$	–	m-dimensional vector of upper bounds on the bars' volumes
$A(t)$	–	symmetric positive semidefinite $n \times n$ matrix, the stiffness matrix.

The matrix $A(t)$ is given in terms of matrices A_i, which are symmetric positive semidefinite $n \times n$ matrices. A_i $(i = 1, 2, \ldots, m)$ determine the geometry of the connection of node i to the other nodes. Typical case:

$$A(t) = \sum_{i=1}^{m} t_i A_i \ ,$$

where A_i are matrices of rank 1.

It was proved in [1] that this problem is equivalent to the following one:

$$\min_{x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}} \left\{ \lambda v - f^T x + \sum_{i=1}^{m} \max \left\{ \left(\frac{1}{2} x^T A_i x - \lambda \right) U_i, \left(\frac{1}{2} x^T A_i x - \lambda \right) L_i \right\} \right\} . \tag{64}$$

Note that (64) is a sum-max problem, and thus can be solved by our algorithm discussed above. The results are given in Table 1. Accuracy in objective function is 6 digits.

7 Conclusions

In this article we have developed Lagrangian duality scheme for sum-max problems, and obtained a new kind of augmented Lagrangian, which use smooth approximation of max-type functions, and does not require artificial variables.

We have demonstrated correspondence of suggested method of multipliers to a proximal point algorithm in dual space, which provides foundation for the convergence proof.

Numerical experiments with large scale problems show practical efficiency of the method.

Appendix

Proposition 3 *Let $\psi(\cdot,\cdot)$ be the conjugate of the quadratic-logarithmic function (3) and one of the following relations takes place*

$$\psi_1'(\bar\lambda,\mu) > \epsilon\ , \quad \alpha \le \mu < \bar\lambda \le \beta - \gamma \tag{65}$$

or

$$\psi_1'(\bar\lambda,\mu) < -\epsilon\ , \quad \alpha + \gamma \le \bar\lambda < \mu \le \beta \tag{66}$$

Then

$$\psi(\bar\lambda,\mu) > \delta > 0\ , \tag{67}$$

where δ is a constant depending on ϵ, γ, and c.

Proof Consider the condition (65) (proof under condition (66) is similar). Suppose that we know (see Proposition 4) that

$$\psi''(\lambda,\mu) < M = \text{const}(\gamma), \quad \mu \le \lambda \le \bar\lambda \tag{68}$$

Then taking into account that $\min_\lambda \psi(\lambda,\mu) = \psi(\mu,\mu) = 0$ (by properties (ψ5) and (ψ6)), we conclude that $\psi(\lambda,\mu)$ majorates the quadratic function $q(\lambda)$ with the following properties

$$q'(\bar\lambda) = \psi'(\bar\lambda,\mu); \quad q''(\lambda) = M; \quad \min q(\lambda) = q(\mu) = 0$$

This gives us a lower bound

$$\psi(\bar\lambda,\mu) \ge q(\bar\lambda) = \frac{\left(q'(\bar\lambda)\right)^2}{2M} > \frac{\epsilon}{2M}$$

This proof is similar to one of Lemma 2 in [4], so reader is refereed there for more details. □

Proposition 4 *Under condition (65) of Proposition 3, the following upper bound on second derivative takes place*

$$\psi''(\lambda, \mu) \leq \frac{1}{\min\{c, \gamma^2/p_2\}} \tag{69}$$

Proof Consider the first and the second derivatives of the function (3) with respect to t

$$\varphi'(t, \mu, c) = \begin{cases} \alpha - p_1/t, & t < \tau_1 \leq 0 \\ ct + \mu, & \tau_1 \leq t \leq \tau_2 \\ \beta - p_2/t, & t > \tau_2 \geq 0\ . \end{cases} \tag{70}$$

$$\varphi''(t, \mu, c) = \begin{cases} p_1/t^2, & t < \tau_1 \leq 0 \\ c, & \tau_1 \leq t \leq \tau_2 \\ p_2/t^2, & t > \tau_2 \geq 0\ . \end{cases} \tag{71}$$

As we already mentioned, if follows directly from the definition of conjugate function, that the derivative of $\varphi(\cdot, \cdot)$ with respect to the first argument is an inverse function of the derivative of $\psi(\cdot, \cdot)$, *i.e.* if for some t, $\lambda = \varphi'(t, \mu)$ then $t = \psi'(\lambda, \mu)$.

Now let $\mu < \lambda \leq \beta - \gamma$ and $\varphi'(t, \mu) = \lambda$. Then taking into account (70), we get

$$\beta - p_2/t \leq \beta - \gamma$$

which gives us

$$t \leq \frac{p_2}{\gamma} \tag{72}$$

On other hand, from $\varphi'(t, \mu) = \lambda > \mu$, convexity of $\varphi(\cdot, \mu)$ and the fact that $\varphi'(0, \mu) = \mu$, we conclude that $t > 0$, *i.e.* only the second and the third branches of $\varphi''(t, \mu)$ in (71) are relevant to our case. Therefore substituting (72) into (71), we obtain

$$\varphi''(t, \mu, c) \geq \min\{c, \gamma^2/p_2\}$$

Taking into account that by property of mutually inverse functions φ' and ψ'

$$\psi''(\lambda, \mu) = \frac{1}{\varphi''(t, \mu)}$$

we complete the proof. □

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