Recovery of the Analytic Center in Perturbed Quadratic Regions and Applications *

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Abstract. We present results to recover an approximate analytic center when a sectional convex quadratic set is perturbed by a finite number of new quadratic inequalities. This kind of restarting may play an important role in some interior-point algorithms that successively refine the region where is the solution of the original problem.

Keywords: analytic center, quadratic regions, restarting, shifting

1. Introduction

One drawback to many interior-point methods is the great computational effort required in each iteration when the search direction is obtained by solving a linear system of equations. Therefore, working with a subset of the constraints of the problem rather than the full system, would save a great deal of computation.

The first attempt to save computations is the so called "build-down" or "column deletion" method proposed by Ye [12]. It was followed by the "build-up" or "column generation" method ; the first papers

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were written by Mitchell [8] and Goffin, Vial [5] for the projective method. Ye [14] proposed a non-interior potential reduction method for the linear feasibility problem which allows column generation. At each iteration, a violated inequality at the current center is added to the system (in a shifted position), until a feasible point has been found. Dantzig and Ye [1] proposed a build-up scheme for the dual affine scaling algorithm. This method differs from the standart affine scalling method in the choice of the ellipsoid that generates the search direction; it is constructed from a set of "promising" constraints. Tone [10] proposed an active-set strategy for the dual potential reduction of the Ye [11], [13] method. In his strategy, the search direction is also constructed from a subset of constraints which have small dual slacks in the current iterate. Den Hertog, Roos and Terlaky [3] proposed a build up and down strategy for the logarithmic barrier method for linear programming. This strategy starts with a small subset of the constraints and follows the correspondig central path until the iterate is close to (or violates) one of the other constraints; at this point it is added to the current system. Moreover, a constraint will be deleted if the corresponding slack value, computed in the current approximately centered iterate, is large. Den Hertog [2] analyzes the effect of shifting, adding and deleting a constraint on the position of center, the distance to the central path and the changing in the barrier function.

In this work we extend the results in [4] for the case of quadratic regions, that is we consider a region in $\Re^n$ given by a finite number of quadratic inequalities and with nonempty interior. We assume a point is given, which is close, in a certain norm, to the analytic center of the region and a new quadratic inequality is added to the ones that define the region. We also show how to obtain a shift such that the point is still close, in the same norm, to the anlalytic center.

This paper is organized as follows. In the next section we present the problem. In section 3 we check the question left in [4] on the recovering of the analytic center in perturbed quadratic regions. Some applications are discussed in section 4. Finally we have the Conclusions and the References.

2. The Problem

We consider a region $S \in \Re^n$ given by

$$S = \{x \in \Re^n : f_i(x) \leq 0; \quad i = 1, 2, \ldots, m\}$$

where $f_i(x) = \frac{1}{2}x^T Q_i x + a_i x - b_i$, $Q_i \in \Re^{n \times n}$ is symmetric and positive semidefinite, $a_i = (a_{i1}, a_{12}, \ldots, a_{in}) \in \Re^n$, $b_i \in R$ for each $i = 1, 2, \ldots, m$. We assume that $m \geq n$ and that the interior of S, denoted by $int(S)$, is nonempty and bounded.

The logarithmic barrier function over $int(S)$ is the function

$$P(x) = -\sum_{i=1}^{m} Ln(-f_i(x)) = -\sum_{i=1}^{m} Ln(b_i - a_i x - \frac{1}{2} x^T Q_i x).$$

It penalizes points x such that $f_i(x)$ is negative and close to zero.

Given $x \in int(S)$, the gradient and the Hessian of $P(.)$ at x are:

$$\begin{aligned} g &= g(x) = \nabla P(x) = \sum_{i=1}^{m} \frac{a_i^T + Q_i x}{-f_i(x)} \\ H &= H(x) = \nabla^2 P(x) = \sum_{i=1}^{m} \frac{(a_i^T + Q_i x)(a_i^T + Q_i x)^T}{(f_i(x))^2} + \frac{Q_i}{-f_i(x)} \end{aligned}$$

Now, in order to guarantee the positive definiteness of H in $int(S)$, we have the following lemma:

Lemma *H is positive definite in $int(S)$ if one of the following properties is true:*

a) *Q_i is positive definite, for some $i \in \{1, 2, \ldots, m\}$,*

b) *the $(m \times n)$ matrix whose i^{th} row is $(a_i^T + Q_i x)$ has full rank for all $x \in int(S)$.*

Then $P(.)$ is strictly convex on $int(S)$. This also implies that $P(.)$ has a unique minimizer x_c in $int(S)$, which is called the analytic center of S, that is

$$x_c = argmin P(x) : x \in int(S),$$

see [7]. The analytic center has played an important role in many interior-point algorithms for linear and quadratic programming [2].

As usual, the polinomiality of the procedure is assured if the analytic center is not exactly computed. Thus, we need some precise practical criteria to decide when a point x is close to it. The matrix $H = H(x)$ is positive definite; then it induces the relative norm,

$$\|y\|_H = \sqrt{y^T H(x) y}$$

for $y \in \Re^n$. The proximity of x to x_c is defined by the norm (induced by H) of the Newton direction of $P(.)$ at x; that is

$$\delta = \delta(x) = \|H^{-1}(x)g(x)\|_{H(x)} = \sqrt{g^T(x)H^{-1}(x)g(x)}.$$

A point x will be considered close to x_c if $\delta(x) < \epsilon$ for some $\epsilon \in (0,1)$. Clearly $\delta(x) = 0$ if and only if $x = x_c$.

Now suppose a point $x^0 \in int(S)$ is given and $(a_i^T + Q_i x^0)^T$ is the i^{th} row of an $(m\,x\,n)$ full-rank matrix $(i = 1, 2, \ldots, m)$, $\delta(x^0) < \epsilon$, for some $\epsilon \in (0,1)$, and also that it is given $Q_0 \in \Re^{m\,x,n}$ symmetric and positive semidefinite, $a_0 = (a_{01}, a_{02}, \ldots, a_{0n}) \in \Re^n$, $b_0 \in \Re$, such that the inequality

$$f_0(x) = \frac{1}{2}x^T Q_0 x + a_0 x - b_0 \leq 0$$

generate a region S^Δ from S,

$$S^\Delta = S \cap \{x \in \Re^n : f_0(x) \leq 0\}.$$

We are interested in finding a "shift" of the rigth hand side of the new inequality which generates a new region where x^0 is still close, in the same norm, to the corresponding analytic center.

3. Results

Given $\gamma \in \Re$ and a positive integer $l \geq m$, we start our search for such region by adding the inequality $f_0(x) \leq \gamma$, repeated l times, to those defining S, that is,

$$S_{l,\gamma} = S \cap \{x \in \Re^n : \underbrace{f_0 \leq \gamma, \ldots, f_0 \leq \gamma}_{l\ times}\}.$$

Note that the integer l, which counts how many times such inequality is repeated, acts as a weigth in computing the associated logarithmic barrier function

$$P_l(x,\gamma) = -lLn(\gamma - f_0(x)) + P(x),$$

see [2]. From a purely geometric point of view, it does not matter how many times an inequality is repeated; the same region is produced. In particular, if $\gamma = 0$, then $S_{l,0} = S^\Delta$.

The analytic center of $S_{l,\gamma}$ is given by

$$x_{c,l}(\gamma) = argmin\{P_l(x,\gamma) : x \in int(S_{l,\gamma})\}.$$

For $\gamma > f_0(x^0)$, $x \in int(S_{l,\gamma})$, let

$$\begin{aligned}
\beta &= \beta(x,\gamma) = \frac{1}{\gamma - f_0(x)} > 0, \\
\beta_0 &= \beta_0(x^0,\gamma) = \frac{1}{\gamma - f_0(x^0)} > 0, \\
d &= d(x) = a_0^T + Q_0 x, \\
d_0(x^0) &= a_0^T + Q_0 x^0.
\end{aligned}$$

Then

$$\begin{aligned}
g_l(x,\gamma) &= \nabla_x P_l(x,\gamma) = l\beta(x,\gamma)d(x) + g(x), \\
H_l(x,\gamma) &= \nabla_x^2 P_l(x,\gamma) = l\beta(x,\gamma)Q_0 + l\beta^2(x,\gamma)d(x)d^T(x) + H(x).
\end{aligned}$$

Note that

$$H_l(x^0,\gamma) = l\beta_0 Q_0 + l\beta_0^2 d_0 d_0^T + H_0, \tag{1}$$

where $H_0 = H_0(x^0)$, is positive definite in $int(S)$. As donne previously, the proximity of x^0 to $x_{c,l}(\gamma)$ is given by

$$\delta_l(x^0,\gamma) = \|H_l^{-1}(x^0,\gamma)g(x^0,\gamma)\|_{H_l(x^0,\gamma)}.$$

Also, from (1),

$$H_0 + l\beta_0^2 d_0 d_0^T \preceq H_l(x^0,\gamma)$$

(or $H_l(x^0,\gamma) - H_0 - l\beta_0^2 d_0 d_0^T$ is positive semidefinite) then (see corollary 7.7.4 of [6])

$$H_l^{-1}(x^0,\gamma) \preceq (H_0 + l\beta_0^2 d_0 d_0^T)^{-1}.$$

Let

$$\theta = d_0^T H_0^{-1} d_0 > 0$$

and

$$g_0 = g(x^0).$$

Then

$$\begin{aligned}
\delta_1^2(x^0,\gamma) &= g_l^T(x^0,\gamma)H_l^{-1}(x^0,\gamma)g_l(x^0,\gamma) \\
&\le g_l^T(x^0,\gamma)(H_0 + l\beta_0^2 d_0 d_0^T)^{-1} g_l(x^0,\gamma) \\
&= (g_0 + l\beta_0 d_0)^T \left(H_0^{-1} - \frac{l\beta_0^2}{1 + l\theta\beta_0^2} H_0^{-1} d_0 d_0^T H_0^{-1}\right)(g_0 + l\beta_0 d_0) \\
&= g_0^T H_0^{-1} g_0 + 2l\beta_0 g_0^T H_0^{-1} d_0 + l^2\beta_0^2 d_0^T H_0^{-1} d_0
\end{aligned}$$

$$-\frac{1}{1+l\theta\beta_0^2}(g_0^T H_0^{-1} d_0 d_0^T H_0^{-1} g_0$$
$$+2l\beta_0 d_0^T H_0^{-1} d_0 d_0^T H_0^{-1} g_0$$
$$+l^2\beta_0^2 d_0 H_0^{-1} d_0 d_0^T H_0^{-1} d_0).$$

Let

$$\alpha = d_0^T H_0^{-1} g_0,$$

then

$$\begin{aligned}\delta_l^2(x^0,\gamma) \;\le\; & g_0^T H_0^{-1} g_0 + 2l\alpha\beta_0 + l^2\theta\beta_0^2 - \frac{l\alpha^2\beta_0^2}{1+l\theta\beta_0^2} \\ & - \frac{2l^2\alpha\theta\beta_0^3}{1+l\theta\beta_0^2} - \frac{l^3\theta^2\beta_0^4}{1+l\theta\beta_0^2}.\end{aligned}$$

Let

$$\delta_0^2 = \delta^2(x^0) = g_0^T H_0^{-1} g_0,$$

then

$$\begin{aligned}\delta_1^2(x^0,\gamma) \;\le\; & \delta_0^2 + \frac{1}{1+l\theta\beta_0^2}(2l\alpha\beta_0 + 2l^2\alpha\theta\beta_0^3 \\ & + l^2\theta\beta_0^2 + l^3\theta^2\beta_0^4 - l\alpha^2\beta_0^2 - 2l^2\alpha\theta\beta_0^3 - l^3\theta^2\beta_0^4) \\ \le\; & \delta_0^2 + \frac{1}{1+l\theta\beta_0^2}(l^2\theta\beta_0^2 + 2l\alpha\beta_0) \\ \le\; & \delta_0^2 + l^2\theta\beta_0^2 + 2l\alpha\beta_0\end{aligned}$$

(since the denominator is greater than or equal to one); or, as $\delta_0 \le \epsilon$:

$$\delta_l^2(x^0,\gamma) \le l^2\theta\beta_0^2 + 2l\alpha\beta_0 + \epsilon^2 = l\beta_0(l\theta\beta_0 + 2\alpha) + \epsilon^2.$$

If

$$\gamma = f_0(x^0) + \frac{l\theta}{\sqrt{\alpha^2+\theta\epsilon(1-\epsilon)} - \alpha}$$

or

$$\beta_0 = \frac{\sqrt{\alpha^2+\theta\epsilon(1-\epsilon)} - \alpha}{l\theta}$$

then

$$\delta_l^2(x^0,\gamma) \le \epsilon$$

or

$$\delta_l^2(x^0, \gamma) \leq \sqrt{\epsilon} < 1.$$

This guarantees that x^0 is close to the analytic center of $S_{l,\gamma}$. Nevertheless, this is not the desired region, since the constraint involving $f_0(.)$ appears with weight $l > 1$ in the associated logarithmic barrier function, and we want x^0 close to the analytic center of such region with $l = 1$ (recall that the constraint $f_0(x) \leq 0$ was added to the ones that define S just once). For each positive integer l , $l \geq 1$, let

$$\gamma = \gamma(l) = f_0(x^0) + \frac{1}{\rho(l)},$$

where

$$\rho(l) = \frac{1}{\theta l}(\sqrt{\alpha^2 + \theta\epsilon} - \alpha).$$

Note that $\rho(l) = \frac{1}{l}\rho(1)$ and

$$\gamma(l) = f_0(x^0) + \frac{1}{\rho(1)};$$

so

$$\gamma(1) = (1 - \frac{1}{l})f_0(x^0) + \frac{1}{l}\gamma(l)$$

or

$$(\gamma(1) - f_0(x^0)) = \frac{1}{l}(\gamma(l) - f_0(x^0)).$$

Let

$$\beta_1 = \beta_1(x^0, \gamma(1)) = \frac{1}{\gamma(1) - f_0(x^0))}.$$

Clearly,

$$\beta_1 = l\beta_0,$$

and

$$\begin{aligned} g_1(x^0, \gamma(1)) &= g_l(x^0, \gamma(l)) \\ H_1(x^0, \gamma(1)) &= H_l(x^0, \gamma(l)) + \beta_0^2(l^2 - l)d_0 d_0^T. \end{aligned}$$

As it was defined,

$$\begin{aligned} \delta_1^2(x^0, \gamma(1)) &= g_1^T(x^0, \gamma(1))H_1^{-1}(x^0, \gamma(1))g_1(x^0, \gamma(1)) \\ &= g_l^T(x^0, \gamma(l))H_1^{-1}(x^0, \gamma(1))g_l(x^0, \gamma(l)). \end{aligned}$$

Now, let

$$\theta_l = d_0^T H_l^{-1}(x^0, \gamma(l)) d_0,$$

then

$$\delta_1^2(x^0, \gamma(1)) = g_l^T(x^0, \gamma(l)) H_l^{-1}(x^0, \gamma(l)) g_l(x^0, \gamma(l)) -$$

$$\frac{l^2 - l}{l + \theta_l (l^2 - l)\beta_0^2} \beta_0^2 g_l^T(x^0, \gamma(l)) H_l^{-1}(x^0, \gamma(l)) d_0 d_0^T H_l^{-1}(x^0, \gamma(l)) g_l(x^0, \gamma(l)),$$

so, we have

$$\delta_l^2(x^0, \gamma(1)) = \delta_l^2(x^0, \gamma(l)) - \frac{(l^2 - l)}{l + \theta_l (l^2 - l)\beta_0^2} \alpha_l^2,$$

where

$$\alpha_l = g_l^T(x^0, \gamma(l)) H_l^{-1}(x^0, \gamma(l)) d_0.$$

It follows that

$$\delta_1(x^0, \gamma(1)) \leq \delta_l(x^0, \gamma(l)) < \sqrt{\epsilon}.$$

It is clear that $S_{1,\gamma(1)}$ is the desired region.

4. Applications

One immediate application of the results is the Interior Feasibility Problem (IFP). Let S as before, $int(S) \neq \emptyset$ and bounded. We assume a point x^0 is known which is close to the analytic center of S and that a new quadratic inequality,

$$f_0(x) \leq 0,$$

is added to the ones that define S. The new region is

$$S^\Delta = S \cap \{x \in \Re^n : f_0(x) = \frac{1}{2} x^T Q_0 x + a_0 x - b_0 \leq 0\},$$

where $Q_0 \in \Re^{m \times n}$ is symmetric, positive semidefinite, $a_0 \in \Re^n, b_0 \in \Re$. The problem is to find (if it exists) a point $x \in int(S^\Delta)$.

The procedure in Table 1 would be used to solve the (IFP). We assume $int(S^\Delta) \neq \emptyset$, S^Δ bounded, $\delta(x^0) < \epsilon$, l an integer such that $l \geq m$, $\mu \in (0,1)$, $d_o = Q_0 x^0 + a_0^T$, $\theta = d_0^T H_0^{-1} d_0$ and $\alpha = d_0^T H_0^{-1} g_0$.

The initial point x^0 is close to the analytic center of the region S_{γ^0}, for

$$\gamma^0 = f_0(x^0) + \frac{l\theta}{\sqrt{\alpha^2 + \theta\epsilon(1 - \epsilon)} - \alpha}$$

and then the analytic center method can be applied to achieve x^{k+1} [1]. The stop criterion $(f_0(x) < 0)$ is satisfied in polynomial time if the problem

$$Minimize\, f_0(x) : f_i(x) \leq 0, i = 1, 2, \ldots, m$$

has a solution [1]. Next, we present the algorithm through its principal steps. The formulation is rather conceptual.

algorithm IFP
 Inicialization:
  Compute x^0 near to the analytic center of S

  Choose $\mu \in (0, 1)$

  Compute $\gamma^0 = f_0(x^0) + \frac{l\theta}{\sqrt{\alpha^2+\theta\epsilon(1-\epsilon)}-\alpha}$

  $k = 0$

 repeat
  if $f_0(x^k) < 0$
   then stop
  end if

  Update the upper bound*:*

$$\gamma^{k+1} = \mu\gamma^k + (1-\mu)f_0(x^k)\}$$

  Compute a new point close to the analytic center of
  $S \cap \{x : \underbrace{f_0(x) \leq \gamma^{k+1}, \ldots, f_0(x) \leq \gamma^{k+1}}_{l\ times}\}$:

$$x^{k+1} = argmin\{-lLn(\gamma^{k+1} - f_0(x)) + P(x) : \\ f_i(x) < 0, i = 1, \ldots, m; f_0(x) < \gamma^{k+1}\}$$

  $k = k + 1$
 end repeat
end algorithm

Table 1
Algorithm to solve the Interior Feasibility Problem (IFP)

Obviously, the IFP procedure could be extended to solve the problem when there is a finite number of new quadratic inequalities to be added.

The ideas in the center method for linear and quadratic programming can be nicely generalized for multi–objective programming problems (MOP) [3], where we have t objective functions. A rough idea is given below. Suppose the (conflicting) quadratic objective functions:

$$f_k(x) = \frac{1}{2}x^T Q_k x + w_k x - \xi_k,\ k = 1, 2, \ldots, t,$$

where $Q_k \in \Re^{m \times n}$ symmetric, positive semidefinite, $w_k \in \Re^n, \xi_k \in \Re$ and the feasibility region given by the constraints:

$$S = \{x \in \Re^n : Ax \leq b\};$$

$A \in \Re^{m \times n}$, $b \in \Re^m$. We assume that the interior of S is nonempty and bounded. Now the problem MOP is:

$$Minimize_{x \in S}\{f_1(x), f_2(x), \ldots, f_t(x)\}.$$

Clearly we must assume that exists an upper bound ρ_j for each objective function $f(.)_j, j = 1, 2, \ldots, t$, and one point x^j close to the analytic center of the region

$$x(\rho_j) = argmin\{-lLn(\rho_j - f_j(x)) - \sum_{i=1}^{m} Ln(b_i - a_i x) :$$
$$Ax < b, \underbrace{\rho_j > f_j(x), \ldots, \rho_j > f_j(x)}_{l\,times}\},$$

where l is a positive integer such that $l \geq m + t$ and a_i is the i–th row of the matrix A for $i = 1, 2, \ldots, m$.

5. Conclusions

We presented a consistent extension (for quadratic case) of the results from [4] to recover the analytic center when a new constraint is added to the constraints set. The results obtained may be applied in the interior feasibility problems, interior–point postoptimality techniques and in the minimization of a non-differentiable convex function in a interior–point cutting planes environment [9].

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