Optimality Conditions for Vector Optimization with Set-Valued Maps

Yong Wei Huang

Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong. (ywhuang@se.cuhk.edu.hk)

Abstract

In this paper, we establish a Farkas-Minkowski type alternative theorem under the supposition of nearly semiconvexlike set-valued maps. Based on the alternative theorem and some other lemmas, we present necessary optimality conditions and sufficient optimality conditions for set-valued vector optimization problems with extended inequality constraints in a sense of weak E-minimizers.

Key Words: Near cone-semiconvexlikeness, set-valued maps, alternative theorems, vector optimization, optimality conditions, weak E-minimizers.

1 Introduction

In recent years, vector optimization with set-valued maps in infinite dimensional spaces has been received an increasing amount of attention (See [2], [3], [4], [6], [7] and references therein), for its extensive applications in many fields such as mathematical programming, optimal control, management science. Vector optimization with set-valued maps, shortly called set-valued vector optimization sometimes, essentially can be considered as an improvement on single-valued vector optimization. Amongst researching topics in optimization problems, optimality condition is a highlight one. For vector optimization with set-valued maps, many authors have published some interesting results on the issue of optimality condition, and most of those results are obtained under different extended cone-convexity assumptions via alternative theorems. For instance, under the supposition of convexlikeness, Li and Chen [3] gave multiplier type and saddle point type optimality conditions for the existence of weak minimizers of set-valued vector optimization both with inequality and equality constraints. Li [6], under the assumption of cone-subconvexlikeness of set-valued maps, established optimality conditions for set-valued vector optimization by using the alternative theorem in ordered linear topological spaces. Yang *et al.* [7] characterized some properties of generalized cone-subconvexlike set-valued maps and showed an alternative theorem, and presented optimality conditions for proper Geoffrion elements of set-valued vector optimization in the Euclidean space.

In this paper, based on near cone-convexity, we introduce the notions of nearly cone-convexlike set-valued maps and nearly cone-semiconvexlike set-valued maps in infinite dimensional spaces, and investigate the relationships between them and give some characterizations of them respectively. Then we establish a Farkas-Minkowski type alternative theorem for set-valued maps under the assumption of near cone-semiconvexlikeness. In the final, we obtain some necessary and sufficient optimality conditions for the existence of weak E-minimizers of set-valued vector optimization with generalized inequality constraints.

The outline of this thesis is as follow. In Section 2, some notations and preliminaries are given. In Section 3, the concepts of nearly cone-convexlike set-valued maps and nearly cone-semiconvexlike set-valued maps are defined, and a Farkas-Minkowski type alternative theorem is established under the supposition of nearly cone-semiconvexlike set-valued maps. In Section 4, weak minimizers for vector optimization is extended to weak E-minimizers, and two main results of optimality conditions for vector optimization with set-valued maps are obtained in the sense of weak E-minimizers.

2 Notations and Preliminaries

Throughout this paper, the scalars of topological vector spaces are always real. Denote by O the null element of every space. Let Z, W be two topological vector spaces with pointed convex cones Z_+, W_+, respectively. Suppose that $intZ_+$, the interior of Z_+, is nonempty, and let $intZ_+ \neq Z_+$. However, the interior of W_+ is not required to be nonempty.

Denote by Z^*, W^* the dual spaces of Z, W, respectively. The dual cone Z_+^* of Z_+ is defined by $Z_+^* = \{z^* \in Z^* | \langle z, z^* \rangle \geq 0, \forall z \in Z_+\}$, where $\langle z, z^* \rangle$ denotes the value of the linear continuous functional z^* at the point z. Analogously, W_+^* is defined. Clearly, if $W_+ = \{O\}$, then we have $W_+^* = W^*$.

Let $B \subset Z$ be a nonempty subset. The closure of B is denoted by clB. The cone hull of B is defined by $cone(B) = \{\alpha b | \alpha > 0, b \in B\}$. The relative interior of B is defined by $riB = \{y \in affB | \exists$ a neighborhood of N of y such that $N \cap affB \subset B\}$, where aff denotes the affine hull operator. We recall the fact that if B is convex, then riB is nonempty and $intM$, the topological interior of B (or shortly, interior), is not necessary to be nonempty.

Denote by R the set of all real numbers. For $A \subset R, b \in R$, write $A \geq (\leq, <, >)b$, if and only if $a \geq (\leq, <, >)b, \forall a \in A$.

Let D be a given nonempty abstract set, and $G : D \to 2^Z$, $H : D \to 2^W$ be set-valued maps such that $G(x) \neq \emptyset$, $H(x) \neq \emptyset, \forall x \in D$. Let $G(D) = \bigcup_{x \in D} G(x), \langle G(x), z^* \rangle = \{\langle z, z^* \rangle | z \in G(x)\}, \langle G(D), z^* \rangle = \bigcup_{x \in D} \langle G(x), z^* \rangle$.

Definition 1 *A subset B in Z is called nearly convex, if there is $\alpha \in (0, 1)$ such that for each $z_1, z_2 \in B$, we have $\alpha z_1 + (1 - \alpha) z_2 \in B$.*

Lemma 1 (See Proposition 2.1 in [8]) *If $B \subset V$ is a nearly convex set, then the set* $\Omega = \{\beta \in [0,1] | \forall y_1, y_2 \in B,\ \beta y_1 + (1-\beta) y_2 \in B\}$ *is dense in* $[0,1]$.

Proposition 1 *If $B \subset Z$ is nearly convex and $riB \neq \emptyset$, then for every $t \in (0,1)$, we have*

$$t(riB) + (1-t)B \subset riB.$$

Proof. Let $t \in (0,1)$, $u_1 \in riB$, $u_2 \in B$. Then by definition there is an open neighborhood N of u_1 such that $N \cap affB \subset B$. Set $u_0 = tu_1 + (1-t)u_2$. Since the map $\varphi : \lambda \to u_0/\lambda + u_2(1 - 1/\lambda)$ is continuous at t, hence noting $\varphi(t) = u_1$, we conclude from Lemma 1 that there is $\beta \in \Omega \setminus \{0\}$ such that $u' := u_0/\beta + u_2(1-1/\beta) \in N$. We notice that $u' \in affB$. Thus $u' \in B$, and hence $u_0 = \beta u' + (1-\beta)u_2 \in B$. Now we show $u_0 \in riB$. Define the map $r : Z \to Z$ by

$$r(x) = x/\beta + u_2(1 - 1/\beta).$$

Since the map r is continuous on Z, then $U := r^{-1}(N)$ is an open neighborhood of u_0. Let $y \in U \cap affB$. Then we have $r(y) \in N$, and $r(y) \in affB$. Hence $y = \beta r(y) + (1-\beta)u_2 \in B$. Thus $U \cap affB \subset B$. Therefore, $u_0 \in riB$. □

Clearly, the Proposition 2, 3 below can be deduced directly by Proposition 1.

Proposition 2 *If $B \subset Z$ be a nearly convex set, then the set riB is convex.*

Proposition 3 *If a nearly convex set $B \subset Z$ is relatively open, namely $riB = B$, then B is convex.*

Proposition 3 given here can be thought of as an extension of Theorem 2.1 in [8].

Proposition 4 *Let $B \subset Z$ be a nearly convex set, and $riB \neq \emptyset$. Let $y^* \in Z^* \setminus \{O\}$. If $\langle u, y^* \rangle > 0, \forall u \in riB$, then $\langle u, y^* \rangle \geq 0, \forall u \in B$.*

Proof. Suppose the contrary. Then there is $u_0 \in B$ such that $\langle u_0, y^* \rangle < 0$. Fix $u_1 \in riB$. Since the function $s(t) = \langle tu_1 + (1-t)u_0, y^* \rangle$ is continuous on R, there is $\alpha \in (0,1)$ such that $s(\alpha) = \langle \alpha u_1 + (1-\alpha)u_0, y^* \rangle = 0$. On the other hand, from Proposition 1, we have $\alpha u_1 + (1-\alpha)u_0 \in riB$. This gives $\langle \alpha u_1 + (1-\alpha)u_0, y^* \rangle > 0$, a contradiction. $\square$

We recall that $riB = intB$ if and only if $intB$ is nonempty(for example, see Theorem 1.2.4 in [12]).

Lemma 2 *If $B \subset Z$ is a nearly convex set with nonempty interior, then for every $t \in (0,1)$ we have*

$$t(intB) + (1-t)clB \subset intB.$$

Proof. According to assumptions and Proposition 1, we obtain that for all $t \in (0,1)$, $t(intB) + (1-t)B \subset intB$. Since $intB$ is nonempty, we suppose $b \in intB$. Then $O \in (b - intB)$, or $\forall t \in (0,1)$, $O \in t(b - intB)$. Hence for every $t \in (0,1)$ we get $cl((1-t)B) \subset (1-t)B - t(b - intB) \subset t(intB) + (1-t)B - tb \subset (intB) - tb$. It follows that

$$\forall t \in (0,1),\ tb + cl((1-t)B) \subset intB.$$

Since $tb + cl((1-t)B) = tb + (1-t)clB$, and $b \in intB$ can be arbitrarily chosen, hence we have $\forall t \in (0,1)$, $t(intB) + (1-t)clB \subset intB$. $\square$

Lemma 3 *If $B \subset Z$ is a nearly convex set, then the set $intB$ is convex.*

The following lemma is the same as Proposition 4 whenever the assumption of $intB \neq \emptyset$ is imposed.

Lemma 4 *Let $B \subset Z$ be a nearly convex set, and $intB \neq \emptyset$. Let $y^* \in Z^* \setminus \{O\}$. If $\langle u, y^* \rangle > 0, \forall u \in intB$, then $\langle u, y^* \rangle \geq 0, \forall u \in B$.*

3 Nearly Cone-semiconvexlike Set-Valued Maps and Farkas-Minkowski Alternative Theorems

For the simplification, we put $U = Z \times W$, $U_+ = Z_+ \times W_+$, $J = (G, H) : D \to 2^U$. The notation $J(x) = (G, H)(x)$ is used for $G(x) \times H(x)$ here. One can easily check that

$U^* = Z^* \times W^*$, and $U^*_+ = Z^*_+ \times W^*_+$.

Definition 2 *A set-valued map $J : D \to 2^U$ is called nearly U_+–convexlike, if there is $\alpha \in (0,1)$ such that for any $x_1, x_2 \in D$, we have*

$$\alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+.$$

Definition 3 *A set-valued map $J : D \to 2^U$ is called nearly U_+–semiconvexlike, if there are $u \in intZ_+$, $\alpha \in (0,1)$ such that for any $x_1, x_2 \in D$, any $\epsilon > 0$, we have*

$$\epsilon(u,O) + \alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+.$$

Next, we give some important characterizations of nearly cone-semiconvexlike set-valued maps and nearly cone-convexlike set-valued maps, and state the relationships between them.

Proposition 5 *The set-valued map $J : D \to 2^U$ is nearly U_+–semiconvexlike, if and only if $M := J(D) + (intZ_+) \times W_+$ is a nearly convex set.*

Proof. "Sufficiency" Since $intZ_+$ is nonempty, and M is nearly convex, hence, $\exists u \in intZ_+$, $\exists \alpha \in (0,1)$, $\forall x_1, x_2 \in D$, $\forall \epsilon > 0$, such that

$$\alpha(J(x_1) + \epsilon(u,O)) + (1-\alpha)(J(x_2) + \epsilon(u,O)) \subset M \subset J(D) + U_+.$$

Therefore, $\epsilon(u,O) + \alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+$, i.e., J is nearly U_+–semiconvexlike.

"Necessity" Let $m_1, m_2 \in M$; then $\exists x_i \in D$, $y_i \in (intZ_+) \times W_+$, $i = 1,2$, such that $m_i \in J(x_i) + y_i$. Since J is nearly U_+–semiconvexlike, there exist $u \in intZ_+$, $\alpha \in (0,1)$, for the previous $x_1, x_2 \in D$, $\forall \epsilon > 0$, we have

$$\epsilon(u,O) + \alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+.$$

Thus $\epsilon(u,O) + \alpha(m_1 - y_1) + (1-\alpha)(m_2 - y_2) \in J(D) + U_+$. Because the set $(intZ_+) \times W_+$ is convex, we have $y_0 := \alpha y_1 + (1-\alpha)y_2 \in (intZ_+) \times W_+$. Thereby,

$$m = \alpha m_1 + (1-\alpha)m_2 \in \alpha J(x_1) + (1-\alpha)J(x_2) + y_0. \tag{1}$$

Let $y_0 = (y_{01}, y_{02}) \in (intZ_+) \times W_+$. Since $y_{01} \in intZ_+$, there is $\epsilon > 0$ such that $y_{01} - \epsilon u \in intZ_+$, Then, $y_0 - \epsilon(u, O) = (y_{01} - \epsilon u, \quad y_{02}) \in (intZ_+) \times W_+$. It follows by (1) that

$$m \in \alpha J(x_1) + (1-\alpha)J(x_2) + \epsilon(u, O) + y_0 - \epsilon(u, O)$$

$$\subset J(D) + U_+ + (intZ_+) \times W_+ \subset J(D) + (intZ_+) \times W_+ = M.$$

Therefore M is nearly convex. □

The following corollaries can be shown similarly.

Corollary 1 *The set-valued map $J : D \to 2^U$ is nearly U_+–convexlike, if and only if $M' = J(D) + Z_+ \times W_+$ is a nearly convex set.*

Corollary 2 *If $M' = J(D)+Z_+ \times W_+$ is nearly convex, then the set $M = J(D)+(intZ_+) \times W_+$ is also nearly convex.*

It follows by Corollary 2 that nearly cone-convexlike set-valued maps imply nearly cone-semiconvexlike set-valued maps. However the example below shows that the converse implication is not always true.

Example 1 *Let $D = \{0, 1\}$, $Z = R^2$, $W = R$. Then $U = Z \times W = R^3$. Let $Z_+ = \{(y_1, y_2) \in R^2 | \ y_1 \geq 0, y_2 > 0\} \cup \{(0, 0)\}$, $W_+ = \{0\}$. Let $G(x) = (G_1(x), G_2(x)) : D \to 2^{R \times R}$, $H(x) : D \to 2^R$. Define $J(x) = (G, H)(x) : D \to 2^U$ by*

$$J(x) = \{(G_1(x), G_2(x), H(x)) \in R \times R \times R | G_1(x) = x, G_2(x) \geq 0, H(x) = 0\}, \forall x \in D.$$

It is easy to check that $M = J(D) + (intZ_+) \times W_+$ is a convex set, so that it is nearly convex. But the set $M' = J(D) + Z_+ \times W_+$ is not nearly convex.

The following corollaries can be deduced directly by definition.

Corollary 3 *Let $J : D \to 2^U$ be a set-valued map. If we can find $\alpha \in (0, 1)$ such that for any $x_i \in D$, $y_i \in J(x_i)$, $i = 1, 2$, there is $x_3 \in D$ satisfying*

$$\alpha y_1 + (1-\alpha)y_2 \in J(x_3) + U_+.$$

Then J is nearly U_+–convexlike.

Corollary 4 *Let $J : D \to 2^U$ be a set-valued map. If we can find $u \in intZ_+$, $\alpha \in (0,1)$ such that for any $x_i \in D$, $y_i \in J(x_i)$, $i = 1,2$, any $\epsilon > 0$, there is $x_3 \in D$ satisfying*

$$\epsilon(u, O) + \alpha y_1 + (1-\alpha)y_2 \in J(x_3) + U_+.$$

Then J is nearly U_+–semiconvexlike.

Next, we give some technical lemmas which will be used in the proof of the alternative theorem.

Lemma 5 *The set $int(cone(J(D)) + Z_+ \times W_+) \neq \emptyset$, if and only if the set $int(cone(J(D)) + (intZ_+) \times W_+) \neq \emptyset$.*

Proof. Sufficiency is trivial. Suppose that $int(cone(J(D)) + Z_+ \times W_+) \neq \emptyset$. Then there are $\alpha \geq 0$, $x_1 \in D$, $z \in Z_+$, $w \in W_+$, $p \in G(x_1)$, $q \in H(x_1)$, such that $(\alpha p + z, \alpha q + w) \in int(cone(J(D)) + Z_+ \times W_+)$. Hence, there are S and T, neighborhoods of the origins in Z and W respectively such that

$$(\alpha p + z + (intZ_+) \cap S) \times (\alpha q + w + T) \subset (\alpha p + z + S) \times (\alpha q + w + T) \subset cone(J(D)) + Z_+ \times W_+.$$

Thus for each $s \in (intZ_+) \cap S$, each $t \in T$, there exist $\beta \geq 0, x' \in D, z' \in Z_+, w' \in W_+$, such that $\alpha p + z + s \in \beta G(x') + z'$, and $\alpha q + w + t \in \beta H(x') + w'$. So, $\alpha p + z + 2s \in \beta G(x') + z' + s \subset \beta G(x') + intZ_+$, and $\alpha q + w + t \in \beta H(x') + W_+$. Therefore,

$$(\alpha p + z + 2((intZ_+) \cap S)) \times (\alpha q + w + T) \subset cone(J(D)) + (intZ_+) \times W_+.$$

Observing the set in the left-hand side of the inclusion is open, we know that $int(cone(J(D)) + (intZ_+) \times W_+)$ is nonempty. □

In a similar way, we can also show the following lemma.

Lemma 6 *The set $int(J(D) + Z_+ \times W_+) \neq \emptyset$, if and only if the set $int(J(D) + (intZ_+) \times W_+) \neq \emptyset$.*

Lemma 7 *If* $u^* = (z^*, w^*) \in U_+^* = Z_+^* \times W_+^*$, *with* $z^* \neq O$, $u = (z, w) \in (intZ_+) \times W_+$, *then* $\langle u, u^* \rangle > 0$.

Proof. According to definition of U_+^*, we have $\langle u, u^* \rangle \geq 0$. Assume that there exists $u_0 = (z_0, w_0) \in (intZ_+) \times W_+$ such that $\langle u_0, u^* \rangle = 0$, i.e.,$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0$. Since $z_0 \in intZ_+$, then there is a neighborhood S of the origin in Z, such that $z_0 + S \subset intZ_+$. Noting that S is absorbing, we obtain that for every $v \in Z$, there is $\epsilon > 0$ such that $z_0 \pm \epsilon v \in intZ_+$. Hence, $\langle z_0 \pm \epsilon v, z^* \rangle + \langle w_0, w^* \rangle \geq 0$, or in other words,

$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle \geq \pm \epsilon \langle v, z^* \rangle.$$

Thus $\langle v, z^* \rangle$=0. Therefore $z^* = O$. However, this contradicts the assumption. Thus the proof is complete. □

In the remainder of this section, we consider the following two systems,

System 1. $\exists x_0 \in D$, *such that* $-G(x_0) \cap intZ_+ \neq \emptyset, -H(x_0) \cap W_+ \neq \emptyset$.

System 2. $\exists u^* = (z^*, w^*) \in Z_+^* \times W_+^* \setminus \{(O, O)\}$, *such that*

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \forall x \in D. \tag{2}$$

In what follows, we use the above two systems to describe the Farkas-Minkowski type alternative theorem under the assumption of nearly cone-semiconvexlike set-valued maps. The proof of this theorem is based on the separation theorems of convex sets in topological vector spaces (for instance, see Theorem 3.8 in [13]).

Theorem 1 *Suppose that the set-valued map* $J = (G, H) : D \to 2^U$ *is nearly* U_+*–semiconvexlike on* D. *Suppose that the interior of the set* $J(D) + U_+$ *is nonempty, Then,*

(i) *If* **System 2** *has a solution* $(z^*, w^*) \in Z_+^* \times W_+^*$, *with* $z^* \neq O$, *then* **System 1** *has no solution.*

(ii) *If* **System 1** *has no solution, then* **System 2** *has a solution* (z^*, w^*).

Proof. (i) Assume that System 2 admits a solution $(z^*, w^*) \in Z_+^* \times W_+^*$, with $z^* \neq O$. If System 1 admits a solution $x_0 \in D$, then there are $p \in G(x_0)$, $q \in H(x_0)$ such that

$-p \in intZ_+$, $-q \in W_+$. It follows by Lemma 7 that $\langle p, z^* \rangle + \langle q, w^* \rangle < 0$. This contradicts (2).

(ii) Set $M = J(D) + (intZ_+) \times W_+$. According to Lemma 6 and the assumption of $int(J(D) + Z_+ \times W_+) \neq \emptyset$, we have $intM \neq \emptyset$. Since J is nearly U_+–semiconvexlike on D, hence M is nearly convex. It follows by Lemma 3 that $intM$ is convex.

Since System 1 has no solution, then $O \notin M$ so that $O \notin intM$. As a matter of fact, assume that $O \in M$; there are $\alpha \geq 0$ and $x' \in D$ such that $O \in \alpha G(x') + intZ_+$, and $O \in \alpha H(x') + W_+$. Since $O \notin intZ_+$, hence $\alpha > 0$. therefore $-G(x') \cap intZ_+ \neq \emptyset$, $-H(x') \cap W_+ \neq \emptyset$. This is irrational since System 1 admits no solution.

Now using the separation theorem of convex sets in topological vector spaces, we have that there is a hyperplane H **properly** separating $\{O\}$ and $intM$, that is, $\exists u^* = (z^*, w^*) \in Z^* \times W^* \setminus \{(O, O)\}$, $a \in R$, such that

$$\langle u, u^* \rangle \geq a \geq 0, \forall u \in intM, \tag{3}$$

where the hyperplane function can be written as $H = \{y \in U | \langle y, u^* \rangle = a\}$.

In the following, we will prove that

$$\langle u, u^* \rangle > 0, \forall u \in intM. \tag{4}$$

There are two cases needed to be considered. One case is $a > 0$. It follows by (3) that the inequality (4) holds.

The other case is $a = 0$. It follows again by (3) that

$$\langle u, u^* \rangle \geq 0, \forall u \in intM. \tag{5}$$

Comparing (4) with (5), we can see that it is sufficient to show $\langle u, u^* \rangle \neq 0$, $\forall u \in intM$. Suppose the contrary; there is $u_0 \in intM$ such that $\langle u_0, u^* \rangle = 0$. Let $v \in intM$ be given arbitrarily. Thus there is $\epsilon > 0$ such that $u_0 - \epsilon v \in intM$. Hence it follows by (5) that $\langle u_0 - \epsilon v, u^* \rangle \geq 0$, that is, $\langle u_0, u^* \rangle \geq \epsilon \langle v, u^* \rangle$. So, $\langle v, u^* \rangle \leq 0$. On the other hand, also by (5), we get $\langle v, u^* \rangle \geq 0$. Therefore,

$$\langle v, u^* \rangle = 0, \forall v \in intM.$$

This illustrates that the hyperplane H does not separate $\{O\}$ and $intM$ **properly**. Then a contradiction is introduced.

Thus the proof that the inequality (4) holds is complete.

It follows by Lemma 4 that

$$\langle u, u^* \rangle \geq 0, \forall u \in M. \tag{6}$$

Next, we check $u^* = (z^*, u^*) \in Z_+^* \times W_+^*$; indeed, assume $z^* \notin Z_+^*$. Then there exists $z_1 \in Z_+$ such that $\langle z_1, z^* \rangle < 0$. Thus, $\lambda \langle z_1, z^* \rangle = \langle \lambda z_1, z^* \rangle < 0$, $\forall \lambda > 0$. According to (6), for each $x \in D$, each $z' \in intZ_+$, each $w' \in W_+$, we have $\langle p + z', z^* \rangle + \langle q + w', w^* \rangle \geq 0$, $\forall p \in G(x)$, $\forall q \in H(x)$. Since $\lambda z_1 \in Z_+$, then $\lambda z_1 + z' \in intZ_+$. Again by (6), we have $\langle p + \lambda z_1 + z', z^* \rangle + \langle q + w', w^* \rangle \geq 0$, that is,

$$\lambda \langle z_1, z^* \rangle + \langle p + z', z^* \rangle + \langle q + w', w^* \rangle \geq 0, \forall \lambda > 0. \tag{7}$$

However, (7) does not hold whenever λ is large enough. Hence we have $z^* \in Z_+^*$. We can analogously show $w^* \in W_+^*$. Thus, $\exists u^* = (z^*, w^*) \in Z_+^* \times W_+^* \setminus \{(O, O)\}$, such that $\langle u, u^* \rangle \geq 0$, $\forall u \in M$, i.e.,

$$\langle J(x) + t, u^* \rangle \geq 0, \forall x \in D, \forall t \in (intZ_+) \times W_+.$$

Take $t_0 \in (intZ_+) \times W_+$, and $\lambda_n > 0$ such that $\lambda_n \to 0 (n \to \infty)$; then we have $\langle J(x) + \lambda_n t_0, u^* \rangle \geq 0$, $\forall x \in D$, $n = 1, 2, \ldots$. Letting $n \to \infty$, we obtain

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \forall x \in D.$$

The proof is thus complete. □

In particular, If we set $W_+ = \{O\}$, the following result is derived directly by Theorem 1.

Corollary 5 *Suppose that the set-valued map $J : D \to 2^U$ is nearly U_+–semiconvexlike on D. Suppose that the interior of the set $J(D) + U_+$ is nonempty. If there is no $x \in D$ such that $-G(x) \cap intZ_+ \neq \emptyset$, $O \in H(x)$. Then $\exists u^* = (z^*, w^*) \in Z_+^* \times W^* \setminus \{(O, O)\}$, such that*

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \forall x \in D.$$

4 Weak E-minimizers and Optimality Conditions

Let Y be a topological vector space with pointed convex cone Y_+ with a nonempty interior. Let $F : D \to 2^Y$ be a set-valued map such that $F(x) \neq \emptyset, \forall x \in D$. Let $E \subset Y$ be a nonempty subset, and let $\epsilon \in Y_+$, $O \in E$.

We consider the following set-valued vector optimization (P),

$$\begin{aligned} &\min F(x), \\ &\text{s.t. } -G(x) \cap Z_+ \neq \emptyset, \\ &\qquad -H(x) \cap W_+ \neq \emptyset. \end{aligned}$$

Whenever we set $W_+ = \{O\}$, (P) reduces to (P′),

$$\begin{aligned} &\min F(x), \\ &\text{s.t. } -G(x) \cap Z_+ \neq \emptyset, \\ &\qquad O \in H(x). \end{aligned}$$

In this section, we work at the optimality conditions for (P). The feasible set of (P) is defined by $K = \{x \in D | -G(x) \cap Z_+ \neq \emptyset, -H(x) \cap W_+ \neq \emptyset\}$.

Definition 4 (i) *$x_0 \in K$ is called a weakly efficient solution of* (P)*, if there is $y_0 \in F(x_0)$ such that $(y_0 - F(K)) \cap intY_+ = \emptyset$. The pair (x_0, y_0) is called a weak minimizer of* (P).

(ii) *$x_0 \in K$ is called a weakly ϵ-efficient solution of* (P)*, if there is $y_0 \in F(x_0)$ such that $(y_0 - F(K) - \epsilon) \cap intY_+ = \emptyset$. The pair (x_0, y_0) is called a weak ϵ-minimizer of* (P).

In [5], the authors defined an H near the minimum solution of vector optimization. In this section, we use their idea to define weakly E-efficient solutions of set-valued vector optimization, and then discuss the existence of weakly E-efficient solutions and weak E-minimizers of set-valued vector optimization.

Definition 5 *A point $x_0 \in K$ is called a weakly E-efficient solution of* (P)*, if and only if $\exists y_0 \in F(x_0)$ such that $(y_0 - F(K) - E) \cap intY_+ = \emptyset$. The pair (x_0, y_0) is called a weak E-minimizer of* (P).

It is clear that the set of weakly efficient solutions contains the set of weakly ϵ-efficient solutions, or the set of E-efficient solutions. Now we investigate the relationships between weakly ϵ-efficient solutions and weakly E-efficient solutions.

Theorem 2 (i) *If $E = \{\epsilon\}$, then weakly E-efficient solutions are equivalent to weakly ϵ-efficient solutions.*

(ii) *If there is $\epsilon' \in E$ such that $\epsilon - \epsilon' \in Y_+$, then weakly E-efficient solutions imply ϵ-efficient solutions.*

(iii) *If $E - \epsilon \subset Y_+$, then weakly ϵ-efficient solutions imply weakly E-efficient solutions.*

Proof. We only show (ii) while (iii) can be proved similarly. Assume there is $\epsilon' \in E$ such that $\epsilon - \epsilon' \in Y_+$. Thus, we have $\epsilon + intY_+ \subset \epsilon' + Y_+ + intY_+ \subset \epsilon' + intY_+ \subset E + intY_+$. Suppose that $x_0 \in K$ is a weakly E-efficient solution. Then $(y_0 - F(K)) \cap (E + intY_+) = \emptyset$. Hence, $(y_0 - F(K)) \cap (\epsilon + intY_+) = \emptyset$. Therefore x_0 is also a weakly ϵ-efficient solution. $\square$

Set $I(x) = F(x) \times G(x) \times H(x) = (F, G, H)(x)$, $\forall x \in D$, $V = Y \times Z \times W$. Hence we have $V_+ = Y_+ \times Z_+ \times W_+$, $V^* = Y^* \times Z^* \times W^*$, and $V_+^* = Y_+^* \times Z_+^* \times W_+^*$. The definition below coincides with Definition 3 when we consider V as the product of $(Y \times Z)$ and W.

Definition 6 *The set-valued map $I = (F, G, H) : D \to 2^V$ is called nearly V_+–semiconvexlike on D, if and only if $\exists t \in intY_+$, $\exists u \in intZ_+$, $\exists \alpha \in (0, 1)$ such that $\forall x_1, x_2 \in D$, $\forall \epsilon > 0$, we have $\epsilon(t, u, O) + \alpha I(x_1) + (1 - \alpha)I(x_2) \subset I(D) + V_+$.*

In view of Proposition 5, we can find that the set-valued map $I : D \to 2^V$ is nearly V_+–semiconvexlike on D if and only if the set $I(D) + (intY_+ \times intZ_+) \times W_+$ is nearly convex.

A set-valued Lagrangian function $L : D \times Y_+^* \times Z_+^* \times W_+^* \to 2^R$ for (P) is defined as,

$$L(x, y^*, z^*, w^*) = \langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle, \quad (x, y^*, x^*, w^*) \in D \times Y_+^* \times Z_+^* \times W_+^*.$$

We consider the following unconstrained scalar optimization problem (UP) with set-valued functions induced by (P),

$$\min_{x \in D} L(x, y^*, z^*, w^*), \qquad (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*.$$

Definition 7 *A point $x_0 \in D$ is called $\langle E, y^* \rangle$-optimal solution of* (UP)*, if and only if $\exists r_0 \in L(x_0, y^*, z^*, w^*)$ such that $r_0 \leq L(x, y^*, z^*, w^*) + \langle E, y^* \rangle$, $\forall x \in D$. The pair (x_0, r_0) is called an $\langle E, y^* \rangle$–optimizer of* (UP)*.*

Now, we establish the optimality conditions in terms of (P) and (UP). For the simplicity, we suppose that the set E, satisfying $O \in E \subset Y$, is convex. It is easy to verify that if the set-valued map H is nearly V_+–semiconvexlike on D, $y_0 \in Y$, then $(F(x)+E-y_0) \times G(x) \times H(x)$ is also nearly V_+–semiconvexlike on D.

Theorem 3 *Let (x_0, y_0) be a weak E-minimizer of* (P)*; assume that*

(i) *$I(x) = F(x) \times G(x) \times H(x)$ is nearly V_+–semiconvexlike on D;*

(ii) *$\exists z_0 \in Y$, such that $(z_0, O, O) \in int(I(D) + V_+)$.*

Then $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$ such that $(x_0, \langle y_0, y^* \rangle)$ is an $\langle E, y^* \rangle$–optimizer of (UP)*, and $inf \langle G(x_0), z^* \rangle = 0$.*

Proof. Let $P(x) = (F(x) + E - y_0) \times G(x) \times H(x)$. It follows by assumption (i) that $P(x)$ is also nearly V_+–semiconvexlike on D. Since (x_0, y_0) is a weak E-minimizer of (P), we have $-(F(K) - y_0 + E) \cap intY_+ = \emptyset$. It is obvious that $-(G(x) \times H(x)) \cap (intZ_+) \times W_+ = \emptyset$, $\forall x \in D \setminus K$. Thus

$$-P(x) \cap ((intY_+) \times (intZ_+) \times W_+) = \emptyset, \quad \forall x \in D.$$

Since $(z_0, O, O) \in int(I(D) + V_+)$, hence $\exists x' \in D$ such that $z_0 \in int(F(x') + Y_+)$, $(O, O) \in int(G(x') \times H(x') + Z_+ \times W_+)$. Thus $z_0 - y + E \subset -y + E + int(F(x') + Y_+) \subset int(F(x') - y + E + Y_+)$. So, $int(P(D) + V_+) \neq \emptyset$.

By applying (ii) in Theorem 1, we have that $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$, such that $\langle P(x), (y^*, z^*, w^*)\rangle \geq 0$, $x \in D$. That is

$$\langle E, y^*\rangle + \langle F(x), y^*\rangle + \langle G(x), z^*\rangle + \langle H(x), w^*\rangle \geq \langle y_0, y^*\rangle, \forall x \in D. \tag{8}$$

Next, we show $y^* \neq O$. Assume the contrary. Then $(z^*, w^*) \neq (O, O)$, and (8) can be rewritten as

$$\langle G(x), z^*\rangle + \langle H(x), w^*\rangle \geq 0, \forall x \in D. \tag{9}$$

Hence

$$\langle G(x) + Z_+, z^*\rangle + \langle H(x) + W_+, w^*\rangle \geq 0, \forall x \in D. \tag{10}$$

We have two cases to be discussed. One case is $z^* \neq O$. Since $(O, O) \in int(G(x') \times H(x') + Z_+ \times W_+)$, then we can take $x_1 \in D$ arbitrarily, and for any $v_1 \in G(x_1)$, $v_2 \in H(x_1)$, $k_1 \in intZ_+$, $k_2 \in W_+$, satisfying $(v_1 + k_1, v_2 + k_2) \in Z \times W$, there is $\epsilon > 0$ such that $\pm\epsilon(v_1 + k_1, v_2 + k_2) \in int(G(x') \times H(x') + Z_+ \times W_+)$. It follows by (10) that $\langle v_1 + k_1, z^*\rangle + \langle v_2 + k_2, w^*\rangle = 0$. Observing (9), we obtain $\langle k_1, z^*\rangle + \langle k_2, w^*\rangle \leq 0$. This is in contradiction to Lemma 7.

The other case is $z^* = O$. Then (10) can be rewritten as $\langle H(x) + W_+, w^*\rangle \geq 0$, $\forall x \in D$. Because of $O \in int(H(x') + W_+)$, we have that for each $v \in W$, there is $\epsilon_0 > 0$ such that $\pm\epsilon_0 v \in int(H(x') + W_+)$. Thus, $\epsilon_0\langle v, w^*\rangle = 0$, $\forall v \in W$, which implies $w^* = O$. This is also a contradiction.

Thus the proof of $y^* \neq O$ is complete.

Observing $O \in E$, we rewrite (8) as

$$\langle F(x), y^*\rangle + \langle G(x), z^*\rangle + \langle H(x), w^*\rangle \geq \langle y_0, y^*\rangle, \forall x \in D. \tag{11}$$

Since $x_0 \in K$, there are $p \in G(x_0)$, $q \in H(x_0)$ such that $p \in -Z_+$, $-q \in W_+$. It follows that $\langle p, z^*\rangle + \langle q, w^*\rangle \leq 0$. On the other hand, setting $x = x_0$ in (11), we get

$$\langle y_0, y^*\rangle + \langle p, z^*\rangle + \langle q, w^*\rangle \geq \langle y_0, y^*\rangle.$$

That is $\langle p, z^* \rangle + \langle q, w^* \rangle \geq 0$. Thus

$$\langle p, z^* \rangle + \langle q, w^* \rangle = 0. \tag{12}$$

Hence $\langle y_0, y^* \rangle \in \langle F(x_0), y^* \rangle + \langle G(x_0), z^* \rangle + \langle H(x_0), w^* \rangle = L(x_0, y^*, z^*, w^*)$. Observing (8), we know that $(x_0, \langle y_0, y^* \rangle)$ is an $\langle E, y^* \rangle$–optimizer of (UP).

Because of $p \in -Z_+$, and $q \in -W_+$, we get $\langle p, z^* \rangle \leq 0$, and $\langle q, w^* \rangle \leq 0$. Noticing (12), we have $\langle p, z^* \rangle = \langle q, w^* \rangle = 0$.

Take $x = x_0$ in (11) again. We obtain

$$\langle y_0, y^* \rangle + \langle G(x_0), z^* \rangle + \langle q, w^* \rangle \geq \langle y_0, y^* \rangle.$$

That is $\langle G(x_0), z^* \rangle \geq 0$. Due to $0 = \langle q, z^* \rangle \in \langle G(x_0), z^* \rangle$, consequently, we have $inf \langle G(x_0), z^* \rangle = 0$. □

Corollary 6 *Let (x_0, y_0) be a weak E-minimizer of* (P)*; assume that*

(i) *$I(x) = F(x) \times G(x) \times H(x)$ is nearly V_+–semiconvexlike on D;*

(ii) *$\exists x' \in D$, such that $-G(x') \cap intZ_+ \neq \emptyset$, $-intH(x') \cap W_+ \neq \emptyset$.*

Then $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$ such that $(x_0, \langle y_0, y^* \rangle)$ is an $\langle E, y^* \rangle$–optimizer of (UP)*, and $inf \langle G(x_0), z^* \rangle = 0$.*

In practice, from assumption (ii) in Corollary 6, one can readily deduce condition (ii) in Theorem 3, thus the proof of Corollary 6 is similar to that of Theorem 3. In the rest of this section, we give some sufficient optimality conditions for Problem (P) under the supposition of generalized constraint qualifications, without any convexity assumptions.

Theorem 4 *Let $x_0 \in K$; assume that,*

(i) *$\exists y_0 \in F(x_0)$, $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$ such that*

$$\min_{x \in D} (\langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle) \geq \langle y_0, y^* \rangle;$$

(ii) $-int(H(D)) \cap W_+ \neq \emptyset$; $\exists x' \in D$, *such that* $-G(x') \cap intZ_+ \neq \emptyset$, $-H(x') \cap W_+ \neq \emptyset$.

Then (x_0, y_0) *is a weak E-minimizer of (P).*

Proof. According to assumption (i), we have

$$\langle F(x) - y_0, y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \forall x \in D. \tag{13}$$

We show $y^* \neq O$ below. Suppose that $y^* = O$. Then

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \forall x \in D. \tag{14}$$

In order to derive a contradiction, we consider the following two cases respectively. One case is $z^* \neq O$. By assumption (ii), there are $x' \in D$, $u_1 \in G(x')$, $u_2 \in H(x')$ such that $-u_1 \in intZ_+$, $-u_2 \in W_+$. Hence $\langle u_1, z^* \rangle + \langle u_2, w^* \rangle < 0$. This contradicts (14).

The other case is $z^* = O$. It follows by assumption (i) that $w^* \neq O$. From assumption (ii), there is $y' \in W_+$ such that $-y' \in intH(D)$. For each $v \in W$, it is not difficult to check $\langle v, w^* \rangle = 0$. This implies $w^* = O$, which is exactly in contradiction.

Therefore the proof of $y^* \neq O$ is complete.

Next we show (x_0, y_0) is a weak E-minimizer of (P). Otherwise, there are $x_1 \in K$, $t \in F(x_1)$, $e \in E$ such that $y_0 - t - e \in intY_+$. By Lemma 1.1 in [6], we have

$$\langle t - y_0 + e, y^* \rangle < 0. \tag{15}$$

Since $x_1 \in K$, there are $p \in G(x_1)$, $q \in H(x_1)$ such that $-p \in Z_+$, $-q \in W_+$. Taking (15) into account, we obtain $\langle t - y_0 + e, y^* \rangle + \langle p, z^* \rangle + \langle q, w^* \rangle < 0$. Seeing the fact $\langle e, y^* \rangle \geq 0$, we again obtain

$$\langle t - y_0, y^* \rangle + \langle p, z^* \rangle + \langle q, w^* \rangle < 0.$$

This conflicts with (13). Thus (x_0, y_0) is a weak E-minimizer of (P). □

The following corollary is very natural.

Corollary 7 *Let $x_0 \in K$; assume that there are $y_0 \in F(x_0)$, $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$, with $y^* \neq O$, such that*

$$\min_{x \in D}(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle) \geq \langle y_0, y^* \rangle.$$

Then (x_0, y_0) is a weak E-minimizer of (P).

References

[1] Yu, P. L., *Cone Convexity, Cone Extreme Points, and Nondominated Solutions in Decision Problems with Multiojective*, Journal of Optimization Theory and Applications, Vol.**14**, pp319-377, 1974.

[2] Borwein, J., *Multiobjective Convexity and Optimization: A Unified Approach to Inequality and Equality Constraints*, Mathematical Programming, Vol.**13**, pp183-199, 1977.

[3] Li, Z. F., and Chen, G. Y., *Lagrangian Multipliers, Saddle Points and Duality in Vector Optimization of Set-Valued Maps*, Journal of Mathematical Analysis and Applications, Vol.**215**, pp297-316, 1997.

[4] Chen, G. Y., and Jahn, J., *Special Issue on Set-Valued Optimization*, Mathematical Methods of Operations Research, Vol.**48**, pp151-285, 1998.

[5] Chen, G. Y., Huang, X. X., and Hou, S. H., *General Ekeland's Variational Principle for Set-Valued Maps* , Journal of Optimization Theory and Applications, Vol.**106**, pp151-164, 2000.

[6] Li, Z. M., *A Theorem of the Alternative and Its Applications to the Optimization with Set-Valued Maps*, Journal of Optimization Theory and Applications, Vol.**100**, pp365-375, 1999.

[7] Yang, X. M., Yang, X. Q., and Chen, G. Y., *Theorems of Alternative and Optimization with Set-Valued Mappings*, Journal of Optimization Theory and Applications, Vol.**27**, pp627-640, 2000.

[8] Paeck, S., *Convexlike and Concavelike Conditions in Alternative, Minimax, and Minimization Theorems*, Journal of Optimization Theory and Applications, Vol.**74**, pp317-332, 1992.

[9] Rong, W. D., and Wu, Y. N., *ϵ-Weak Minimal Solutions of Vector Optimization Problems with Set-Valued Maps*, Journal of Optimization Theory and Applications, Vol.**106**, pp581-591 2000.

[10] Illes, T., and Kassay, G., *Theorems of the Alternative and Optimality Conditions for Convexlike and General Convexlike Programming*, Journal of Optimization Theory and Applications, Vol.**101**, pp243-257, 1999.

[11] Song, W., *Lagrangian Duality for Minimization of Nonconvex Multifunctions*, Journal of Optimization Theory and Applications, Vol. **93**, pp167-182, 1997.

[12] Hu, Y.D, and Meng, Z.Q., *Convex analysis and nonsmooth analysis*(in Chinese), Shanghai Science and Technology Press, Shanghai, 2000.

[13] Tiel, J. V., *Convex Analysis: An Introductory Text*, John Wiley and Sons, New York, New York, 1984.