Clique Family Inequalities for the Stable Set Polytope of Quasi-Line Graphs

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Abstract

In one of fundamental work in combinatorial optimization Edmonds gave a complete linear description of the matching polytope. Matchings in a graph are equivalent to stable sets its line graph. Also the neighborhood of any vertex in a line graph partitions into two cliques: graphs with this latter property are called quasi-line graphs. Quasi-line graphs are a subclass of claw-free graphs, and as for claw-free graphs, there exists a polynomial algorithm for finding a maximum weighted stable set on such graphs, but we do not have a complete characterization of their stable set polytope (SSP).

In the paper we introduce a class of inequalities, called clique-family inequalities, which are valid for the SSP of any graph and match the odd set inequalities defined by Edmonds for the matching polytope. This class of inequalities unifies all the known (non-trivial) facet inducing inequalities for the SSP of a quasi-line graph. We therefore conjecture that all the non-trivial facets of the SSP of a quasi-line graph belong to this class. We show that the conjecture is indeed correct for the classes of quasi-line graphs for which we have a complete description of the SSP. We discuss some approaches for solving the conjecture and a related problem.

Keywords: *polyhedral combinatorics, matching polytope, claw-free graphs, quasi-line graphs.*

Premise. This paper has been inspired by the work of Abdellah Ben Rebea [1] and is dedicated to his memory.

Ben Rebea died shortly after discussing his Ph.D thesis at the University of Grenoble (France), in 1980. The subject of the thesis was the description of the stable set polytope (SSP) of a relevant subclass of claw-free graphs, named quasi-line graphs. Even if Ben Rebea claims a linear description for this polytope, up to now all the efforts to reorganize and publish the results of his thesis have been unsuccessful. This is because many parts of the thesis are obscure and very technical. In addition, there are some errors; it is possible to correct them in a former part, but arguments in the latter part are very fallacious.

Nevertheless, it is this author's opinion that the thesis forms an impressive piece of work. Especially in the first part, results are non-trivial and proofs are skilful. And, therefore, I do not believe that such a skilful researcher could ignore errors and gaps in his arguments. Very likely, he had a very good intuition of the problem, but he had not enough time to let his intuition entirely come to surface and solve all the related technical problems.

In the paper I propose a conjecture on a linear description of the SSP of a quasi-line graph. Even if this conjecture is original, I believe it has the spirit of what Ben Rebea was doing, so I would like to call it the Ben Rebea's Conjecture.

1 Introduction

The basic properties of the stable set polytope (SSP) of a graph $STAB(G)$ and the crucial connections with the theory of perfect graphs have been studied by several authors (see [12] for a survey, and [6, 21]). The task of characterizing the facets of the SSP can be greatly simplified by restricting our analysis to some special subclass of graphs. For instance, a linear description of the SSP of a line graph follows from matching theory [8].

Another important and well-studied class of graphs is that of *claw-free* graphs (definitions come at the end of this section). Claw-free graphs are a superclass of line graphs and many crucial properties of the matching problem extend to stable set problems in claw-free graphs. In particular, there exist polynomially bounded algorithms for finding a maximum (weighted) stable set in a claw-free graph [18, 19, 20, 25]. Conversely, the nice polyhedral properties of the matching polytope do *not* extend to the polytope $STAB(G)$ associated with a claw-free graph. In fact, as shown in 1980 by Giles and Trotter [11], when G is claw-free, the minimal defining system for $STAB(G)$ contains facets that have a much more complex structure than those defining the matching polytope.

This apparent asymmetry between the algorithmic and the polyhedral status of the stable set problem in claw-free graphs gives rise to the challenging problem of providing a "...decent linear description of $STAB(G)$" [12].

On the other hand, the result by Giles and Trotter was quite negative in a sense, and discouraged researchers for several years from working on the problem. One exception was a controversial result by Abdellah Ben Rebea who proposed in his Ph.D thesis [1] a linear description of $STAB(G)$ for a subclass of claw-free graphs named *quasi-line*. In a quasi-line graph the neighborhood of any vertex partitions into two cliques; note that this is the case with every vertex of a line graph: quasi-line graphs are therefore a superclass of line graphs.

Unfortunately, a number of critical mistakes are present in Ben Rebea's thesis. To this date, no representation of it thesis appears in print (see the Premise), and, we are indeed far from having a linear description of the SSP of quasi-line or claw-free graphs.

Nevertheless, some positive related results came in the last years. In 1993 Pulleyblank and Shepherd [23] considered another subclass of claw-free graphs, the *distance claw-free* graphs (namely, those claw-free graphs for which, for each vertex v, $\alpha(N^2(v)) \leq 2$, where $N^2(v) = N(N(v)) \setminus v$), and gave a compact projective formulation of the SSP for such graphs, i.e. a formulation with a polynomially bounded number of variables and constraints from which $STAB(G)$ can be obtained by projection. Also, in 1995 Shepherd [27] gave a complete description of the SSP of a *near-bipartite graph*, the complement of a quasi-line graph. In 1997 Galluccio and Sassano [10] gave a characterization of the *rank* facets of the SSP of any claw-free graph. In 1999 Dahl proved that the SSP-s of circulants with the clique number equal to three have only rank facets and characterized these facets. Note that any circulant is quasi-line. In 2000 Kind and Jacobs [15], by means of the PORTA tool - by PORTA it is possible to generate all the facets of the convex hull of a given set of points -, showed that circulants with clique number greater than four have, in general, non-rank facets. Finally, in 2001 Cao and Nemhauser [4] gave a polyhedral characterization of line graphs.

In this paper, we first review results on the SSP of claw-free and quasi-line graphs. They show that quasi-line are quite a rich and challenging subclass of claw-free graphs. We discuss therefore how to simplify the description of $STAB(G)$ by some properties of *minimality* for facets.

We introduce a new class of inequalities, called clique-family inequalities, which are valid for the SSP of any graph and, in some sense, match the odd set inequalities for the matching polytope (see Theorem 4.1). These inequalities are not obtained through standard rounding arguments and, in fact, they include facets which cannot be obtained through a single application of the Chvátal-Gomory procedure [5] to the fractional stable set polytope. We show that clique family inequalities include, and therefore unify, *all* the known non-trivial facets for the SSP of a quasi-line graph (including the "complex" facets by Giles and Trotter and other non-rank facets).

We introduce a conjecture, which we call Ben Rebea's Conjecture (Conjecture 5.1), claiming that all the non-trivial facets of the SSP of a quasi-line graph are clique-family inequalities. We verify some basic properties, i.e. we show that Ben Rebea's Conjecture is indeed correct for the classes of quasi-line graphs for which we already have a linear description of the SSP.

We discuss approaches for proving (or disproving) the conjecture and other open questions. In particular, we exploit connections between an algorithmic proof and the design of a primal-dual algorithm for the minimum weighted clique cover problem on a claw-free perfect graph.

The paper is organized as follows. We close this section by some definitions. In Sections 2 we review some results on the SSP of claw-free graphs and quasi-line graphs. In Section 3 we discuss some properties of minimality for facets of the SSP of a quasi-line graph. In Section 4 we introduce clique-family inequalities and discuss relations with known facet inducing inequalities for the SSP of a quasi-line graph. In Section 5 we introduce Ben Rebea's Conjecture and show that it is correct for basic classes of quasi-line graphs. Section 6 is dedicated to open questions and possible approaches for proving or disproving the conjecture.

1.1 Notations and definitions

Let $G(V, E)$ be a graph. Unless otherwise specified, a graph will always be a finite, undirected and loopless. When confusion may arise, we use subscripts to specify the graph we are referring to. We respectively denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G. Let W be a subset of V: we denote by $G[W]$ the subgraph of G induced by W, and by $E[W]$ the set of edges of $G[W]$ ($E[W] = \{uv \in E : u \text{ and } v \in W\}$). In the following, unless otherwise specified, when referring to a subgraph, we mean an *induced* one.

For a vertex $v \in V$ we denote $G \setminus v := G[V \setminus v]$; for an edge e we denote $G \setminus e := G(V, E \setminus e)$. Conversely, if H is an induced subgraph of G and $u \in V(G) \setminus V(H)$, we denote $H \cup u := G[V(H) \cup u]$. We denote by $N(v)$ the *neighborhood* of a vertex $v \in V$, i.e. the set of vertices that are adjacent to v, and by $\delta(v)$ the *star* of v, the set of edges that are incident to v. Analogously, if W is a subset of V, we denote by $\delta(W)$ the subset of edges of E joining vertices of W to the vertices outside W, that is $\delta(W) = \{uv \in E : u \in W \text{ and } v \notin W\}$.

We denote by $\nu(G)$ the *matching number* of G, the size of a maximum matching; by $\omega(G)$ the *clique number*, the size of a maximum clique; by $\alpha(G)$ the *stability number*, the size of a maximum stable set. When no confusion may arise, we write $\nu(W)$ ($\alpha(W)$) to denote the size of a maximum matching (stable set) of $G[W]$. G is *hypomatchable* if $\nu(G) = \nu(G \setminus v)$ for each $v \in V$. A subset W of V is *α-maximal* in G if $\alpha(W \cup v) > \alpha(W)$ for each $v \notin W$.

Given a graph G, the *line graph* of G, $L(G)$, is constructed as follows. The vertex set of $L(G)$ is $E(G)$ and two vertices of $L(G)$ are adjacent if and only if they are adjacent edges in G. Vice versa, we say that a graph L is *line* if there exists a graph G such that $L = L(G)$. A *claw* is a graph with vertex set $\{u, v, w, z\}$ and edge set $\{uv, uw, uz\}$. A graph is said to be *claw-free* if it does not contain a claw as an induced subgraph. A graph is *quasi-line* if the neighborhood of any vertex partitions into two cliques. A *(n,p) circulant*, $p < \frac{n}{2} - 1$, is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+j} : 1 \leq i \leq n \textit{ and } 1 \leq j \leq p\}$ (sums are taken *modulo* n). Any $(n, 1)$ circulant with $n \geq 4$ is a *hole*: n is the *length* of the hole, and the hole is *odd* if n is odd. An (odd) *antihole* is the complement of a (odd) hole.

Let $STAB(G)$ be the convex hull of the incidence vectors of all stable sets of G. $STAB(G)$ is a full-dimensional polytope and a vector x is a vertex of $STAB(G)$ if and only if it is the incidence vector of a stable set in G (see, for instance, [12]). A linear inequality $\sum_{j \in V} a_j x_j \leq b$ is said to be *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. A valid inequality for $STAB(G)$ is a *facet inducing* inequality if and only if it is satisfied as an equality by $\mid V \mid$ affinely independent incidence vectors of stable sets of G. For shortness, in the following, we often say *facet* meaning facet inducing inequality. The inequalities $-x_j \leq 0$ are facets for $STAB(G)$ and are called *trivial*. If an inequality $ax \leq b$ is a non-trivial facet for $STAB(G)$, then $b \neq 0$; if in particular $b > 0$, then the vector a is non-negative. Among valid inequalities for the SSP, a special attention is given to *rank* inequalities, i.e. inequalitise of the form $\sum_{j \in T} x_j \leq \alpha(T)$, for some $T \subseteq V$.

Let $\mathcal{K}_G$ denote the family of maximal cliques of G and B the matrix whose rows are the incidence vectors of elements of $\mathcal{K}_G$; the *fractional* stable set polytope of G is the polytope: $Fract(G) = \{x \in \mathcal{Q}^n_+ : Bx \leq \bar{1}\}$, $\bar{1}$ is the vector of all 1-s. If a non-negative weight w_v is given for each vertex $v \in V(G)$, a *weighted clique cover* is a vector of the polyhedron $DualFract(G) = \{y \in \mathcal{Q}^{\mathcal{K}_\mathcal{G}}_+ : yB \geq w\}$, and the *minimum weighted clique cover problem* is that of finding a weighted clique cover y minimizing $\sum_{k \in \mathcal{K}_G} y_k$.

2 From line to quasi-line graphs

A linear description of the matching polytope is a celebrated result by Edmonds [8] of 1965. Later on, Edmonds and Pulleyblank found a *minimal* description for the matching polytope, by characterizing all the facets. Define a vertex v of a graph G *essential* if either $\mid N(v) \mid > 2$ or $N(v) = \{x, y\}$ and $xy \notin E(G)$. They proved the following theorem.

Theorem 2.1 [9] *The facets of the matching polytope $M(G)$ of a connected graph $G(V, E)$, $\mid V \mid \geq 3$, are given by the following inequalities:*

(i) $x_e \geq 0$ *for each* $e \in E$

(ii) $x(\delta(v)) \leq 1$ *for each* $v \in V$, v *essential*

(iii) $x(E[Q]) \leq \frac{|Q|-1}{2}$ *where* $Q \subseteq V$ *spans a 2-connected hypomatchable graph.*

Let L be the line graph of a connected graph $G(V, E)$, $\mid V \mid \geq 3$. Note that stable sets of L correspond to matchings of G. Hence, a linear minimal description of $STAB(L)$ follows from Theorem 2.1; this is shown in detail in the following. We assume L to be simple but allow G to have parallel edges. (Actually, we might assume G simple because of the following argument: if G has parallel edges, then there are adjacent vertices u and v of L such that $N_L(u) = N_L(v)$; in this case, it follows from [6] that a characterization of $STAB(L)$ can be easily derived from a characterization of $STAB(L \setminus u)$.)

Observe that, by definition, for each vertex $v \in G$ the star $\delta(v)$ is a clique of $L(G)$ which is maximal if and only if v is essential. Vice versa, if a maximal clique of $L(G)$ does not correspond to the star of an essential vertex of G, then its vertices correspond to the edges of a *triangle* of G (possibly with parallel edges).

Let $Q \subseteq V(G)$ span a 2-connected hypomatchable subgraph of G. It is easy to see that the subgraph $H \subseteq L(G)$ induced by the set of vertices $E[Q]$ is α-maximal in $L(G)$ and $\alpha(H) = \nu(Q) = \frac{|Q|-1}{2}$. Also, observe that triangles are the only 2-connected hypomatchable graphs whose matching number is equal to one. We can therefore state the following corollary of Theorem 2.1.

Corollary 2.2 *The facets of the stable set polytope of a connected line graph L with at least two vertices are given by the following inequalities:*

(j) $x_v \geq 0$ *for each vertex* v

(jj) $\sum_{v \in K} x_v \leq 1$ *for each maximal clique* K

(jjj) $\sum_{v \in H} x_v \leq \alpha(H)$ *for each $H \subseteq L$ such that H is α-maximal in L, H is the line graph of a 2-connected hypomatchable graph and $\alpha(H) \geq 2$.*

There are several characterizations for the line graphs and some directly yield efficient tests for deciding if a graph is line [3, 13, 16, 17, 24]. In [3, 13, 17] a forbidden subgraph characterization is given; for instance, a line graph does not admit a claw as an induced subgraph.

Claw-free graphs are therefore a superclass of line graphs. Actually, many crucial properties of the matching problem extend to the stable set problems in claw-free graphs. In particular, as it was shown by Berge [2], it is possible to extend to stable sets the augmenting path theorem for matchings. Due to this strong analogy, there exist polynomially bounded algorithms for finding a maximum stable set in a claw-free graph for the unweighted case [18, 19, 25] and the classical algorithm by Minty for the weighted case. Nakamura and Tamura [20] recently revised Minty's algorithm.

But, do the strong analogies between matchings in general graphs and stable sets in claw-free graphs extend to polyhedral characterizations? A first (negative) answer was given by Giles and Trotter. Let $q \geq 2$ be an integer and let $n = 2q^2 - 1$. Let W be a (n, q) circulant with vertices $V(W) = \{1, 2, \ldots, n\}$, and W' be a $(n, q-1)$ circulant with vertices $V(W') = \{1', 2', \ldots, n'\}$. Let G^q be the graph such that $V(G^q) = V(W) \cup V(W')$ and $E(G^q) = E(W) \cup E(W') \cup \{ij' : i \in \{1, \ldots, n\}$ and $j' \in \{(i-2)', \ldots, (i+2q-3)'\}\}$. They proved the following theorem.

Theorem 2.3 [11] *For each $q \geq 2$, the inequality:*

$$q \sum_{i=1..n} x_i + (q-1) \sum_{i'=1..n} x_{i'} \leq 2q(q-1) \tag{1}$$

is a facet of $STAB(G^q)$.

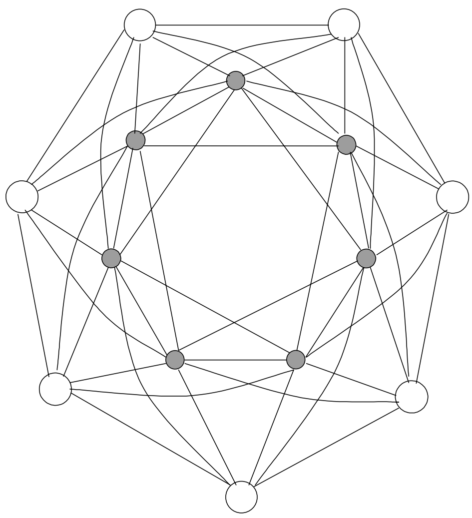


Figure 1: The Graph G^2

Recall that a graph G is *quasi-line* if the neighborhood of any vertex partitions into two cliques. Note that quasi-line graphs are claw-free and properly include the class of line graphs. Also, for each $q \geq 2$, the graph G^q is quasi-line. Therefore, a linear description of the SSP of a quasi-line graphs should include both the facets arising from Corollary 2.2, but also more "complex" facets as (1).

We close this section with a simple lemma, whose proof is omitted.

Lemma 2.4 *Let $G(V, E)$ be a graph. The following statements are equivalent:*

1. *G is quasi-line;*
2. *for each $v \in V$ the complement of $G[N(v)]$ is bipartite;*
3. *G is claw-free and no vertex is (totally) joined to an odd-antihole;*
4. *G is claw-free and for each $v \in V$ the graph $G[N(v)]$ is perfect.*

3 Facets for quasi-line graphs

Providing a linear description of the SSP of a quasi-line graphs seems to be a challenging task. In this section we exploit the idea of simplifying it by restricting to some class of *minimal* facets.

There are several standard techniques for producing facets of the SSP from "smaller" ones [6, 21]. When dealing with claw-free graphs, (i) sequential lifting and (ii) complete join seem to be the most interesting.

(i) In [21] Padberg proved the following. Let $\sum_{j\in T} a_j x_j \leq b$ be a facet of $STAB(G[T])$, for some $T \subset V$. If $u \in V \setminus T$ and

$$a_u = b - \max_{S \text{ stable set}, \ S \subseteq T \setminus N(u)} \sum_{j \in S} a_j, \tag{2}$$

then the inequality $\sum_{j\in T\cup u} a_j x_j \leq b$ is a facet of $STAB(G[T \cup u])$; a_u is the *lifting coefficient* of variable x_u. By iteration, after $\mid V \setminus T \mid$ steps, we come up with an inequality $\sum_{j\in V} a_j x_j \leq b$ which is a facet of $STAB(G)$ and is called *sequential lifting* of $\sum_{j\in T} a_j x_j \leq b$.

(ii) A *complete join* $G_1 + G_2$ is the (disjoint) union of G_1 and G_2 together with additional edges that join each vertex of G_1 to each vertex of G_2; Chvátal [6] proved that a complete description of $STAB(G_1+G_2)$ can be easily derived from one of $STAB(G_1)$ and one of $STAB(G_2)$. In particular, if $\sum_{j\in V(G_1)} a_j x_j \leq b_1$ is a facet of $STAB(G_1)$ and $\sum_{j\in V(G_2)} a_j x_j \leq b_2$ is a facet of $STAB(G_2)$ then $b_2 \sum_{j\in V(G_1)} a_j x_j + b_1 \sum_{j\in V(G_2)} a_j x_j \leq b_1 b_2$ is a facet of $STAB(G)$.

It is therefore reasonable to characterize facets which do not arise from sequential lifting and/or complete join. Galluccio and Sassano have exploited this approach for the class of *rank* facets of claw-free graphs [10]. They define a rank facet $\sum_{j\in V} x_j \leq \alpha(G)$ to be *rank-minimal* if, for any $T \subset V$, the inequality $\sum_{j\in T} x_j \leq \alpha(G)$ is not a facet of $STAB(G[T])$. They show that every rank facet can be constructed from a rank-minimal one by means of sequential lifting and/or complete join. Finally, they characterize rank minimal facets by the following theorem (a hypomatchable graph G is *minimal*, if $G \setminus e$ is no more hypomatchable for any $e \in E(G)$).

Theorem 3.1 [10] *Let $G(V, E)$ be a claw-free graph such that $\sum_{j\in V} x_j \leq \alpha(G)$ is facet inducing and rank-minimal. Then one of the following statements holds:*

(i) G is a singleton;

(ii) G is the line graph of a minimal 2-connected hypomatchable graph;

(iii) G is a $(\alpha\omega + 1, \omega - 1)$ circulant, $\omega \geq 3$.

Observe that there are rank (but non rank-minimal) facets of the SSP of a claw-free graph which arise from complete joins and *not* from sequential lifting: for instance, the inequality $\sum_{v \in V(G)} x_j \leq 2$ for the graph $G = C_5 + C_5$. As we are going to show, this is not the case with quasi-line graphs.

Lemma 3.2 *Let a quasi-line graph G be the complete join of graphs G_1 and G_2. Then G is perfect.*

Proof. Since the strong perfect graph conjecture holds for claw-free graphs [22], if G is not perfect, then either G_1 or G_2 is not perfect. Assume that G_1 is not perfect: it admits either an odd hole or antihole as an induced subgraph. Therefore, the neighborhood of any vertex of G_2 contains either an odd hole or an odd antihole: this is in contradiction with Lemma 2.4. □

Hence, we define *minimal* a (either rank or not) facet if it does not arise from *sequential* lifting: namely, a facet inducing inequality $\sum_{j \in V} a_j x_j \leq b$ of $STAB(G)$ such that $\sum_{j \in V \setminus u} a_j x_j \leq b$ is not a facet of $STAB(G \setminus u)$, for any $u : a_u > 0$.

In order to describe the SSP of a quasi-line graph, a sensible task is therefore that of characterizing minimal facets. Observe that, according to Lemma 3.2 and a result from [10] stated above, a rank facet of the SSP of a quasi-line graphs is minimal if and only if it is rank-minimal. Since rank-minimal facets for quasi-line graphs are characterized by Theorem 3.1, this task reduces to the description of minimal non-rank facets.

Actually, it is non-trivial that the set of minimal non-rank facets for quasi-line graphs is non-empty: for instance, one might wonder if inequalities (1) can be obtained by sequential lifting of *rank* inequalities. This is not the case, as we show in the following, where - adapting an argument from [11] and [26] - we prove that if G is quasi-line, then the sequential lifting of any rank facet of $STAB(G[T])$, for some $T \subset V$, produces facets of $STAB(G)$ which are still rank.

Let $G(V, E)$ be a quasi-line graph, T a subset of V and suppose that $\sum_{j \in T} x_j \leq \alpha(T)$ is a rank facet of $STAB(G[T])$. Consider any vertex $u \in V \backslash T$. We show that $\alpha(T \backslash N(u)) \geq \alpha(T) - 1$: then from (2) the lifting coefficient of a_u is either 0 or 1 and therefore our claim follows by induction. Suppose, vice versa, that there exists a vertex $u \in V - T$ such that $\alpha(T \setminus N(u)) \leq \alpha(T) - 2$; since G is claw-free, then each stable set in $G[T]$ of size $\alpha(T)$ must contain 2 vertices adjacent to u. Since $\sum_{j \in T} x_j \leq \alpha(T)$ is a facet of $STAB(G[T])$, there exists a $\mid T \mid \times \mid T \mid$ invertible matrix A whose rows are incidence vectors of stable sets in $G[T]$ of size $\alpha(T)$. Thus the sum of the columns of A corresponding to neighbors of u is the vector of all 2's and the sum of remaining columns is the vector whose components are $\alpha(T) - 2$. This contradicts invertibility of A unless u is adjacent to each vertex in T, in which case $\alpha(T) = 2$. And since $\alpha(T) = 2$, one sees (by an appropriate rearrangement of the rows and columns of A) that $G[T]$ contains an odd antihole. But, since $T \subseteq N(u)$, this is a contradiction.

4 Clique-family inequalities

In this section we introduce a new class of valid inequalities for the SSP of a graph. They are called *clique-family* inequalities and play a crucial role in the forthcoming conjecture on a linear description of the SSP of quasi-line graphs.

Let $G(V,E)$ be a graph and let $\mathcal{F}$ be a family of n (maximal) cliques of G, $n \geq 3$. Let $p \leq n$ be an integer. We define the following sets:

$I(\mathcal{F},p) = \{v \in V :\mid \{F \in \mathcal{F} : v \in F\} \mid \geq p\}$

$O(\mathcal{F},p) = \{v \in V :\mid \{F \in \mathcal{F} : v \in F\} \mid = p-1\}$;

and we set $\lambda(n,p) = \frac{p-r-1}{p-r}$, where $r = n - p\lfloor \frac{n}{p} \rfloor$.

Theorem 4.1 *The following inequality holds for $STAB(G)$:*

$$\sum_{v \in I(\mathcal{F},p)} x_v + \lambda(n,p) \sum_{v \in O(\mathcal{F},p)} x_v \leq \lfloor \frac{n}{p} \rfloor. \tag{3}$$

Proof. For shortness, we will write $I(\mathcal{F})$ $(O(\mathcal{F}))$ instead of $I(\mathcal{F},p)$ $(O(\mathcal{F},p))$. We must show that, for each stable set S of G, the following holds:

$$\mid S \cap I(\mathcal{F}) \mid + \frac{p-r-1}{p-r} \mid S \cap O(\mathcal{F}) \mid \leq \lfloor \frac{n}{p} \rfloor.$$

Let $V(\mathcal{F}) = I(\mathcal{F}) \cup O(\mathcal{F})$. The previous statement trivially holds for each stable set S such that $\mid S \cap V(\mathcal{F}) \mid \leq \lfloor \frac{n}{p} \rfloor$. Hence, from now on, let S be a stable set such that $\mid S \cap V(\mathcal{F}) \mid > \lfloor \frac{n}{p} \rfloor$. Observe that, by definition, for any stable set S we have:

$$p \mid S \cap I(\mathcal{F}) \mid + (p-1) \mid S \cap O(\mathcal{F}) \mid \leq n. \tag{4}$$

Claim. If $\mid S \cap V(\mathcal{F}) \mid > \lfloor \frac{n}{p} \rfloor$, then $\mid S \cap O(\mathcal{F}) \mid \geq p-r$. The next inequalities follow from (4):

$$\begin{aligned} p \mid S \cap V(\mathcal{F}) \mid &\leq n + \mid S \cap O(\mathcal{F}) \mid \\ \mid S \cap V(\mathcal{F}) \mid &\leq \frac{\lfloor \frac{n}{p} \rfloor p + r + \mid S \cap O(\mathcal{F}) \mid}{p} \\ \mid S \cap V(\mathcal{F}) \mid &\leq \lfloor \frac{n}{p} \rfloor + \frac{r + \mid S \cap O(\mathcal{F}) \mid}{p}. \end{aligned}$$

On the other hand, since $\mid S \cap V(\mathcal{F}) \mid \geq \lfloor \frac{n}{p} \rfloor + 1$, we have that:

$$\frac{r + \mid S \cap O(\mathcal{F}) \mid}{p} \geq 1 \rightarrow \mid S \cap O(\mathcal{F}) \mid \geq p-r. \qquad \textit{(End of the claim.)}$$

Now, let us consider again (4):

$$\begin{aligned} p \mid S \cap I(\mathcal{F}) \mid + (p-1) \mid S \cap O(\mathcal{F}) \mid &\leq n \\ (p-r) \mid S \cap I(\mathcal{F}) \mid + \frac{(p-r)(p-1)}{p} \mid S \cap O(\mathcal{F}) \mid &\leq \frac{n(p-r)}{p} \\ (p-r) \mid S \cap I(\mathcal{F}) \mid + (p-r-1+\frac{r}{p}) \mid S \cap O(\mathcal{F}) \mid &\leq \lfloor \frac{n}{p} \rfloor (p-r) + \frac{r(p-r)}{p} \\ (p-r) \mid S \cap I(\mathcal{F}) \mid + (p-r-1) \mid S \cap O(\mathcal{F}) \mid &\leq \lfloor \frac{n}{p} \rfloor (p-r) + \varepsilon(S) \end{aligned} \tag{5}$$

where: $\varepsilon(S) = \frac{r(p-r)}{p} - \frac{r}{p} \mid S \cap O(\mathcal{F}) \mid = \frac{r}{p}(p - r - \mid S \cap O(\mathcal{F}) \mid).$

Finally, since from our claim $\varepsilon(S) \leq 0$:

$$\begin{aligned}(p-r) \mid S \cap I(\mathcal{F}) \mid + (p-r-1) \mid S \cap O(\mathcal{F}) \mid &\leq \lfloor \frac{n}{p} \rfloor (p-r) \\ \mid S \cap I(\mathcal{F}) \mid + \frac{p-r-1}{p-r} \mid S \cap O(\mathcal{F}) \mid &\leq \lfloor \frac{n}{p} \rfloor \qquad \square\end{aligned}$$

We point out that clique family inequalities are valid for the SSP of *any* graph. Also, in order to get "stronger" inequalities, we may assume, without loss of generality, that the cliques in the family $\mathcal{F}$ are maximal. In fact, let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a family of cliques and let $\mathcal{F}' = \{K'_1, \ldots, K'_n\}$ be a family of *maximal* cliques such that, for any i, $K_i \subseteq K'_i$. It follows that, for any p, the inequality induced by $\mathcal{F}$ is dominated by the inequality induced by $\mathcal{F}'$.

Also note that the proof of Theorem 4.1 is *not* based on standard rounding arguments. This is not by chance, in fact we will show (point 5 below) that there are some clique family inequalities which are facets of the SSP of a (quasi-line) graph and yet can not be obtained through a single application of the Chvátal-Gomory procedure [5] to the fractional stable set polytope. This is not the case with facets arising from the matching polytope (see Corollary 2.2).

In the rest of this section, we show that clique family inequalities contain all the known classes of facets of the SSP of quasi-line graphs.

1. Let $G(V, E)$ be the line graph of a 2-connected hypomatchable graph; the inequality $\sum_{v \in V} x_v \leq \alpha(G)$ is therefore a facet of $STAB(G)$. As we shall show in the next section, in this case there exists a family $\mathcal{F}$ of $2\alpha(G) + 1$ cliques of G such that $I(\mathcal{F}, 2) = V$.

2. Suppose that $G[V, E]$ is a (n, q) circulant (recall that circulants are quasi-line). It is known [28] that the inequality $\sum_{v \in V} x_v \leq \alpha(G)$ is a facet of $STAB(G)$ if and only if n is not a multiple of $q + 1$. This inequality can be derived by Theorem 4.1. In fact, let $\mathcal{F}$ be the family of maximal cliques of G and $p = q + 1$; then $\mid \mathcal{F} \mid = n$, $I(\mathcal{F}, p) = V$ and $\lfloor \frac{|\mathcal{F}|}{p} \rfloor = \alpha(G)$. Hence, inequality (3) returns:

$$\sum_{v \in V} x_v \leq \alpha(G).$$

3. Consider the graphs G^q defined in Section 2. We show that inequalities (1) can be derived by Theorem 4.1. For each $1 \leq i \leq n$, the set $K_i = \{i, \ldots, i + q, i', \ldots, i' + q - 1\}$, is a (maximal) clique of G^q. Let $\mathcal{F} = \{K_i,\ i = 1..n\}$ and $p = q + 1$; it follows that $I(\mathcal{F}, p) = V(W)$ and $O(\mathcal{F}, p) = V(W')$. Also, since $\mid \mathcal{F} \mid = n = 2(p-1)^2 - 1$, then $\lfloor \frac{|\mathcal{F}|}{p} \rfloor = 2(p-2)$, $r = \mid \mathcal{F} \mid - p \lfloor \frac{|\mathcal{F}|}{p} \rfloor = 1$, $\lambda(\mid \mathcal{F} \mid, p) = \frac{p-r-1}{p-r} = \frac{p-2}{p-1}$. Hence, inequality (3) returns:

$$\sum_{v \in V(W))} x_v + \frac{p-2}{p-1} \sum_{v \in V(W')} x_v \leq 2(p-2).$$

4. In [11] Giles and Trotter provided other examples of non-rank facets of the SSP of quasi-line graphs, different from inequalities (1). Even if we do not go into details, all these inequalities are clique family inequalities. Other non-rank facets were found by Kind and Niessen [15] for the SSP of a circulant graph; also these inequalities are clique family inequalities.

5. Giles and Trotter [11] consider the following (quasi-line) graph. Let W be a $(37,7)$ circulant with vertices $V(W) = \{1, 2, \ldots, n\}$, and W' be a $(37,6)$ circulant with vertices $V(W') = \{1', 2', \ldots, n'\}$. G is the graph such that $V(G) = V(W) \cup V(W')$ and $E(G) = E(W) \cup E(W') \cup \{ij' : i \in \{1, \ldots, n\} \text{ and } j' \in \{i', \ldots, (i + 2k + 1)'\}\}$. It is a routine to check that the inequality

$$\gamma x \equiv 3 \sum_{v \in V(W))} x_v + 2 \sum_{v \in V(W')} x_v \leq 12 \tag{6}$$

is a facet of $STAB(G)$. In fact, there exists a family $\mathcal{F}$ of 37 (maximal) cliques such that $I(\mathcal{F}, 8) = V(W)$ and $O(\mathcal{F}, 8) = V(W')$ so that (6) is the clique-family inequality defined by $\mathcal{F}$ and $p = 8$. What is more interesting is that, as we show in the following, this inequality cannot be obtained through a single application of the Chvátal-Gomory procedure to $Fract(G)$. Let B denote the matrix whose rows are the incidence vectors of maximal cliques of G and suppose to the contrary that there exists a non-negative vector y such that $yB \geq \gamma$ and $\lfloor y\bar{1} \rfloor \leq 12$, where $\bar{1}$ is the vector of all 1-s. Let $\bar{x}$ be a vector defined as follows: $\bar{x}_j = \frac{1}{8}$ if $j \in V(W)$; $\bar{x}_j = 0$ else. The vector $\bar{x}$ satisfies $B\bar{x} \leq \bar{1}$, and so if $y \geq 0$ and $yB \geq \gamma$, then $\lfloor y\bar{1} \rfloor \geq \lfloor yB\bar{x} \rfloor \geq \lfloor \gamma\bar{x} \rfloor = 13$; which is a contradiction.

The previous argument was introduced by Giles and Trotter [11] for disproving a conjecture by Edmonds, namely that all the facets of the SSP of a claw-free graph G can be obtained through a single application of the Chvátal-Gomory procedure to $Fract(G)$. Interestingly, the graph given by Giles and Trotter in [11] is claw-free but *not* quasi-line; however, inequality (6) shows that Edmonds' conjecture does not hold even if restricted to quasi-line graphs.

5 Ben Rebea's conjecture

We are now ready for introducing a conjecture on a linear description of the SSP of a quasi-line graph. We would like to call this conjecture the Ben Rebea's Conjecture (see the Premise).

Conjecture 5.1 (Ben Rebea's Conjecture) *The stable set polytope of a quasi-line graph $G(V, E)$ may be described by the following inequalities:*

- *(i)* $x_v \geq 0$ *for each* $v \in V$
- *(ii)* $\sum_{v \in K} x_v \leq 1$ *for each maximal clique* K
- *(iii)* $\sum_{v \in I(\mathcal{F},p)} x_v + \lambda(|\mathcal{F}|, p) \sum_{v \in O(\mathcal{F},p)} x_v \leq \lfloor \frac{|\mathcal{F}|}{p} \rfloor$ *for each family of maximal cliques* $\mathcal{F}$ *such that* $|\mathcal{F}|$ *is not a multiple of* p, $|\mathcal{F}| > 2p$ *and* $p \geq 2$.

Inequalities (i) and (ii) trivially hold, while inequalities (iii) hold because of Theorem 4.1. We point out that, in general this set of inequalities is not *minimal*, i.e. they are not facet.

There are two non-trivial classes of (non-perfect) quasi-line graphs for which we have a linear description of the SSP: line graphs and circulants with the clique number equal to three. In the following, we show that Ben Rebea's Conjecture holds for both classes of graphs. We start with line graphs and with a simple lemma.

Lemma 5.2 *Let $G(V,E)$ be a graph, $L(G)$ the line graph of G and $Q \subseteq V$ such that $G[Q]$ is a 2-connected hypomatchable subgraph. There exists a family $\mathcal{L}_Q$ of $\mid Q \mid$ (maximal) cliques of $L(G)$ such that $I(\mathcal{L}_Q, 2) = E[Q]$; $O(\mathcal{L}_Q, 2) = \delta(Q)$.*

Proof. In the proof we allow G to have parallel edges. Recall that a vertex v of G is essential if either $\mid N_G(v) \mid > 2$ or $N_G(v) = \{x, y\}$ and x and y are not adjacent; also, since $G[Q]$ is a 2-connected hypomatchable subgraph, $\mid N_G(u) \cap Q \mid \geq 2$, for each $u \in Q$.

We associate with each vertex $v \in Q$ a maximal clique K_v of $L(G)$. If $v \in Q$ is essential, then let $K_v = \delta_G(v)$. If $v \in Q$ is non-essential, then there exist x and y in Q such that $N(v) = \{x, y\}$ and $xy \in E(G)$: let $K_v = \delta_G(v) \cup (\delta_G(x) \cap \delta_G(y))$.

Consider the family of cliques of $L(G)$ $\mathcal{L}_Q = \{K_v, v \in Q\}$: by construction $\mathcal{L}_Q$ is a family of maximal cliques. The set of vertices of $L(G)$ may be divided into three classes: V_1, vertices corresponding to edges uv of G with u and $v \notin Q$; V_2, vertices corresponding to edges uv of G with $u \in Q$ and $v \notin Q$; V_3, vertices corresponding to edges uv of G with u and $v \in Q$.

By construction, no vertex of class V_1 will belong to any clique of the family $\mathcal{L}_Q$, any vertex of class V_2 will belong to exactly one clique of the family and a vertex of class V_3 will belong to either two or three cliques of the family (it belongs to three cliques if and only if it corresponds to an edge xy, x and $y \in Q$, and there exists $w \in Q$ such that $N_G(w) = \{x, y\}$). □

Corollary 5.3 *Ben Rebea's Conjecture holds for line graphs.*

Proof. We need to show that the family of inequalities (iii) of Conjecture 5.1 contains inequalities (jjj) of Corollary 2.2. Let L be the line graph of a graph G. Let H be an α-maximal subgraph of L such that H is the line graph of a 2-connected hypomatchable graph and $\alpha(H) \geq 2$. Hence, there exists a subset Q of $V(G)$, such that $G[Q]$ is a 2-connected and hypomatchable subgraph of G and $H = L(G[Q])$. Also $\mid Q \mid = 2\nu(G[Q]) + 1 = 2\alpha(H) + 1 \geq 5$.

From Lemma 5.2, it follows that in L there exists a family $\mathcal{L}_Q$ of $\mid Q \mid$ (maximal) cliques of L such that $I(\mathcal{L}_Q, 2) = E_G[Q] = V_L(H)$, $O(\mathcal{L}_Q, 2) = \delta(Q)$. Also, since $\lambda(\mid \mathcal{L}_Q \mid, 2) = 0$ and $\lfloor \frac{|\mathcal{L}_Q|}{2} \rfloor = \frac{|Q|-1}{2} = \nu(Q) = \alpha(H)$, the clique family inequality associated with $\mathcal{L}_Q$ and $p = 2$ is:

$$\sum_{v \in H} x_v \leq \alpha(H).$$

Finally, observe that $\mid \mathcal{L}_Q \mid$ is odd and greater than 4. □

We now consider circulants with the clique number equal to three. Let W be the $(n, 2)$ circulant with vertices $\{1, 2, \ldots, n\}$. We call a subset of consecutive vertices of W an *interval.* Computation is modulo n, so, for instance, $\{n-1, n, n+1\}$ is an interval. A *1-interval set* is a subset $T \subseteq V(W)$ which is the union of vertex disjoint intervals $I_1, \ldots, I_t$, separated by just one vertex; for instance, if $n = 7$ then $\{\{1\}, \{3, 4\}, \{6\}\}$ is a 1-interval set. Finally, a 1-interval set is *regular* if t is odd, $t \geq 3$ and $\mid I_j \mid \equiv 1 \; mod \; 3$, for any j. Dahl proved the following theorem.

Theorem 5.4 [7] *The facets of the stable set polytope of W are given by the following inequalities:*

(j) $x_v \geq 0$ *for each vertex v*

(jj) $\sum_{v \in K} x_v \leq 1$ *for each maximal clique K*

(jjj) $\sum_{v \in V(W)} x_v \leq \lfloor \frac{n}{3} \rfloor$ *if n is not a multiple of 3.*

(jv) $\sum_{v \in T} x_v \leq \alpha(T)$ *for each $T \subset V(W)$ such that T is a regular 1-interval set.*

We have already shown (point 2 in the previous section) that the inequality (jjj) is a clique-family inequality, for a given pair $(\mathcal{F}, p)$ with $\mid \mathcal{F} \mid= n$ and $p = 3$: since n is not a multiple of 3, (jjj) belongs to the family of inequalities (iii) of Conjecture 5.1. We now show that also this family contains inequalities (jv). Let $T = I_1 \cup \ldots \cup I_t$ be a regular 1-interval set. For each $j \in \{1, \ldots, t\}$, denote $s_j := \frac{|I_j|-1}{3}$ (hence $s_j = 0$ if $\mid I_j \mid= 1$). It is easy to see that $\alpha(T) = \lfloor \frac{t}{2} \rfloor + \sum_{j=1..t} s_j$ and that there exists a family $\mathcal{F}$ of $(t + 2\sum_{j=1..t} s_j)$ cliques such that $I(\mathcal{F}, 2) = T$. Hence, the clique family inequality associated with $\mathcal{F}$ and $p = 2$ is:

$$\sum_{v \in T} x_v \leq \alpha(T).$$

Finally, observe that $\mid \mathcal{F} \mid$ is odd and without loss of generality greater than 4 (if $\mid \mathcal{F} \mid= 3$ then $n = 6$ and T induces a triangle of W). □

6 Open questions

The main open question is, of course, the solution of Ben Rebea's Conjecture. Observe that, for disproving the conjecture, it would be enough to show a facet $\sum_{v \in V(G)} a_j x_j \leq b$ for the SSP of a quasi-line graph G and three vertices u, v, z of $V(G)$ such that $a_u > a_v > a_z > 0$. Examples of such facets are known for claw-free graphs but not for quasi-line graphs [11].

Vice versa, proving the conjecture seems to be more challenging. Unfortunately, the existing (polynomial time) algorithms for solving the (weighted) stable set problem in quasi-line graphs give no clue how to describe the SSP of these graphs. For instance, the algorithm for the weighted case by Minty [19] (recently revised by Nakamura and Tamura [20]) is based on the solution of a polynomial number of matching problems on some auxiliary graphs, and therefore gives no hints on the polyhedral structure of the SSP of the original graph. Similar remarks hold for the algorithms for the unweighted case by Lovász and Plummer [18] and by Sbihi [25]. (Incidentally, we point out that Minty's algorithm works for the larger claw-free graphs, but the restriction to quasi-line graphs does not bring any simplification.)

A sensible approach would be that of extending existing proofs for the matching polytope: either a *direct* proof, such as the proof by Lovász and Plummer [18], or an *algorithmic* proof, such as the primal-dual algorithm by Edmonds [8, 18]. Going for a direct proof, a reasonable task would be that of characterizing minimal facets (see Section 3).

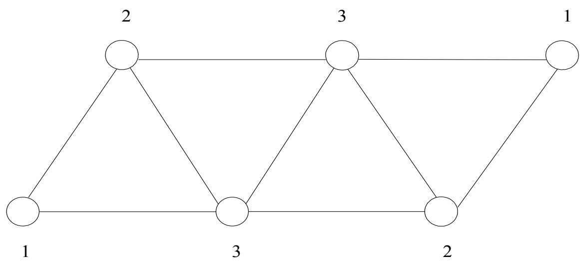
On the other hand, since Conjecture 5.1 does not provide a *minimal* description of the polytope, we have investigated more in depth the idea of a primal-dual algorithm: Ben Rebea also took this approach. We prefer not to go into (long) details but discuss a crucial issue still to be solved (and actually missed by Ben Rebea). We assume the reader to be familiar with the structure of Edmonds' primal-dual algorithm.

The idea behind the conjecture is that families of cliques play for the stable set problem in quasi-line graphs the same role as odd sets of vertices for the matching problem. Therefore, in a primal-dual approach, one should extend to families of cliques crucial operations like shrinking.

When dealing with the matching problem (polytope), the dual variables are associated with single vertices and odd sets of vertices and without loss of generality, one may assume that a dual solution y is *laminar*: i.e. if S and T are two odd sets of vertices (single vertices) such that $y_S > 0$ and $y_T > 0$, then either $S \subset T$, or $T \subset S$ or $S \cap T = \emptyset$. In fact, if y is not laminar, then by simple *uncrossing* operations we get an appropriate laminar solution (see for instance [18]). Observe that laminar solutions are crucial for performing *sequences* of shrinking operations.

Unfortunately, when dealing with the stable set problem (polytope) in quasi-line graphs, laminar solutions for the dual problem seem elusive, even if we restrict to a class of graphs for which a linear description of $STAB(G)$ is known. In fact, consider the class of quasi-line *perfect* graphs: then $STAB(G) = Fract(G)$ [6] and a dual solution correspond to a weighted clique cover of G (and the maximum weight of a stable set is equal to the minimum value of a weighted clique cover). We wonder if a minimum weighted clique cover for these graphs is, without loss of generality, laminar.

Consider the weighted graph in the following figure. There exists only one minimum weighted clique cover: namely, $y_K = 1$ for each maximal clique K. But the structure of this solution seems far from being laminar! Observe that the graph in the figure is a forbidden subgraph for line graphs [3] (but not for quasi-line graphs).



Hence, we believe that finding a primal-dual algorithm for the maximum weighted stable set (minimum weighted clique cover) problem in a quasi-line perfect graph is a relevant open problem, since the solution of this problem could give clues how to design a primal-dual algorithm for the general (non-perfect) case. Observe that a graph is quasi-line and perfect if and only if it is claw-free and perfect. The only efficient algorithm for finding a minimum weighted clique cover on a claw-free perfect graph has been given by Hsu and Nemhauser [14]; in fact, this algorithm is neither a dual nor a primal-dual one: first, a maximum weighted stable set is found by Minty's algorithm and then, by complementarity slackness, an optimal clique cover is built.

Other open questions concern clique-family inequalities. We do not know the complexity of separating these inequalities. On the other hand, if the conjecture was true, a natural question would be that of characterizing clique family inequalities that are facets of the SSP.

Finally, we would like to know more about quasi-line graphs. Consider the following question. Let G be a quasi-line graph and suppose that there exists a subgraph H of G such that H is the line graph of a 2-connected hypomatchable graph and $\alpha(H) = \alpha(G)$. We know (see Section 4) that there exists a family $\mathcal{F}$ of $2\alpha(H) + 1$ cliques of H such that $I(\mathcal{F}, 2) = V(H)$. On the other hand, by sequential lifting, $\sum_{v \in G} x_v \leq \alpha(G)$ is a facet of $STAB(G)$; therefore we expect that there exists a pair ($\mathcal{F}$', p'), where $\mathcal{F}$' is a family of cliques of G, such that $I(\mathcal{F}', p') = V(G)$ and $\lfloor \frac{|\mathcal{F}'|}{p'} \rfloor = \alpha(G)$. How is it possible to prove that such a pair always exists if G is quasi-line?

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