Geometry of Homogeneous Convex Cones, Duality Mapping, and Optimal Self-Concordant Barriers *

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Abstract

We study homogeneous convex cones. We first characterize the extreme rays of such cones in the context of their primal construction (due to Vinberg) and also in the context of their dual construction (due to Rothaus). Then, using these results, we prove that every homogeneous cone is facially exposed. We provide an alternative proof of a result of Güler and Tunçel that the Siegel rank of a symmetric cone is equal to its Carathéodory number. Our proof does not use the Jordan-vonNeumann-Wigner characterization of the symmetric cones but it easily follows from the primal construction of the homogeneous cones and our results on the geometry of homogeneous cones in primal and dual forms. We study optimal self-concordant barriers in this context. We briefly discuss the duality mapping in the context of automorphisms of convex cones and prove, using numerical integration, that the duality mapping is not an involution on certain self-dual cones.

Keywords: convex optimization, self-concordant barriers, homogeneous cones, symmetric cones, Siegel domains, facially exposed, Carathéodory number, interior-point methods

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1 Introduction

An elegant, powerful and modern theory of interior-point methods treats convex optimization problems in the conic form, as the problem of optimizing a linear function over a convex cone intersected with an affine space (see Nesterov and Nemirovskii [12], Nesterov and Todd [13, 14] and the exposition by Renegar [17]). Also see [24] where symmetric primal-dual interior-point methods are generalized to all convex optimization problems in conic form.

In such formulations, we interpret the convex cone constraint as the "difficult" constraint and deal with the cone constraint by utilizing a smooth, strictly convex barrier function for it. Successful modern theories imposed rather sophisticated conditions on these barriers (*self-concordance* by Nesterov and Nemirovskii, *self-scaledness* by Nesterov and Todd). In return, powerful theoretical convergence results as well as effective computational procedures emerged.

Even though the fundamental theory developed by Nesterov and Nemirovskii applies to all convex cones, the more specialized and sophisticated theory developed by Nesterov and Todd deals with *symmetric cones* (see the next section for a definition). Such cones include the cones of symmetric positive semidefinite matrices, second order cones and cones of Hermitian positive semidefinite matrices over complex and quaternion numbers as well as arbitrary compositions of the above under direct sums and nonsingular linear maps. *Homogeneous cones* properly include symmetric cones. Güler [5] was the first to point out connections between the theory of self-concordant barriers and the theory of homogeneous cones. This connection was further investigated by Güler and the second author in [7].

While not necessarily self-dual, homogeneous cones have very rich automorphisms. Such cones are interesting not only for their elegant structure but also for the potential they have for more sophisticated interior-point-method theories and duality theories for convex optimization problems involving them as convex cone constraints. For instance, Güler [6] proved that certain long-step estimation property of self-scaled barriers of Nesterov-Todd extends to hyperbolic barriers (every homogeneus cone admits a hyperbolic barrier). A nice example of homogeneous cones is the epigraph of matrix norms. Such cones also arise in applications (see for instance Ben-Tal and Nemirovskii's book on engineering applications [1]) as well as in strong duality theories (see Ramana [15] and [16]).

In the next section, we describe some background results and establish various definitions and notation. In Section 3, we characterize the extreme rays of homogeneous convex cones in the primal form (as described by Vinberg) as well as in the dual form (as described by Rothaus). Using these characterizations and other techniques, we prove that every homogeneous cone is facially exposed. This generalizes the well-known property of the cone of symmetric positive semidefinite matrices and the polyhedral cones. We also provide an alternative proof of an earlier result of Güler and the second author that for symmetric cones, the Carathéodory number and the Siegel rank coincide. Our proof does not use the Jordan-vonNeumann-Wigner classification of the irreducible, symmetric cones but it easily follows from Vinberg and Rothaus' descriptions of homogeneous cones and our geometric results. In Section 4, we study optimal self-concordant barriers for homogeneous convex cones. In Section 5, we prove that for self-dual cones that are not homogeneous, the duality mapping is not necessarily an involution. We conclude with a preliminary discussion of interesting linear maps on convex cones.

2 Notation and Background

Let S be a subset of $\mathbb{R}^n$. Then, $\mathrm{int}(S)$, $\mathrm{cl}(S)$, $\partial(S)$ denote the interior, closure and the boundary of S respectively. For a given linear operator A, $\mathcal{R}(A)$, $\mathcal{N}(A)$ denote the range and the null space of A respectively.

In this paper, K denotes a pointed, closed, convex cone in $\mathbb{R}^n$ with nonempty interior. The set of extreme rays of K are denoted by $\mathrm{Ext}(K)$. Sometimes it is more convenient to normalize those extreme rays; $\mathrm{ext}(K)$ denotes the set of normalized extreme rays of K. For a given inner product $\langle \cdot, \cdot \rangle$, the *dual cone* of K is given by

$$K^* := \{s \in \mathbb{R}^n : \langle s, x \rangle \geq 0, \ \forall x \in K\}.$$

K is *self-dual* if an inner product can be chosen so that $K = K^*$. Self-duality is indeed of interest since in the setting of convex conic optimization problems, the primal and the dual problems have the same "domain" (cone) defining the "difficult" constraints. The cone is *homogeneous* if the group $\mathrm{Aut}(K)$ of nonsingular, linear maps on $\mathbb{R}^n$ keeping K invariant acts transitively on $\mathrm{int}(K)$. Homogeneity is also of great interest, since it allows interior-point methods to use "scalings" without changing the "difficult" cone constraints. Finally, K is *symmetric* if it is both homogeneous and self-dual.

Our approach will involve geometric, algebraic and analytic structures related to convex cones. The following analytic description of convex cones is fundamental in various areas of mathematics. The *characteristic function* of K is defined to be

$$\phi_K(x) := \int_{K^*} \mathbf{e}^{-\langle x, s \rangle} \, ds,$$

where $\mathbf{e}$ denotes the logarithmic constant. The logarithm of the characteristic function, F, is a *self-concordant barrier function* for K. That is, F is a

$$\begin{aligned}
&\mathcal{C}^3\text{-smooth, strictly convex function on } \mathrm{int}(K); && (1)\\
&F(x^{(k)}) \to \infty \text{ as } \{x^{(k)}\} \subset \mathrm{int}(K) \text{ approaches } \partial(K); && (2)\\
&\exists \text{parameter } \vartheta \geq 1 \text{ such that } \forall \alpha > 0, \ \ F(\alpha x) = F(x) - \vartheta \ln(\alpha); && (3)\\
\text{and} \quad &\forall x \in \mathrm{int}(K), \forall h \in \mathbb{R}^n, \ \ \left| D^3F(x)[h,h,h] \right| \leq 2(D^2F(x)[h,h])^{\frac{3}{2}}. && (4)
\end{aligned}$$

The last condition ensures that the change in the third derivative of F is bounded by the size of its Hessian (hence, the name *self-concordance*). The definition we gave above corresponds to what Nesterov and Nemirovskii call a *ϑ-normal (self-concordant) barrier for K*.

Barrier functions have been studied extensively because they are a very useful tool in convex optimization algorithms. In addition to proposing the usage of self-concordant barriers in the general, modern theory of interior-point methods, Nesterov and Nemirovskii [12] proved the exceptional result that every convex cone in $\mathbb{R}^n$ admits a self-concordant barrier with parameter $\vartheta = O(n)$. In their constructive proof, they used the following function

$$\Phi_K(x) := \ln\left(\mathrm{vol}_n\{s \in K^* : \langle s, x\rangle \le 1\}\right)$$

(more precisely, a constant multiple of it) which they called *the universal barrier for K*. Here, vol_n denotes the n-dimensional Lebesgue measure.

The barrier F coming from the characteristic function, which we will subsequently refer to only as *the barrier of K*, is in fact equal to the universal barrier (modulo a constant), a result of Güler [5]. Moreover, the negation of the gradient of F, i.e., $-F'(\cdot)$, always determines a bijection from $\mathrm{int}(K)$ to $\mathrm{int}(K^*)$. Its inverse is $-{F_*}'(\cdot)$ where F_* is the slightly modified Legendre-Fenchel conjugate of F:

$$F_*(s) := \sup\{-\langle s, x\rangle - F(x) : x \in \mathrm{int}(K)\}.$$

Some additional properties of general self-concordant barrier functions are given below.

Proposition 2.1 (Nesterov and Nemirovskii [12]) *Let F be a self-concordant barrier for K with parameter ϑ. Then F_* is a self-concordant barrier for K^* with parameter ϑ. Also, for all $x \in \mathrm{int}(K)$ and $s \in \mathrm{int}(K^*)$, F and F^* satisfy the following*

$$\forall k \in \mathbb{Z}_{++} \text{ and } \alpha > 0, D^kF(\alpha x) = \frac{1}{\alpha^k} D^kF(x); \tag{5}$$

$$\langle -F'(x), x\rangle = \vartheta; \tag{6}$$

$$F''(x)x = -F'(x); \tag{7}$$

$$F_*(-F'(x)) = -\vartheta - F(x) \text{ and } F(-F_*'(s)) = -\vartheta - F_*(s); \tag{8}$$

$$F_*''(-F'(x)) = \left[F''(x)\right]^{-1} \text{ and } F''(-F_*'(s)) = \left[F_*''(s)\right]^{-1}; \tag{9}$$

$$D^kF(Ax)[Ah, Ah, \ldots, Ah] = D^kF(x)[h, h, \ldots, h], \forall h \in \mathbb{R}^n, k \in \mathbb{Z}_{++}, A \in \mathrm{Aut}(K); \tag{10}$$

□

We denote the set of $n \times n$ symmetric matrices over reals by Σ^n. The closed convex cone of $n \times n$ symmetric positive semidefinite matrices is denoted by Σ^n_+. The interior of Σ^n_+ is the symmetric positive definite matrices of the same order, denoted by Σ^n_{++}. For $x, y \in \Sigma^n$ we write $x \succeq y$ to mean that $(x - y) \in \Sigma^n_+$.

3 Homogeneous cones, their geometry, and Siegel domains

3.1 Elements of the algebraic theory of homogeneous cones

For a more detailed exposure to this theory see Güler [5] and Güler and Tunçel [7]. In this subsection below, we give a brief summary of that exposure to serve our purposes in this paper.

Definition 3.1 *Let K be a closed convex cone in $\mathbb{R}^n$. A K-bilinear symmetric form $B(u,v)$ in $\mathbb{R}^p$ is a mapping from $\mathbb{R}^p \oplus \mathbb{R}^p$ to $\mathbb{R}^n$ satisfying the following properties (here the trivial bilinear form, that is $p = 0$, is allowed)*

1. $B(\alpha_1 u^{(1)} + \alpha_2 u^{(2)}, v) = \alpha_1 B(u^{(1)}, v) + \alpha_2 B(u^{(2)}, v), \quad \forall u^{(1)}, u^{(2)} \in \mathbb{R}^p,$ *and* $\forall \alpha_1, \alpha_2 \in \mathbb{R}$;
2. $B(u,v) = B(v,u), \quad \forall u, v \in \mathbb{R}^p$;
3. $B(u,u) \in K, \quad \forall u \in \mathbb{R}^p$;
4. $B(u,u) = 0$ *implies* $u = 0, \qquad \forall u \in \mathbb{R}^p$.

Let B and K satisfy the conditions (1)-(4) in Definition 3.1. Then the *Siegel domain* corresponding to K and B is the set

$$SD(K,B) := \{(x,u) \in \mathbb{R}^n \oplus \mathbb{R}^p : x - B(u,u) \in K\}.$$

The *Siegel cone* corresponding to the Siegel domain $SD(K,B)$ is

$$SC(K,B) := \mathrm{cl}\left\{(x,u,t) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R} : t > 0, \left[x - \frac{B(u,u)}{t}\right] \in K\right\}.$$

Definition 3.2 *A K-bilinear symmetric form B is called* homogeneous *if K is a homogeneous cone and there exists a transitive subset $G \subseteq \mathrm{Aut}(K)$ such that for every $g \in G$, there exists a linear transformation of $\bar{g}$ of $\mathbb{R}^p$ such that*

$$g\,B(u,v) = B(\bar{g}u, \bar{g}v), \text{ for all } u, v \in \mathbb{R}^p.$$

The above definition isolates an important property of K and the bilinear form B. It means that as far as the bilinear form B is concerned, the effect on $\mathbb{R}^n$ of every element g in the transitive subset G can be simulated by a corresponding linear transformation $\bar{g}$ acting on $\mathbb{R}^p$.

The following lemma describes how to construct a transitive subset of the automorphism group of $SC(K,B)$ based on a transitive subset of the automorphism group of K (and of course the bilinear form B).

Lemma 3.1 *(Vinberg [27]) If K is a homogeneous cone and B is a homogeneous K-bilinear symmetric form, then the cone $SC(K,B)$ is homogeneous, and the following linear maps generate a transitive subgroup of* $\mathrm{Aut}(SC(K,B))$, *(for each $(x,u,t) \in \mathrm{int}(SC(K,B))$)*

$$\begin{aligned}
T_1(x,u,t) &:= (x, \sqrt{\alpha}u, \alpha t), \quad \alpha > 0, \\
T_2(x,u,t) &:= (x + 2B(u,v) + tB(v,v), u + tv, t), \quad v \in \mathbb{R}^p, \\
T_3(x,u,t) &:= (gx, \bar{g}u, t), \quad g \in G \subseteq \mathrm{Aut}(K),
\end{aligned}$$

where G is the transitive subset of $\mathrm{Aut}(K)$ *from Definition 3.2 and $\bar{g}$ is the corresponding linear transformation from the same definition.*

□

The above lemma describes a recursive construction of homogeneous cones. A homogeneous cone K and a homogeneous K-bilinear symmetric form B together give rise to a homogeneous cone $SC(K,B)$ in a higher dimensional space. A more impressive fact is that the converse is also true. For every homogeneous cone K of dimension at least 2, there exists a lower dimensional cone $\bar{K}$ and a homogeneous $\bar{K}$-bilinear symmetric form B such that K is linearly isomorphic to $SC(\bar{K},B)$, see for example Gindikin [4] (page 75). Therefore, an arbitrary homogeneous cone can be constructed, recursively, using lower dimensional homogeneous cones, starting from the real half-line $\mathbb{R}_+$. The minimum number of steps required to construct K in this way is called *the Siegel rank of K*. We denote that integer invariant of the cone by $\mathrm{rank}(K)$ and we define $\mathrm{rank}(\mathbb{R}_+) := 1$.

K is homogeneous if and only if K^* is. So, the above classification theory of Vinberg also applies to K^*. In fact, the Siegel ranks of K and K^* coincide. Rothaus [20] worked out the theory from the dual side, paralleling the algebraic structures in the primal construction of Vinberg. Rothaus' dual Siegel cone construction begins by defining, for each $y \in \mathbb{R}^n$, the symmetric linear mapping $U(y) : \mathbb{R}^p \to \mathbb{R}^p$,

$$\langle U(y)u, v\rangle := \langle B(u,v), y\rangle, \text{ for all } u, v \in \mathbb{R}^p. \tag{11}$$

Remark 3.1 *Note that according to the above definition, the inner product on $\mathbb{R}^p$ and the linear mapping $U(y)$ are determined by the inner product on $\mathbb{R}^n$ (which is central to our study) and the bilinear form B. We can easily choose any inner product on $\mathbb{R}^p$ and redefine $U(y)$ via the above equations. These changes do not affect the following results (i.e., the following results are valid for any choice of the inner product on $\mathbb{R}^p$ so long as the definition of $U(y)$ is consistent with the above equations).*

Proposition 3.1 *(Rothaus [20]) Let G be a transitive subset of* $\mathrm{Aut}(K)$ *such that for every $g \in G$, there exists a linear mapping $\bar{g} : \mathbb{R}^p \to \mathbb{R}^p$ satisfying $gB(u,v) = B(\bar{g}u, \bar{g}v)$ whenever*

$u, v \in \mathbb{R}^p$. Then, for every $g \in G$ there exists a linear operator $T_{\bar{g}} : \Sigma_p \to \Sigma_p$ such that the following diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{g^*} & \mathbb{R}^n \\ U\big\downarrow & & \big\downarrow U \\ \Sigma_p & \xrightarrow{T_{\bar{g}}} & \Sigma_p \end{array}$$

commutes. If $y \in K^*$, then the operator $U(y)$ is positive semidefinite. Moreover, if $y \in \mathrm{int}(K^*)$, then $U(y)$ is positive definite.

□

Rothaus proves that

$$\mathrm{int}\,(SC(K,B)^*) = \{(y,v,s) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R} \;:\; y \in \mathrm{int}(K^*),\ s > \langle U(y)^{-1}v, v\rangle\},$$

where the inner product on $(\mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R})$ is defined by

$$\langle (x,u,t),(y,v,s)\rangle := \langle x,y\rangle + 2\langle u,v\rangle + st.$$

We now describe the generators of a transitive subgroup of $\mathrm{Aut}(SC(K,B)^*)$ derived from a transitive subset of $\mathrm{Aut}(K)$.

Lemma 3.2 *(Rothaus [20]) Let $G \subseteq \mathrm{Aut}(K)$ be a transitive subset. Then for every point $(\bar{y}, \bar{v}, \bar{s}) \in \mathrm{int}\,(SC(K,B)^*)$ the following maps are in $\mathrm{Aut}(SC(K,B)^*)$:*

$$\begin{array}{rcl} T_1(y,v,s) & = & \left(y, v - \bar{v}, s - 2\langle U(y)^{-1}\bar{v}, v\rangle + \langle U(y)^{-1}\bar{v}, \bar{v}\rangle\right), \\ T_2(y,v,s) & = & (g^*y, \bar{g}^*v, s), \quad g \in G, \\ T_3(y,v,s) & = & \left(y, \dfrac{v}{\sqrt{\alpha}}, \dfrac{s}{\alpha}\right), \quad \alpha > 0, \end{array}$$

where the linear transformation $\bar{g}$ is the one described in Proposition 3.1.

□

3.2 Geometry of homogeneous cones

A face P of K is called a *d-face of K* if the dimension of the affine hull of P is d. An *exposed d-face of K* is a d-face of K which is exposed.

Theorem 3.1 *Let K be a pointed, closed convex cone with nonempty interior. Also let $A \in \mathrm{Aut}(K)$. Then*

(a) $v \in \mathrm{Ext}(K) \iff A(v) \in \mathrm{Ext}(K)$;

(b) $v \in \mathrm{Ext}(K)$ *is exposed* $\iff A(v) \in \mathrm{Ext}(K)$ *is exposed;*

(c) $\{0\} \subset P \subset K$ *is an exposed d-face of K* $\iff A(P)$ *is an exposed d-face of K.*

Proof. For part (a) see [7], for (b) see [25]. We give below a proof of (c) which generalizes the arguments of [25].

Let $\{0\} \subset P \subset K$ be an exposed d-face of K. Then there exists a supporting hyperplane

$$H := \{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\}$$

such that $K \cap H = P$ and

$$\langle a, x\rangle \quad \leq \quad \alpha, \quad \forall x \in K. \tag{12}$$

Let $A \in \mathrm{Aut}(K)$. Then we have

$$\begin{aligned} A(H) &= \{Ax \in \mathbb{R}^n : \langle a, x\rangle = \alpha\} \\ &= \{x \in \mathbb{R}^n : \langle a, A^{-1}x\rangle = \alpha\} \\ &= \{x \in \mathbb{R}^n : \langle A^{-*}a, x\rangle = \alpha\}. \end{aligned}$$

Moreover, $\langle A^{-*}a, x\rangle = \langle a, A^{-1}x\rangle \leq \alpha$ for all $x \in K$ by (12) and the fact that "$A \in \mathrm{Aut}(K)$ iff $A^{-1} \in \mathrm{Aut}(K)$."

$$\begin{aligned} A(H) \cap K &= \{x \in K : \langle A^{-*}a, x\rangle = \alpha\} \\ &= \{x \in K : \langle a, A^{-1}x\rangle = \alpha\} \\ &= A(P). \end{aligned}$$

We proved that $A(H)$ is a hyperplane supporting K at $A(P)$. Since P is a face of K, for all $x, y \in K$, $(x+y) \in P$ implies x and y are both in P. Let $x, y \in K$ such that $(x+y) \in A(P)$. Then using $A(P) = A(H) \cap K$, we find $A^{-1}(x), A^{-1}(y) \in P$. Thus, $x, y \in A(P)$. Clearly, $\dim(A(P)) = \dim(P)$. Therefore, $\{0\} \subset A(P) \subset K$ is an exposed d-face of K. The converse also follows from the above argument, since $A \in \mathrm{Aut}(K) \iff A^{-1} \in \mathrm{Aut}(K)$. $\square$

Using the algebraic construction for homogeneous cones described in the previous section, we seek a more detailed description of homogeneous cones in terms of the lower-dimensional cones and the symmetric bilinear forms from which they arise. First, we describe the set $\mathrm{ext}(SC(K, B))$ of normalized extreme rays of $SC(K, B)$. This next theorem generalizes Theorem 5.1 of [25] and its proof.

Theorem 3.2 *Let K be a homogeneous cone and let B be a homogeneous K-bilinear symmetric form. Then*

$$\mathrm{ext}(SC(K,B)) = \{(x,0,0) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R} \ : \ x \in \mathrm{ext}(K)\} \cup \left\{ \frac{(B(u,u),u,1)}{\|(B(u,u),u,1)\|} \ : u \in \mathbb{R}^p \right\}.$$

Proof. It can be readily checked that the two types of rays above are in $SC(K,B)$ and have norm 1. To show that $(x,0,0)$ is an extreme ray, suppose that we can write

$$(x,0,0) = (\delta x, u, t) + (\lambda x, -u, -t),$$

as a sum of two vectors in $SC(K,B)$, where $\delta, \lambda \geq 0$. Both

$$[\delta t x - B(u,u)] \in K$$

and

$$-\lambda t x - B(-u,-u) = [-\lambda t x - B(u,u)] \in K$$

imply that $\pm x \in K$; thus, $x = 0$ because K is pointed. But now, $\pm B(u,u) \in K$ so $u = 0$ as well. Therefore, $(x,0,0)$ must be an extreme ray.

To show that $(B(u,u),u,1)$ is an extreme ray, by Theorem 3.1, it suffices to consider the case $u = 0$, for every vector $(B(u,u),u,1)$ is the image of $(B(0,0),0,1) = (0,0,1)$ under the automorphism T_1 described in Lemma 3.1, with $v := u$. So, suppose that

$$(0,0,1) = (x,u,t_1) + (-x,-u,t_2)$$

for $t_1, t_2 > 0$. Since

$$[t_1 x - B(u,u)]\,,\ [-t_2 x - B(u,u)] \in K,$$

we must have $\pm x \in K$. Therefore, $x = 0$ because K is pointed. But now $\pm B(u,u) \in K$, so that $u = 0$ for the same reason. Thus, $(0,0,1)$ and every vector of the form $(B(u,u),u,1)$ is an extreme ray.

Finally, we must show that every extreme ray of $SC(K,B)$ is one of the two types described. Let $r := (x,u,t) \in \mathrm{Ext}(K)$. If $t = 0$ then, since $\pm B(u,u) \in K$, $u = 0$. Thus, $r = (x,0,0)$ for some $x \in \mathrm{Ext}(K)$. Otherwise, we can assume without loss of generality that $t = 1$. We write $x = B(u,u) + w$ for $w \in K$ to obtain

$$r = (B(u,u),u,1) + (w,0,0) \in \mathrm{cone}\{(B(u,u),u,1),\ (x,0,0) \ : \ x \in \mathrm{ext}(K)\},$$

so that r must be a positive multiple of one of these rays. $\square$

The dual characterization for $SC(K,B)^*$ given in Proposition 3.1 allows us to give a parallel description of the extreme rays of $SC(K,B)^*$ in terms of K^* and B. Recall that

$$SC(K,B)^* \ = \ \mathrm{cl}\{(y,v,s) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R} \ : \ y \in \mathrm{int}(K^*),\ s > \langle U(y)^{-1} v, v\rangle\}. \qquad (13)$$

However, this characterization describes only the interior of $SC(K,B)^*$ explicitly, relying on the positive-definiteness of the map $U(y)$ for $y \in \text{int}(K^*)$. When $y \in \partial(K^*)$, $U(y)$ may not be invertible. To proceed, we first address the definability of the quantity $\langle U(y)^{-1}v, v\rangle$ when $y \in \partial(K^*)$.

Remark 3.2 *Let $y \in \partial(K^*)$ and $v \in \mathcal{R}(U(y))$. Then $\langle U(y)^{-1}v, v\rangle$ exists and is well-defined.*

Proof. By assumption, there exists $\tilde{u} \in \mathcal{R}(U(y))$ with $U(y)\tilde{u} = v$. Since $U(y)$ is linear, every u satisfying $U(y)u = v$ has form $u = \tilde{u} + w$ where $w \in \mathcal{N}(U(y))$. Now,

$$\langle u, v\rangle = \langle \tilde{u}, v\rangle + \langle w, v\rangle = \langle \tilde{u}, v\rangle + \langle w, U(y)u\rangle = \langle \tilde{u}, v\rangle + \langle U(y)w, u\rangle = \langle \tilde{u}, v\rangle.$$

Therefore, we are justified in writing $\langle U(y)^{-1}v, v\rangle$. This quantity has a unique interpretation as

$$\begin{aligned} \langle U(y)^{-1}v, v\rangle &= \langle u, v\rangle, \forall u \text{ such that } U(y)u = v \\ &= \langle \tilde{u}, v\rangle. \end{aligned}$$

□

Henceforth we will write $\langle U(y)^{-1}v, v\rangle$ without justification whenever $v \in \mathcal{R}(U(y))$. As we shall see, this happens for all $(y, v, s) \in SC(K,B)^*$, so that we obtain an explicit description for $SC(K,B)^*$.

Theorem 3.3 *Let K be a homogeneous cone and let B be a homogeneous K-bilinear symmetric form. Also let U be defined as in (11). Then*

$$SC(K,B)^* = \{(y, v, s) \ : \ y \in K^*, \ v \in \mathcal{R}(U(y)), \ s \geq \langle U(y)^{-1}v, v\rangle\}.$$

Proof. Let $(\bar{y}, \bar{v}, \bar{s})$ be in $SC(K,B)^*$ and suppose that $\bar{v} \notin \mathcal{R}(U(\bar{y}))$. Let us write $\bar{v} = v_R + v_N$ where $v_N \neq 0$ is the orthogonal projection of $\bar{v}$ onto $\mathcal{N}(U(\bar{y}))$. For all $\lambda > 0$,

$$(B(\lambda v_N, \lambda v_N), -\lambda v_N, 1) \in SC(K,B).$$

However,

$$\begin{aligned} \langle (\bar{y}, \bar{v}, \bar{s}), (B(\lambda v_N, \lambda v_N), -\lambda v_N, 1)\rangle &= \langle \bar{y}, B(\lambda v_N, \lambda v_N)\rangle + 2\lambda\langle \bar{v}, -v_N\rangle + \bar{s} \\ &= \lambda^2\langle U(\bar{y})v_N, v_N)\rangle - 2\lambda\|v_N\|^2 + \bar{s} \\ &= -2\lambda\|v_N\|^2 + \bar{s} \\ &< 0 \end{aligned}$$

for large enough λ. This is a contradiction to the fact that $(\bar{y}, \bar{v}, \bar{s}) \in SC(K,B)^*$. We have established that $v \in \mathcal{R}(U(y))$ for all $(y, v, s) \in SC(K,B)^*$.

Now, let (y, v, s) satisfy $v \in \mathcal{R}(U(y))$. Then

$$(y, v, s) \in SC(K,B)^* \text{ iff } \langle (x,u,t), (y,v,s) \rangle \geq 0 \text{ for all } (x,u,t) \in SC(K,B).$$

By substituting for (x, u, t) each of the extreme rays found in Theorem 3.2, we obtain that $(y, v, s) \in SC(K,B)^*$ iff $y \in K^*$ and

$$\langle U(y)v, v\rangle + 2\langle u, v\rangle + s \geq 0 \text{ for all } v \in \mathbb{R}^p. \tag{14}$$

This happens iff $y \in K^*$ and over all v's, the minimum value of (14) is non-negative:

$$-\langle U(y)^{-1}v, v\rangle + s \geq 0.$$

□

We are now ready to describe the extreme rays of $SC(K,B)^*$.

Theorem 3.4 *Let K be a homogeneous cone and let B be a homogeneous K-bilinear symmetric form. Also let U be defined as in (11). Then*

$$\begin{aligned}\mathrm{Ext}(SC(K,B)^*) &= \{(y, v, \langle U(y)^{-1}v, v\rangle) \;:\; y \in \mathrm{Ext}(K^*),\ v \in \mathcal{R}(U(y))\} \\ &\quad \cup \{(0,0,s) \;:\; s \in \mathbb{R}_{++}\}.\end{aligned}$$

Proof. The above rays are clearly in $SC(K,B)^*$ by Theorem 3.3. Suppose that

$$(0,0,s) = (y, v, s_1) + (-y, -v, s_2)$$

is a sum of two vectors in $SC(K,B)^*$. Then,

$$\langle (x,0,0), (y,v,s_1)\rangle,\ \langle (x,0,0), (-y,-v,s_2)\rangle \geq 0,\ \forall x \in K$$

imply that $\pm y \in K^*$; thus, $y = 0$. Now, $\pm v \in \mathcal{R}(U(y)) = \{0\}$, so $v = 0$ as well. Therefore, $(0,0,s)$ is an extreme ray.

To see that a vector of the form $(y, v, \langle U(y)^{-1}v, v\rangle)$, $y \in \mathrm{Ext}(K^*)$, $v \in \mathcal{R}(U(y))$ is an extreme ray, we note that such a vector can be mapped to $(y, 0, 0)$ via the automorphism T_1 of Lemma 3.2. Therefore, it suffices to prove that $(y, 0, 0)$ is an extreme ray of $SC(K,B)^*$ for $y \in \mathrm{Ext}(K^*)$. Again, suppose that we can write

$$(y, 0, 0) \;=\; (\lambda y, v, s) + (\delta y, -v, -s) \tag{15}$$

as a sum of vectors in $SC(K,B)^*$. Then the relations

$$\langle (0,0,1), (\lambda y, v, s)\rangle, \langle (0,0,1), (\delta y, -v, -s)\rangle \geq 0$$

imply that $s = 0$ and

$$\langle U(y)^{-1}v, v\rangle \leq 0,$$

so that the decomposition in (15) holds when we replace y with any positive multiple of itself. But

$$\langle (B(v,v), -v, 1), (\lambda y, v, 0)\rangle, \langle (B(v,v), v, 1), (\delta y, -v, 0)\rangle \geq 0$$

imply that

$$\langle B(v,v), y\rangle - 4\|v\|^2 \geq 0. \tag{16}$$

Unless $v = 0$, this inequality fails when y is contracted (by multiplication with a positive scalar) to an appropriately small magnitude while v is kept constant. Therefore, $(y,0,0)$ is an extreme ray.

Finally, let $r := (y, v, s)$ be an extreme ray of $SC(K,B)^*$. Writing

$$\begin{aligned} r &= (y, v, \langle U(y)^{-1}v, v\rangle) + (0,0,s - \langle U(y)^{-1}v, v\rangle) \\ &\in \operatorname{cone}\{(y, v, \langle U(y)^{-1}v, v\rangle), (0,0,s) \ : \ s > 0, \\ &\qquad y \in \operatorname{Ext}(K^*),\ v \in \mathcal{R}(U(y))\} \end{aligned}$$

we immediately see that r must coincide with one of the rays in the generating set. $\square$

The proof of the following theorem illustrates the power of the algebraic construction for homogeneous cones.

Theorem 3.5 *All extreme rays of every homogeneous cone are exposed.*

Proof. We use induction on the Siegel rank of a cone. When the rank is 1, the cone is, by definition, the real half-line $\mathbb{R}_+$ and the statement holds trivially. Assume that our cone is $SC(K,B)$. Let $r \in \operatorname{ext}(SC(K,B))$.

First, if r has the form $(0,0,1)$ then we choose $x^{(0)} \in -\operatorname{int}(K)$ and define

$$H := \{(x,u,t) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \ \mathbb{R} \ : \ \langle (x,u,t), (x^{(0)}, 0, 0)\rangle = 0\}.$$

Clearly, H is a supporting hyperplane for $SC(K,B)$; for all $(x,u,t) \in SC(K,B)$,

$$\langle (x,u,t), (x^{(0)}, 0, 0)\rangle = \langle x, x^{(0)}\rangle \leq 0,$$

with equality holding iff $x = 0$, iff $(x,u,t) = (0,0,t) = tr$ for $t \geq 0$. Therefore, r is exposed:

$$\{\lambda r \ : \ \lambda \geq 0\} = H \cap SC(K,B).$$

Moreover, every extreme ray of the form $(B(v,v), v, 1)$ is an automorphic image of $(0,0,1)$, and hence, must also be exposed (see Theorem 3.1).

It remains to show that r is exposed if it has the form $(\tilde{x}, 0, 0)$ for $\tilde{x} \in \mathrm{Ext}(K)$. Since $\mathrm{rank}(K) < \mathrm{rank}(SC(K,B))$, we can use the induction hypothesis to obtain $x^{(0)}$ such that

$$\{\lambda\tilde{x} \ : \ \lambda \geq 0\} = K \cap \{x \in \mathbb{R}^n \ : \ \langle x, x^{(0)}\rangle = 0\}$$

and

$$K \subseteq \{x \in \mathbb{R}^n \ : \ \langle x, x^{(0)}\rangle \leq 0\}.$$

Now, define

$$H := \{(x,u,t) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \ \mathbb{R} \ : \ \langle (x,u,t), (x^{(0)}, 0, -1)\rangle = 0\}$$

as above. Then for all $(x,u,t) \in SC(K,B)$,

$$\langle (x,u,t), (x^{(0)}, 0, -1)\rangle = \langle x, x^{(0)}\rangle - t \leq 0.$$

Equality occurs iff $t = 0$ (implying $u = 0$) and $x \in \{\lambda\tilde{x} \ : \ \lambda \geq 0\}$. In other words,

$$H \cap SC(K,B) = \{\lambda r \ : \ \lambda \geq 0\}.$$

Therefore, r is an exposed ray. By induction, the theorem is established. □

The above result does not immediately imply that all proper faces of all homogeneous cones are exposed. There are convex cones having every one of their extreme rays exposed, but also having some higher dimensional proper face unexposed (for this, one has to consider at least four-dimensional convex cones). See, for instance, the example in Tam [22] (page 50).

In the case of homogeneous convex cones however, we do have every proper face exposed. The next result generalizes the corresponding well-known result on the cone Σ_+^n (and of course, it implies the preceding theorem).

Theorem 3.6 *All proper faces of every homogeneous cone are exposed.*

Proof. Again, we use induction on the Siegel rank of a cone. When the rank is 1, the cone is the real half-line $\mathbb{R}_+$, which has all of its faces exposed (if the reader is bothered by the fact that $\mathbb{R}_+$ has no nontrivial exposed proper face, then it is possible to start with $\mathbb{R}_+^2$). Assume that the cone is $SC(K,B)$ where K is a homogenous cone and B is a homogeneous K-bilinear symmetric form. Let P be a face of $SC(K,B)$ and define a set Q in K by

$$Q := \{x \in K \ : \ (x,0,0) \in P\}.$$

Q is clearly a face of K because whenever $x = y + z$ for y, $z \in K$, we have

$$(x,0,0) = (y,0,0) + (z,0,0).$$

Both vectors on the right hand side are in $SC(K,B)$, implying that y and z belong to Q. Since $\text{rank}(K) < \text{rank}(SC(K,B))$ and Q is a face of K, Q is exposed by the induction hypothesis. Therefore, there is a $\bar{y} \in K^*$ such that

$$\langle \bar{y}, x \rangle \geq 0 \ \forall x \in K,$$

$$\langle \bar{y}, x \rangle = 0 \text{ for } x \in K \Longrightarrow x \in Q.$$

We distinguish between two cases. First, suppose that P does not contain a ray of the form $(B(u,u),u,1)$. In this case,

$$H := \{(x,u,t) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R} \ : \ \langle (\bar{y},0,1),(x,u,t) \rangle = 0\}$$

is a supporting hyperplane for $SC(K,B)$ exposing the face P. That H is a supporting hyperplane follows from the fact that $(\bar{y},0,1) \in SC(K,B)^*$. We check the extreme rays of $SC(K,B)$. If $x \in \text{Ext}(K)$ then

$$\begin{aligned}
\langle (\bar{y},0,1),(x,0,0) \rangle = 0 \quad &\Longleftrightarrow \quad \langle \bar{y}, x \rangle = 0 \\
&\Longleftrightarrow \quad x \in Q \\
&\Longleftrightarrow \quad (x,0,0) \in P.
\end{aligned}$$

If $u \in \mathbb{R}^p$ then

$$\langle (\bar{y},0,1),(B(u,u),u,1) \rangle > 0 \quad \Longleftrightarrow \quad (B(u,u),u,1) \notin P.$$

Note that the trivial case $Q = \{(0,0,0)\}$ is included in the above.

In the second case, P contains a ray of form $(B(u,u),u,1)$. By considering an automorphic image of P and applying Theorem 3.1, we may assume without loss of generality that $(0,0,1) \in P$. Then we claim that $(B(u,u),0,0) \in P \iff (B(u,u),u,1) \in P$:

If $(B(u,u),0,0) \in P$ then

$$\begin{aligned}
P &\ni 2(B(u,u),0,0) + 2(0,0,1) \\
&= (B(u,u),u,1) + (B(-u,-u),-u,1).
\end{aligned}$$

Both of the above are in $SC(K,B)$, so $(B(u,u),u,1) \in P$. On the other hand, if $(B(u,u),u,1) \in P$ then

$$\begin{aligned}
P &\ni (B(u,u),u,1) + (0,0,1) \\
&= \left(\frac{1}{2}B(u,u),u,2\right) + \frac{1}{2}(B(u,u),0,0) \\
&= 2\left(B\left(\frac{u}{2},\frac{u}{2}\right),\frac{u}{2},1\right) + \frac{1}{2}(B(u,u),0,0).
\end{aligned}$$

Both of the above are in $SC(K,B)$; thus $(B(u,u),0,0) \in P$.

Now, it is easy to show that

$$H := \{(x,u,t) \in \mathbb{R}^n \oplus \mathbb{R}^p \oplus \mathbb{R}_+ \ : \ \langle (\bar{y},0,0),(x,u,t)\rangle = 0\}$$

is a supporting hyperplane for $SC(K,B)$ which exposes P:

If $x \in \mathrm{Ext}(K)$ then

$$\begin{aligned} \langle (\bar{y},0,0),(x,0,0)\rangle = 0 &\iff \langle \bar{y}, x\rangle = 0 \\ &\iff x \in Q \\ &\iff (x,0,0) \in P. \end{aligned}$$

If $u \in \mathbb{R}^p$ then

$$\begin{aligned} \langle (\bar{y},0,0),(B(u,u),u,1)\rangle = 0 &\iff \langle \bar{y}, B(u,u)\rangle = 0 \\ &\iff B(u,u) \in Q \\ &\iff (B(u,u),0,0) \in P \\ &\iff (B(u,u),u,1) \in P, \end{aligned}$$

where the last statement follows from the claim. We have shown that every face of $SC(K,B)$ is exposed. By induction, the theorem is proved for all homogeneous cones. □

While this paper was in preparation, Chua [2], using the classification theory of Vinberg, showed that homogoneous cones are representable as feasible regions of semidefinite programming (SDP) problems (yet illustrating another powerful application of Vinberg's classification theory). Hence, any conic optimization problem with a homogenous cone as the cone constraint can be expressed as an SDP problem in principle. This result is a step towards the important question of "what are the (convex) optimization problems that can be efficiently formulated as SDP problems?"

3.3 Applications to the geometry of symmetric cones

As an application of Theorems 3.2 and 3.4, we ask what condition must be imposed on K and B for $SC(K,B)$ to be symmetric.

Theorem 3.7 *$SC(K,B)$ is symmetric iff K is symmetric and B satisfies*

$$\langle x,y\rangle U(y)^{-1} - U(x) \succeq 0 \ \forall \ y \in \mathrm{int}(K), \forall \ x \in K.$$

Proof. By considering the extreme rays of $SC(K,B)$ and its dual, we see that $SC(K,B)$ is symmetric iff

$$\langle y,x\rangle \geq 0$$

and

$$\langle U(y)^{-1}v, v\rangle y - B(v,v) \in K$$

for all $x \in K$, $y \in \text{int}(K^*)$ and $v \in \mathbb{R}^p$. In other words, $SC(K,B)$ is symmetric iff K is symmetric and

$$\langle U(y)^{-1}v, v\rangle\langle y, x\rangle - \langle U(x)v, v\rangle \geq 0$$

for all $x \in K$, $y \in \text{int}(K^*)$ and $v \in \mathbb{R}^p$. This is precisely the statement of the proposition. $\square$

Note that the neccessary and sufficient condition of the above fact can be easily verified for $K := \Sigma_+^n$, $SC(K,B) := \Sigma_+^{n+1}$. For $x \in \Sigma^n$, let $\text{tr}(x)$ denote the trace of x. Then for all $y \in \text{int}(K), \forall\, x \in K$, we have

$$\begin{aligned}\langle x, y\rangle y^{-1} - x \succeq 0 \quad &\iff \quad h^T x h \leq (h^T y^{-1} h)\text{tr}(y^{1/2} x y^{1/2}), \forall\, h \in \mathbb{R}^n \\ &\iff \quad h^T (y^{1/2} x y^{1/2}) h \leq (h^T h)\text{tr}(y^{1/2} x y^{1/2}), \forall\, h \in \mathbb{R}^n, \|h\| = 1.\end{aligned}$$

The last statement is clearly true. For the equivalence of the second and third statements, we used the isomorphism $h \to y^{1/2}h$ (and $y \in \text{int}(K)$).

Lemma 3.3 *If $SC(K,B)$ is symmetric then for every $u \in \mathbb{R}^p$, either $B(u,u) = 0$ or $B(u,u)$ is an extreme ray of K.*

Proof. Suppose that $B(u,u) \neq 0$. Then $(B(u,u), u, 1)$ is an extreme ray of $SC(K,B)$. Now, the extreme rays of $SC(K,B)$ coincide with those of $SC(K,B)^*$ because $SC(K,B) = SC(K,B)^*$. (Note that by Remark 3.1, we can assume that $SC(K,B) = SC(K,B)^*$ under the current inner product.) Thus, $(B(u,u), u, 1)$ must be either $(0, 0, \lambda)$ for some $\lambda > 0$ or $(y, v, \langle U(y)^{-1}v, v\rangle)$ for $y \in \text{Ext}(K^*) = \text{Ext}(K)$. It is clear that the latter case must be true because we assumed that $B(u,u) \neq 0$. Therefore, $B(u,u) = y \in \text{Ext}(K)$. $\square$

Given K in $\mathbb{R}^n$, we define the Carathéodory number of K as the minimum number of extreme rays of K needed to express any interior point of K as a convex combination. We denote this invariant of K by $\kappa(K)$. By a classical theorem of Carathéodory, this number is at most n; however, in many cases it can be much less. Güler and the second author noticed that this number is equal to the algebraic invariant $\text{rank}(K)$ when K is symmetric. They note that both $\kappa(K)$ and $\text{rank}(K)$ are invariant under linear isomorphisms of $\mathbb{R}^n$ and that for any pair of homogeneous convex cones K_1, K_2,

$$\kappa(K_1 \oplus K_2) = \kappa(K_1) + \kappa(K_2) \text{ and } \text{rank}(K_1 \oplus K_2) = \text{rank}(K_1) + \text{rank}(K_2)$$

hold. Therefore, they proceed to prove the claim using the classification of irreducible symmetric cones (this is the classification based on the Jordan-vonNeumann-Wigner classification of Euclidean Jordan Algebras, see Faraut and Korányi [3]). Four of the cases in the proof are rather elementary but they rely on the existence of generalized eigenvalue decompositions and the fifth utilizes a result of Freudenthal on the existence of certain automorphisms in the algebra of Albert. Our new proof below is based on the geometric insights that we provided for homogeneous cones. First, we need an elementary fact. Let $\bar{x} \in K$. We define $\kappa(\bar{x})$ as the minimum number of extreme rays of K required to express $\bar{x}$ as a convex combination. Then Proposition 2.3 of [7] establishes that $\kappa(x) = \kappa(K)$ for every $x \in \text{int}(K)$, for every homogeneous convex cone K.

Theorem 3.8 *(Güler and Tunçel [7]) For all symmetric cones K, $\kappa(K) = \text{rank}(K)$.*

Proof. We will proceed by induction on $\text{rank}(K)$. If $\text{rank}(K) = 1$ then K is $\mathbb{R}_+$ and the statement of the theorem is true. Suppose $\kappa(K) = \text{rank}(K)$ for all symmetric cones K with $\text{rank}(K) \leq k$. Let K be a symmetric cone with $\text{rank}(K) = k+1$. Then there exists a symmetric cone $\bar{K}$ and a homogeneous $\bar{K}$-bilinear symmetric form B such that $\text{rank}(\bar{K}) = k$ and $K = SC(\bar{K}, B)$. By the induction hypothesis $\kappa(\bar{K}) = k$. Consider $e \in \text{int}(\bar{K})$. Then $\kappa(e) = k$, $(e, 0, 1) \in \text{int}(K)$ and

$$\kappa(e,0,1) \leq k+1 \text{ since } (e,0,1) = (0,0,1) + \sum_{i=1}^{k} (v^{(i)}, 0, 0) \text{ for some } v^{(i)} \in \text{Ext}(\bar{K}).$$

Suppose for a contradiction

$$(e,0,1) = \sum_{i=1}^{q} (w^{(i)}, 0, 0) + \sum_{i=q+1}^{k} \lambda_i \left(B(u^{(i)}, u^{(i)}), u^{(i)}, 1 \right),$$

where $\lambda \geq 0$, $w^{(i)} \in \text{Ext}(\bar{K})$ and $u^{(i)} \in \mathbb{R}^p \backslash \{0\}$. Since $B(u^{(i)}, u^{(i)}) \in \text{Ext}(\bar{K})$ and $\kappa(e) = k$, we have $\lambda > 0$. We also have

$$q \leq (k-2), \quad \sum_{i=q+1}^{k} \lambda_i = 1, \quad \sum_{i=q+1}^{k} \lambda_i u^{(i)} = 0.$$

Without loss of generality (by redefining $u^{(i)}$ if necessary) we have

$$e = \sum_{i=1}^{q} w^{(i)} + \sum_{i=q+1}^{k} B(u^{(i)}, u^{(i)}) \text{ and } \sum_{i=q+1}^{k-1} \mu_i u^{(i)} = u^{(k)} \text{ for some } \mu \neq 0.$$

Using the linear dependence on $u^{(i)}$ and the properties of the bilinear form, we compute

$$\sum_{i=q+1}^{k} B(u^{(i)}, u^{(i)}) = \sum_{i=q+1}^{k-1} (1+\mu_i^2) B(u^{(i)}, u^{(i)}) + 2 \sum_{q+1 \leq i < j \leq k-1} \mu_i \mu_j B(u^{(i)}, u^{(j)}).$$

We claim that we can find $\bar{u}^{(q+1)}, \bar{u}^{(q+2)}, \ldots, \bar{u}^{(k-1)} \in \mathbb{R}^p$ such that

$$\sum_{i=1}^{q} w^{(i)} + \sum_{i=q+1}^{k-1} B(\bar{u}^{(i)}, \bar{u}^{(i)}) = e$$

which would be a contradiction. In fact, we claim that we can choose $\bar{u}^{(i)} \in \operatorname{span}\{u^{(q+1)}, u^{(q+2)}, \ldots, u^{(k-1)}\}$. If we can find $\Gamma \in \mathbb{R}^{(k-q-1)\times(k-q-1)}$ such that

$$\sum_{l=q+1}^{k-1} B\left(\sum_{i=q+1}^{k-1} \gamma_{li} u^{(i)}, \sum_{i=q+1}^{k-1} \gamma_{li} u^{(i)}\right) = \sum_{i=q+1}^{k} B(u^{(i)}, u^{(i)})$$

then the claim would follow. Expanding the left hand side and comparing terms, we see that if

$$\Gamma^T \Gamma = I + \mu\mu^T$$

then the desired equality above holds. Since $(I+\mu\mu^T)$ is symmetric positive definite, such matrix Γ always exists. Thus, we expressed e as a convex combination of $(k-1)$ extreme rays of $\bar{K}$, a contradiction. This completes the induction and the proof. □

4 Optimal Barriers

In this section, we are interested in the structure of the *optimal barriers* for homogeneous convex cones (for a fixed cone K, those ϑ-normal barriers with the smallest possible parameter value ϑ). Available evidence suggests that perhaps there is a unique way to construct such barriers. For example, all of the well-known optimal barriers for homogeneous cones arise, in the context of the algebraic construction by Vinberg, from the extension of the optimal barriers on homogeneous cones of lower ranks. More precisely, an optimal barrier F on $SC(K,B)$ is generally constructed from an optimal barrier $\tilde{F}$ on K by setting

$$F(x,u,t) := \tilde{F}\left(x - \frac{B(u,u)}{t}\right) - \ln(t), \ \forall\ (x,u,t) \in \operatorname{int}\left(SC(K,B)\right). \tag{17}$$

This construction always yields optimal barriers for $SC(K,B)$ (see [12]). Whether the construction accounts for all optimal barriers is less well-understood. The other most relevant results are those given by first Nesterov-Todd [13, 14] (on the foundations of self-scaled barriers) then by Hauser [8], Schmieta [21], Hauser-Güler [9], and Hauser-Lim [10] (also see [23] about a geometric-mean like characterization of self-scaled barriers). It follows from these works that an optimal self-scaled barrier is unique up to an additive constant. Here, we first formalize the necessary and sufficient conditions for an optimal barrier F to be derivable from (17). Then, we exhibit an example showing that these conditions are not satisfied by a general function with properties (1), (2) and (3) (with optimal homogeneity parameter). We do not know whether homogeneous cones admit any optimal barriers which do not arise from the recursive formula (17).

Proposition 4.1 *Let K be a homogeneous convex cone and B be a homogeneous K-bilinear symmetric form such that* $\mathrm{rank}(SC(K,B)) = \mathrm{rank}(K) + 1 \geq 2$. *Also let F be a k-normal, optimal barrier for $SC(K,B)$. Then there is a $(k-1)$-normal optimal barrier $\tilde{F}$ of K such that*

$$F(x,u,t) = \tilde{F}\left(x - \frac{B(u,u)}{t}\right) - \ln(t), \;\forall\; (x,u,t) \in \mathrm{int}\,(SC(K,B))$$

if and only if

$$F(\alpha x, \sqrt{\alpha} u, t) = F(x,u,t) - (k-1)\ln(\alpha), \;\forall\; \alpha > 0 \tag{18}$$

and

$$F(x + B(u,u), u, 1) = F(x,0,1), \;\forall\; (x,0,1) \in \mathrm{int}\,(SC(K,B)), \;\forall\; u \in \mathbb{R}^p. \tag{19}$$

Proof. If there is $\tilde{F}$ as in the statement of the proposition then the above two conditions are necessarily true, (18) by the $(k-1)$-logarithmic homogeneity of $\tilde{F}$ and (19) by the well-definedness of $\tilde{F}$. On the other hand, if both (18) and (19) hold for F then we can specify $\tilde{F}$ by letting

$$\tilde{F}(x) := F(x,0,1), \;\forall\; x \in \mathrm{int}(K).$$

The properties required for $\tilde{F}$ to be an optimal self-concordant barrier for K can be easily proved from the corresponding properties for F. Also, we can utilize the general, affine restriction result of Nesterov and Nemirovskii [12]. □

Note that only (18) is needed if we wish merely to define an optimal barrier for K in terms of F as we showed in the proof. It is instructive to consider what happens when we produce $\tilde{F}$ from F in this way, then extend $\tilde{F}$ again by (17) to form an optimal barrier $\hat{F}$ on $SC(K,B)$. In this case, $\hat{F}$ will satisfy (19) and $\hat{F} = F$ if and only if F satisfies (19) as well.

The following theorem describes a convex function, satisfying the conditions (1), (2) and (3) with $\vartheta = 2$ for Σ_+^2 which satisfies (18), but not (19). Recall that in our present framework,

$$\Sigma_+^2 = \{(x,u,t) \in \mathbb{R}_+ \oplus \mathbb{R} \oplus \mathbb{R}_+ \;:\; tx - u^2 \geq 0\},$$

and the corresponding optimal self-scaled barrier is $-\ln\left(tx - u^2\right)$. When restricted to the set

$$\{(x,0,1) \;:\; (x,0,1) \in \Sigma_+^2\},$$

$-\ln\det\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = -\ln(x)$, we obtain the unique optimal barrier for $\mathbb{R}_+$.

Theorem 4.1 $F(x,u,t) := -\ln\left(x - \frac{u^2}{t}\right) - \ln\left(t - \frac{u^2}{x}\right) = -2\ln\left(\sqrt{tx} - \frac{u^2}{\sqrt{tx}}\right)$ *satisfies (1), (2) and (3) (with $\vartheta = 2$), but not (4) for Σ_+^2.*

Proof. Note that F is clearly $\mathcal{C}^3$-smooth and as a convergent sequence $(x^{(k)}, u^{(k)}, t^{(k)}) \in \Sigma^2_{++}$ approaches $\partial(\Sigma^2_+)$, we have

$$t^{(k)}x^{(k)} - \left(u^{(k)}\right)^2 \to 0,$$

so that

$$F(x^{(k)}, u^{(k)}, t^{(k)}) \to \infty.$$

F is also logarithmicly homogeneous with parameter 2.

Next, we prove strict convexity of F. Let $z := (x, u, t) \in \Sigma^2_{++}$, $h =: (h_1, h_2, h_3)$. Also let

$$A_\alpha := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \sqrt{\alpha_1\alpha_2} & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}.$$

Then for every $\alpha \in \mathbb{R}^2_{++}$, A_α is an automorphism of Σ^2_+. Moreover, $F(A_\alpha z) = F(z) - \ln(\alpha_1\alpha_2)$. Therefore,

$$D^k F(A_\alpha z)[A_\alpha h, A_\alpha h, \ldots, A_\alpha h] = D^k F(z)[h, h, \ldots, h] \text{ when } k \in \mathbb{Z}_{++}.$$

By choosing an appropriate α and switching to the pair $(A_\alpha z, A_\alpha h)$ if necessary, we can make the assumption that $x = 1$ and $t = 1$. We have

$$D^2F(z)[h,h] = 2\left(\lambda_1^2 + \lambda_2^2\right) - \left(h_1^2 + h_3^2\right),$$

where λ_1, λ_2 are the eigenvalues of $\begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}^{-1} \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$. If we apply the automorphism $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to the matrices $\begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}$ and $\begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$, then we effectively multiply by (-1) the off-diagonal elements of both matrices. Since this does not change the value of $D^2F(z)[h,h]$ and the domain of u is symmetric about the origin, we can assume without loss of generality that $h_2 \geq 0$. We will consider two cases: $h_2 \in \{0, 1\}$. First we compute

$$\lambda_1^2 + \lambda_2^2 \quad = \quad \frac{1}{(1-u^2)^2}\left[h_1^2 + 2(1+u^2)h_2^2 + h_3^2 + 2u^2h_1h_3 - 4uh_2(h_1+h_3)\right].$$

Case 1: $h_2 = 0$.

In this case, checking $D^2F(z)[h,h] > 0$ for all $h \in \mathbb{R}^3 \setminus \{0\}$ is accomplished by noting that

$$(1 + 2u^2 - u^4)(h_1^2 + h_3^2) + 4u^2h_1h_3 = (1 - u^4)(h_1^2 + h_3^2) + 2u^2(h_1 + h_3)^2$$

and that the latter is positive for every $u \in (-1, 1)$ and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$.

Case 2: $h_2 = 1$.

In this case, it suffices to prove that

$$(1-u^4)(h_1^2+h_3^2)+2u^2(h_1+h_3)^2-8u(h_1+h_3)+4(1+u^2)>0,$$

for every $u \in (-1,1)$ and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2$. The Hessian of the above quadratic (in $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$) is

$$\begin{pmatrix} 2(1-u^4)+4u^2 & 4u^2 \\ 4u^2 & 2(1-u^4)+4u^2 \end{pmatrix}$$

which is clearly positive definite for every $u \in (-1,1)$. Its unique minimizer satisfies

$$h_1 = h_3 = \frac{4u}{1+4u^2-u^4}.$$

Substituting this back to the inequality we are trying to prove in Case 2, and multiplying by $(1+4u^2-u^4)^2$ (which is positive for every $u \in (-1,1)$), we obtain $4(1-u)^3(1+u)^3(1+4u^2-u^4)$ which is clearly positive for every $u \in (-1,1)$. Hence, we complete the proof that F is strictly convex on Σ^2_{++}.

We also checked with equal ease (and a similar computation) that the Hessian of F is positive definite by explicitly showing that every principal minor is positive.

We show that the self-concordance inequality (4) fails. Recall the condition:

$$2(D^2F(z)[h,h])^{\frac{3}{2}} - \mid D^3F(z)[h,h,h] \mid \geq 0$$

for all $z \in \Sigma^2_{++}$, $h \in \mathbb{R}^3$. Equivalently,

$$H(z,h) := 4(D^2F(z)[h,h])^3 - \left(D^3F(z)[h,h,h]\right)^2 \geq 0,$$

for all $z \in \Sigma^2_{++}$, $h \in \mathbb{R}^3$. We compute

$$\frac{1}{4}H\left((1,u,1),h\right) = \left[2\left(\lambda_1^2+\lambda_2^2\right) - \left(h_1^2+h_3^2\right)\right]^3 - \left[2\left(\lambda_1^3+\lambda_2^3\right) - \left(h_1^3+h_3^3\right)\right]^2.$$

Let $u := 1/2$, $h_1 := h_2 := h_3 := 1$. Then, $\lambda_1 = 4/3$, $\lambda_2 = 0$ and

$$\frac{1}{4}H\left((1,1/2,1),(1,1,1)\right) = -683\left(\frac{4}{3^6}\right) < 0.$$

$\square$

We note here that as $\{z^{(k)}\} \subset \Sigma^2_{++}$ approaches $\partial(\Sigma^2_+)$, $\|DF(z^{(k)})\| \to +\infty$ and hence F is *essentially smooth* in the sense of Rockafellar [18] (page 251). The Legendre-Fenchel conjugate of the above convex function gives another convex function for Σ^2_+ that is different from the negation of the logarithm of the determinant (also, $F_* \neq F$ in this case). Also, by Theorem

26.3 of [18], F_* is strictly convex. In fact, in Rockafellar's terminology $\left(\Sigma^2_{++}, F\right)$ and $\left(\Sigma^2_{++}, F_*\right)$ are *Legendre type* (modulo some sign changes because of our slight difference in the choice of notation). The function F_* is defined by

$$F_*(y,v,s) := \sup\{-\langle (y,v,s),(x,u,t)\rangle + 2\ln(tx - u^2) - \ln(tx) \;:\; (x,u,t) \in \Sigma^2_{++}\}.$$

For a given $(y,v,s) \in \Sigma^2_{++}$, the supremum occurs at a point (x,u,t) where the derivative of the above expression vanishes. This point must be a global maximizer because the expression is a strictly concave function. We solve for the critical point, using for convenience the matrix representation for the points of Σ^2_+:

$$-\begin{pmatrix} s & v \\ v & y \end{pmatrix} + 2\begin{pmatrix} t & u \\ u & x \end{pmatrix}^{-1} - \begin{pmatrix} t & 0 \\ 0 & x \end{pmatrix}^{-1} = 0.$$

We multiply both sides of the above given system by the nonsingular matrix $\begin{pmatrix} t & u \\ u & x \end{pmatrix}$ and arrive at the following system of three equations

$$\begin{aligned} uv + xy &= 1, \\ u\left(s + \frac{1}{t}\right) + xv &= 0, \\ ts + uv &= 1. \end{aligned}$$

This system has a unique solution

$$\tilde{x} := \frac{1}{y} + \frac{v^2}{y^2\sqrt{s - \frac{v^2}{y}}\left(\sqrt{s - \frac{v^2}{y}} + \sqrt{s}\right)},\ \tilde{u} := -\frac{v}{y\sqrt{s - \frac{v^2}{y}}\left(\sqrt{s - \frac{v^2}{y}} + \sqrt{s}\right)},\ \tilde{t} := \frac{1}{\sqrt{s\left(s - \frac{v^2}{y}\right)}}.$$

Thus,

$$\begin{aligned} F_*(y,v,s) &= -\langle (y,v,s),(\tilde{x},\tilde{u},\tilde{t})\rangle + 2\ln(\tilde{t}\tilde{x} - \tilde{u}^2) - \ln(\tilde{t}\tilde{x}) \\ &= 2\ln 2 - 2 - 2\ln\left(\sqrt{s - \frac{v^2}{y}} + \sqrt{s}\right) - \ln(y) \\ &= -2\ln\left(\sqrt{sy - v^2} + \sqrt{sy}\right) + (2\ln 2 - 2). \end{aligned}$$

F_* satisfies the properties (1) and (3) (but neither (2) nor (4)) for Σ^2_+.

We can extend F to Σ^n_+ for any $n \geq 3$ via the construction (17). These extended functions can be further manupilated by the (barrier calculus) tools of Nesterov-Nemirovskii [12] to give interesting functions for more complicated sets. We give below an elementary example. Note that *the second order cone in* $\mathbb{R}^{n+1}$ is defined as

$$SO^n := \{(x,t) \in \mathbb{R}^n \oplus \mathbb{R} : t \geq \|x\|_2\}.$$

Corollary 4.1 *The following functions satisfy (1), (2) and (3) (with $\vartheta = 2$), but not (4) for the second order cone in $\mathbb{R}^3$:*

$$F_1(x_1, x_2, t) := -2\ln\left(t^2 - x_1^2 - x_2^2\right) + \ln\left(t^2 - x_1^2\right),$$
$$F_2(x_1, x_2, t) := -2\ln\left(t^2 - x_1^2 - x_2^2\right) + \ln\left(t^2 - x_2^2\right).$$

Proof. We used the representation

$$\begin{pmatrix} t - x_1 & x_2 \\ x_2 & t + x_1 \end{pmatrix} \in \Sigma_+^2$$

for the second order cone and then applied the previous theorem and simplified. For F_2 we just used the apparent symmetry of the above argument. □

5 Duality Mapping and Homogeneity

Recall the (universal) barrier Φ_K of a cone K. Let $x^* := -\Phi_K'(x)$ for $x \in \operatorname{int}(K)$. Faraut and Korányi [3] showed that for homogeneous cones, the relation

$$(x^*)^* = -\Phi_{K^*}'(-\Phi_K'(x)) = x \tag{20}$$

holds for every $x \in \operatorname{int}(K)$. They asked whether the relation is true in general. The second author and Xu [25] answered the question in the negative, giving as counter example the cone of the L_1-norm. From their result, it appears that involutive property of the duality mapping may depend in a fundamental way on the homogeneity of the cone. We would like to investigate whether this connection is sufficiently strong that (20), alone or in combination with other properties, can be used to formulate a new (analytic) characterization for homogeneous cones.

5.1 Self-dual cones

Here, we consider self-dual cones for which the duality mapping is an involution. This class of cones is a useful starting point because a candidate for the set of automorphisms of such cones acting transitively on the interiors of these cones is already known. This is the set

$$\mathcal{H}(K) := \{D^2\Phi(x) : x \in K\}$$

of images of the Hessian of the barrier function evaluated at points in K. Rothaus [19] noted that $\mathcal{H}(K)$ is a transitive subset of $\operatorname{Aut}(K)$ iff K is symmetric. To present his argument, we first state three preliminary results:

Proposition 5.1 *$D^2\Phi(x)$ is positive definite for each $x \in \text{int}(K)$. Moreover, for every pair $(y, s) \in (\text{int}(K) \oplus \text{int}(K^*))$, $\exists x \in \text{int}(K)$ such that $D^2\Phi(x)y = s$.*

The first part follows easily from the definition of self-concordance (see [12]). The second part is due to Rothaus and Nesterov-Todd [13]. The following result is well-known (see for instance [3]).

Proposition 5.2 *The map $x \to x^*$ has a unique fixed point.*

Let $e \in \text{int}(K)$ denote the fixed point of the map $x \to x^*$.

Theorem 5.1 (Rothaus [19]) *$\mathcal{H}(K)$ is a transitive subset of $\text{Aut}(K)$ iff K is symmetric.*

Proof. The forward direction is clear. Let K be symmetric. Then the duality mapping is an involution with unique fixed point e. Differentiating equality (20), we obtain

$$D^2\Phi(x)D^2\Phi(x^*) = I, \; \forall x \in \text{int}(K). \tag{21}$$

Thus, $[D^2\Phi(e)]^2 = I$. Because $D^2\Phi(e)$ is symmetric and positive definite, it is easy to see that $D^2\Phi(e) = I \in \text{Aut}(K)$.

Now, for every $x \in K$, there exists $A \in \text{Aut}(K)$ such that $x = A(e)$. By property (10) of general barrier functions,

$$D^2\Phi(x) = D^2\Phi(Ae) = A^{-1}D^2\Phi(e)A^* \in \text{Aut}(K).$$

Hence, $\mathcal{H}(K) \subseteq \text{Aut}(K)$. That $\mathcal{H}(K)$ forms a transitive subset follows from Proposition 5.1.
$\Box$

In view of Theorem 5.1, to show that equality (20) implies homogeneity for self-dual cones, we can show that $\mathcal{H}(K)$ is a transitive subset of $\text{Aut}(K)$. Proposition 5.1 already proves that elements of $\mathcal{H}(K)$ are linear, positive definite, and can be chosen to map any point in K to any other point. Moreover, if we assume (20), (21) gives, for every $x \in K$,

$$[D^2\Phi(x)]^{-1} = D^2\Phi(x^*) \in \mathcal{H}(K).$$

So, it suffices to show that $D^2\Phi(x)$ maps K onto K. That is,

$$\langle D^2\Phi(x)u, v\rangle \geq 0, \text{ for all } x \in \text{int}(K), \text{ and for all } u, v \in K. \tag{22}$$

Note that (22) does not, in general, hold for self-dual cones. To this end, consider the following family of convex cones presented by Koecher [11]:

$$K_\rho := \mathrm{cl}\left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : u > 0,\ v > 0,\ |w| < u^\rho v^{1-\rho} \right\}. \tag{23}$$

These cones, parameterized by $\rho \in (0,1)$, are self-dual under the inner product

$$\langle x, y\rangle := x_1 y_1 + x_2 y_2 + \alpha x_3 y_3,$$

where

$$\alpha := \frac{1}{\rho}\left(\frac{1}{1-\rho}\right)^{1-\rho}. \tag{24}$$

However, K_ρ is not homogeneous unless $\rho = \frac{1}{2}$. See the Appendix for the details of the numerical calculations showing that when $\rho \neq \frac{1}{2}$, the condition (22) can fail. Therefore, we established that the duality mapping is not an involution even if $K = K^*$.

5.2 Simple Linear Maps and Automorphisms

In this section, we explore candidates for the automorphism group of K, from a very simple linear algebraic viewpoint. Let

$$\mathcal{G}(K) := \mathrm{cone}\{hh^T : h \in K\} \cap \Sigma^n_{++}.$$

Recall that Σ^n_{++} denotes the cone of $n \times n$, symmetric, positive definite matrices with dimension $p = n\left(\frac{n+1}{2}\right)$. The set $\mathcal{G}(K)$ has the property that for every pair $(x,s) \in \mathrm{int}(K) \oplus \mathrm{int}(K^*)$, there is a map $T \in \mathcal{G}(K)$ satisfying $T(s) = x$, see [24]. As such, $\mathcal{G}(K)$ may yield a transitive subset of $\mathrm{Aut}(K)$ when $K = K^*$. We asked when is an element $T \in \mathcal{G}(K)$ an automorphism of K?

Proposition 5.3 *Let $K \subset \mathbb{R}^n$ be as above. Assume $K = K^*$ under the inner product $\langle x, y\rangle := x^T S y$, where $S \in \Sigma^n_{++}$. Let $T := \sum_{i=1}^m h^{(i)}(h^{(i)})^T$ for $m \in \{n, n+1, \ldots, p\}$ and assume that T is positive definite. Then $TS \in \mathrm{Aut}(K)$ iff $\{h^{(1)}, \ldots, h^{(m)}\}$ forms an orthonormal system with respect to the inner product $\langle \cdot, \cdot \rangle$.*

Proof. Let $T \in \mathcal{G}(K)$ be as above. Then, for every $x,\ y \in \mathbb{R}^n$

$$\langle TSx, y\rangle = \sum_{i=1}^m \langle h^{(i)}, y\rangle \langle h^{(i)}, x\rangle. \tag{25}$$

It is easy to see that TS maps K into itself. Therefore, $TS \in \mathrm{Aut}(K)$ iff it maps K onto K. That is, whenever $x \notin K$ there must be some $y \in K$ such that $\langle TSx, y\rangle < 0$.

Suppose $\{h^{(1)}, h^{(2)}, \ldots, h^{(m)}\}$ is an orthonormal system. Let $x \notin K$. If

$$\alpha_i := \langle x, h^{(i)}\rangle \geq 0$$

for every i, then $x = \sum_{i=1}^m \alpha_i h^{(i)} \in K$, a contradiction. Thus, there exists j such that $\alpha_j < 0$. But

$$\begin{aligned}
\langle TSx, h^{(j)}\rangle &= \sum_{i=1}^{m} \langle h^{(i)}, h^{(j)}\rangle\langle h^{(i)}, x\rangle \\
&= \alpha_j \langle h^{(j)}, h^{(j)}\rangle \\
&< 0
\end{aligned}$$

implying that $TSx \notin K$. Therefore, T is an automorphism.

Now, suppose that T is an automorphism. We define the half-spaces

$$H_i := \{x \ : \ \langle h^{(i)}, x\rangle \geq 0\}, \quad i \in \{1, 2, \ldots, m\}.$$

Then, since each $h^{(i)} \in K$, we must have that

$$K \subseteq \bigcap_{i=1}^{m} H_i.$$

If the inclusion is strict then there must be an $x \notin K$ satisfying $\langle h^{(i)}, x\rangle \geq 0$ for all i, forcing $TSx \in K$ by (25). This is a contradiction because TS is an automorphism. Hence,

$$K = \bigcap_{i=1}^{m} H_i,$$

and K is polyhedral.

First, assume that $n = m$. For each i, find $x^{(i)}$ such that

$$x^{(i)} \in \mathrm{span}\{h^{(j)} \ : \ j \neq i\}^{\perp}$$

(in the above, orthogonality is with respect to the inner product $\langle\cdot,\cdot\rangle$) and $\langle h^{(i)}, x^{(i)}\rangle > 0$. Since $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ can be taken as the extreme rays of K, we can write, for each i,

$$h^{(i)} =: \sum_{j=1}^{n} \alpha_{ij} x^{(j)}, \text{ for some } \alpha_{ij} \geq 0 \ \forall j.$$

Since for $i \neq k$,

$$\begin{aligned} 0 &= \langle h^{(i)}, x^{(k)} \rangle \\ &= \sum_{j=1}^{n} \alpha_{ij} \langle x^{(i)}, x^{(k)} \rangle \\ &\geq \alpha_{ik} \langle x^{(k)}, x^{(k)} \rangle \\ &\geq 0, \end{aligned}$$

we must have that $\alpha_{ik} = 0$ for all $i \neq k$. Therefore, each $h^{(i)}$ is a positive multiple of $x^{(i)}$. By the construction of $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$, $\{h^{(1)}, h^{(2)}, \ldots, h^{(m)}\}$ are orthonormal.

Finally, assume that $m > n$. We may assume without loss of generality that $\{h^{(1)}, h^{(2)}, \ldots, h^{(n)}\}$ forms a basis for $\mathbb{R}^n$ and that the inequality $\langle h^{(m)}, x \rangle \geq 0$ is a facet of the polyhedral cone K. Find a sequence $\{x^{(k)}\}_{k=1}^{\infty}$ satisfying

$$\langle h^{(m)}, x^{(k)} \rangle < 0, \ \forall k,$$

$$\langle h^{(i)}, x^{(k)} \rangle \geq 0, \ \forall i \in \{1, 2, \ldots, m-1\},$$

and $x^{(k)} \to \bar{x} \in \partial(K)$ where

$$\langle h^{(m)}, x^{(k)} \rangle = 0, \ \forall k,$$

$$\langle h^{(i)}, x^{(k)} \rangle > 0, \ \forall i \in \{1, 2, \ldots, m-1\}.$$

We have

$$TSx^{(k)} = \underbrace{\langle h^{(m)}, x^{(k)} \rangle}_{\to 0} h^{(m)} + \sum_{i=1}^{m-1} \underbrace{\langle h^{(i)}, x^{(k)} \rangle}_{\to \langle h^{(i)}, \bar{x} \rangle > 0} h^{(i)}.$$

But,

$$\sum_{i=1}^{m-1} \langle h^{(i)}, x^{(k)} \rangle h^{(i)} \in \operatorname{int}(K).$$

To establish the above claim, let $y \in K$. If

$$\langle y, \sum_{i=1}^{m-1} \langle h^{(i)}, \bar{x} \rangle h^{(i)} \rangle = 0$$

then, since each $\langle h^{(i)}, \bar{x} \rangle > 0$, we must have that each $\langle h^{(i)}, y \rangle = 0$, implying $y \in \operatorname{span}\{h^{(1)}, h^{(2)}, \ldots, h^{(n)}\}^{\perp} = \{0\}$. Thus, for large enough k, $TSx^{(k)} \in \operatorname{int}(K)$ while for every k, $x^{(k)} \notin K$. Therefore, TS cannot be in $\operatorname{Aut}(K)$ as assumed, a contradiction. $\square$

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APPENDIX

Let $x = (x_1, x_2, x_3) := (1, 1, \frac{1}{2})$. Set $\rho := \frac{1}{3}$. The characteristic function for Koecher's cone is given by

$$\phi(x) = \int_{u=0}^{\infty} \int_{v=0}^{\infty} \int_{w=-u^\rho v^{1-\rho}}^{w=u^\rho v^{1-\rho}} \mathbf{e}^{-u-v-\frac{\alpha w}{2}} dw dv du$$

where α is as defined in (24). We show that

$$\phi(x)\phi(x^*) \geq 1.263548762 > \phi(1,1,0)\phi(1,1,0)^* \doteq 0.7560. \qquad (26)$$

Since the property

$$(x^*)^* = x, \ \forall \ x \in \text{int}(K)$$

is equivalent to (see [3])

$$\phi(x)\phi(x^*) = \text{constant}, \ \forall \ x \in \text{int}(K),$$

(26) shows that the duality mapping is not an involution for the Koecher cone having the chosen ρ as parameter.

The value 0.7560 can be obtained by direct integration. We calculated the bound 1.263548762 by a series of upper and lower estimates as shown below.

Term	Lower bound	Upper bound
$\phi(x)$	2.403166504	2.524866182
$-\frac{\partial \phi}{\partial x_1}(x)$	4.002586762	4.201473719
$-\frac{\partial \phi}{\partial x_2}(x)$	5.553655036	5.904058796
$-\frac{\partial \phi}{\partial x_3}(x)$	-5.305267296	-4.245920848
${x_1}^* = \frac{-\frac{\partial \phi}{\partial x_1}(x)}{\phi(x)}$	1.585266891	1.748307373
${x_2}^* = \frac{-\frac{\partial \phi}{\partial x_2}(x)}{\phi(x)}$	2.199583914	2.456783076
${x_3}^* = \frac{-\frac{\partial \phi}{\partial x_3}(x)}{\phi(x)}$	-2.207615364	-1.681641934
$\phi(x^*)$	0.5257849423	
$\phi(x)\phi(x^*)$	1.263548762	

The following is a sample calculation showing how the bounds on $\phi(x)$ were obtained:

$$
\begin{aligned}
\phi(x) &= \int_{u=0}^{\infty} \int_{v=0}^{\infty} \int_{w=-u^{\rho}v^{1-\rho}}^{w=u^{\rho}v^{1-\rho}} \mathbf{e}^{-u-v-\frac{\alpha w}{2}}\, dw dv du \\
&= \frac{1}{\alpha x_3} \int_{u=0}^{\infty} \int_{v=0}^{\infty} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}u^{\rho}v^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}u^{\rho}v^{1-\rho}} \right) dv du \\
&= \frac{1}{\alpha x_3} \int_{u=0}^{\infty} \int_{v=0}^{u} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}u^{\rho}v^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}u^{\rho}v^{1-\rho}} \right) dv du \\
&\quad + \frac{1}{\alpha x_3} \int_{v=0}^{\infty} \int_{u=0}^{v} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}u^{\rho}v^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}u^{\rho}v^{1-\rho}} \right) du dv.
\end{aligned}
$$

Put $n := 70$.

$$
\begin{aligned}
\phi(x) &\leq \sum_{k=1}^{n} \int_{u=0}^{\infty} \int_{v=\frac{k-1}{n}u}^{\frac{k}{n}u} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}u^{\rho}\left(\frac{ku}{n}\right)^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}u^{\rho}\left(\frac{ku}{n}\right)^{1-\rho}} \right) dv du \\
&\quad + \sum_{k=1}^{n} \int_{v=0}^{\infty} \int_{u=\frac{k-1}{n}v}^{\frac{k}{n}v} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}\left(\frac{kv}{n}\right)^{\rho}v^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}\left(\frac{kv}{n}\right)^{\rho}v^{1-\rho}} \right) du dv \\
&\doteq 2.524866182. \\
\phi(x) &\geq \sum_{k=1}^{n} \int_{u=0}^{\infty} \int_{v=\frac{k-1}{n}u}^{\frac{k}{n}u} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}u^{\rho}\left(\frac{(k-1)u}{n}\right)^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}u^{\rho}\left(\frac{(k-1)u}{n}\right)^{1-\rho}} \right) dv du \\
&\quad + \sum_{k=1}^{n} \int_{v=0}^{\infty} \int_{u=\frac{k-1}{n}v}^{\frac{k}{n}v} \left(\mathbf{e}^{-u-v+\frac{\alpha}{2}\left(\frac{(k-1)v}{n}\right)^{\rho}v^{1-\rho}} - \mathbf{e}^{-u-v-\frac{\alpha}{2}\left(\frac{(k-1)v}{n}\right)^{\rho}v^{1-\rho}} \right) du dv \\
&\doteq 2.403166504.
\end{aligned}
$$