Sufficient Global Optimality Conditions for Bivalent Quadratic Optimization

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Abstract

We prove a sufficient global optimality condition for the problem of minimizing a quadratic function subject to quadratic equality constraints where the variables are allowed to take values -1 and 1. We extend the condition to quadratic problems with matrix variables and orthonormality constraints, and in particular, to the quadratic assignment problem.

We consider the bivalent quadratic optimization problem
[**QP**]

$$\begin{array}{ll} \min & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} & x^T E_i x + d_i^T x = f_i, \forall\, i = 1, \ldots, m \\ & x \in \{-1, 1\}^n \end{array}$$

where $Q \in \mathcal{S}^n$, and $E_i \in \mathcal{S}^n$, $\forall i = 1, \ldots, m$, $c, d_i \in \mathbb{R}^n$, and $f_i \in \mathbb{R}$ for all $i = 1, \ldots, m$ ($\mathcal{S}^n$ denotes the space of $n \times n$ symmetric real matrices). These problems are known to be NP hard even when the quadratic constraints are absent; see [2].

The purpose of this note is to present a sufficient condition for global optimality in **QP** and to give a natural extension to nonconvex quadratic programs in matrix variables, and in particular, to the quadratic assignment problem. The result is inspired by the work of Beck and Teboulle [1] which gave a sufficient condition for optimality in the problem

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx + c^Tx \\ \text{s.t.} & x \in \{-1,1\}^n \end{array}$$

Let D^T denote the $n \times m$ matrix with columns d_i, $i = 1,\ldots,m$. We define $X = \text{Diag}(x)$ to be the $n \times n$ diagonal matrix with diagonal equal to the vector x. Naturally, $x = Xe$, where e represents the n-dimensional vector of ones. We use $\otimes$ to denote Kronecker product.

Our main result is the following.

Theorem 1 *Let x be a feasible point for* **QP**. *If there exists $z \in \mathbb{R}^m$ which solves*

$$Q + Diag(-XQx - X(\sum_{i=1}^m z_iE_i)x - Xc - \frac{1}{2}XD^Tz) + \sum_{i=1}^m z_iE_i \succeq 0$$

then x is a global optimal solution for **QP**.

Proof: We rewrite **QP** as

$$\begin{array}{lrl} \min & \frac{1}{2}x^TQx + c^Tx & \\ \text{s.t.} & x^TE_ix + d_i^Tx &= f_i, \forall\, i = 1,\ldots,m \\ & x_j^2 &= 1, \forall\, j = 1,\ldots,n. \end{array}$$

Now, consider the Lagrangian function associated with **QP**

$$L(x,y,z) = \frac{1}{2}x^T(Q + Y + \sum_{i=1}^m z_iE_i)x - \frac{1}{2}y^Te + c^Tx - \frac{1}{2}z^Tf + \frac{1}{2}x^TD^Tz$$

where we have introduced multipliers $y \in \mathbb{R}^n$ $(Y = \text{Diag}(y))$ for the bivalency constraints, and $z \in \mathbb{R}^m$ for the first set of quadratic constraints after multiplying all constraints by one half, and have rearranged the expression of the function L to regroup quadratic and linear terms together. It is well-known that we have

$$\inf_x L(x, y, z) > -\infty$$

if and only if there exist y and z such that

$$Q + Y + \sum_{i=1}^{m} z_i E_i \succeq 0 \qquad (1)$$

and

$$(Q + Y + \sum_{i=1}^{m} z_i E_i)x + c + \frac{1}{2} D^T z = 0. \qquad (2)$$

For a feasible x, define

$$y := -XQx - X(\sum_{i=1}^{m} z_i E_i)x - Xc - \frac{1}{2} X D^T z.$$

for some $z \in \mathbb{R}^m$. It is easily verified using the fact that $XX = I$ that the vector y so defined satisfies (2) along with x and z.

Consider now the dual problem:

$$\sup_{y,z} h(y, z),$$

where

$$h(y, z) = \inf_x L(x, y, z).$$

Using (2), we immediately write $h(y, z)$ as

$$h(y, z) = \frac{1}{2} x^T (Q + Y + \sum_{i=1}^{m} z_i E_i)x - \frac{1}{2} y^T e - \frac{1}{2} z^T f.$$

Now, evaluate h at the point (x, y, z) defined above. A simple calculation using the fact that $XX = I$ shows that this yields

$$h(y, z) = \frac{1}{2} x^T Q x + \frac{1}{2} x^T (\sum_{i=1}^{m} z_i E_i)x + c^T x + \frac{1}{2} x^T D^T z - \frac{1}{2} z^T f.$$

But, since x is feasible, the second, fourth and fifth terms sum up to zero. Therefore, we see that the value of dual function equals the value of the primal objective function, which is sufficient to ensure global optimality of x from basic duality theory (c.f. Rockafellar [4]). ∎

Notice that the sufficient condition involves the solution of a linear matrix inequality (LMI), and as such can be checked using polynomial time interior point methods; see [3].

When one has only linear constraints, the sufficient condition becomes simpler. Consider the linearly constrained problem [**LCQP**]:

$$\begin{array}{ll} \min & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} & Ax = b \\ & x \in \{-1, 1\}^n \end{array}$$

where $Q \in \mathcal{S}^n$, and $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Corollary 2 *Let x be a feasible point for* **LCQP**. *If there exists $z \in \mathbb{R}^m$ which solves*

$$\lambda_{min}(Q)e \geq XQx + Xc + XAz$$

then x is a global optimal solution for **LCQP**.

Proof: The sufficient condition reduces to

$$Q + \mathrm{Diag}(-XQx - Xc - XAz) \succeq 0.$$

Since we always have $\lambda_{\min}(Q+Y) \geq \lambda_{\min}(Q) + \lambda_{\min}(Y)$, the above condition is satisfied if $\lambda_{\min}(Q) \geq -\lambda_{\min}(\mathrm{Diag}(-XQx - Xc - XAz))$. But since the right hand matrix is diagonal, the result follows. ∎

When one deals with with a linear bivalent program $(Q \equiv 0)$ we have the following corollary.

Corollary 3 *Let x be a feasible point. If there exists $z \in \mathbb{R}^m$ satisfying*

$$Xc + XAz \leq 0$$

then x is a global optimal solution.

Note that it is equally easy to treat inequality constraints by restricting the sign of the multiplier; see Theorem 4 below.

The above results admit natural extensions to nonconvex quadratic programs with matrix variables and orthonormality constraints. In particular, consider the quadratic assignment problem [**QAP**]:

$$\begin{array}{ll} \min & \text{Tr}AXBX^T + \text{Tr}CX^T \\ \text{s.t.} & XX^T = I \\ & Xe = e \\ & X^Te = e \\ & X \geq 0 \end{array}$$

where A, B are symmetric $n \times n$ matrices, C and X are an $n \times n$ matrices.

We use $\mathbb{R}_+^{n\times n}$ to denote the space of $n \times n$ real nonnegative matrices.

Theorem 4 *Let X be a feasible point for QAP. If there exist $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ and $T \in \mathbb{R}_+^{n\times n}$ with $T_{ij} = 0$ for all (i,j) such that $X_{ij} > 0$ satisfying the LMI*

$$B \otimes A - I \otimes \left(AXBX^T + CX^T - (ue^T + ew^T + T)X^T\right) \succeq 0$$

then X is global optimal in QAP.

Proof: The proof is essentially identical to the proof of Theorem 1, with the necessary modifications. ∎

The sufficient condition remains a linear matrix inequality with some linear side conditions.

A well-known relaxation of the QAP is the following nonconvex quadratic program defined over orthonormal matrices (Stiefel manifold) known as the **Eigenvalue Bounds** (for $C \equiv 0$).

$$\begin{array}{ll} \min & \text{Tr}AXBX^T + \text{Tr}CX^T \\ \text{s.t.} & XX^T = I \end{array}$$

The sufficient condition for optimality is simplified in this case.

Corollary 5 *Let X be an orthonormal matrix. If*

$$\lambda_{min}(B \otimes A) \geq \lambda_{max}(AXBX^T + CX^T)$$

then X is global optimal.

References

[1] A. Beck and M. Teboulle, Global Optimality Conditions for Quadratic Optimization with Binary Constraints, *SIAM J. on Optimization*, 11 (2000), 179–188.

[2] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, San Fransisco, CA, 1979.

[3] Yu. Nesterov and A. Nemirovski, Interior Point Polynomial Algorithms in Convex Programming, SIAM, 1993.

[4] T.R. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N.J., 1970.