

Facets of a polyhedron closely related to the integer knapsack-cover problem

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Abstract

We investigate the polyhedral structure of an integer program with a single functional constraint: the *integer capacity-cover polyhedron*. Such constraints arise in telecommunications planning and facility location applications, and feature the use of general integer (rather than just binary) variables. We derive a large class of facet-defining inequalities by using an *augmenting* technique that builds upon the facets of a family of related knapsack-cover polyhedra. To the best of our knowledge, this technique is new, and in particular it differs from sequential or simultaneous lifting. It demonstrates an interesting theoretical connection between the facial structure of two families of polyhedra important in applications. Additionally, we derive another class of facet-defining inequalities via coefficient reduction.

1 Introduction

We study the structure of a polyhedron generated by a single functional constraint involving both integer and binary variables, and parametrized by a positive integer m :

$$P_{CC}(m) := \text{conv} \left\{ (x, y) \in \mathbb{B}^m \times \mathbb{Z}_+^t : \sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k \right\},$$

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where $C_1 \leq C_2 \leq \dots \leq C_t$ are positive integers. We call this the *integer capacity-cover (ICC) polyhedron* because it models a situation in which customers with unit demands (represented by the binary decision variables x_i) need to be covered by capacity which can be constructed using integer multiples of the C_k . Such situations arise in telecommunications planning problems, among them the *optical network unit (ONU) placement problem* which is studied in Carpenter *et al.* [5]. Examination of the ONU placement problem led to the study of the more general *multi-facility location problem* in Mazur [13].

The main result of this paper is that there exists an intimate relationship between the polyhedral structure of the ICC polyhedron and that of the family of related *integer knapsack-cover (IKC) polyhedra*, parametrized by a positive integer b :

$$P_{\text{KC}}(b) := \text{conv} \left\{ y \in \mathbb{Z}_+^t : \sum_{k=1}^t C_k y_k \geq b \right\}.$$

Specifically, for appropriate values of b , one may take a certain valid inequality of $P_{\text{CC}}(m)$ and augment it by a nontrivial facet-representing inequality of $P_{\text{KC}}(b)$, to obtain a facet-representing inequality of $P_{\text{CC}}(m)$. Typically, there are several different values of b that are appropriate for a single $P_{\text{CC}}(m)$ polyhedron, thus leading to a large family of valid inequalities and facets for the ICC polyhedron.

1.1 Overview and placement

We study the ICC polyhedron for two reasons. First, we want to further the knowledge of the polyhedral structure of small integer programs, especially those with general integer variables. We believe that the augmenting technique detailed in Section 4 is an interesting theoretical result, in that it describes a close relationship between the facial structure of two polyhedra on general integer variables. Second, we want to understand the structure of larger integer programs (like those in the applications mentioned earlier) in which the ICC problem appears as a subproblem.

The seminal work of Crowder, Johnson & Padberg [8] on solving large scale binary integer programs demonstrated the importance of subproblem polyhedral structure in the solution of integer programs via branch-and-bound and cutting plane algorithms. In particular, they offered the first computational evidence that the theoretical results on the facial structure of the knapsack polytope could be useful in such algorithms (see Balas [4] or the survey by Padberg [15]). Soon after, Padberg, Van Roy & Wolsey [16] investigated the facial structure of certain fixed-charge problems occurring in network applications.

Their ideas proved particularly fruitful in studying facility location problems. For the *capacitated facility location problem*, the residual capacity inequality of Leung

& Magnanti [11] essentially comes from the flow cover inequality for a fixed charge problem. For the same facility location problem, Aardal, Pochet & Wolsey [2] derive knapsack and flow cover inequalities, and then extend the flow cover idea to a family of effective capacity inequalities. For the closely-related *capacitated concentrator location problem*, Deng & Simchi-Levi [9] use knapsack and bin-packing subproblems to develop valid inequalities and facets.

The aforementioned papers involve pure binary or mixed binary settings, while our work involves general integer variables. Recently there has been interest in understanding the polyhedral structure of general integer and mixed integer programs. Four relevant papers are those by Marcotte [12] on the integer knapsack problem; by Pochet & Wolsey [17] on the integer knapsack cover problem; by Ceria, Cordier, Marchand & Wolsey [6] on cutting planes for general IPs; and by Günlük & Pochet [10] on mixed-integer inequalities.

Although our facets of the ICC polyhedron depend directly on having facets of the IKC polyhedron, facet identification for the latter appears to be difficult. Pochet & Wolsey [17] give a complete description of the IKC polyhedron in the special case of divisible coefficients: $C_k | C_{k+1}$ for all k . While their results directly apply to our work, we did not necessarily encounter divisible coefficients in the application we examined; typical capacity values encountered are 4-8-16-32 which are divisible, but also 2-5-12-20-50 and 4-8-12-24-48 which are not. The importance of studying problems with general, not necessarily divisible, capacity types has most recently been underscored by Atamtürk [3] in his study of capacitated network design polyhedra.

The good news is that in many capacitated network design problems, including the application that inspired this work, the number t of capacity types is typically small, usually between four and seven. In this case, one could use software such as PORTA [7] to find all facet-representing inequalities of the required IKC polyhedra, and then build the facets of the ICC polyhedron via the augmenting technique.

1.2 Some notation

We use \mathbf{e}_k and $\mathbf{1}$ to denote the k -th unit vector and the all-1s vector, respectively, of appropriate dimension. Also, the notation $[n]$ denotes the set of integers $\{1, 2, \dots, n\}$.

2 The integer knapsack-cover polyhedron

We now list some useful results for the integer knapsack-cover polyhedron $P_{\text{KC}}(b)$, for any positive integer b . In general, by a *nontrivial valid inequality* we mean one which is neither dominated by nor expressible as a nonnegative linear combination of nonnegativity constraints.

Proposition 1 For polyhedron $P_{\text{KC}}(b)$, $\dim(P_{\text{KC}}(b)) = t$. ■

Proposition 2 If $\sum_{k=1}^t \alpha_k y_k \geq \beta$ is a nontrivial valid inequality for $P_{\text{KC}}(b)$, then $\alpha_k > 0$ for all $k \in [t]$, and $\beta > 0$. ■

Propositions 1 and 2 follow from the fact that $P_{\text{KC}}(b)$ is of anti-blocking type; see Chapter 9 of Schrijver [18].

The following theorem provides a simple class of facets for the IKC polyhedron; for a proof see Mazur [13].

Theorem 1 Define $C_{t+1} := +\infty$. If $P_{\text{KC}}(b)$ is an integer knapsack-cover polyhedron with data satisfying $C_{k'} < b \leq C_{k'+1}$ for some $k' \in [t]$, then the inequality

$$\sum_{k=1}^{k'} y_k + \left\lceil \frac{b}{C_{k'}} \right\rceil \sum_{k=k'+1}^t y_k \geq \left\lceil \frac{b}{C_{k'}} \right\rceil$$

is valid for $P_{\text{KC}}(b)$, and represents a facet if and only if

$$b \leq \left(\left\lceil \frac{b}{C_{k'}} \right\rceil - 1 \right) C_{k'} + C_1.$$

■

In the case $k' = t$, the inequality is just $\sum_{k=1}^t y_k \geq \lceil b/C_t \rceil$.

Example. Consider

$$P_{\text{KC}}(9) := \text{conv} \left\{ y \in \mathbb{Z}_+^3 : 4y_1 + 8y_2 + 12y_3 \geq 9 \right\}.$$

Choosing $k' = 2$, the resulting inequality $y_1 + y_2 + 2y_3 \geq 2$ is valid for $P_{\text{KC}}(9)$. It also represents a facet. ■

Finally, we observe that optimization over $P_{\text{KC}}(b)$ may be performed in pseudo-polynomial time. Let

$$F(b) := \min \left\{ \sum_{k=1}^t a_k y_k : y \in \mathbb{Z}_+^t \text{ and } \sum_{k=1}^t C_k y_k \geq b \right\}.$$

Then $F(b)$ obeys

$$F(b) = \begin{cases} \min \{ a_k + F(b - C_k) : k \in [t] \} & \text{if } b > 0 \\ 0 & \text{if } b \leq 0. \end{cases}$$

This leads to a dynamic programming algorithm: set $F(b) := 0$ for $b \leq 0$, and then sequentially compute $F(1), F(2), \dots, F(b)$. Although this takes $O(tb)$ time, it is polynomial if b is bounded by a polynomial function of the input size. We shall use this observation in Section 4.

3 The integer capacity-cover polyhedron

We now undertake a study of the polyhedral structure of the integer capacity-cover polyhedron

$$P_{\text{CC}}(m) := \text{conv} \left\{ (x, y) \in \mathbb{B}^m \times \mathbb{Z}_+^t : \sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k \right\}.$$

We remind the reader that the C_k are positive integers satisfying $C_1 \leq C_2 \leq \dots \leq C_t$.

3.1 Dimension and trivial facets

The following proposition, whose proof we omit, catalogs some basic results.

Proposition 3 *For the polyhedron $P_{\text{CC}}(m)$, $\dim(P_{\text{CC}}(m)) = m + t$. In addition, the valid inequalities*

- $x_i \geq 0$ for all $i \in [m]$;
- $x_i \leq 1$ for all $i \in [m]$; and
- (provided $t \geq 2$) $y_k \geq 0$ for all $k \in [t]$

represent facets of $P_{\text{CC}}(m)$. ■

If $t = 1$, then a complete description of $P_{\text{CC}}(m)$ is given by $0 \leq x_i \leq 1$ for all $i \in [m]$ and $x_i \leq y_1$ for all $i \in [m]$. Also, if $m = 1$, then a complete description of $P_{\text{CC}}(1)$ is given by $0 \leq x_1 \leq 1$ and $x_1 \leq \sum_{k=1}^t y_k$.

3.2 Coefficient reduction inequalities

Let $I \subseteq [m]$ be nonempty. Since

$$\sum_{i \in I} x_i \leq \sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k,$$

the inequality $\sum_{i \in I} x_i \leq \sum_{k=1}^t C_k y_k$ is valid for $P_{\text{CC}}(m)$. In addition, since $\sum_{i \in I} x_i \leq |I|$, the inequality

$$\sum_{i \in I} x_i \leq \sum_{k=1}^t \min\{C_k, |I|\} y_k$$

is also valid and potentially tighter. We term such inequalities *coefficient reduction (CR) inequalities*.

The following theorem provides necessary and sufficient conditions under which a CR inequality represents a facet of $P_{CC}(m)$. The proof is straightforward but technical; we refer the interested reader to Chapter 2 of Mazur [13].

Theorem 2 *For any nonempty set $I \subseteq [m]$, the CR inequality*

$$\sum_{i \in I} x_i \leq \sum_{k=1}^t \min\{C_k, |I|\} y_k \quad (1)$$

is valid for $P_{CC}(m)$. The following statements provide exact conditions under which it represents a facet.

1. *If $m = 1$ then (1) always represents a facet, and if $m \geq 2$ and $|I| = 1$ then it represents a facet if and only if $C_t > 1$.*
2. *If $m \geq 2$ and $|I| = m$ then (1) represents a facet if and only if $C_1 < m$.*
3. *If $m \geq 3$ and $1 < |I| < m$, then (1) represents a facet if and only if $C_1 < |I| < C_t$.*

■

Example. We illustrate by listing all of the facet-representing CR inequalities of the integer capacity cover polyhedron

$$P_{CC}(10) := \left\{ (x, y) \in \mathbb{B}^{10} \times \mathbb{Z}_+^3 : \sum_{i=1}^{10} x_i \leq 4y_1 + 8y_2 + 12y_3 \right\}.$$

Using Theorem 2, the values of $|I|$ which lead to facet-representing CR inequalities are 1, 5, 6, 7, 8, 9, and 10. The inequalities are:

- $x_i \leq y_1 + y_2 + y_3$ for all $i \in [10]$;
- $\sum_{i \in I} x_i \leq 4y_1 + 5y_2 + 5y_3$ for all $I \subseteq [10]$ with $|I| = 5$;
- $\sum_{i \in I} x_i \leq 4y_1 + 6y_2 + 6y_3$ for all $I \subseteq [10]$ with $|I| = 6$;
- $\sum_{i \in I} x_i \leq 4y_1 + 7y_2 + 7y_3$ for all $I \subseteq [10]$ with $|I| = 7$;
- $\sum_{i \in I} x_i \leq 4y_1 + 8y_2 + 8y_3$ for all $I \subseteq [10]$ with $|I| = 8$;

- $\sum_{i \in I} x_i \leq 4y_1 + 8y_2 + 9y_3$ for all $I \subseteq [10]$ with $|I| = 9$; and
- $\sum_{i=1}^{10} x_i \leq 4y_1 + 8y_2 + 10y_3$.

There are $\binom{10}{1} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 648$ such inequalities. ■

In fact, every facet-representing inequality of the form $\sum_{i \in I} x_i \leq \sum_{k=1}^t \alpha_k y_k$ must be a CR inequality. We shall invoke this result, stated in the following proposition, in the next section.

Proposition 4 *If $I \subseteq [m]$ is nonempty and the inequality*

$$\sum_{i \in I} x_i \leq \sum_{k=1}^t \alpha_k y_k \tag{2}$$

represents a facet of $P_{CC}(m)$, then $\alpha_k = \min\{C_k, |I|\}$ for all $k \in [t]$.

Proof. Choose any $k' \in [t]$, and select any $I' \subseteq I$ such that $|I'| = \min\{C_{k'}, |I|\}$. Define $(\hat{x}, \hat{y}) \in P_{CC}(m)$ by

- $\hat{x}_i = \begin{cases} 1 & \text{if } i \in I' \\ 0 & \text{otherwise, and} \end{cases}$
- $\hat{y} = \mathbf{e}_{k'}$.

Since (2) is valid for $P_{CC}(m)$, we have

$$\alpha_{k'} = \sum_{k=1}^t \alpha_k \hat{y}_k \geq \sum_{i \in I} \hat{x}_i = \sum_{i \in I'} \hat{x}_i = |I'| = \min\{C_{k'}, |I|\}.$$

Therefore, since we chose $k' \in [t]$ arbitrarily, it follows that $\alpha_k \geq \min\{C_k, |I|\}$ holds for all k . Using this fact, along with validity of the CR inequality for $P_{CC}(m)$, we see that

$$\sum_{i \in I} x_i \leq \sum_{k=1}^t \min\{C_k, |I|\} y_k \leq \sum_{k=1}^t \alpha_k y_k$$

holds for all $(x, y) \in P_{CC}(m)$. By hypothesis, (2) represents a facet, and hence it must be the case that $\alpha_k = \min\{C_k, |I|\}$ for all $k \in [t]$. ■

Finally, we observe that the separation problem for the CR inequalities may be solved in polynomial time. Given a point (x^*, y^*) , we may decide whether

$$\sum_{i \in I} x_i^* > \sum_{k=1}^t \min\{C_k, |I|\} y_k^* \quad \text{for some } I \subseteq [m]$$

as follows:

- Sort the components of x^* ; for convenience, assume that $x_1^* \geq x_2^* \geq \dots \geq x_m^*$.
- For $r = 1, 2, \dots, m$, iteratively check whether

$$\sum_{i=1}^r x_i^* > \sum_{k=1}^t \min\{C_k, r\} y_k^*.$$

Notice that if (x^*, y^*) violates at least one CR inequality, then the procedure will find at least one (possibly stronger) violation. This is because $\sum_{i=1}^{|I|} x_i^* \geq \sum_{i \in I} x_i^*$ for any $I \subseteq [m]$, assuming x^* is sorted as indicated above.

4 Augmented knapsack-cover inequalities for the ICC polyhedron

We now provide the details of the relationship between facets of integer capacity-cover polyhedra and those of integer knapsack-cover polyhedra. Our technique augments a trivial valid inequality of $P_{CC}(m)$ by a valid inequality of $P_{KC}(b)$, resulting in a stronger valid inequality for $P_{CC}(m)$. We then provide conditions under which such inequalities represent facets of $P_{CC}(m)$.

4.1 A motivating example

Let $C_1 = 4$, $C_2 = 8$, and $C_3 = 12$ be the capacities available. Consider

$$P_{CC}(m) := \text{conv} \left\{ (x, y) \in \mathbb{B}^m \times \mathbb{Z}_+^3 : \sum_{i=1}^m x_i \leq 4y_1 + 8y_2 + 12y_3 \right\}.$$

Provided m is large enough in each case (i.e., $m \geq 5$, 6, or 7, respectively), the following inequalities are valid for $P_{CC}(m)$:

$$\sum_{i \in I} x_i \leq 3 + y_1 + 2y_2 + 2y_3 \quad \text{for any } I \subseteq [m] \text{ with } |I| = 5;$$

$$\sum_{i \in I} x_i \leq 2 + 2y_1 + 4y_2 + 4y_3 \quad \text{for any } I \subseteq [m] \text{ with } |I| = 6;$$

$$\sum_{i \in I} x_i \leq 1 + 3y_1 + 6y_2 + 6y_3 \quad \text{for any } I \subseteq [m] \text{ with } |I| = 7.$$

To see that the first is valid, observe that if $y = (0, 0, 0)$ or $(1, 0, 0)$, then the inequality implies $\sum_{i \in I} x_i \leq 3$ or $\sum_{i \in I} x_i \leq 4$, respectively. Both are true because no capacity

is placed in the first case, and four units are placed in the second case. For all other $y \in \mathbb{Z}_+^3$, the implied inequality is equal to or dominated by the trivial valid inequality $\sum_{i \in I} x_i \leq 5$. Validity of the other two inequalities follows similarly.

The question is how the coefficients on the y_k 's arise. Rewrite each of the inequalities as

$$\sum_{i \in I} x_i \leq 5 + 1(y_1 + 2y_2 + 2y_3 - 2) \quad \text{for any } I \subseteq [m] \text{ with } |I| = 5;$$

$$\sum_{i \in I} x_i \leq 6 + 2(y_1 + 2y_2 + 2y_3 - 2) \quad \text{for any } I \subseteq [m] \text{ with } |I| = 6;$$

$$\sum_{i \in I} x_i \leq 7 + 3(y_1 + 2y_2 + 2y_3 - 2) \quad \text{for any } I \subseteq [m] \text{ with } |I| = 7;$$

respectively. The expression $y_1 + 2y_2 + 2y_3 - 2$ in each comes from the valid inequality (actually, Theorem 1 implies that it represents a facet)

$$y_1 + 2y_2 + 2y_3 \geq 2 \tag{3}$$

for the integer knapsack-cover polyhedron

$$P_{\text{KC}}(b) := \{y \in \mathbb{Z}_+^3 : 4y_1 + 8y_2 + 12y_3 \geq b\}$$

for $b = 5, 6, 7$.¹ From the ICC point of view, the amount by which the inequality (3) is violated tightens the trivial upper bound of $|I|$ on $\sum_{i \in I} x_i$. This is the key observation in describing the relationship between the ICC and IKC polyhedra. Below, we show how to determine the multiplier on the expression $y_1 + 2y_2 + 2y_3 - 2$.

4.2 Validity of the augmented knapsack-cover inequality

Let $I \subseteq [m]$ and suppose that $\sum_{k=1}^t \alpha_k y_k \geq \beta$ is valid for $P_{\text{KC}}(|I|)$. Certainly $\sum_{i \in I} x_i \leq |I|$ is valid for $P_{\text{CC}}(m)$. We now determine conditions under which we can augment that inequality by a multiple of $\sum_{k=1}^t \alpha_k y_k - \beta$ in such a way that the resulting inequality, of the form

$$\sum_{i \in I} x_i \leq |I| + \rho \left(\sum_{k=1}^t \alpha_k y_k - \beta \right),$$

is stronger than $\sum_{i \in I} x_i \leq |I|$ while remaining valid. Define

¹Notice that in this case, $P_{\text{KC}}(5) = P_{\text{KC}}(6) = P_{\text{KC}}(7)$.

$$Q_{\text{CC}}(m) := \left\{ (x, y) \in \mathbb{B}^m \times \mathbb{Z}_+^t : \sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k \right\}$$

$$Q_{\text{KC}}(b) := \left\{ y \in \mathbb{Z}_+^t : \sum_{k=1}^t C_k y_k \geq b \right\}$$

so that $P_{\text{CC}}(m) = \text{conv}(Q_{\text{CC}}(m))$ and $P_{\text{KC}}(b) = \text{conv}(Q_{\text{KC}}(b))$.

Proposition 5 *Let $I \subseteq [m]$ be nonempty. If $\sum_{k=1}^t \alpha_k y_k \geq \beta$ is a nontrivial valid inequality for $P_{\text{KC}}(|I|)$, then the augmented knapsack-cover (AKC) inequality*

$$\sum_{i \in I} x_i \leq |I| + \rho \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \quad (4)$$

is valid for $P_{\text{CC}}(m)$ if and only if ρ satisfies $0 \leq \rho \leq \rho^*$. Here,

$$\rho^* := \min \left\{ \frac{|I| - \sum_{i \in I} x_i}{\beta - \sum_{k=1}^t \alpha_k y_k} : (x, y) \in Q_{\text{CC}}(m) \text{ and } \sum_{k=1}^t \alpha_k y_k < \beta \right\}$$

is well-defined and positive.

Proof. For any $\rho \in \mathbb{R}$, inequality (4) is valid for $P_{\text{CC}}(m)$ if and only if

$$\sum_{i \in I} x_i - |I| \leq \rho \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \quad \text{for all } (x, y) \in Q_{\text{CC}}(m).$$

In turn, by considering the sign of $\sum_{k=1}^t \alpha_k y_k - \beta$, it is true if and only if

- $\sum_{i \in I} x_i \leq |I|$ for all $(x, y) \in Q_{\text{CC}}(m)$ such that $\sum_{k=1}^t \alpha_k y_k = \beta$;
- $\rho \geq \frac{\sum_{i \in I} x_i - |I|}{\sum_{k=1}^t \alpha_k y_k - \beta}$ for all $(x, y) \in Q_{\text{CC}}(m)$ such that $\sum_{k=1}^t \alpha_k y_k > \beta$;
- $\rho \leq \frac{|I| - \sum_{i \in I} x_i}{\beta - \sum_{k=1}^t \alpha_k y_k}$ for all $(x, y) \in Q_{\text{CC}}(m)$ such that $\sum_{k=1}^t \alpha_k y_k < \beta$.

The first bulleted condition is always true, and the second reduces to $\rho \geq 0$. This leaves the third condition, and thus that ρ must satisfy $0 \leq \rho \leq \rho^*$ for ρ^* as defined in the statement of the theorem.

That ρ^* is well-defined follows from Proposition 2: since $\beta > 0$, the point $(x, y) = (0, 0)$ lies in $Q_{CC}(m)$ and satisfies $\sum_{k=1}^t \alpha_k y_k < \beta$; moreover, $\alpha_k > 0$ for all k ensures that the minimization occurs over a finite set.

Finally, from the definition of ρ^* it is clear that $\rho^* \geq 0$. If it were the case that $\rho^* = 0$, then there would exist some $(x, y) \in Q_{CC}(m)$ with $\sum_{i \in I} x_i = |I|$ and $\sum_{k=1}^t \alpha_k y_k < \beta$. However, this would contradict the validity of $\sum_{k=1}^t \alpha_k y_k \geq \beta$ for $P_{KC}(|I|)$. This completes the proof. \blacksquare

In fact, we can compute ρ^* in polynomial time. First, fix the value of $\sum_{i \in I} x_i$ to some $\ell \in \{0, 1, \dots, |I| - 1\}$.² For this fixed value of ℓ , we then compute

$$\min \left\{ \frac{|I| - \ell}{\beta - \sum_{k=1}^t \alpha_k y_k} : y \in Q_{KC}(\ell) \text{ and } \sum_{k=1}^t \alpha_k y_k < \beta \right\}.$$

We can accomplish this by making the denominator as large as possible, which is equivalent to solving

$$\min \left\{ \sum_{k=1}^t \alpha_k y_k : y \in Q_{KC}(\ell) \right\}.$$

This problem can be solved in $O(t\ell)$ time via the dynamic program for integer knapsack-cover given in Section 2. We can therefore compute ρ^* by first solving such a problem for each value of ℓ , and then taking the minimum over all appropriate values of the ratio. Such a procedure takes $O(t \cdot |I|^2)$ time.

4.3 Augmented knapsack-cover facets

We now show how to generate facet-representing inequalities of $P_{CC}(m)$ using the AKC idea. We shall start with a nontrivial facet-representing, rather than merely valid, IKC inequality and use ρ^* as the multiplier. Most of the work is contained in proving Theorem 3, which concerns the case $|I| = m$. From there, we quickly extend to other values of $|I|$ in Theorem 4.

Before proving these theorems, we carry out some preliminary analysis. First, we dispense with the special case $m = 1$. A straightforward verification shows that the only nontrivial facet-representing inequality of

$$P_{KC}(1) = \text{conv} \left\{ y \in \mathbb{Z}_+^t : \sum_{k=1}^t C_k y_k \geq 1 \right\}$$

²We needn't consider $\ell = |I|$ by the argument which shows $\rho^* > 0$.

is $\sum_{k=1}^t y_k \geq 1$. In this case,

$$\rho^* = \min \left\{ \frac{1 - x_1}{1 - \sum_{k=1}^t y_k} : (x, y) \in Q_{CC}(1) \text{ and } \sum_{k=1}^t y_k < 1 \right\} = 1,$$

so the level- ρ^* AKC inequality we obtain is $x_1 \leq 1 + 1(\sum_{k=1}^t y_k - 1)$ or $x_1 \leq \sum_{k=1}^t y_k$. This is a CR inequality with $|I| = 1$, and Theorem 2 tells us that it always represents a facet. It hence suffices to prove our theorem for $m \geq 2$.

Second, in constructing the AKC inequality studied in Theorem 3, namely

$$\sum_{i=1}^m x_i \leq m + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right),$$

it suffices to consider the case $m > \rho^* \beta$. It is easy to see that $m \geq \rho^* \beta$ always holds: evaluate the feasible point $(x, y) = (0, 0)$ in the valid inequality. If $m = \rho^* \beta$, the inequality becomes

$$\sum_{i=1}^m x_i \leq \sum_{k=1}^t (\rho^* \alpha_k) y_k.$$

If this is to represent a facet, Proposition 4 implies that it must be a CR inequality. Therefore, it represents a facet if and only if

- $\rho^* \alpha_k = \min\{C_k, m\}$ for all $k \in [t]$, and
- it meets the appropriate conditions from Theorem 2.

Thus, it suffices to consider the case $m > \rho^* \beta$.

Theorem 3 *Suppose $\sum_{k=1}^t \alpha_k y_k \geq \beta$ represents a nontrivial facet of $P_{KC}(m)$, where $m \geq 2$. If $m > \rho^* \beta$ then the AKC inequality*

$$\sum_{i=1}^m x_i \leq m + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \tag{5}$$

represents a facet of $P_{CC}(m)$, where

$$\rho^* := \min \left\{ \frac{m - \sum_{i=1}^m x_i}{\beta - \sum_{k=1}^t \alpha_k y_k} : (x, y) \in Q_{CC}(m) \text{ and } \sum_{k=1}^t \alpha_k y_k < \beta \right\}.$$

Proof. By Proposition 5, the inequality (5) is valid for $P_{\text{CC}}(m)$. Fix any

$$(\bar{x}, \bar{y}) \in \arg \min \left\{ \frac{m - \sum_{i=1}^m x_i}{\beta - \sum_{k=1}^t \alpha_k y_k} : (x, y) \in Q_{\text{CC}}(m) \text{ and } \sum_{k=1}^t \alpha_k y_k < \beta \right\}$$

and define

$$F := \left\{ (x, y) \in P_{\text{CC}}(m) : \sum_{i=1}^m x_i = m + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \right\}.$$

A direct verification shows that (\bar{x}, \bar{y}) lies in F ; therefore $F \neq \emptyset$. Noting $\rho^* > 0$ (by Proposition 5) and $\alpha_1 > 0$ (by Proposition 2), it follows that $(\bar{x}, \bar{y} + \mathbf{e}_1) \in P_{\text{CC}}(m) \setminus F$, and hence that F is a proper face of $P_{\text{CC}}(m)$.

Before proving that F is a facet, we claim that $0 < \sum_{i=1}^m \bar{x}_i < m$. To see this, define

$$\sigma := \rho^* \left(\beta - \sum_{k=1}^t \alpha_k \bar{y}_k \right). \quad (6)$$

Since $\sum_{i=1}^m \bar{x}_i = m - \sigma$, it then follows that σ must be some positive integer (where positivity follows from $\rho^* > 0$ and $\sum_{k=1}^t \alpha_k \bar{y}_k < \beta$). This implies $\sum_{i=1}^m \bar{x}_i < m$. In fact, since $m - \rho^* \beta > 0$ by hypothesis,

$$\sum_{i=1}^m \bar{x}_i = m - \rho^* \beta + \rho^* \sum_{k=1}^t \alpha_k \bar{y}_k > \rho^* \sum_{k=1}^t \alpha_k \bar{y}_k \geq 0.$$

Therefore, (\bar{x}, \bar{y}) is a point in F which satisfies

$$0 < \sum_{i=1}^m \bar{x}_i < m. \quad (7)$$

To show that F is a facet of $P_{\text{CC}}(m)$, suppose that

$$\sum_{i=1}^m \lambda_i x_i = \lambda_0 + \sum_{k=1}^t \mu_k y_k \quad (8)$$

holds for all $(x, y) \in F$. It then suffices to show that there exists $\gamma \in \mathbb{R}$ such that

- $\lambda_i = \gamma$ for all $i \in [m]$;
- $\lambda_0 = \gamma(m - \rho^* \beta)$; and

- $\mu_k = \gamma \rho^* \alpha_k$ for all $k \in [t]$.

(a) There exists $\gamma \in \mathbb{R}$ such that $\lambda_i = \gamma$ for all $i \in [m]$. Fix any $i_1, i_2 \in [m]$. Select any subset $I \subseteq [m]$ with the properties (i) $|I| = m - \sigma$, (ii) $i_1 \in I$, and (iii) $i_2 \notin I$. Such a set exists because $0 < m - \sigma < m$, by (7) and (6). Now define the point $(x', y') \in Q_{CC}(m)$ by

- $x'_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise,} \end{cases}$
- $y' = \bar{y}$.

Then $(x', y') \in F$ because

$$\sum_{i=1}^m x'_i = |I| = m - \sigma = m + \rho^* \left(\sum_{k=1}^t \alpha_k \bar{y}_k - \beta \right) = m + \rho^* \left(\sum_{k=1}^t \alpha_k y'_k - \beta \right).$$

Moreover, $(x' - \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, y') \in F$ as well. Evaluating each of these in (8) and subtracting one equation from the other implies $\lambda_{i_1} = \lambda_{i_2}$. Since we chose i_1 and i_2 arbitrarily, there exists some $\gamma \in \mathbb{R}$ such that $\lambda_i = \gamma$ for all $i \in [m]$. This shows that (8) must look like

$$\sum_{i=1}^m \gamma x_i = \lambda_0 + \sum_{k=1}^t \mu_k y_k \tag{9}$$

for the just-found γ (with λ_0 and the μ_k as before).

(b) $\lambda_0 = \gamma(m - \rho^* \beta)$ and $\mu_k = \gamma \rho^* \alpha_k$ for all $k \in [t]$. Since $\sum_{k=1}^t \alpha_k y_k \geq \beta$ represents a facet of $P_{KC}(m)$, which has dimension t by Proposition 1, there exist t affinely independent points $y^1, y^2, \dots, y^t \in Q_{KC}(m)$ satisfying the inequality at equality. We claim that the $t+1$ points $y^1, y^2, \dots, y^t, \bar{y}$ are affinely independent: the first t points all lie in the affine space defined by $\sum_{k=1}^t \alpha_k y_k = \beta$, while \bar{y} does not, since $\sum_{k=1}^t \alpha_k \bar{y}_k < \beta$.

Now consider the points

$$(\mathbf{1}, y^1), (\mathbf{1}, y^2), \dots, (\mathbf{1}, y^t), (\bar{x}, \bar{y}). \tag{10}$$

Each of these points lies in F . Indeed, for $\ell \in [t]$, $y^\ell \in Q_{KC}(m)$, and thus $(\mathbf{1}, y^\ell) \in Q_{CC}(m)$ because

$$\sum_{k=1}^t C_k y_k^\ell \geq m = \sum_{i=1}^m 1;$$

in addition, $(\mathbf{1}, y^\ell) \in F$ because

$$\sum_{i=1}^m 1 = m = m + \rho^* \cdot 0 = m + \rho^* \left(\sum_{k=1}^t \alpha_k y_k^\ell - \beta \right).$$

Therefore each point in (10) satisfies (9), from which we obtain the linear system

$$\underbrace{\begin{bmatrix} (y^1)^T & 1 \\ (y^2)^T & 1 \\ \vdots & \vdots \\ (y^t)^T & 1 \\ \bar{y}^T & 1 \end{bmatrix}}_{=:A} \begin{bmatrix} \mu \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} \gamma m \\ \gamma m \\ \vdots \\ \gamma m \\ \gamma(m - \sigma) \end{bmatrix}$$

Since the points $y^1, y^2, \dots, y^t, \bar{y}$ are affinely independent, the $(t+1) \times (t+1)$ matrix A is nonsingular and so this system has a unique solution. Moreover, a quick verification shows that $(\mu, \lambda_0) = (\gamma\rho^*\alpha, \gamma(m - \rho^*\beta))$ solves the system. Thus, once we determine γ , the coefficients λ_0 and μ are uniquely determined as claimed. This completes the proof that F is a facet of $P_{CC}(m)$. \blacksquare

The following theorem shows that practically every facet-representing AKC inequality of $P_{CC}(m)$ is also a facet-representing AKC inequality of $P_{CC}(m')$ for all $m' > m$. The sole exception occurs in the degenerate case when the inequality for $P_{CC}(m)$ is actually the original defining inequality $\sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k$. In fact, it is sufficient to start with an AKC inequality which satisfies the condition $m > \rho^*\beta$ from Theorem 3; such an inequality would have a nonzero constant term, and therefore no chance of being the original defining inequality.

Theorem 4 *Suppose*

$$\sum_{i=1}^m x_i \leq m + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \tag{11}$$

is a facet-representing AKC inequality for $P_{CC}(m)$, where ρ^ is as defined in the statement of Theorem 3. Then*

$$\sum_{i \in I} x_i \leq |I| + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right) \tag{12}$$

is a facet-representing inequality of $P_{CC}(m')$ for all $m' > m$ and for all $I \subseteq [m']$ with $|I| = m$, provided (11) is not identically the inequality $\sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k$.

Proof. Validity of (12) for $P_{\text{CC}}(m')$ follows from Proposition 5. Let F and F' denote the facet of $P_{\text{CC}}(m)$ and the face of $P_{\text{CC}}(m')$ represented by (11) and (12), respectively. That F' is a proper face of $P_{\text{CC}}(m')$ follows directly from the fact that F is a proper face of $P_{\text{CC}}(m)$.

Let (\hat{x}, \hat{y}) be any point in $F \cap Q_{\text{CC}}(m)$ satisfying $\sum_{i=1}^m \hat{x}_i < \sum_{k=1}^t C_k \hat{y}_k$. If it were the case that $\sum_{i=1}^m x_i = \sum_{k=1}^t C_k y_k$ for all $(x, y) \in F \cap Q_{\text{CC}}(m)$, then the inequality (11) would in fact be $\sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k$. We have assumed the contrary, and hence such a (\hat{x}, \hat{y}) exists.

The proof that F' is a facet of $P_{\text{CC}}(m')$ then proceeds much as the proof of Theorem 5, so we will not repeat the details of the entire argument. The only additional condition we need to check is

- $\lambda_i = 0$ for all $i \in [m'] \setminus I$,

which we may accomplish as follows. For convenience, and without loss of generality, assume that $I = [m]$. Fix any $i' \in [m'] \setminus I$, and define the point (x, y) by

- $x_i = \begin{cases} \hat{x}_i & \text{for } i = 1, \dots, m, \\ 0 & \text{for } i = m+1, \dots, m'; \end{cases}$
- $y = \hat{y}$.

Clearly, $(x, y) \in P_{\text{CC}}(m')$ since $(\hat{x}, \hat{y}) \in P_{\text{CC}}(m)$. In addition, $(\hat{x}, \hat{y}) \in F$ implies

$$\sum_{i=1}^{m'} x_i = \sum_{i=1}^m \hat{x}_i = m + \rho^* \left(\sum_{k=1}^t \alpha_k \hat{y}_k - \beta \right) = |I| + \rho^* \left(\sum_{k=1}^t \alpha_k y_k - \beta \right),$$

so $(x, y) \in F'$. By our choice of (\hat{x}, \hat{y}) , it follows that $(x + \mathbf{e}_{i'}, y) \in F'$ as well; in particular, $(x + \mathbf{e}_{i'}, y) \in P_{\text{CC}}(m')$ since $\sum_{i=1}^m x_i = \sum_{i=1}^{m'} \hat{x}_i < \sum_{k=1}^t C_k \hat{y}_k = \sum_{k=1}^t C_k y_k$. Evaluating each of these points in an appropriate equation (analogous to (8)) and subtracting one from the other implies $\lambda_{i'} = 0$. This completes the proof. ■

It is easy to see why we do not obtain a facet-representing inequality if the original facet-representing inequality for $P_{\text{CC}}(m)$ is $\sum_{i=1}^m x_i \leq \sum_{k=1}^t C_k y_k$. For $m' > m$, the new inequality would be

$$\sum_{i \in I} x_i \leq \sum_{k=1}^t C_k y_k \quad \text{for } I \subseteq [m'] \text{ with } |I| = m,$$

which is strictly dominated by $\sum_{i=1}^{m'} x_i \leq \sum_{k=1}^t C_k y_k$.

Example. We continue the example of the previous subsection by listing all of the facet-representing AKC inequalities for the capacity-cover polyhedron

$$P_{\text{CC}}(10) := \left\{ (x, y) \in \mathbb{B}^{10} \times \mathbb{Z}_+^3 : \sum_{i=1}^{10} x_i \leq 4y_1 + 8y_2 + 12y_3 \right\}.$$

We first examine appropriate knapsack-cover polyhedra. Starting with $P_{\text{KC}}(b) := \{y \in \mathbb{Z}_+^3 : 4y_1 + 8y_2 + 12y_3 \geq b\}$, Theorem 1 implies that $y_1 + y_2 + y_3 \geq 1$ represents a facet; moreover, one can verify that it is the only nontrivial facet-representing inequality of $P_{\text{KC}}(b)$ for $b \in \{1, 2, 3, 4\}$. The level- ρ^* inequalities that arise in $P_{\text{CC}}(m)$ for $m \in \{1, 2, 3, 4\}$ are exactly the CR inequalities with $|I| = 1, 2, 3, 4$; the only one that represents a facet arises when $m = 1$.

The only nontrivial facet-representing inequality of $P_{\text{KC}}(b)$ for $b \in \{5, 6, 7, 8\}$ is $y_1 + 2y_2 + 2y_3 \geq 2$. A quick computation shows that

$$\begin{aligned} P_{\text{CC}}(5) &\implies \rho^* = 1 \\ P_{\text{CC}}(6) &\implies \rho^* = 2 \\ P_{\text{CC}}(7) &\implies \rho^* = 3 \\ P_{\text{CC}}(8) &\implies \rho^* = 4. \end{aligned}$$

Except when $m = 8$ (there the AKC inequality reduces to the facet-representing $|I| = 8$ CR inequality), this generates the following families of facet-representing inequalities:

- $\sum_{i \in I} x_i \leq 3 + y_1 + 2y_2 + 2y_3$ for all $I \subseteq [10]$ with $|I| = 5$;
- $\sum_{i \in I} x_i \leq 2 + 2y_1 + 4y_2 + 4y_3$ for all $I \subseteq [10]$ with $|I| = 6$; and
- $\sum_{i \in I} x_i \leq 1 + 3y_1 + 6y_2 + 6y_3$ for all $I \subseteq [10]$ with $|I| = 7$.

These are precisely the inequalities we mentioned in the motivating example at the beginning of this section.

Finally, the only two nontrivial facet-representing inequalities of $P_{\text{KC}}(b)$ for $b \in \{9, 10\}$ are (notice $P_{\text{KC}}(9) = P_{\text{KC}}(10)$)

$$\begin{aligned} y_1 + 2y_2 + 3y_3 &\geq 3 \\ y_1 + y_2 + 2y_3 &\geq 2. \end{aligned}$$

(The second one can be obtained from Theorem 1.) For the first inequality, we compute

$$\begin{aligned} P_{CC}(9) &\implies \rho^* = 1 \\ P_{CC}(10) &\implies \rho^* = 2, \end{aligned}$$

which leads to the facet-representing inequalities

- $\sum_{i \in I} x_i \leq 6 + y_1 + 2y_2 + 3y_3$ for all $I \subseteq [10]$ with $|I| = 9$; and
- $\sum_{i=1}^{10} x_i \leq 4 + 2y_1 + 4y_2 + 6y_3$.

For the second inequality, we compute

$$\begin{aligned} P_{CC}(9) &\implies \rho^* = 1 \\ P_{CC}(10) &\implies \rho^* = 2, \end{aligned}$$

which leads to the facet-representing inequalities

- $\sum_{i \in I} x_i \leq 7 + y_1 + y_2 + 2y_3$ for all $I \subseteq [10]$ with $|I| = 9$; and
- $\sum_{i=1}^{10} x_i \leq 6 + 2y_1 + 2y_2 + 4y_3$.

Notice that all the bulleted inequalities meet the conditions of Theorems 3 and 4. In total, we have generated $\binom{10}{5} + \binom{10}{6} + \binom{10}{7} + 2 \left(\binom{10}{9} + \binom{10}{10} \right) = 604$ facet-representing AKC inequalities which are not CR inequalities. ■

The example illustrates that some AKC inequalities could be CR inequalities. Clearly, though, not all CR inequalities are AKC inequalities. For example, consider an $|I| = 7$ CR inequality $\sum_{i \in I} x_i \leq 4y_1 + 7y_2 + 7y_3$. It is not possible to write it in the form

$$\sum_{i \in I} x_i \leq 7 + \rho(y_1 + 2y_2 + 2y_3 - 2)$$

for any ρ .

We used the software PORTA [7] to determine the facet-representing inequalities for the IKC polyhedra used in the examples. As long as the number t of capacity types is relatively small, this could be a viable off-line preprocessing step in a cutting plane or branch-and-cut algorithm. As mentioned in Section 1, if the coefficients C_k exhibit the divisibility property then the results of Pochet & Wolsey [17] may be used to obtain all relevant IKC facets.

Regarding the separation problem for the AKC facets, it can be solved quickly *provided* we know the appropriate facet-defining inequalities for the related IKC polyhedra, and there aren't too many of them. That is, for each value of $|I|$ from 1 to m , we need to find the nontrivial facets of $P_{\text{KC}}(|I|)$, find each corresponding ρ^* value, and then check to see whether a violation occurs (using a similar approach to the separation problem for the CR inequalities given earlier). This hinges on having a "black box" which can generate IKC facets. In many applications, either PORTA or the work of Pochet & Wolsey can serve as such a device.

5 Conclusions

We have provided two families of facets for the integer capacity-cover polyhedron, which arises as a subproblem in integer programs for telecommunications and facility location problems. Although the augmented knapsack-cover facets depend on knowledge of the integer knapsack-cover polyhedron, of which little is available, this is not a restriction in many practical situations.

Much remains to be known about the structure of the IKC polyhedron. The work of Pochet & Wolsey [17] provides a starting point, and Mazur [13] has derived some properties of facet-representing inequalities beyond those listed in Section 2.

An interesting open question is whether the CR and AKC facets, together with the trivial facets, give a complete description of the ICC polyhedron. In our computational investigations with PORTA, we have not encountered any facets other than those three types. Although the instances we investigated were relatively small in size ($m \leq 11$ and $t \leq 3$; the time requirement is prohibitive on larger instances), we attempted to mimic the relationships, that might occur in larger instances, between the relative sizes of the capacities and the number of customers.

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