# Optimal portfolios using Linear Programming models

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## Abstract

The classical Quadratic Programming formulation of the well known portfolio selection problem, is cumbersome, time consuming and relies on two important assumptions: (a) the expected return is multivariate normally distributed; (b) the investor is risk averter. This paper formulates two alternative models, (i) maximin, and (ii) minimization of absolute deviation. Data from a very simple problem, consisting of five securities over twelve months, is used, to examine if these various formulations provide similar portfolios or not. As expected, the *maximin* formulation has the highest return and risk, while the $min\ s$ (quadratic programming) has the lowest risk and return, with the $min\ |s|$ formulation being closed to $min\ s$ formulation.

**Keywords:** Linear Programming; optimal portfolios; return and risk

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## Introduction

Expected return and risk are the most important parameters with regard to optimal portfolios. Two well-known approaches to formulate optimal portfolios are (i) risk minimization, given some minimum return, and (ii) return maximization, given a maximum risk investors wish to tolerate. These formulations do not necessarily lead to efficient portfolios. One might find other efficient portfolios that yield higher expected return for the same risk, or lower risk for the same expected return.

Harry Markowitz[1] was the first to apply variance or standard deviation as a measure of risk. His classical formulation is the following:

$$\textbf{min} \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} x_i x_j$$

s.t. the following conditions:

$$\sum_{j=1}^{n} \overline{r_j} x_j \geq a * B, \qquad (1)$$

$$\sum_{j=1}^{n} x_j = B, \qquad (2)$$

$$0 \leq x_j \leq u_j, \quad j = 1, ..., n. \qquad (3)$$

with $i$ and $j$ securities, over $T$ periods;

$s_{ij} = \frac{1}{T} \sum_{t=1}^{T} (x_{it} - \overline{r_i})(x_{jt} - \overline{r_j})$, is the covariance of these securities;

$x_{jt}$ = per krona return, invested in security $j$ over period $t$;

$\overline{r_j}$ = average return in security $j$ over the entire period $T$;

$x_j$ = portfolio allocation of security $j$, that should not be larger than an upper bound $u$;

$a$ = the minimum return demanded by a particular investor;

$B$ = total budget that is invested in portfolio;

This classical model is always valid given two important assumptions: (a) the expected return is multivariate normally distributed; (b) the investor is risk averter and prefers lower risk.

Notice that constraint (3) does not allow short selling of securities. If that constraint is excluded and short selling is allowed, a different solution will be obtained, by using perhaps all securities, either with positive or negative weights.

However, the following simple example shows that the minimum variance portfolio might be inefficient.

Portfolio A, over a specific period, yields a return of either 8 % (even months) or of 16 % (odd months). Portfolio B on the other hand, for the same period, yields returns of either 7 % or 8 %. Given $a$ = 7 %, this formulation would select only portfolio B, because it has lower variance, despite the fact that A outperforms B in all months! Portfolio A should be selected, only if $a$ was high enough to eliminate B's lower return.

Another problem with the classical formulation is its complexity. Since the objective function is non-linear (often quadratic), it is very hard to find optimal solutions when the number of securities is large. For instance, for 300 securities, one needs to calculate a variance-covariance matrix of (n*(n+1))/2 = 44,850 combinations.

Not only are these calculations cumbersome, but even the implementation of the (hopefully) optimal solution is hard. In reality, one is satisfied with local minima, or sub optimal solutions. If for instance the number of securities is larger than 500, there might be up to 200 of them that take a positive value. That forces the investor to allocate a part of his budget into a large number of small blocks of shares. Given the transaction costs, it might be unprofitable to split the budget into many small blocks of shares. If we reformulate the problem as integer (for instance a minimum block of 100 shares), the difficulty increases dramatically. Experts argue that Integer Quadratic Programming with more than 50 securities, might be very difficult to solve!

In the following sections we will present the simple problem and formulate it as (a) Quadratic Programming, (b) maximin, (c) minimization of absolute deviation. The three portfolios will then be compared with various utility functions and with out of sample data.

## The problem

The portfolio manager Sigma wants to construct an optimal portfolio for a customer. Instead of considering a large number of potential shares to be included in the portfolio, let us consider only five OMX-shares from the Stockholm stock exchange: (A, B, C, D, and E). The models developed are of course very flexible and could include as many shares as one wishes. To simplify calculations of our parameters and speed up the solution though, we stick with these five shares.

Based on historical data over the last 12 months, (see Appendix, Table A1) the variance-covariance matrix is depicted in the following tables. Notice that the variance-covariance matrix is already multiplied by two.

**Table 1** Variance-covariance matrix over these 5-shares´ returns

| | $\sigma_A$ | $\sigma_B$ | $\sigma_C$ | $\sigma_D$ | $\sigma_E$ |
|---|---|---|---|---|---|
| $\sigma_A$ | 0.00146739 | 0.00026179 | 0.00052103 | 0.00017382 | 0.00043272 |
| $\sigma_B$ | | 0.00111640 | -0.00001133 | 0.00059010 | 0.00048699 |
| $\sigma_C$ | | | 0.00105790 | -0.00002990 | -0.00017064 |
| $\sigma_D$ | | | | 0.00108543 | 0.00055264 |
| $\sigma_E$ | | | | | 0.00100371 |

The customer plans to invest at most 100,000 SEK, demands a monthly return of at least 3 %, (or 3,000 SEK) and wishes that no share will receive more than 75 %, of his budget (that is at most 75,000 SEK).

Sigma's problem is to minimize his customer's risk, by neglecting the risk-free interest rate, and not allowing short selling.

We start by formulating it as a quadratic programming.

**(a) Quadratic Programming**

The objective function is to minimize this variance-covariance matrix, i.e.:

$$\textbf{min } 0.00146739\, x_A^2 + 0.0011164\, x_B^2 + 0.0010579\, x_C^2 + 0.00108543\, x_D^2 + 0.00100371 x_E^2 + 0.00026179 x_A x_B + \ldots + 0.00055264\, x_D\, x_E$$

s.t. the budget constraint,

$$x_A + x_B + x_C + x_D + x_E = 1 \qquad (1)$$

the return demand constraint,

$$0.0207\, x_A + 0.0316\, x_B + 0.0323\, x_C + 0.0337\, x_D + 0.0376\, x_E \geq 0.03 \qquad (2)$$

plus, the lower and upper limits for all shares,

$$0 \leq x_A , x_B , x_C, x_D , x_E \leq 0.75 \qquad (3)$$

A quadratic programming algorithm, such as QSB+[2] in its third edition, provides the following solution.

$$x_B = 0.16416,\ x_C = 0.33148,\ x_D = 0.18841,\ x_E = 0.31595,$$
$$E(r) = 3.412\ \%,\ s = 1.847\%$$

**(b) Maximin formulation**

The standard return/risk formulation, as above, regards variance as a measure of risk volatility. There are many researchers (and traders as well) who question if $s$ is an appropriate measure of risk. They assume that the normal investor's view regarding risk, is far away from symmetry or normal distribution. Very often, a small loss is enough to make one very sad. On the other hand, one gets very happy if the profit is considerable. This implies that the Markowitz classical model should be considered as an approximation to rather complex problems that every investor meats.

An alternative formulation is to maximize the minimum return demanded by the investor. According to Young[3] such a formulation, based on monthly returns on the stock indices from 8 countries, from January 1991 until December 1995, as well as from a simulation study, performs similarly with the classical Markowitz model.

In addition, Young argues that, for certain distributions, for instance when data is log-normally distributed, or skewed, the maximin formulation might be more appropriate method, compared to the classical minimization of variance, which is optimal for normally distributed data. The maximin formulation might also be preferable, if the portfolio optimisation problem involves a large number of decision variables, including integer variables, or if the utility function is more risk averse than it is implied by the classical minimization of variance.

To formulate the problem as a maximin, we use the same 5 variables, $x_A$, $x_B$, $x_C$, $x_D$, $x_E$. In addition, we define the minimum return from the optimal portfolio as, $Z \geq 0$. Notice that $Z$ is not decided ex ante, but it will be a part of the optimal solution and might differ from the customer's explicit return demand of 3 %. The objective function is then to maximize that minimum return, i.e.

**max** $Z$

Regarding the constraints, it is required that every month's return will be at least equal to Z. For January for instance, this constraint can be formulated as:

$$0.054\, x_A + 0.032\, x_B + 0.064\, x_C + 0.038\, x_D + 0.049\, x_E - Z \geq 0 \qquad (1)$$

Because the objective function is to maximize $Z$, this constraint will be $\geq 0$. only if $0.054\ x_A + 0.032\ x_B + 0.064\ x_C + 0.038\ x_D + 0.049\ x_E \geq Z$. This is obviously possible if there were not a budget constraint. It is easy to show that it is also possible even if the budget constraint is included.

Similarly, we can formulate for all other months, i.e.,

$$0.045\, x_A + 0.055\, x_B + 0.056\, x_C + 0.062\, x_D + 0.067\, x_E - Z \geq 0 \qquad (2)$$

... ... ... ... ... ... ... ... ...

$$0.052\, x_A - 0.017\, x_B + 0.032\, x_C + 0.025\, x_D + 0.040\, x_E - Z \geq 0 \qquad (12)$$

The main point with such a formulation is that all these constraints capture the interesting “downside” risk in portfolios volatility. We simply do not allow this “downside” risk to be below $Z$. “Upside” risk on the other hand, is not of interest and is free to vary.

In addition, we formulate the budget constraint as:

$$x_A + x_B + x_C + x_D + x_E \leq 100{,}000 \qquad (13)$$

Similarly, as before, the return demand constraint as:

$$0.020\, x_A + 0.0316\, x_B + 0.0323\, x_C + 0.0337\, x_D + 0.0376\, x_E \geq 3{,}000 \qquad (14)$$

And finally, as before, the lower and upper bounds as:

$$0 \leq x_A, x_B, x_C, x_D, x_E \leq 75{,}000 \qquad (15)$$

The optimal solution to this formulation is: $Z = 98.5$, $x_C = 45{,}959.6$, $x_E = 54{,}040.4$, and a slack for constraint (14) of *516.4*, i.e., this portfolio yields a monthly return of *0.5164* % more than the customer’s minimum demand.

Although this formulation is very simple, it leads to a rather satisfactory portfolio. This formulation would lead to an infeasible solution though, if all shares during a certain month yielded a negative return. If for example during the month of March, even share *C* had a negative return, precisely as all others, there would not exist any portfolio with positive shares. One possibility to achieve a feasible solution would be to allow $Z$ to have negative values as well. Such a modification creates some other problems. For instance, if the March return of share C was –0.03, the portfolio changes significantly, and the new solution is:

$Z = -3{,}155$, $x_A = 28{,}321$, $x_B = 25{,}929$, $x_C = 45{,}750$, *with* zero slack in constraint (14).

## (c) Absolute deviation minimization

Portfolio analysts and researchers have always regarded $s$ as the main risk measure. Another alternative to simplify the Markowitz classic formulation is to use the absolute deviation as a risk measure. Konno & Yamazaki[4] and later Speranza[5] Mansini & Speranza[6] and Rudolf, Wolter & Zimmermann[7] formulated similar problems.

According to Konno & Yamazaki, if the return is multivariate normally distributed, the minimization of the absolute deviation provides similar results with the classical Markowitz formulation. Also, according to Rudolf, Wolter & Zimmermann, minimization of the absolute deviation, or of the absolute downside deviation, is equivalent to expected utility maximization under risk aversion. Mansini & Speranza developed a mixed integer LP algorithm to solve a relatively small problem (with less than 20 securities), they formulated it as a general mean semi-absolute deviation, and applied to the Milan Stock Exchange.

As known, the (average) absolute deviation, is defined as:

$$|s| = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (x_{it} - \overline{r_i}) x_i \right| .$$

This function replaces the Markowitz variance-covariance objective function and will be minimized.

Since this objective function is not linear, we must first linearize it. We are going to follow Konno & Yamazaki in the transformation procedure, using of course the same example of 5 shares over 12 months.

We calculate first the absolute deviation for every share. All calculations are presented in Appendix, Table A2. Thereafter we linearize the objective function and simultaneously formulate our model.

### *Formulation*

We define first 12 $Y_t \geq 0$ variables, $t = 1,2, \ldots , 12$, i.e. one for every month. These $Y_t$ variables can be interpreted as linear mappings of the non-linear $\left| \sum_{i=1}^{n} (x_{it} - \overline{r_i}) x_i \right|$. We define then 5 variables, $A, B, C, D,$ *and* $E$, i.e. one for every share.

Thus, the objective function is to minimize the average absolute deviation, i.e.:

$$\textbf{min} = \frac{1}{12}\{Y_1 + Y_2 + \ \dots \ + Y_{12}\}$$

Regarding the budget constraint, we formulate it as equality, i.e.:

$$A + B + C + D + E = 100{,}000 \qquad (1)$$

The return demand constraint on the other hand, is formulated as before:

$$0.020\,A + 0.0316B + 0.0323C + 0.0337D + 0.0376E \geq 3{,}000 \qquad (2)$$

It is now time to relate the objective function's variables $Y_t$, with those appeared in other constraints $A, B, C, D, and\ E$.

Consider the first month. Because $Y_t$ is equal to $\left|\sum_{i=1}^{n}(x_{it} - \overline{r_i})x_i\right|$, $Y_1$ can be either, at least equal to $-\sum_{i=1}^{5}(x_i - \overline{r_i})x_i$ , or at least equal to $+\sum_{i=1}^{5}(x_i - \overline{r_i})x_i$ . This can be formulated as:

$$Y_1 \geq -\{0.0333A + 0.0004B + 0.0083C + 0.0043D + 0.0114E\} \Rightarrow$$
$$Y_1 + 0.0333A + 0.0004B + 0.0083C + 0.0043D + 0.0114E \geq 0 \qquad (3)$$

and,

$$Y_1 \geq 0.0333A + 0.0004B + 0.0083C + 0.0043D + 0.0114E \Rightarrow$$
$$Y_1 - 0.0333A - 0.0004B - 0.0083C - 0.0043D - 0.0114E \geq 0 \qquad (4)$$

Due to the fact that all five deviations for this month are positive, it is not possible for $Y_1$ to be zero, if constraint (4) is valid. Constraints (1) and (2) will ensure that, and at least one share should be purchased. Therefore, $Y_1$ is going to be positive.

Obviously, $Y_1$ would be positive too, even if all five deviations for that month were negative. That would be ensured by constraint (3) instead, together with constraints (1) and (2). It must be stressed however, that it is not necessary that all $Y_t$ must be positive. They will be positive if all deviations for the same month are either positive or negative. Some of them can be equal to zero, in case of mixed plus and minus signs for the same month. As we will see in the optimal solution below, there are three months (April, June and August) which lack absolute deviations, because the respective $Y_4 = Y_6 = Y_8 = 0$.

We continue with all remaining months:

$$Y_2 + 0.0243A + 0.0234B + 0.0203C + 0.0283D + 0.0294E \geq 0 \qquad (5)$$
$$Y_2 - 0.0243A - 0.0234B - 0.0203C - 0.0283D - 0.0294E \geq 0 \qquad (6)$$

.....
.....
$$Y_{12} + 0.0313A - 0.0486B - 0.0037C - 0.0087D + 0.0024E \geq 0 \quad (25)$$
$$Y_{12} - 0.0313A + 0.0486B + 0.0037C + 0.0087D - 0.0024E \geq 0 \quad (26)$$

Finally, we need lower and upper limits for all shares as before:

$$0 \leq A, B, C, D, E \leq 75{,}000 \quad (27)$$

The optimal solution to this formulation is: *B = 8,568.9, C = 36,295.2, D=1,600.2 E = 53,535.7, min = 1,286.7,* and slack for the return demand constraint, almost *0.510 %*.

This formulation has many advantages. For instance, we did not need to estimate the variance-covariance matrix and the optimal solution of this model is faster compared to the non-linear formulation. In fact it is $t$ that decides the number of constraints, no matter if the number of shares is very large. The number of constraints is simply $2t + 2$, if we count the budget and the return demand constraints. The optimal solution cannot include more than $2t + 2$ shares in portfolio, no matter if the number of shares is very large. One can therefore use $t$ as a control variable, if one wishes to limit the number of shares in portfolio. The solution exists always, even if all possible shares happen to yield a negative return during the same month. Finally, this model can easily reformulated as an Integer LP, to take into account fixed and variable costs or other decision variables.

**Risk aversion and choice of portfolios**

A standard method to determine which portfolios will be selected for different risk measures, is to assume various utility functions. Let us assume the following simple form that represents the investor's indifference curves:

$$U = E(r) - ws^2$$

where $U$ is the constant level of utility, and $w$ is a positive constant that indicates the investor's risk aversion. If $w = 0$, the investor is risk-neutral, because the utility level the specific portfolio provides is independent from its risk. If the value of $w$ approaches infinite, the investor will never invest in risky assets, and prefer the risk-free interest rate.

Instead of using variance, and therefore obtaining linear indifference curves upward sloping, one can use standard deviation and obtain standard parabolas that open upwards.

Let us now estimate the indifference levels provided by these portfolios, for various values of $w$. The following table summarizes the estimates, based on standard deviation.

It is clear that the highest the risk aversion the lower the indifference level for all three portfolios. As expected, the Markowitz classical formulation provides higher indifference levels when the risk aversion is high, while for rather low degrees of risk aversion, or risk neutrality, the portfolio obtained from the *min* $|s|$ formulation seems to provide higher indifference levels. To ensure that the *min* $|s|$ formulation might be a satisfactory method to linearize the quadratic objective function, more examples based on a larger data set are required.

**Table 2** Utility levels from these three portfolios

| ***w*** | ***min*** $s$ | ***Maximin*** | ***min*** $\|s\|$ |
|---|---|---|---|
| 0 | 3.412 | 3.516 | 3.510 |
| 0.1 | 3.2273 | 3.1718 | 3.2840 |
| 0.3 | 2.8579 | 2.4834 | 2.8320 |
| 0.5 | 2.4885 | 1.7950 | 2.3800 |
| 0.7 | 2.1191 | 1.1066 | 1.9280 |
| 0.9 | 1.7497 | 0.4182 | 1.4760 |
| 1 | 1.5650 | 0.0740 | 1.2500 |
| 1.5 | 0.6415 | -1.6470 | 0.1200 |

**Out of sample performance of portfolios**

Let us now examine the performance of these portfolios using additional data. We assume that these portfolios were constructed based on these 12 months and check how well they performed during the next 3- and 6-months. Real data for these shares are given in Appendix, Table A3.

The performance is summarized on Table 3. The third column shows the return and risk for these three models, based on the 12-months data, i.e. the optimal portfolios in these formulations, while the last two columns show the respective performance of these portfolios over the next 3- and 6-months. Both returns and risk are expressed in percentage units.

**Table 3** Optimal portfolios and performances

| **model** | **weights** | **12-months** | | **Next 3-months** | | **Next 6-months** | |
|---|---|---|---|---|---|---|---|
| | | ***E(r)*** | $s$ | ***E(r)*** | $s$ | ***E(r)*** | $s$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ***min*** $s$ | *B = 0.16416*<br>*C = 0.33148*<br>*D = 0.18841*<br>*E = 0.31595* | *3.412* | *1.847* | *-.302* | *.901* | *-.656* | *.745* |
| ***maximin*** | *C = 0.4596*<br>*E = 0.5404* | *3.516* | *2.286* | *.079* | *1.72* | *-.685* | *1.551* |
| ***min*** $\lvert s \rvert$ | *B = 0.08569*<br>*C = 0.36295*<br>*D = 0.01600*<br>*E = 0.53535* | *3.510* | *2.260* | *.043* | *1.49* | *-.651* | *1.345* |

Again, as expected, the *maximin* has the highest return, the *min* $s$ has the lowest risk and return, while the *min* $\lvert s \rvert$ lies somewhere between. This applies for all the periods. Notice however that the *min* $s$ portfolio performs worse during the next 3-months, compared to the other two portfolios. Similarly, despite the fact that the differences are marginal, the *min* $\lvert s \rvert$ portfolio provides the lowest losses during the next 6-months.

Before a final word is said, more examples, based on a larger data set are required.

**Conclusions**

Despite the fact that computers' capacity and speed has increased considerably over recent years, optimal solutions to very large quadratic problems might be hard to achieve, especially if one wants integer solutions as well. Alternative linear approximations, as those presented in this paper might provide satisfactory returns and risks, cheaper and faster. The example, although the data is real, was extremely simple to decide if these alternative models are to be strong candidates to the classical Markowitz model. On the other hand, these findings show that one can do much more with LP than many think of!

A possible way to go is to reformulate this problem as an Integer LP, for instance by taking into account fixed and variable costs associated with the purchase of shares. Very often, if the fixed costs are high, it is unprofitable to purchase small posts. Large ILP, although they are hard to solve, they are rather "easy" compared to Integer Quadratic problems of the Markowitz type. The other way to go is to use these simple LP models for larger data sets.

**Appendix**

**Table A1** Historic monthly returns for five selected shares over one year

| | $x_A$ | $x_B$ | $x_C$ | $x_D$ | $x_E$ |
|---|---|---|---|---|---|
| *Jan.* | *0.054* | *0.032* | *0.064* | *0.038* | *0.049* |

| | | | | | |
|---|---|---|---|---|---|
| *Feb.* | *0.045* | *0.055* | *0.056* | *0.062* | *0.067* |
| *March* | *-0.030* | *-0.036* | *0.048* | *-0.037* | *-0.039* |
| *April* | *-0.018* | *0.052* | *0.007* | *0.050* | *0.051* |
| *May* | *0.043* | *0.047* | *0.053* | *0.065* | *0.049* |
| *June* | *0.047* | *0.034* | *0.036* | *-0.043* | *0.037* |
| *July* | *0.055* | *0.063* | *0.017* | *0.062* | *0.055* |
| *Aug.* | *0.036* | *0.048* | *0.047* | *0.034* | *0.025* |
| *Sept.* | *-0.039* | *0.025* | *-0.059* | *0.035* | *0.052* |
| *Oct.* | *-0.043* | *0.040* | *0.047* | *0.056* | *0.020* |
| *Nov.* | *0.046* | *0.036* | *0.040* | *0.057* | *0.045* |
| *Dec.* | *0.052* | *-0.017* | *0.032* | *0.025* | *0.040* |
| *Average* | *0.0207* | *0.0316* | *0.0323* | *0.0337* | *0.0376* |

**Table A2** Absolute deviation per month

| ***A*** | ***B*** | ***C*** | ***D*** | ***E*** |
|---|---|---|---|---|
| *0.0333* | *0.0004* | *0.0083* | *0.0043* | *0.0114* |
| *0.0243* | *0.0234* | *0.0203* | *0.0283* | *0.0294* |
| *-0.0507* | *-0.0676* | *0.0123* | *-0.0707* | *-0.0766* |
| *-0.0387* | *0.0204* | *-0.0087* | *0.0163* | *0.0134* |
| *0.0223* | *0.0154* | *0.0173* | *0.0313* | *0.0114* |
| *0.0263* | *0.0024* | *0.0003* | *-0.0767* | *-0.0006* |
| *0.0343* | *0.0314* | *0.0013* | *0.0283* | *0.0174* |
| *0.0153* | *0.0164* | *0.0113* | *0.0003* | *-0.0126* |
| *-0.0597* | *-0.0066* | *-0.0747* | *0.0013* | *0.0144* |
| *-0.0637* | *0.0084* | *0.0113* | *0.0223* | *-0.0176* |
| *0.0253* | *0.0044* | *0.0043* | *0.0233* | *0.0074* |
| *0.0313* | *-0.0486* | *-0.0037* | *-0.0087* | *0.0024* |

**Table A3** Real returns over the next 6-months

| | $x_A$ | $x_B$ | $x_C$ | $x_D$ | $x_E$ |
|---|---|---|---|---|---|
| *J* | -0,028 | -0,023 | 0,018 | -0,021 | 0,013 |
| *F* | -0,011 | 0,018 | 0,009 | -0,034 | 0,007 |
| *M* | 0,016 | 0,011 | -0,016 | -0,011 | -0,024 |
| *A* | 0,023 | -0,026 | -0,021 | 0,018 | -0,018 |
| *M* | -0,033 | -0,015 | -0,011 | -0,023 | 0,005 |
| *J* | 0,019 | 0,021 | -0,019 | 0,015 | -0,025 |

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