

# A Spring-Embedding Approach for the Facility Layout Problem

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## Abstract

The facility layout problem is concerned with finding the most efficient arrangement of a given number of departments with unequal area requirements within a facility. The facility layout problem is a hard problem, and therefore, exact solution methods are only feasible for small or greatly restricted problems. In this paper, we propose a spring-embedding approach that unlike previous approaches results in a model that is convex. Numerical results demonstrating the potential of our model and the efficiency of our solution procedure are presented.

**Keywords:** facility layout; non-linear programming; optimisation; augmented Lagrangian multiplier method.

## 1 Introduction

The facility layout problem is concerned with finding the most efficient arrangement of  $N$  indivisible departments with unequal area requirements within a facility. As defined in the literature, such an arrangement is subject to two sets of constraints: (1) department and floor area requirements, and (2) department locational restrictions; that is, departments cannot overlap, must be placed within the facility, and some must be fixed to a location or cannot be placed in specific regions.

The facility layout problem has been studied extensively in the literature. For the most recent surveys the reader is referred to Kusiak and Heragu<sup>1</sup> and Meller and Gau.<sup>2</sup> We note that the facility layout problem is a hard problem. Even the version known as the Quadratic Assignment Problem, where the optimisation is carried over a set of possible department locations, is NP-hard and therefore, exact solution methods are only feasible for small or greatly restricted problems.<sup>3</sup> For this reason, most of the solution methodologies in the literature are based on heuristics.

In general, the facility layout problems found in the literature deal with the arrangement of rectangular departments. There are applications, however, in which an orthogonal arrangement of departments is not necessarily a requirement; for example, planning of an airplane's dashboard, a city or a neighbourhood, a layout of office buildings, and integrated circuits. This paper is aimed at modelling and solving facility layout problems for such applications.

In order to put our contribution into context, we begin by reviewing two previous methods that are related to our new approach: the DISCON (DISpersion-CONcentration) and AR (Attractor-Repeller) methods. The major shortcoming of these methods is that they provide facility layout models that are non-convex, which impacts the quality and, more importantly, the feasibility of the final layout solutions. In contrast, our proposed SE (Spring-Embedding) model results in a model that is convex, from which final solutions are optimal or near-optimal.

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## 1.1 The DISCON method

To describe the DISCON<sup>4,5</sup> (DISpersion-CONcentration) method, let us suppose that the departments are labelled  $v_i$ ,  $i = 1, \dots, N$ , where  $N$  is the total number of departments, and that:

- (1) Each department is a circle (or can be approximated by a circle) of given radius  $r_i$ ,  $i = 1, \dots, N$ .
- (2) The position of department  $v_i$  is expressed by the coordinates of its centre and is denoted by  $(x_i, y_i)$ ,  $i = 1, \dots, N$ .
- (3) The distance between two departments is measured as the  $l_2$  norm (Euclidean distance) between the centres of the departments; that is,

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad \forall 1 \leq i < j \leq N. \quad (1)$$

- (4) The non-negative material handling cost per unit distance between departments  $v_i$  and  $v_j$  is given and denoted by  $c_{ij}$ , where  $c_{ij} = c_{ji}$  and  $c_{ii} = 0$  (which we also assume throughout the paper).

The DISCON method uses a formulation equivalent to:

$$\text{DISCON:} \quad \min \sum_{i=1}^{N-1} \sum_{j=i+1}^N c_{ij} d_{ij} \quad (2)$$

$$\text{s.t.} \quad r_i + r_j - d_{ij} \leq 0, \quad \forall 1 \leq i < j \leq N, \quad (3)$$

where the optimisation is carried over  $x_i$  and  $y_i$ ,  $i = 1, \dots, N$ . The objective function (2) minimises the total material handling cost inside a facility. The constraints (3) require that the areas for each pair of departments do not overlap.

Drezner<sup>4</sup> solves the DISCON model using a penalty algorithm and a two-phase approach as follows:

- Phase 1 (DISpersion phase): all the departments are dispersed far from the origin providing a starting point for the next phase.
- Phase 2 (CONcentration phase): all the departments are concentrated and the resulting arrangement is the final solution.

A significant difficulty with the DISCON method is that the overlapping constraints (3) make the feasible set of the facility layout model non-convex. Furthermore, the model does not require all departments to be placed within the facility nor require the floor area of the facility to satisfy dimensional requirements.

## 1.2 The AR method

In contrast to the DISCON model, the AR<sup>6</sup> (Attractor-Repeller) model employs the concept of target distances in order to generate a final solution that attempts to satisfy department locational restrictions. Let the target distance between two departments be denoted by the parameters

$$t_{ij} = \alpha(r_i + r_j), \quad \forall 1 \leq i < j \leq N, \quad (4)$$

where  $\alpha > 0$ . The parameters  $t_{ij}$  attempt to enforce the separation of the departments within the facility. The idea is that the value  $t_{ij}$  is the target distance between departments  $v_i$  and  $v_j$ . The parameter  $\alpha$  is introduced to provide control over department area overlapping and some flexibility as to how tightly the user would like to enforce department separation. Presumably, the AR model will ensure that  $d_{ij} = t_{ij}$  at optimality, so choosing  $\alpha < 1$  sets a target value  $t_{ij}$  that allows some overlap of the areas of the respective departments, which means that a relaxed version of the overlapping constraints (3) on the distances between departments is enforced. Similarly,  $\alpha = 1$  means that there should be no overlap and the department areas should intersect at exactly one point on their boundaries.

The target distances are enforced as follows: for each pair of departments  $v_i$  and  $v_j$ , the overlap with respect to the target distance  $t_{ij}$  is penalised by introducing the term  $f(d_{ij}/t_{ij})$  in the objective function of the AR model, where

$$f(z) = \frac{1}{z} - 1, z > 0. \quad (5)$$

The AR model is stated as follows:

$$\mathbf{AR:} \quad \min \quad \sum_{i=1}^{N-1} \sum_{j=i+1}^N c_{ij} d_{ij} + \kappa \sum_{i=1}^{N-1} \sum_{j=i+1}^N f\left(\frac{d_{ij}}{t_{ij}}\right) \quad (6)$$

$$\text{s.t.} \quad x_i + r_i - \frac{1}{2}w \leq 0, \quad i = 1, \dots, N, \quad (7)$$

$$r_i - x_i - \frac{1}{2}w \leq 0, \quad i = 1, \dots, N, \quad (8)$$

$$y_i + r_i - \frac{1}{2}h \leq 0, \quad i = 1, \dots, N, \quad (9)$$

$$r_i - y_i - \frac{1}{2}h \leq 0, \quad i = 1, \dots, N, \quad (10)$$

$$w^{\text{low}} \leq w \leq w^{\text{up}}, \quad (11)$$

$$h^{\text{low}} \leq h \leq h^{\text{up}}, \quad (12)$$

where  $\kappa > 0$ ,  $w$  and  $h$  are the width and height of the floor area of the facility,  $w^{\text{low}}$  and  $w^{\text{up}}$  are the floor area width requirements with  $w^{\text{low}} \leq w^{\text{up}}$ , and  $h^{\text{low}}$  and  $h^{\text{up}}$  are the floor area height requirements with  $h^{\text{low}} \leq h^{\text{up}}$ .

Recall that the DISCON model does not require all departments to be placed within the facility nor require the floor area of the facility to satisfy dimensional requirements. In the AR model, floor area requirements and department locational restrictions are enforced via the constraints (7)-(10), which require that all departments are placed within the facility, and the constraints (11) and (12), which ensure that floor area dimensions satisfy the given height and width requirements. Facility layout models that are concerned with finding orthogonal arrangements with constraints similar to the ones used in the AR model have been developed by others, among them Heragu and Kusiak<sup>7</sup> and van Camp et al.<sup>8</sup>

Although all the constraints are linear (and hence convex), the AR model is not a convex problem because the objective function is not convex. Anjos and Vannelli<sup>6</sup> solve the AR model using the optimisation package MINOS<sup>9</sup> accessed via the modelling language GAMS. The authors choose  $\alpha$  empirically in such a way that a reasonable separation between all pairs of departments is achieved. Surprisingly, for three of the four problem instances presented, the authors find that setting  $\alpha = 1$  results in excessive separation of the departments, which challenges the optimality criterion stated above. In order to obtain pleasing facility layouts, the authors are required to reduce  $\alpha$  to 0.005. These facility layouts, however, contained some overlap of the department areas; thus, providing solutions that are infeasible with respect to the department locational restrictions. Clearly, the results presented by Anjos and Vannelli<sup>6</sup> show that the choice of the parameter  $\alpha$  in the AR model has a significant influence on the final solution. Hence, the question of how to choose  $\alpha$  appropriately deserves further study.

### 1.3 Contribution of this paper

Our contribution is the Spring-Embedding (SE) model: a convex programming model which is designed to improve upon the DISCON and AR models. The spring-embedding concept is based on a dynamic system in which  $N$  particles of a given area are mutually connected by springs. Assume that each of the springs connecting pairs of the  $N$  particles have differing strengths. From an initial arrangement of the particle locations, the system oscillates until it stabilises at a minimum-energy arrangement resulting in a dynamically balanced system. Substituting the particles for the  $N$  departments and replacing the strength of the springs by the material handling costs per unit distance between departments  $c_{ij}$ , we have our SE approach to modelling the facility layout problem.

This paper is organised as follows: in Section 2, we motivate and derive the SE model. We present an augmented Lagrangian multiplier method for constrained non-linear optimisation in Section 3 that is implemented in order to efficiently solve the SE model. Numerical results demonstrating the potential of our SE model and

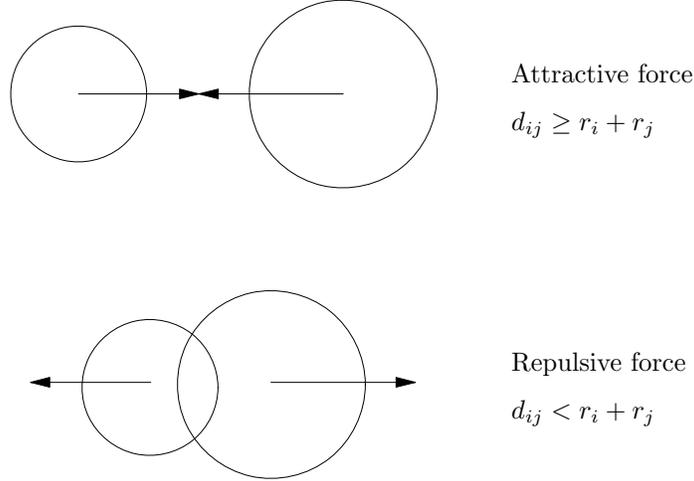


Figure 1: Repulsive and attractive forces between particles.

solution procedure are provided in Section 4. Finally, we present our conclusions in Section 5 and discuss a possible direction for future research.

## 2 Model Description

### 2.1 The spring-embedding concept

Let  $p_i$ ,  $i = 1, \dots, N$ , be the particles on a plane corresponding to departments  $v_i$ ,  $i = 1, \dots, N$ , respectively. Inspired by the work of Kamada and Kawai<sup>10</sup> on graph drawing, we relate the arrangement of departments within a facility to a dynamically balanced spring system. As a result, the degree of imbalance inside a facility can be formulated as the total energy of springs; that is,

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{1}{2} c_{ij} \|p_i - p_j\|_2^2, \quad (13)$$

where  $\|\cdot\|_2^2$  represents the square of the  $l_2$  norm between the centres of the particles and the non-negative material handling cost per unit distance between departments  $c_{ij}$  represents the strength of the spring between the particles.

Note that if the constraints (3) are removed from the DISCON model, the optimal solution is simply to place all the departments with their centres at the same point since  $c_{ij} \geq 0$ ,  $\forall 1 \leq i < j \leq N$ . This happens since the objective function (2) seeks to make the distances between departments  $d_{ij}$  as small as possible by attracting all pairs of departments to each other. The same holds for the total energy function (13). To prevent area overlapping and in order to satisfy particle (department) locational restrictions, we enforce the constraints (3) by introducing a penalty function to the total energy of springs. The non-overlapping condition is enforced as follows: for each pair of particles  $p_i$  and  $p_j$ , if their areas overlap (i.e.,  $d_{ij} < r_i + r_j$ ), the total energy function is penalised by introducing the term  $K_{ij}(r_i + r_j - d_{ij})$ ,  $K_{ij} > 0$ ,  $\forall 1 \leq i < j \leq N$ . That is, the penalty function will assume a positive value proportional to the magnitude of the area overlap of the particles resulting in a repulsive force. On the other hand, if there is no overlap (i.e.,  $d_{ij} \geq r_i + r_j$ ), the total energy function is not penalised, thus creating an attractive force (see Figure 1).

Therefore, the spring-embedding total energy function can be stated as follows:

$$E = \sum_i \sum_{j>i} \frac{1}{2} c_{ij} \|p_i - p_j\|_2^2 + \sum_i \sum_{j>i} \max\{0, K_{ij}(r_i + r_j - d_{ij})\}, \quad (14)$$

where  $K_{ij} > 0, \forall 1 \leq i < j \leq N$ . Optimal or near-optimal facility layouts can be obtained by decreasing  $E$ , where the most efficient arrangement of departments within a facility is represented by the state of the particles with minimum  $E$ .

## 2.2 The spring-embedding model

Let  $(x_i, y_i), i = 1, \dots, N$ , be the coordinates of departments  $v_i, i = 1, \dots, N$ , respectively. The ideas above yield the SE (Spring-Embedding) model:

$$\begin{aligned} \text{SE: } \min \quad & \sum_i \sum_{j>i} \frac{1}{2} c_{ij} d_{ij}^2 + \sum_i \sum_{j>i} \max\{0, K_{ij}(r_i + r_j - d_{ij})\} \\ \text{s.t. } \quad & (7) - (12), \end{aligned} \tag{15}$$

where the optimisation is carried over  $x_i$  and  $y_i, i = 1, \dots, N$ , and  $K_{ij} > 0, \forall 1 \leq i < j \leq N$ . The objective function (15) represents the spring-embedding total energy function and minimises the degree of imbalance inside a facility. In order to preserve the spring-embedding concept, we work with the squares of the Euclidean distances between each pair of departments. It is clear that this, as well as the use of the coefficient  $1/2$ , does not change the optimal facility layout.<sup>11</sup> Thus, the economical interpretation of the final facility layout solution is that of minimising the total material handling cost inside a facility.

Note that the ability to specify bounds on the desired dimensions of the floor area of the facility, which is available in the AR model, is also available in the SE model in the sense that it allows the user to specify bounds  $w^{\text{low}} \leq w^{\text{up}}$  on the width and  $h^{\text{low}} \leq h^{\text{up}}$  on the height of the floor area. In particular, if the actual width  $\bar{w}$  and height  $\bar{h}$  of the floor area are known in advance, such dimensions can be enforced by setting  $w^{\text{low}} = w^{\text{up}} = \bar{w}$  and  $h^{\text{low}} = h^{\text{up}} = \bar{h}$ . Furthermore, under the assumption that  $c_{ij} > 0$  for all pairs of departments  $v_i, v_j, \forall 1 \leq i < j \leq N$ , we can show that the SE model yields a convex problem.

In the SE model, all the constraints are linear, hence forming a non-empty convex feasible set. We next focus our attention on the objective function (15). The following two lemmas are well-known in the operational research literature.

**Lemma 1** *If  $f(x), g(x)$  are convex functions and  $c, d \geq 0$  are scalars, then the function  $c \cdot f(x) + d \cdot g(x)$  is convex.*

**Lemma 2** *If  $f(x), g(x)$  are convex functions, then the function  $\max\{f(x), g(x)\}$  is convex.*

**Proposition 1** *Objective function (15) is convex.*

**Proof.** Consider the first part of (15): since  $d_{ij}^2$  is convex (in fact, quadratic), the function  $c_{ij}d_{ij}^2, c_{ij} > 0$ , is convex due to Lemma 1. Now consider the second part of (15): since  $r_i + r_j - d_{ij}$  is convex, the function  $K_{ij}(r_i + r_j - d_{ij}), K_{ij} \geq 0$ , is convex due to Lemma 1. Moreover, the function  $\max\{0, K_{ij}(r_i + r_j - d_{ij})\}$  is also convex due to Lemma 2. Since the sum of convex functions is convex, objective function (15) is convex.  $\square$

Figure 2 illustrates the objective function (15) for various values of  $c_{ij}$ . Note that we set  $r_i + r_j = 1$  and  $K_{ij} = c_{ij}(r_i + r_j)$ . The figure was generated by taking random vectors  $\mathbf{x}, \mathbf{y}, \Delta\mathbf{x}$ , and  $\Delta\mathbf{y}$  and using  $\lambda$  to parameterise the change in the objective function; that is, for each step  $\lambda, \lambda = 0, 0.005, 0.010, \dots, 2, d = \|x - y + \lambda(\Delta x - \Delta y)\|_2$ .

We conclude this section by noting that the SE model can be generalised to distinguish between two kinds of departments: those that are mobile and those that are fixed to a location within a facility (e.g., heavy machinery, elevator shafts, etc.). This can be accomplished by simply fixing the coordinates of the centres of the fixed departments *a priori*. Another feature is that by fixing the location of some departments, the SE model allows for a multiple-round strategy for solving large-scale problems. That is, in each round, the model is solved and some departments are chosen to be fixed *a priori* before re-solving in the next round. The reader is referred to Kamada and Kawai<sup>10</sup> and Rowe et al.<sup>12</sup> for details of such possible strategy using the Newton-Raphson method on unconstrained models for graph drawing. In addition, locational (linear) constraints can be added to the model to forbid departments to be placed in specific regions within the facility.

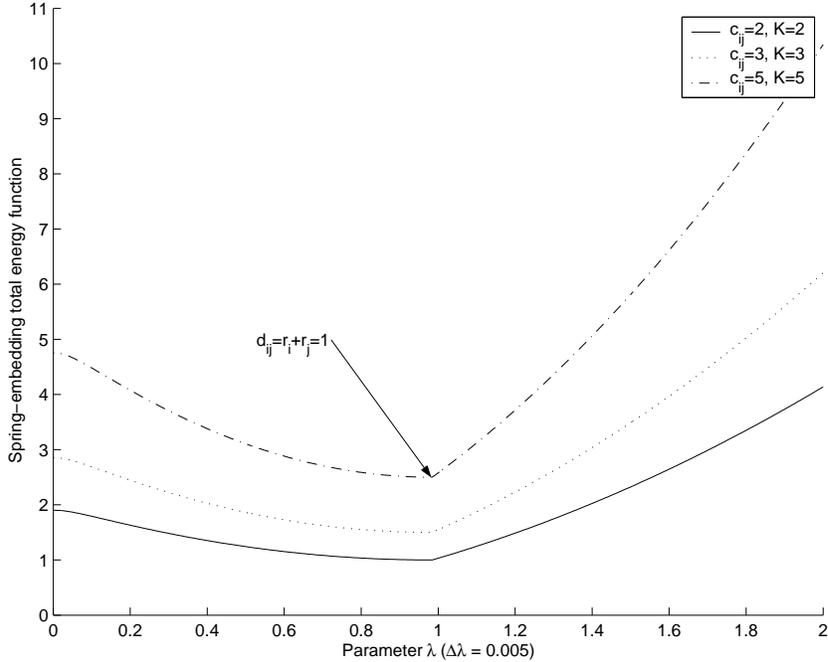


Figure 2: Graph of the objective function (15) for several values of  $c_{ij}$ .

### 3 Solution Procedure

The SE model is transformed from a constrained problem into an unconstrained problem by an augmented Lagrangian multiplier (ALM) method. The transformation is accomplished by augmenting the objective function with additional functions that reflect the violation of the constraints very significantly. These functions are referred to as penalty functions. The ALM method is the most robust of the penalty function methods. It assumes no prior knowledge regarding bounds on the optimal value and it manipulates an augmented objective function to induce convergence. More importantly, the method provides information on the Lagrange multipliers at the solution. This is achieved by not solving for the multipliers but merely updating them during successive ALM iterations.<sup>13</sup> The method also overcomes several difficulties associated with exterior penalty function methods without significant overhead. In particular, it avoids the need to make the penalty parameter infinitely large in a limiting sense to recover an optimal solution, which is known to cause numerical difficulties and ill-conditioning effects.<sup>14</sup>

The SE model has  $2N + 2$  variables and  $4N$  inequality constraints, all of which are linear. Only the objective function is non-linear. The ALM method exploits this structure by applying a gradient-based technique which handles simple bounds via projections that are easy to compute.

Let  $\mathbf{X} = (x_1, \dots, x_N, y_1, \dots, y_N, w, h)$  and  $M$  be the number of inequality constraints (i.e.,  $4N$ ). The standard format of our non-linear programming problem in vector notation is:

$$\text{NLP: } \min E(\mathbf{X}) \tag{16}$$

$$\text{s.t. } g_m(\mathbf{X}) \leq 0, \quad m = 1, \dots, M, \tag{17}$$

$$\mathbf{X}^{\text{low}} \leq \mathbf{X} \leq \mathbf{X}^{\text{up}}. \tag{18}$$

In the method of Lagrange multipliers, the NLP model is transformed as follows (see, for example, Bertekas<sup>13</sup>

for details of such a transformation):

$$\begin{aligned} \min \quad & F(\mathbf{X}, \boldsymbol{\beta}, \phi) : \\ & E(\mathbf{X}) + \phi \sum_{m=1}^M \left( \max \left[ g_m(\mathbf{X}), -\frac{\beta_m}{2\phi} \right] \right)^2 + \sum_{m=1}^M \beta_m \left( \max \left[ g_m(\mathbf{X}), \frac{\beta_m}{2\phi} \right] \right) \\ \text{s.t.} \quad & \mathbf{X}^{\text{low}} \leq \mathbf{X} \leq \mathbf{X}^{\text{up}}. \end{aligned} \quad (19)$$

$$(20)$$

Here  $\boldsymbol{\beta}$  is the multiplier vector associated to the inequality constraints and  $\phi$  is the penalty multiplier.  $F$  is solved as an unconstrained problem for predetermined values of  $\boldsymbol{\beta}$  and  $\phi$ . Therefore, the solution for each ALM iteration is  $\mathbf{X}^* = \mathbf{X}^*(\boldsymbol{\beta}, \phi)$ . At the end of each iteration, the values of the multiplier vector and penalty multiplier are updated. The latter is geometrically scaled but unlike exterior penalty function methods, it does not have to be driven to infinity to recover an optimal solution. We use the gradient-based method, originally due to Davidon,<sup>15</sup> to solve the unconstrained problem. Unlike other gradient-based methods, the aforementioned method benefits from the robustness provided by carrying information from the previous iterations.

Our algorithmic implementation of the ALM method is as follows:

**Step 1.** Let

- $I$  be the maximum number of ALM iterations;
- $q = 1$  be the ALM iteration counter;
- $\epsilon > 0$  be the termination criteria tolerance;
- $\mathbf{X}^1$  be a null vector;
- $\phi^1$  be the initial penalty multiplier;
- $\theta$  be the scaling value for the penalty multiplier; and
- $\boldsymbol{\beta}^1$  be the initial multiplier vector.

**Step 2.** Solve the unconstrained problem to minimise  $F(\mathbf{X}^q, \boldsymbol{\beta}^q, \phi^q)$  subject to  $\mathbf{X}^{\text{low}} \leq \mathbf{X} \leq \mathbf{X}^{\text{up}}$ , and let  $\mathbf{X}^{q*}$  denote the solution obtained using the gradient-based method due to Davidon.<sup>15</sup>

**Step 3.** Termination criteria for ALM:

- Compute  $\Delta F = F^q - F^{q-1}$  and  $\Delta \mathbf{X} = \mathbf{X}^{q*} - \mathbf{X}^{(q-1)*}$ ;
- If  $(\Delta F)^2 \leq \epsilon$  then stop: objective function not changing;
- Else If  $\Delta \mathbf{X}^T \Delta \mathbf{X} \leq \epsilon$  then stop: incumbent solution not changing;
- Else If  $q = I$  then stop: maximum number of ALM iterations reached;
- Continue: let
  - $\boldsymbol{\beta}^{q+1} = \boldsymbol{\beta}^q + 2\phi^q(\max[\mathbf{g}(\mathbf{X}^{q*}), -\boldsymbol{\beta}^q/2\phi^q])$ ;
  - $\phi^{q+1} = \theta\phi^q$ ;
  - $\mathbf{X}^{q+1} = \mathbf{X}^{q*}$ ;
  - $q = q + 1$ ; and
  - go to Step 2.

Note that the termination criteria contributes to the early stopping of the algorithm within, presumably, an acceptable percentage of optimality.

The augmented Lagrangian multiplier method described above was implemented in MATLAB. See Venkataraman<sup>16</sup> for more details of a possible MATLAB implementation. The implementation platform is a Windows-based PC with a Pentium III 933MHz processor and 512MB of RAM. For our numerical results, we chose  $I = 15$ ,  $\epsilon = 1 \times 10^{-16}$ ,  $\phi^1 = 1$ ,  $\theta = 10$ , and  $\beta_m^1 = 1$ ,  $m = 1, \dots, M$ .

## 4 Numerical Results

The SE model is based on a dynamic system in which springs of a given strength are replaced by the material handling costs per unit distance between departments and particles are replaced by departments of a given area. From the initial arrangement of departments, the system oscillates until it stabilises at a minimum-energy arrangement. The SE model is transformed from a constrained problem into an unconstrained problem by an ALM method. Since the ALM method is iterative, the user is required to supply an initial arrangement. It is not clear *a priori* what is the ‘best’ initial arrangement so we place the centres of the  $N$  departments at the origin. In order to trigger oscillation, we place two of the departments very close to the origin but not centred there. The department with  $\min_i\{\sum_j c_{ij}\}$  is placed at the left of the origin, while the department with  $\max_i\{\sum_j c_{ij}\}$  is placed at the right of the origin. Also, we set  $K_{ij} = c_{ij}(r_i + r_j)$ ,  $\forall 1 \leq i < j \leq N$ .

For the class of equal area problems, 6 data sets based on the material handling costs per unit distance between departments given in Nugent et al.<sup>17</sup> were used to study the SE model and the performance of our algorithmic implementation of the ALM method. For this class of problems, all departments were assumed to have an area of 1 square unit. The radii of the departments were chosen so that  $r_i = \sqrt{a_i}/2$ ,  $i = 1, \dots, N$ . Note that no floor area requirements or facility dimensions were imposed on these problems. The numerical results are summarised in Table 1.

The performance of our algorithmic implementation of the ALM method is compared to that of DISCON and AR. Since the DISCON, AR, and SE models are formulated with different objective functions, we used the final arrangement of departments and computed the material handling cost of the same arrangement using the  $l_1$  norm. The DISCON method applies the CRAFT exchange algorithm due to Armour and Buffa<sup>18</sup> ten times in order to improve the final arrangement of departments. However, the objective function values reported for the DISCON method in Table 1 correspond to the objective function values before the application of the CRAFT exchange algorithm. Thus, providing a more meaningful comparison of the SE model with DISCON. Note that the objective function values reported for DISCON are the average objective function values provided in Drezner<sup>5</sup> for the  $l_1$  norm.

The initial solutions for the AR method were generated as suggested by Anjos and Vannelli.<sup>6</sup> In our experiments, we used  $\kappa = 10 \sum_i \sum_{j>i} c_{ij}$  and values of 0.005, 0.01, 0.5, and 1 for the parameter  $\alpha$ . No attempt was made to optimise the parameter  $\alpha$ , and we simply used several of the values reported by Anjos and Vannelli.<sup>6</sup> The AR model was solved using MINOS<sup>9</sup> accessed via the NEOS<sup>19</sup> server. Since the final solution using the AR model is likely to contain some overlap of the department areas, thus providing an infeasible solution, we computed the percentage overlap (i.e., degree of infeasibility) of each layout solution as follows:

$$\% \text{ overlap} = \frac{\sum_{i=1}^{N-1} \sum_{j=i+1}^N \max\{0, r_i + r_j - d_{ij}\}}{\sum_{i=1}^{N-1} \sum_{j=i+1}^N (r_i + r_j)}. \quad (21)$$

In Table 1, we report the  $l_1$  cost and the  $\alpha$  values that provide the lowest degree of infeasibility. Thus, providing a more meaningful comparison of the SE model with AR as well.

From Table 1, the  $l_1$  cost values obtained from our SE method improve upon those from DISCON and AR for all 6 data sets. Furthermore, we note that our SE results for problems 2, 3, 5, and 6 are optimal.<sup>5</sup> In the DISCON method, multiple optimisation runs were carried out and only in a small fraction of these optimisation runs did the author obtain the optimal solutions, which are not known *a priori* and must be verified by other techniques. Although our SE approach has longer run-times, facility layout problems are generally planned on a long-term horizon. Therefore, it is more critical to obtain optimal or near-optimal solutions consistently albeit via a slower method such as our spring-embedding approach than to use faster methods that do not guarantee the quality nor the feasibility (in the case of the AR model) of the final solutions.

The number of ALM iterations executed by our algorithm for all equal area problems were either 10 or 11. Thus, termination criteria tolerances were reached before the maximum number of ALM iterations were reached. The minimum-energy arrangement for problem 6 is shown in Figure 3.

For the class of unequal area problems, 12 data sets based on the material handling costs per unit distance between departments given in Nugent et al.<sup>17</sup> were used. In contrast to the class of equal area problems, the departments were assumed to have areas that are uniformly distributed between 1 and 10 square units (see Table 2). The large variety of areas make these data sets quite representative of real-world problems and hence

Table 1: Numerical results for equal area problems.

Problem number	Number of departments	DISCON		AR*			SE	
		$l_1$ cost	CPU secs.†	$l_1$ cost	% overlap	$\alpha$	$l_1$ cost	CPU secs.‡
1	6	47.5	0.06	65.5	1.0%	0.01	44.7	0.93
2	8	118.8	0.08	148.8	0.2%	0.005	107.1	1.94
3	12	322.2	0.16	528.9	0.0%	0.005	294.4	6.40
4	15	630.8	0.32	1154.0	0.0%	0.005	586.6	19.27
5	20	1416.4	0.86	3064.8	0.0%	0.005	1285.0	53.82
6	30	3436.4	4.86	9208.0	0.0%	0.005	3063.7	246.13

\* All problems solved in less than 2 seconds using MINOS accessed via the NEOS server.

† CPU time in seconds required on an AMDAHL 470/v8 workstation.

‡ CPU time in seconds required on a Pentium III Windows-based PC.

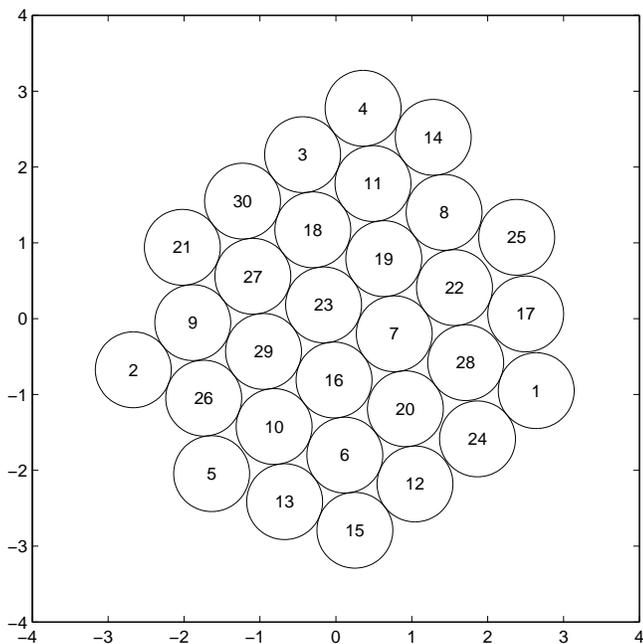


Figure 3: Minimum-energy arrangement for problem 6 (30 equal area departments with no floor area requirements).

Table 2: Departmental areas for unequal area problems.

Department	Area	Department	Area
1	8	16	1
2	10	17	9
3	5	18	1
4	6	19	4
5	6	20	6
6	2	21	3
7	1	22	2
8	8	23	1
9	4	24	9
10	8	25	8
11	10	26	5
12	2	27	4
13	8	28	1
14	6	29	6
15	5	30	1

Table 3: Dimensions of the existing facility.

Problem number	Number of departments	Width $\bar{w}$	Height $\bar{h}$
13	6	4.0	10.0
14	8	5.0	10.0
15	12	7.0	10.0
16	15	8.5	11.0
17	20	10.0	11.5
18	30	10.5	15.0

difficult to solve by other methods. As above, the radii of the departments were chosen so that  $r_i = \sqrt{a_i}/2$ ,  $i = 1, \dots, N$ . For problems 7-12, no floor area requirements or facility dimensions were imposed. However, problems 13-18 were assumed to come from an existing production plant and hence the dimensions of the facility were fixed in advance. Therefore, we set  $w^{\text{low}} = w^{\text{up}} = \bar{w}$  and  $h^{\text{low}} = h^{\text{up}} = \bar{h}$  (see Table 3).

The numerical results for the class of unequal area problems are summarised in Table 4. The performance of our algorithmic implementation of the ALM method is compared only with that of AR and not with DISCON for two reasons: there are no published results of the application of the DISCON model to these data sets, and more importantly, the DISCON model is not capable of handling the floor area requirements and department locational restrictions of problems 13 – 18. Since the AR and SE models are formulated with different objective functions, we used the final arrangement of departments and computed the material handling cost of such an arrangement using the  $l_2$  norm. As with the equal area problems above, we experimented with four different values of the AR parameter  $\alpha$  (0.005, 0.01, 0.5, and 1) and set  $\kappa = 10 \sum_i \sum_{j>i} c_{ij}$ . In Table 4, we report the  $l_2$  cost, the % overlap values (see equation (21)), and the  $\alpha$  values that provide the lowest degree of infeasibility to provide a more meaningful comparison of the SE model with AR.

From Table 4, the  $l_2$  cost values obtained from our SE method improve upon those from AR for all 12 data sets although SE takes longer to run. We remark again that it is more critical to obtain optimal or near-optimal solutions albeit via a slower method than to use a faster method that does not guarantee the quality nor the feasibility of the final solutions. Note that for problems 13-18, which included facility dimensions, the AR model did not provide any final solution without department area overlap for any of the  $\alpha$  values used. We could have searched for more suitable values of  $\alpha$  to obtain more pleasing solutions but this was neither the intent nor purpose of our study.

Table 4: Numerical results for unequal area problems.

Problem number	Number of departments	AR*			SE		
		$l_2$ cost	% overlap	$\alpha$	$l_2$ cost	SE energy	CPU secs.†
7	6	84.4	5.6	0.010	89.8	117.1	1.22
8	8	246.0	0.8	0.010	195.4	276.1	2.14
9	12	632.6	0.6	0.005	530.0	943.9	11.07
10	15	1389.9	0.2	0.005	1046.2	2175.0	21.71
11	20	3605.2	0.0	0.005	2273.7	5456.5	68.71
12	30	10516.3	0.0	0.005	4651.9	11873.0	260.96
13	6	136.7	3.9	1.000	103.2	158.9	3.07
14	8	338.7	1.9	1.000	235.2	424.6	7.11
15	12	789.1	1.5	0.010	548.8	1002.3	20.97
16	15	1358.7	2.0	0.005	1119.6	2456.2	42.09
17	20	3163.5	1.1	0.005	2354.7	5907.9	113.2
18	30	8695.2	0.9	0.005	5032.6	14053.0	375.84

\* All problems solved in less than 2 seconds using MINOS accessed via the NEOS server.

† CPU time in seconds required on a Pentium III Windows-based PC.

The number of ALM iterations executed by our algorithm for all unequal area problems were between 10-12. As expected, we see that the energy of the final arrangements increases when the oscillation of departments is reduced due to the incorporation of floor area requirements and facility dimensions. The effect of such ‘freedom’ reduction is illustrated in Figures 4 and 5 where the minimum-energy arrangements for problems 12 and 18 are shown. For comparison purposes, Figure 6 shows the final arrangement using the AR method with  $\alpha = 0.005$  for problem 18. Such a value of  $\alpha$  results in an infeasible layout and challenges the optimality criterion stated in Section 1.2.

We conclude this section by noting that no attempt was made to optimise the performance of our algorithmic implementation of the ALM method at this stage of our research. Optimal or near-optimal facility layouts were reached in at most 6 minutes. Note that even several hours of run-time can be acceptable since facility layout decisions are planned on a long-term horizon (e.g., 5 to 10 years).

## 5 Conclusions

The SE model is based on a dynamic system in which springs of a given strength are replaced by the material handling costs per unit distance between departments and particles are replaced by departments of a given area. From the initial arrangement of department locations, the system oscillates until it stabilises at a minimum-energy arrangement.

The SE model, as expected from a convex programming model, places the departments in optimal or near-optimal locations within the facility; hence, improving upon previous non-convex models and their solutions for the facility layout problem. Moreover, our numerical results show that our algorithmic implementation of an augmented Lagrangian multiplier method is efficient and robust on several facility layout problems irrespective of the number of departments to arrange. Although our solution run-times were slower, we believe that it is more critical to obtain optimal or near-optimal solutions via a slower method than to use faster methods that guarantee neither the quality nor the feasibility of the final solutions. Moreover, the exceedingly simple implementation of our solution procedure and its solution quality allow it to seriously challenge other methods in this important class of hard optimisation problems.

Finally, we believe that the ideas presented in this paper can be used to heuristically solve facility layout problems in which an orthogonal arrangement of departments is a requirement. That is, the minimum-energy arrangement of departments within a facility that satisfies department and floor area requirements and department locational restrictions can be used to fix a subset of the disjunctive boolean variables in the mixed-integer programming model and then optimise the reduced problem. The development of such heuristic solution for

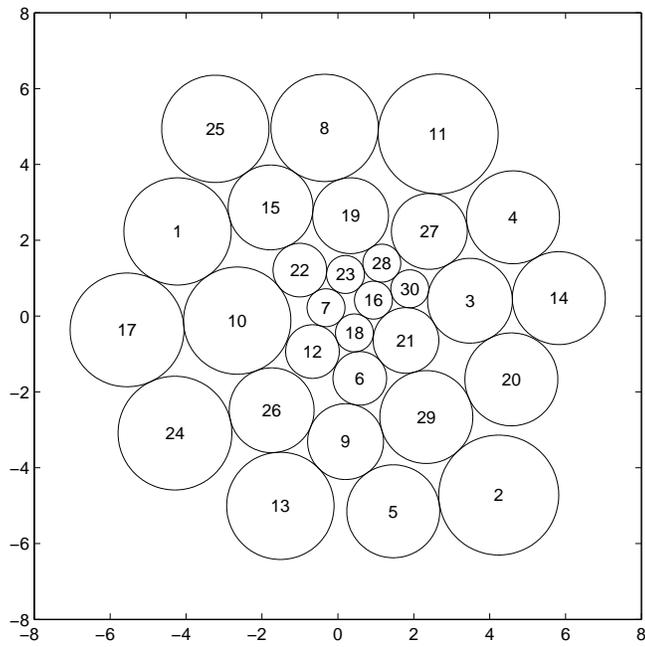


Figure 4: Minimum-energy arrangement using the SE model for problem 12 (30 unequal area departments without floor area requirements).

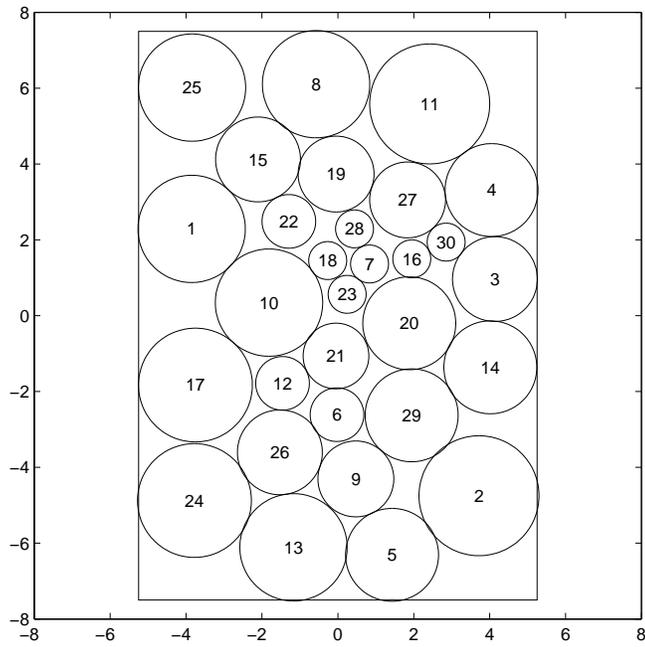


Figure 5: Minimum-energy arrangement using the SE model for problem 18 (30 unequal area departments with floor area requirements).

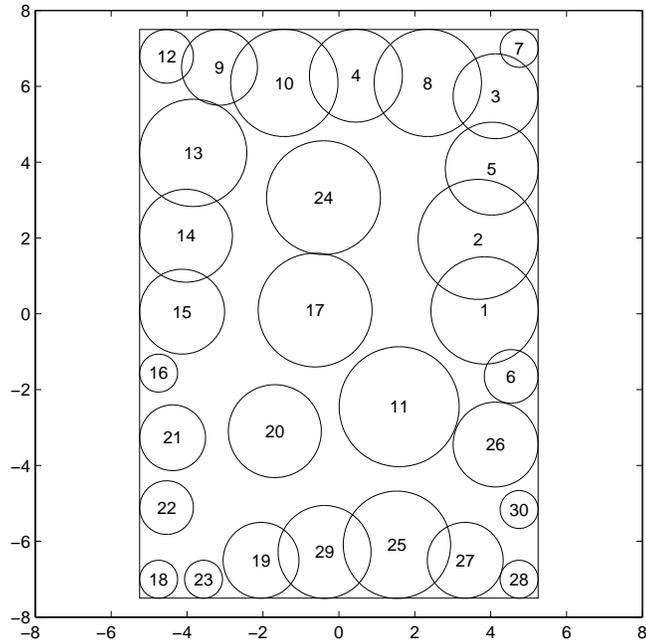


Figure 6: Final arrangement using the AR model with  $\alpha = 0.005$  for problem 18 (30 unequal area departments with floor area requirements).

orthogonal arrangements is in progress.

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