Location and design of a competitive facility for profit maximisation

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Abstract

A single facility has to be located in competition with fixed existing facilities of similar type. Demand is supposed to be concentrated at a finite number of points, and consumers patronise the facility to which they are attracted most. Attraction is expressed by some function of the quality of the facility and its distance to demand. For existing facilities quality is fixed, while quality of the new facility may be freely chosen at known costs. The total demand captured by the new facility generates income. The question is to find that location and quality for the new facility which maximises the resulting profits.

It is shown that this problem is well posed as soon as consumers are novelty oriented, i.e. attraction ties are resolved in favor of the new facility. Solution of the problem then may be reduced to a bicriterion maxcovering-minquantile problem for which solution methods are known. In the planar case with Euclidean distances and a variety of attraction functions this leads to a finite algorithm polynomial in the number of consumers, whereas, for more general instances, the search of a maximal profit solution is reduced to solving a series of small-scale nonlinear optimisation problems. Alternative tie-resolution rules are finally shown to result in ill-posed problems.

Keywords: Competitive location, Consumer behaviour, Facility design, Maxcovering, Minquantile, Biobjective.

1 Introduction

This paper addresses the location of a new facility in a competitive environment. Competition consists of a number of existing facilities having a known fixed location within the market. Typically the expected income the new facility will generate directly depends upon the market share it captures. This market share will be determined by several factors, among which we single out its location and its quality as compared to the competing facilities. These two factors are both controllable for a new facility and are considered as decision variables.

With respect to spatial consumer distribution, we assume that demand is concentrated at a finite number of known fixed points in some metric space. Thinking of individual customers this is the most correct description of reality, although one might have to be more precise by considering customers not only as persons, but rather as "persons at a particular time period"— a same person at home does not necessarily behave the same way as at work, and usually has different locations in these two situations. The number of individuals to be considered is, however, usually too

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large for such a precise description of reality to be feasible, both in terms of amount of necessary data as in terms of complexity for practical solution. Therefore one often resorts to either a statistical description, in many cases pragmatically oversimplified into some uniform distributions, as in Vaughan (1987), or to aggregation of demand into a few 'conglomerate' consumers, see e.g. Goodchild (1979) and Francis-Lowe (1992). It is this latter approach that is followed here. However, we allow for possible presence of several customer-groups at a same site, each with their own particular behaviour towards facility choice. This enables the modeller to split the population at a given site into several groups, e.g. by time period and/or, as is often done, by income, activity and/or age. In what follows we will call each such customer-groups simply a *customer*, having its own individual behaviour and location.

Spatial consumer behaviour has been studied in several disciplines such as geography, economics and marketing, see Eiselt-Laporte-Thisse (1993). Generalising many of these models we start by considering some measure of the *attraction* a consumer feels for a facility, often also called the *utility* of the facility for this consumer. This attraction is some function of the *distance* between facility and customer on the one hand, and on the other hand of internal characteristics of the facility, which we express as one global positive measure we will call the *quality*. The particular function describing attraction may differ from one customer to the other, but is always nonincreasing with distance and nondecreasing with quality. Two typical examples of such attraction functions are additive ones, i.e. a weighted difference of quality and distance (compare with T. Drezner (1994)), with weights possibly differing between customers, or multiplicative ones, leading to gravity type attraction, given by quality divided by some strictly positive power of distance (see Plastria (1997)).

We restrict our attention here to *deterministic* behavioural models. In these models each customer is supposed to patronise that facility to which it is attracted most, in contrast to in probabilistic behaviour models where attraction is interpreted as (proportional to) a probability, and the expected value of the demand attracted to each facility is considered (see e.g. T. Drezner (1995)).

When setting up the new facility the main decisions relate to its site and to its design. In our models they consist of two choices: the location and the quality, and these decisions directly influence both the level of sales at the facility and the operational costs. Sales or income are generated in an increasing way by the total demand attracted to the facility. This will depend upon the actual site to be chosen, but also on the quality of the facility. The costs involved in starting up and running the facility are evidently related to its quality in an increasing way. Indeed, quality is determined by a mixture of several facility attributes, e.g. floor area, number of check-counters, point of sales-system (bar coding, bank-card readers, ...), product mix, price-level, marketing in a retail context, and raising the level of any of these attributes always involves higher costs.

Our aim in this paper is to *maximise profit.* We allow profit to be any indicator of profitability with the minimal properties one may expect of such: it should increase with income and decrease with cost. The standard examples are sales minus cost or sales divided by cost. We will show that profit-maximisation with respect to both the location and the quality of the new facility may be obtained under some mild conditions by inspecting only a finite number of solutions, obtained after solving some biobjective maxcovering-minquantile location problem, as studied in Carrizosa and Plastria (1995). More specific assumptions on the nature of the distance and attraction function used will allow to obtain quite efficient algorithms for this task.

The first competitive location model involving both the choice of location and facility's characteristic simultaneously seems to have been Plastria (1997), of which the present study is a broad generalisation. All other papers we are aware of either consider a fixed site and attempt to maximize profit by an adequate choice of quality (also called attractiveness), or consider the quality fixed and construct an optimal site. Models of the first type are studied by Eiselt and Laporte (1988a,b) with demand uniformly distributed on a linear market, while T. Drezner (1994) discusses a model of the second type in a planar situation with discrete demand and Euclidean distance. As such our model may be considered as a simultaneous generalization of both these studies. For a general overview of competitive location models we refer to the survey of Eiselt, Laporte and

Thisse (1993) and to the more restricted but more recent survey by Plastria (2001).

The paper is organised as follows. Section 2 gives the formal description of the model and introduces the notations to be used. In Section 3 we show how to reduce the solution of the model to the biobjective location problem. Section 4 is devoted to the planar case, emphasising gravity-type attraction models with Euclidean distances, illustrated with a small-size example showing the power of our methodology to obtain optimal solutions and perform sensitivity analysis. Extensions to different types of attraction functions or distances are also discussed, ending with a discussion about alternative tie-breaking rules in Section 5.

2 The general profit-maximising competitive location model

A finite set of *consumer(group)s* is denoted by A. Each consumer $a \in A$ has a known location x_a and a strictly positive *weight* ω_a, supposed to be an indicator of its buying power, e.g. the population or total wealth represented by consumer(group) a.

A finite set of *competing facilities* with which our new facility is to compete is denoted by CF. Competing facility $f \in CF$ is located at site x_f and has a quality α_f considered to be known and fixed.

Any consumer $a \in A$ feels an attraction $attr(a, f)$ towards facility f at x_f, which depends on factors such as the distance from x_a to x_f, the facility (and its firm's) attractiveness, tradition, etc.

Consider a *new facility* with unknown site x and unknown quality α, of at least some minimal quality $\alpha_0 > 0$. Its attraction on consumer $a \in A$ is given by $\mathcal{A}_a(\alpha, \mathrm{dist}_a(x))$, a function of its quality α and the distance $\mathrm{dist}_a(x)$ from the consumer to the facility. The possibly $+\infty$-valued function $\mathcal{A}_a : [\alpha_0, +\infty[\times[0, +\infty[\longrightarrow [0, +\infty]$ satisfies

1. For each fixed $d \geq 0$, the function $\mathcal{A}_a(\cdot, d)$ is nondecreasing and upper-semicontinuous
2. For each fixed $\alpha \geq \alpha_0$, the function $\mathcal{A}_a(\alpha, \cdot)$ is nonincreasing and upper-semicontinuous.

Note that these functions are allowed to differ from one customer to the other, enabling differentiation in their spatial behaviour.

Examples 1

Example 1.1
A typical example is attraction of (generalised) *gravity type*, given by

$$\mathcal{A}_a^{grav}(\alpha, d) = \frac{\alpha k_a}{d^p} \qquad \forall d \geq 0,\ \alpha \geq \alpha_0 > 0, \tag{1}$$

where p is any strictly positive exponent, and $k_a > 0$ represents some proportionality constant depending on a.

The exponent p allows the analyst to finetune the sensitivity of attraction to distance. In pure gravity type models $p = 2$, by similarity with the gravitational law in Physics, see the iso-attraction curves in Figure 1. The models considered by Eiselt and Laporte (1988a,b) use $p = 1$, whereas the case of general $p > 0$ was considered in Plastria (1997).

One small technical point should be raised here, which is apparently ignored, but probably implicitly assumed in all literature on these types of consumer behaviour models : gravity-type attraction is infinite as soon as $\mathrm{dist}_a(x) = 0$, e.g. when the facility location coincides with the location x_a of consumer a. This might also happen with other types of attraction functions. In general we will evidently consider that in such a case the attraction is infinite ($+\infty$). In other words, it is impossible for a to be attracted more by some other facility located elsewhere than at zero distance of x_a.

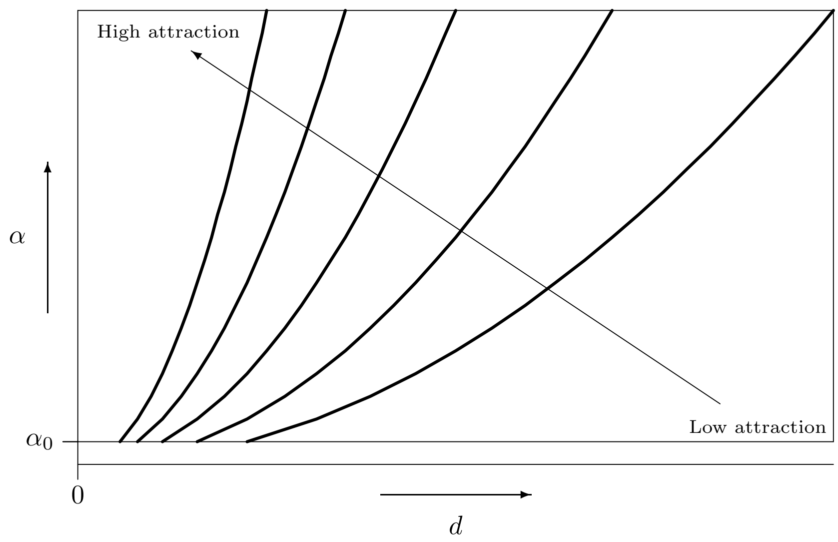

Figure 1: Iso-attraction curves for $\mathcal{A}^{grav}$ $(p = 2)$

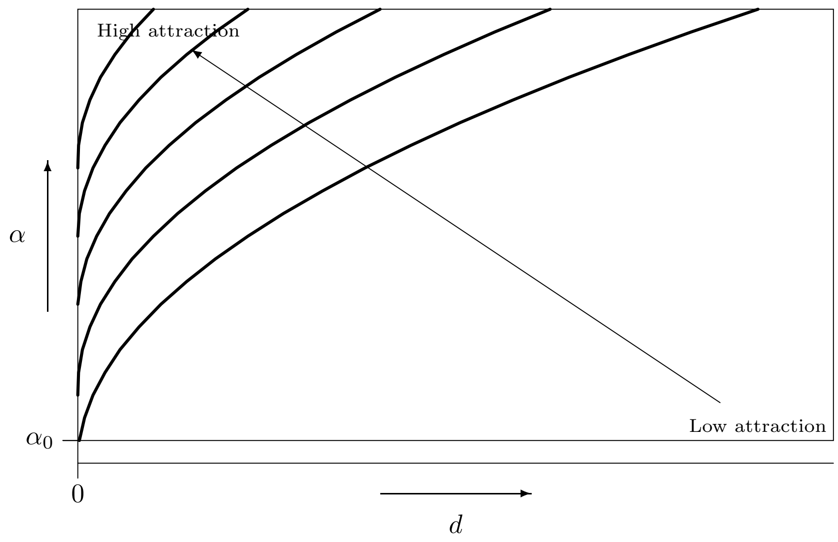

Figure 2: Iso-attraction curves for $\mathcal{A}^{add}$

Example 1.2
Note, however, that for attraction functions with a finite value for $d = 0$, it might be possible for a consumer to be attracted less to a facility located at its own site than to some facility elsewhere, as soon as the quality of the latter is sufficiently high. Such a situation might arise with an *additive attraction function*,

$$\mathcal{A}_a^{add}(\alpha, d) = \max\{0, \alpha k_a - c_a(d)\} \tag{2}$$

where $k_a > 0$ and c_a any nondecreasing lower-semicontinuous function. One example of an additive attraction function is obtained when considering the real prices carried by a customer in a mill-pricing system (see e.g. Hansen et al., 1995). In this interpretation k_a represents the normal price to pay for the quantity of some good consumer a needs to obtain per trip, $c_a(d)$ denotes the transport cost carried by the consumer a when travelling over distance d, and α is the price-reduction factor (per unit) offered. The attraction $\mathcal{A}_a^{add}(\alpha, d)$ then expresses the reduction on the total cost the consumer obtains at a facility with price reduction factor α at distance d. Obviously it might be better to go to a cheaper shop further (but not too far) away than to buy at high price right here. See in Figure 2 iso-attraction curves for $c(d) = \sqrt{d}$.

Example 1.3
Quality and distances might also be combined through an attraction of the *minimum disutility* form

$$\mathcal{A}_a^{min}(\alpha, d) = \min\{\alpha k_a, c_a(d)\} \tag{3}$$

for some upper-semicontinuous nonincreasing c_a, in which consumers measure facility attraction through its least favourable feature: either the utility due to its quality ($k_a \alpha$) or the utility due to travel cost ($c_a(d)$). See in Figure 3 iso-attraction curves for $c(d) = d^{-\frac{1}{2}}$.

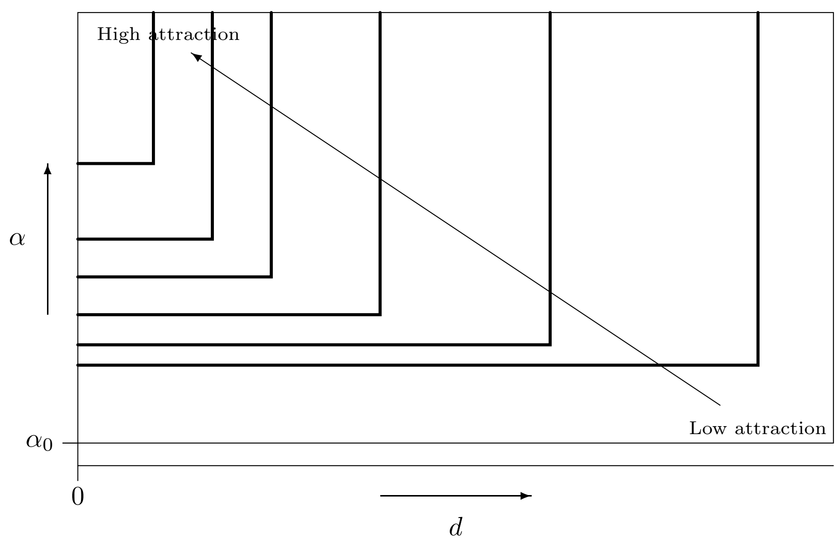

Figure 3: Iso-attraction curves for $\mathcal{A}^{min}$

Example 1.4
In fact one might also consider max-type functions as

$$\mathcal{A}_a^{max}(\alpha, d) = \max\{\alpha k_a, c_a(d)\} \tag{4}$$

with shape something like in next figure.

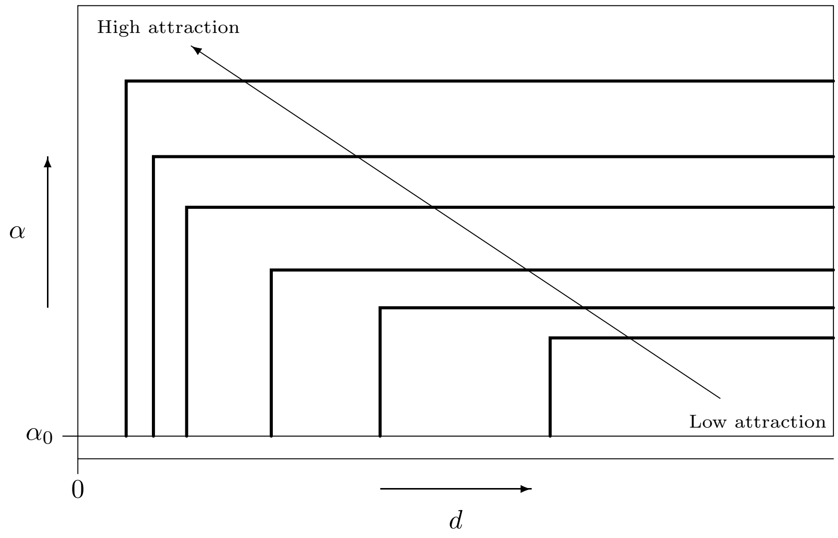

Figure 4: Iso-attraction curves for $\mathcal{A}^{max}$

Example 1.5
Observe also that, since functions $\mathcal{A}_a$ are not assumed to be continuous, one can also consider within this framework non-compensatory models, such as

$$\mathcal{A}_a^{step}(\alpha, d) = \begin{cases} \beta_a, & \text{if } \alpha \geq \underline{\alpha}_a \text{ and } d \leq \overline{d}_a \\ 0, & \text{else} \end{cases} \tag{5}$$

for which the attraction felt is null unless the quality is sufficiently high ($\alpha \geq \underline{\alpha}_a$) and the facility is not too distant ($d \leq \overline{d}_a$), see e.g. Roberts and Lilien (1993). See figure 5 for its shape.

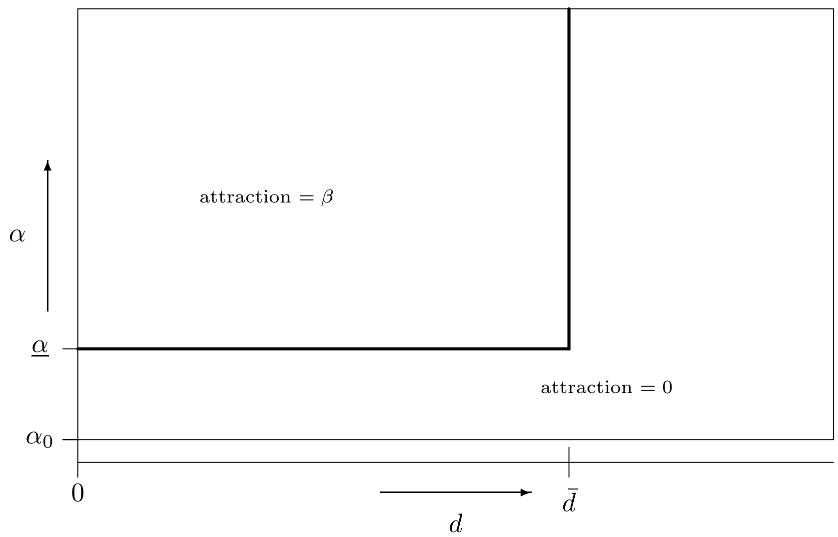

Figure 5: Iso-attraction curves for $\mathcal{A}^{step}$

Example 1.6
Partially non-compensatory models can also be considered; indeed, define $\mathcal{A}_a^{part}$ as

$$\mathcal{A}_a^{part}(\alpha, d) = \begin{cases} \mathcal{A}_a(\alpha, d), & \text{if } d \leq \overline{d}_a \\ 0, & \text{else} \end{cases} \tag{6}$$

where $\mathcal{A}_a$ is, e.g., of types (1)-(4) and $\overline{d}_a$ is a given non-negative threshold value. This yields an attraction function which behaves exactly as $\mathcal{A}_a$ when distances are not too high (up to the threshold distance $\overline{d}_a$), but drops to zero (thus no compensation is possible) as soon as distances exceed this threshold. In Figure 6 the reader can see the effect produced when altering (1), as depicted in Figure 1, by introducing a threshold distance $\overline{d}$.

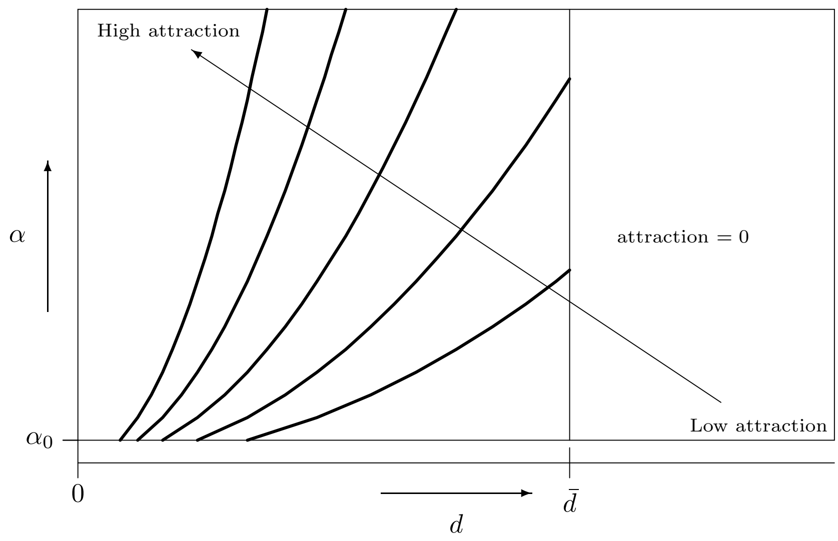

Figure 6: Iso-attraction curves for $\mathcal{A}^{part}$

We denote the attraction felt by consumer $a \in A$ towards an existing facility $f \in CF$ by $attr(a, f)$, whereas the attraction towards the new facility at x with quality α is given by $\mathcal{A}_a(\alpha, \text{dist}_a(x))$, where $\text{dist}_a(x)$ denotes the distance between x_a and x. Hence, with the deterministic consumer choice rule, it will capture those consumers attracted more to the new facility than to any competing facility in CF. The set of captured consumers is thus given by

$$Capt(\alpha, x) = \{a \in A \mid \forall f \in CF : \mathcal{A}_a(\alpha, \text{dist}_a(x)) \geq attr(a, f)\}.$$

Note the use of $\geq$ in this definition, which means that for consumers equally attracted by the new facility and any existing facility they patronised before, we assume they will start patronising the new facility as soon as it arrives on the market. In other words we consider consumers to be *novelty oriented*. It will turn out in Section 3 that this assumption leads to clear results in our model, while, as shown in Section 5, any other assumption on the resolution of attraction ties leads to technical difficulties, more precisely non-existence of optimal solutions (strictu sensu) when α may vary continuously.

The total weight captured by the new facility is now given by

$$CW(\alpha, x) = \sum_{a \in Capt(\alpha, x)} \omega_a \tag{7}$$

The *sales income* at the new facility is given as some *strictly increasing* function σ of the captured weight. The *operating costs* of the new facility are given by a *strictly increasing* function γ of its quality α. Profit is expressed by some *profit-indicator* function π which is *strictly increasing in sales and strictly decreasing in costs*, yielding a profit indicator

$$\Pi(\alpha, x) = \pi(\ \sigma(CW(\alpha, x))\ ,\ \gamma(\alpha)\ ) \tag{8}$$

Typical examples of such indicator functions are the standard notion of profit, i.e. the difference of sales minus costs, $\pi(s, c) = s - c$, yielding

$$\Pi(\alpha, x) = \sigma(CW(\alpha, x)) - \gamma(\alpha), \tag{9}$$

or a profitability ratio like sales divided by costs, $\pi(s, c) = \frac{s}{c}$, giving rise to

$$\Pi(\alpha, x) = \frac{\sigma(CW(\alpha, x))}{\gamma(\alpha)}$$

It is the profit-indicator Π in (8) we want to optimise by an adequate choice of both the quality $\alpha \geq \alpha_0$ and the site x within some set of feasible sites S :

$$\max\{\Pi(\alpha, x) \mid\ x \in S, \quad \alpha \geq \alpha_0\}. \tag{10}$$

Given a fixed quality α, profit maximisation is achieved by maximisation of the total captured weight, in other words we obtain a *maximal covering problem* as studied in the planar context by Z. Drezner (1981) and Mehrez and Stulman (1982). This is in fact what T. Drezner (1994) proposes. Since to us α is a variable, we would have to solve such a maximal covering problem for each possible value of α. At first glance this seems possible only when no more than a finite number of feasible quality values are available.

This is however not the approach we want to use here: we want to optimise over the full range of positive α-values no less than α_0. Instead, we show in Section 3 how to solve Problem (10) by reducing it to a bicriterion problem, which was addressed by the authors in Carrizosa and Plastria (1995). In particular, this will enable to solve (10) in a number of cases by inspecting a finite number of points, polynomial in the cardinality of A.

3 Bicriterion minquantile/maxcovering view

We now take a closer look at the capturing process. Consumer a is captured by the new facility given by (α, x) iff for any facility $f \in CF$ we have $\mathcal{A}_a(\alpha, \text{dist}_a(x)) \geq attr(a, f)$. Let us define $\mu_a \in [0, +\infty]$, the *decisive attraction* of a, as the highest attraction consumer a felt before the advent of the new facility, i.e.,

$$\mu_a = \max\{attr(a, f) \mid f \in CF\}, \tag{11}$$

We may now write the capturing of a by the new facility at x as

$$\begin{aligned} a \in Capt(\alpha, x) \quad &\iff \quad \mathcal{A}_a(\alpha, \text{dist}_a(x)) \geq \mu_a \\ &\iff \quad \alpha \in \{\bar{\alpha} \geq \alpha_0 : \mathcal{A}_a(\bar{\alpha}, \text{dist}_a(x)) \geq \mu_a\} \end{aligned}$$

Since, by assumption, for each $a \in A$ and $d \geq 0$, the function $\mathcal{A}_a(\cdot, d)$ is nondecreasing and upper semicontinuous, the set $\{\bar{\alpha} \geq \alpha_0 : \mathcal{A}_a(\bar{\alpha}, d) \geq \mu_a\}$ is a closed (possibly empty) subinterval of $[\alpha_0, +\infty]$. It then has either the form $[\mathcal{B}_a(d), +\infty]$, or it is empty, in which case we define $\mathcal{B}_a(d)$ as $+\infty$.

¿From this definition one immediately obtains that

$$\mathcal{B}_a(d) \leq \alpha \iff \mathcal{A}_a(\alpha, d) \geq \mu_a, \tag{12}$$

This means that $\mathcal{B}_a(d)$ indicates, as a function of d, the quality threshold above which customer a at distance d is captured. We call this partial inverse function of $\mathcal{A}_a$ the *decisive quality function.*

We have the following property

Theorem 2 *For each $a \in A$, $\mathcal{B}_a$ is a nondecreasing lower-semicontinuous function satisfying*

$$a \in Capt(\alpha, x) \quad \text{iff} \quad \alpha \geq \mathcal{B}_a(\text{dist}_a(x)) \tag{13}$$

Proof.
Since, for any $d_2 \geq d_1 \geq 0$, one has for each α that

$$\mathcal{A}_a(\alpha, d_1) \geq \mathcal{A}_a(\alpha, d_2),$$

it follows that

$$\{\alpha \geq \alpha_0 \mid \mathcal{A}_a(\alpha, d_1) \geq \mu_a\} \supset \{\alpha \geq \alpha_0 \mid \mathcal{A}_a(\alpha, d_2) \geq \mu_a\},$$

thus $\mathcal{B}_a(d_1) \leq \mathcal{B}_a(d_2)$.

Moreover, by (12), we have

$$\{d \geq 0 \mid \mathcal{B}_a(d) \leq \alpha\} = \{d \geq 0 \mid \mathcal{A}_a(\alpha, d) \geq \mu_a\}$$

and this is a closed set by the upper semicontinuity of $\mathcal{A}_a(\alpha, \cdot)$. Hence, $\mathcal{B}_a$ is lower semicontinuous.

Finally, (13) follows from the definition. □

Observe that by (12) the graph of $\mathcal{B}_a$ is shown in the figures 1-6 as the iso-attraction curve at attraction level μ_a. Analytically $\mathcal{B}_a$ can be easily constructed for the attraction functions in the examples of previous section, as discussed below.

Examples 3

Example 3.1
For the gravity model (1), we see that, for any $d > 0$ and $\mu_a \in [0, +\infty]$,

$$\begin{aligned}\{\bar{\alpha} \geq \alpha_0 : \mathcal{A}_a^{grav}(\bar{\alpha}, d) \geq \mu_a\} &= \{\bar{\alpha} \geq \alpha_0 : \frac{k_a \bar{\alpha}}{d^p} \geq \mu_a\} \\ &= \{\bar{\alpha} \geq \alpha_0 : \bar{\alpha} \geq \frac{\mu_a d^p}{k_a}\} \\ &= [\max\{\alpha_0, \frac{\mu_a d^p}{k_a}\}, +\infty[,\end{aligned}$$

whilst for $d = 0$, one has that

$$\{\bar{\alpha} \geq \alpha_0 : \mathcal{A}_a^{grav}(\bar{\alpha}, 0) \geq \mu_a\} = [\alpha_0, +\infty[$$

In other words, for finite μ_a, $\mathcal{B}_a^{grav}$ is given by

$$\mathcal{B}_a^{grav}(d) = \max\{\alpha_0, \frac{\mu_a d^p}{k_a}\}$$

whereas for $\mu_a = +\infty$, $\mathcal{B}_a^{grav}$ is given by

$$\mathcal{B}_a^{grav}(d) = \begin{cases} \alpha_0, & \text{if } d = 0 \\ +\infty, & \text{if } d > 0 \end{cases}$$

□

Example 3.2
For the additive model (2) we obtain for $\mu_a = 0$ that $\mathcal{B}_a^{add}(d) = \alpha_0$ for each $d \geq 0$, whereas for each $\mu_a > 0$ it follows that

$$\begin{aligned}\{\bar{\alpha} \geq \alpha_0 : \mathcal{A}_a^{add}(\bar{\alpha}, d) \geq \mu_a\} &= \{\bar{\alpha} \geq \alpha_0 : k_a \bar{\alpha} - c_a(d) \geq \mu_a\} \\ &= [\max\{\alpha_0, \frac{\mu_a + c_a(d)}{k_a}\}, +\infty[\end{aligned}$$

Hence, if $\mu_a = 0$ then $\mathcal{B}_a^{add}(d) = \alpha_0$ for all d whereas, for $\mu_a > 0$, $\mathcal{B}_a^{add}$ has the form

$$\mathcal{B}_a^{add}(d) = \max\{\alpha_0, \frac{\mu_a + c_a(d)}{k_a}\}$$

□

Example 3.3
For the minimum disutility model (3), $\mathcal{B}_a^{min}$ takes the form

$$\mathcal{B}_a^{min}(d) = \begin{cases} \max\{\alpha_0, \frac{\mu_a}{k_a}\}, & \text{if } c_a(d) \geq \mu_a \\ +\infty, & \text{else} \end{cases}$$

Since c_a is assumed to be non-increasing, it follows that, for $\mu_a > c_a(0)$, $\mathcal{B}_a^{min} = +\infty$. For $\mu_a \leq c_a(0)$, since c_a is upper-semicontinuous, we can define $\overline{d}_a$ as

$$\overline{d}_a := \max\{d \geq 0 : \, c_a(d) \geq \mu_a\} \in [0, +\infty], \tag{14}$$

yielding the following equivalent expression of $\mathcal{B}_a^{min}$

$$\mathcal{B}_a^{min}(d) = \begin{cases} \max\{\alpha_0, \frac{\mu_a}{k_a}\}, & \text{if } d \leq \overline{d}_a \\ +\infty, & \text{else,} \end{cases} \tag{15}$$

Example 3.4
Similarly for the max-type model (4),

$$\mathcal{B}_a^{max}(d) = \begin{cases} \max\{\alpha_0, \frac{\mu_a}{k_a}\}, & \text{if } c_a(d) < \mu_a \\ \alpha_0, & \text{else} \end{cases}$$

yielding

$$\mathcal{B}_a^{max}(d) = \begin{cases} \max\{\alpha_0, \frac{\mu_a}{k_a}\}, & \text{if } d > \overline{d}_a \\ \alpha_0, & \text{else} \end{cases} \tag{16}$$

with $\overline{d}_a$ defined by (14).

Example 3.5
For the step model (5), one has that, for $\mu_a \leq \beta_a$,

$$\mathcal{B}_a^{step}(d) = \begin{cases} \underline{\alpha}_a, & \text{if } d \leq \overline{d}_a \\ +\infty, & \text{else,} \end{cases} \tag{17}$$

whereas, for $\mu_a > \beta_a$, one obtains that $\mathcal{B}_a^{step}$ is constantly $+\infty$. Observe that this expression has the same form as (15), obtained for a different attraction model.

Example 3.6
For $\mathcal{A}_a^{part}$ given in (6) for some given $\mathcal{A}_a$ with corresponding partial inverse $\mathcal{B}_a$, the construction of $\mathcal{B}_a^{part}$ is straightforward; indeed, since, for $\mu_a = 0$,

$$\{\alpha \geq \alpha_0 : \, \mathcal{A}_a^{part}(\alpha, d) \geq 0\} = [\alpha_0, +\infty[,$$

we get for $\mu_a = 0$ that $\mathcal{B}_a^{part}(d) = \alpha_0$ for all $d \geq 0$. On the other hand, for $\mu_a > 0$,

$$\{\alpha \geq \alpha_0 : \, \mathcal{A}_a^{part}(\alpha, d) \geq \mu_a\} = \begin{cases} \{\alpha \geq \alpha_0 : \, \mathcal{A}_a(\alpha, d) \geq \mu_a\} & \text{if } d \leq \overline{d}_a \\ \emptyset, & \text{else} \end{cases}$$

Hence, for $\mu_a > 0$ we obtain

$$\mathcal{B}_a^{part}(d) = \begin{cases} \mathcal{B}_a(d), & \text{if } d \leq \overline{d}_a \\ +\infty, & \text{else} \end{cases}$$

□

Our optimisation problem may now be stated as follows. For each $a \in A$ we have a function $\text{dist}_a(\cdot)$, a weight ω_a and a threshold value μ_a as defined in (11). For any quality $\alpha \geq \alpha_0 \geq 0$, the captured weight $CW(\alpha, x)$ is defined in (7) as

$$\begin{aligned} CW(\alpha, x) &= \sum \{\omega_a \mid a \in Capt(\alpha, x)\} \\ &= \sum_a \{\omega_a \mid \mathcal{A}_a(\alpha, \text{dist}_a(x)) \geq \mu_a\} \\ &= \sum_a \{\omega_a \mid \mathcal{B}_a(\text{dist}_a(x)) \leq \alpha\} \end{aligned}$$

and the maximisation of the profit-indicator function Π defined in (8) yields

$$\max_{\alpha \geq \alpha_0, x \in S} \quad \Pi(\alpha, x) := \pi(\sigma(CW(\alpha, x)), \gamma(\alpha)) \tag{18}$$

Following Plastria (1997), we propose to find a maximal profit solution (i.e., to solve (18)) through the determination of the efficient (or nondominated, or Pareto-optimal) solutions of the bi-objective problem

$$\begin{array}{ll} \min & \alpha \\ \max & CW(\alpha, x) \\ & \alpha \geq \alpha_0 \ ; \ x \in S \end{array} \tag{19}$$

We first recall that a feasible solution (α, x) is said to be efficient for (19) iff there exists no feasible pair (α^*, x^*) satisfying

$$\begin{aligned} \alpha^* &\leq \alpha \\ CW(\alpha^*, x^*) &\geq CW(\alpha, x), \end{aligned}$$

with at least one of the two inequalities above as strict (see e.g. Steuer, 1986).

As a direct consequence of this definition of efficient solutions, and the fact that, by assumption, σ, γ, π are strictly monotonic in their arguments, one obtains the following

Theorem 4 *Any maximal profit solution is an efficient solution for (19).*

Problems of type (19), called *biobjective minquantile-maxcovering problems* were defined and studied in Carrizosa and Plastria (1995) in a general theoretical setting. In the next section we show that, under mild conditions, the general theory developed there leads, in the planar context, to a finite number of candidates for being maximal profit solutions, obtained through a geometrical procedure in the most common model (gravity-type attraction, Euclidean distances) or after solving a finite number of small-scale nonlinear optimisation problems.

Theorem 4 asserts that, *if* maximal profit solutions exist, then they all are efficient for the biobjective problem (19). Sufficient conditions for the existence of such maximal profit solutions are given in the next result.

Theorem 5 *Suppose that, for each $a \in A$, dist_a is a continuous function of the location x, and, for any $\beta \geq 0$, the set $\{x \in S : \text{dist}_a(x) \leq \beta\}$ is compact. Then, under novelty orientation, there exists a maximal profit solution.*

Proof.
For a given $\alpha \geq \alpha_0$, let $\mathcal{Q}(\alpha)$ denote the highest market capture which can be obtained for a quality α, if the facility is properly located,

$$\mathcal{Q}(\alpha) = \sup_{x \in S} CW(\alpha, x) = \max_{x \in S} CW(\alpha, x)$$

where the last equality is a consequence of the fact that CW can take only finitely many values. Then,

$$\begin{aligned}\mathcal{Q}(\alpha) &= \max_{A^* \subseteq A} \{ \sum_{a \in A^*} w_a : \text{ for some } x \in S \text{ we have } (\forall a \in A^* : \mathcal{B}_a(\text{dist}_a(x)) \leq \alpha\ ) \} \\ &= \max_{A^* \subseteq A} \{ \sum_{a \in A^*} w_a : \text{ for some } x \in S \text{ we have } \max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x)) \leq \alpha \}\end{aligned}$$

Since each function dist_a is assumed to be continuous, and, by Theorem 2, $\mathcal{B}_a$ is lower-semicontinuous, then, for any $A^* \subseteq A$, the function $x \longmapsto \max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x))$ is also lower-semicontinuous.

By Theorem 2, each $\mathcal{B}_a$ is non-decreasing, and, by assumption, the sets $\{x \in S : \text{dist}_a(x) \leq \beta\}$ are compact, thus, given $x_0 \in S$, for each $A^* \subseteq A$,

$$\begin{aligned}&\inf \{\max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x)) : x \in S\} = \\ &= \inf \{\max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x)) : x \in S, \text{dist}_a(x) \leq \text{dist}_a(x_0) \text{ for some } a \in A^*\},\end{aligned}$$

and the latter infimum is attained, since it is the infimum of a lower-semicontinuous function over the compact set

$$\bigcap_{a \in A^*} \{x \in S : \text{dist}_a(x) \leq \text{dist}_a(x_0)\}$$

Hence, for each non-empty $A^* \subseteq A$, there exists some $x(A^*)$ such that

$$\inf \{\max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x)) : x \in S\} = \max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x(A^*)))$$

Therefore,

$$\begin{aligned}\mathcal{Q}(\alpha) &= \max_{A^* \subseteq A} \{ \sum_{a \in A^*} w_a : \max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x(A^*))) \leq \alpha\} \\ &= \sum_{a \in A^*_\alpha} w_a\end{aligned} \tag{20}$$

for some $A^*_\alpha \subseteq A$ satisfying

$$\max_{a \in A^*_\alpha} \mathcal{B}_a(\text{dist}_a(x(A^*_\alpha))) \leq \alpha$$

Hence, for any $\alpha \geq \alpha_0$,

$$\mathcal{Q}(\alpha) \geq \beta \text{ iff } \alpha \geq \max \left\{ \min_{x \in S} \max_{a \in A^*} \mathcal{B}_a(\text{dist}_a(x)) : A^* \subseteq A, \sum_{a \in A^*} w_a \geq \beta \right\} \tag{21}$$

Now observe that, by the finiteness of A, the set of values taken by $CW(\alpha, x)$ when α ranges $[\alpha_0, +\infty[$ and x ranges S is finite: $c_1, \ldots, c_N$. Then,

$$\max_{\alpha \geq \alpha_0,\, x \in S} \pi(\sigma(CW(\alpha, x), \gamma(\alpha))) = \max_{1 \leq i \leq N} \max_{\alpha \geq \alpha_0, \mathcal{Q}(\alpha) \geq c_i} \pi(\sigma(c_i), \gamma(\alpha)),$$

thus, since by assumption, γ is strictly increasing, and π is strictly decreasing in its second argument, it follows from (21) that

$$\max_{\alpha \geq \alpha_0,\, x \in S} \pi(\sigma(CW(\alpha, x), \gamma(\alpha))) = \max_{1 \leq i \leq N} \pi(\sigma(c_i), \gamma(\min\{\alpha \geq \alpha_0 : \mathcal{Q}(\alpha) \geq c_i\})),$$

and this value is attained at a certain quality level α^*, from which an optimal location x^* is obtained by (20). □

Hence, under mild assumptions (which are automatically satisfied, e.g., under the assumptions of the model in Section 4, namely, S is a closed set in the plane and each dist_a is induced by a norm), novelty orientation leads to optimization problems which are well behaved, in the sense that an optimal solution (α, x) exists.

4 Location in the plane

4.1 General attraction and distance measure

When the facility is to be located in a closed convex subset S of the plane $\mathbb{R}^2$, distances are usually assumed to be measured by some norm (or gauge, when non-symmetric), possibly differing with demand point, see e.g. Plastria (1995). It follows that the distance function $\text{dist}_a(x)$ to a fixed point a is convex in x, see e.g. Michelot (1993) for this and further results on distances. This key property induces (quasi)convexity properties on the objective function of (19). We refer the reader to Avriel et al. (1988) for concepts and properties on quasiconvex functions.

Lemma 6 *Let $\mathcal{D}_a : \mathbb{R}^2 \longrightarrow [0, +\infty]$ be defined as*

$$\mathcal{D}_a : x \longmapsto \mathcal{B}_a(\text{dist}_a(x))$$

One then has:

1. *$\mathcal{D}_a$ is quasiconvex and lower-semicontinuous.*
2. *If $\mathcal{A}_a$ is quasiconcave, then $\mathcal{B}_a$ and $\mathcal{D}_a$ are convex.*

Proof.
Since distances dist_a are assumed to be induced by gauges, they are convex functions and therefore continuous. By Theorem 2, $\mathcal{B}_a$ is non-decreasing and lower-semicontinuous, thus, by composition, $\mathcal{D}_a$ is quasiconvex and lower-semicontinuous, showing part 1.

To show part 2, we first show that $\mathcal{B}_a$ is convex. Indeed, let $d_1, d_2 \geq 0$ and $0 < \lambda < 1$. By definition, for $i = 1, 2$, $\mathcal{B}_a(d_i)$ is the optimal solution to the problem

$$\begin{array}{lll} \inf & \alpha & \\ \text{s.t.} & \alpha & \geq \alpha_0 \\ & \mathcal{A}_a(\alpha, d_i) & \geq \mu_a \end{array} \tag{22}$$

Obviously, if any of these problems is infeasible, then the inequality

$$\mathcal{B}_a((1-\lambda)d_1 + \lambda d_2) \leq (1-\lambda)\mathcal{B}_a(d_1) + \lambda \mathcal{B}_a(d_2) \tag{23}$$

immediately follows since the right hand side equals $+\infty$.

If both problems are feasible, the upper semicontinuity of $\mathcal{A}_a(\cdot, d_i)$ implies that their feasible regions are closed intervals, thus their optimal values, $\mathcal{B}_a(d_1), \mathcal{B}_a(d_2)$, are attained. In other words,

$$\mathcal{A}_a(\mathcal{B}_a(d_i), d_i) \geq \mu_a \qquad \text{for } i = 1, 2$$

The quasiconcavity of $\mathcal{A}_a$ implies

$$\begin{aligned} \mu_a &\leq \min\{\mathcal{A}_a(\mathcal{B}_a(d_1), d_1), \mathcal{A}_a(\mathcal{B}_a(d_2), d_2)\} \\ &\leq \mathcal{A}_a((1-\lambda)(\mathcal{B}_a(d_1), d_1) + \lambda(\mathcal{B}_a(d_2), d_2)) \\ &= \mathcal{A}_a((1-\lambda)\mathcal{B}_a(d_1) + \lambda\mathcal{B}_a(d_2), (1-\lambda)d_1 + \lambda d_2), \end{aligned}$$

thus $(1-\lambda)\mathcal{B}_a(d_1) + \lambda\mathcal{B}_a(d_2)$ is feasible for the optimization problem

$$\min\{\alpha \geq \alpha_0 : \mathcal{A}_a(\alpha, (1-\lambda)d_1 + \lambda d_2) \geq \mu_a\},$$

the optimal solution of which is, by definition, $\mathcal{B}_a((1-\lambda)d_1 + \lambda d_2)$. Therefore (23) follows, thus $\mathcal{B}_a$ is convex.

Finally, $\mathcal{D}_a$ is the composition of the convex nondecreasing function $\mathcal{B}_a$ with the convex function dist_a, thus $\mathcal{D}_a$ is convex, which concludes part 2. □

Remark 7 Quasiconcavity of $\mathcal{A}_a$ (thus convexity of $\mathcal{D}_a$) is easily checked for the gravity-type model (1), the additive attraction model (2) for convex c_a, models (3)–(5), and for model (6) induced by an attraction function $\mathcal{A}_a$ which is quasiconcave.

One can then use the application of Helly-Drezner theorem presented in Carrizosa and Plastria (1995) to conclude that efficient solutions for (19) are optimal solutions to simple single-objective problems:

Theorem 8 *Let (α^*, x^*) be an efficient solution for (19). Then, one has:*

1. *When $Capt(\alpha^*, x^*) \neq \emptyset$ there exists a nonempty subset $T \subset A$, with cardinality at most 3 such that*
   (a) x^* solves the generalized single-facility minmax location problem
   $$\min_{x \in S} \max_{a \in T} \mathcal{D}_a(x) \qquad (P_T)$$
   (b) α^* is the optimal value of (P_T), which is finite.
2. *In case $Capt(\alpha^*, x^*) = \emptyset$ (i.e. $\mu_a \geq \mathcal{A}(\alpha_0, 0)$ for all a) one must have $\alpha^* = \alpha_0$, and any other pair (α_0, x) is then also efficient for (19).*

By Theorem 4, any profit-maximising solution (α^*, x^*) is also efficient for (19). Hence, Theorem 8 implies that, after finding the optimal value α_T of each of the $O(n^3)$ problems of the form (P_T), and finding the set S_T of optimal solutions for (P_T) we end up with a list $\mathcal{L}$ of pairs (α_T, x_T) known to contain the set of efficient solutions for (19). Therefore we obtain the following

General Algorithm in the Plane

Step 1 Initialise the list of candidate solutions $\mathcal{L}$ with the singleton $\{(\alpha_0, x_0)\}$, with $x_0 \in S$ arbitrarily chosen.

Step 2 For all $T \subset A$, with cardinality $1, 2$ or 3, do

1. Compute α_T, the optimal value of (P_T)
2. If $\alpha_T < +\infty$ then find the set S_T of optimal solutions for (P_T) and add $\{(\alpha_T, x_T) : x_T \in S_T\}$ to $\mathcal{L}$.

Step 3 For each $(\hat{\alpha}, \hat{x}) \in \mathcal{L}$ evaluate $\Pi(\hat{\alpha}, \hat{x})$ and select the one yielding the maximal value.

Observe that in step 2.2 all optimal solutions $x_T \in S_T$ will have the same corresponding quality value α_T. In step 3 only the ones with maximal captured weight will be retained. Therefore we may replace step 2.2 by

2. If $\alpha_T < +\infty$ then find the set S_T of optimal solutions for (P_T)
3. find the set S_T^* of optimal solutions to the subproblem
   $$\max\{CW(\alpha_T, x) \quad | \quad x \in S_T\} \qquad (24)$$
   and add $\{(\alpha_T, x_T) : x_T \in S_T^*\}$ to $\mathcal{L}$.

In general the problems (P_T), or its subproblem (24), may have an infinite number of optimal solutions, thus the algorithm as such does not always yield a finite procedure. When S_T, or S_T^*, is infinite we may retain just one solution in S_T^* which will still guarantee finding at least one optimal solution.

Observe that the subproblem (24) to be solved is just a maximal covering location problem, but with a peculiar locational constraint $x \in S_T$. If the corresponding solution (α_T, x_T^*) turns out to be the optimal profit-mix, then all optimal solutions to this subproblem will be optimal profit locations, all combined with the same quality α_T.

As we will see in the next section, in several important cases the problems (P_T) have a finite, often unique solution. Although (quasi)convex (see Lemma 6 and Theorem 8)), solving (P_T) is usually not straightforward, and will involve a convergent iterative search, see e.g. Plastria (1988) and Section 4.4. Step 2 involves solving $O(n^3)$ problems (P_T), so this will take quite some time.

For a given solution (α, x) the captured weight can be computed in $O(n)$ time in a straightforward manner. Therefore, when S_T (or S'_T) is finite for all T, Step 3 can be performed in $O(n|\mathcal{L}|)$, where $|\mathcal{L}|$ denotes the cardinality of the list $\mathcal{L}$ generated in Step 2.

In the following we show how in some very relevant cases, the structure of the function $\mathcal{A}_a$ and the geometry of the distances dist_a can be used to obtain finite and low-complexity procedures.

4.2 Gravity-type attraction and Euclidean distances

Let us now focus on the case of gravity-type attraction functions and Euclidean distances in the Euclidean plane. This means that for any $a \in A$ we have, as introduced in (1),

$$\mathcal{A}_a(\alpha, x) = \frac{k_a \alpha}{\|x - x_a\|^p}, \tag{25}$$

($\|x - x_a\|$ standing here for the Euclidean distance from x_a to x), which leads to (see Example 3.1)

$$\mathcal{D}_a(x) = \max\left\{\alpha_0, \left(\frac{\mu_a}{k_a}\right)\|x - x_a\|^p\right\}$$

Under these assumptions, the problems (P_T) introduced in Section 4.1 have a rich structure which will allow us to easily identify their optima. First we have that (P_T) can be re-written as

$$\min_{x\in S} \max\left(\alpha_0, \max_{a\in T} \mathcal{D}_a(x)\right)$$

Defining Problem (Q_T) as

$$\min_{x\in S} \max_{a\in T} \frac{\mu_a}{k_a}\|x - x_a\|^p \tag{Q_T}$$

we see that both problems are equivalent (same optimal solutions, same objective value), except for the degenerate case in which (Q_T) has an optimal value not greater than α_0. This enables us to rephrase Theorem 8 as

Theorem 9 *Let (α^*, x^*) be an efficient solution for (19). Then, one has:*

1. *If $Capt(\alpha^*, x^*) \neq \emptyset$ then there exists a nonempty subset $T \subset A$, with cardinality at most 3 such that*
   - *(a) x^* solves the single-facility minmax location problem (Q_T)*
   - *(b) α^* is the maximum between α_0 and the optimal value of (Q_T), which is finite.*
2. *If $Capt(\alpha^*, x^*) = \emptyset$ (i.e. $\mu_a \geq \mathcal{A}(\alpha_0, 0)$ for all a) then $\alpha^* = \alpha_0$, and any pair (α_0, x) is also efficient for (19).*

For Problem (Q_T) we first have:

Lemma 10 *For each $T \subset A$, Problem (Q_T) has exactly one optimal solution x_T.*

Proof.
With the current assumptions $\mathcal{B}_a$ is strictly increasing and $\|\cdot\|$ is a *round* norm, thus the uniqueness of solution follows from the general results on minmax problems with round norms described in Pelegrín-Michelot-Plastria (1985). □

Since, at *the* optimal solution x_T of (Q_T), at least one function $\mathcal{D}_a$ is *active*, i.e. $\mathcal{D}_a(x_T) = \max_{b\in T} \mathcal{D}_b(x_T)$, in order to find x_T we can search for it sequentially in the locus of points where exactly one, then where exactly two, and finally where exactly three of the functions are active.

It follows that we will need to consider sets of points of following form. Define the *mediatrix* of two consumers a, b located at different sites ($x_a \neq x_b$) — it would be empty otherwise — as the following set of points of the plane

$$\mathrm{med}(a, b) = \{x \in \mathbb{R}^2 : \frac{\mu_a}{k_a}\|x - x_a\|^p = \frac{\mu_b}{k_b}\|x - x_b\|^p\}$$

Denoting $\lambda_a \equiv (\mu_a/k_a)^{1/p}$ and by $d_a(x) \equiv \lambda_a\|x - x_a\|$ the Euclidean distance between consumer's a's site and x, inflated by the factor λ_a, we may also write

$$\mathrm{med}(a, b) = \{x \in \mathbb{R}^2 : d_a(x) = d_b(x)\}$$

which is much better known. This set is a circle, the Appolonius circle (also called equicircle by Hearn and Vijay, 1982), which degenerates into a straight line when inflation factors are equal. As shown e.g. in Okabe-Boots-Sugihara (1992), for $\lambda_a \neq \lambda_b$ the mediatrix med(a, b) is the circle with centre the point

$$m_{ab} = \frac{\lambda_a^2 x_a - \lambda_b^2 x_b}{\lambda_a^2 - \lambda_b^2}$$

of the line joining x_a and x_b, and radius

$$\rho_{ab} = \frac{\lambda_a\lambda_b \parallel x_a - x_b \parallel}{|\lambda_a^2 - \lambda_b^2|}.$$

For $\lambda_a = \lambda_b$ we obtain as med(a, b) the straight line of equation

$$\langle\ x_a - x_b\ ,\ x - \frac{x_a + x_b}{2}\ \rangle\ = 0$$

(where $\langle\cdot, \cdot\rangle$ denotes scalar product), which is the well known perpendicular bisector of the segment $[x_a, x_b]$.

In order to describe the geometric properties of this mediatrix, it is convenient to choose momentarily a new orthonormal coordinate axis with origin x_a and $x_b - x_a$ as first unit vector, and to define $\lambda_{ab} = \lambda_a/\lambda_b > 0$. In this coordinate system $x_a = (0, 0)$, $x_b = (1, 0)$ and, for $\lambda_{ab} \neq 1$, med(a, b) is the circle with centre

$$m_{ab} = (\frac{1}{1 - \lambda_{ab}^2}, 0)$$

and radius

$$\rho_{ab} =| \frac{\lambda_{ab}}{1 - \lambda_{ab}^2} |,$$

while for $\lambda_{ab} = 1$ it is the vertical line through $(1/2, 0)$. The intersection points of med(a, b) with the horizontal axis are

$$z_{ab} = (\frac{1}{1 + \lambda_{ab}}, 0)$$

and

$$Z_{ab} = (\frac{1}{1 - \lambda_{ab}}, 0)$$

(note that these formulae remain valid for $\lambda_{ab} = 1$, pushing the latter Z_{ab} to infinity). Observe that the first of these intersection-points always lies on the segment $[x_a, x_b]$. Note also that for

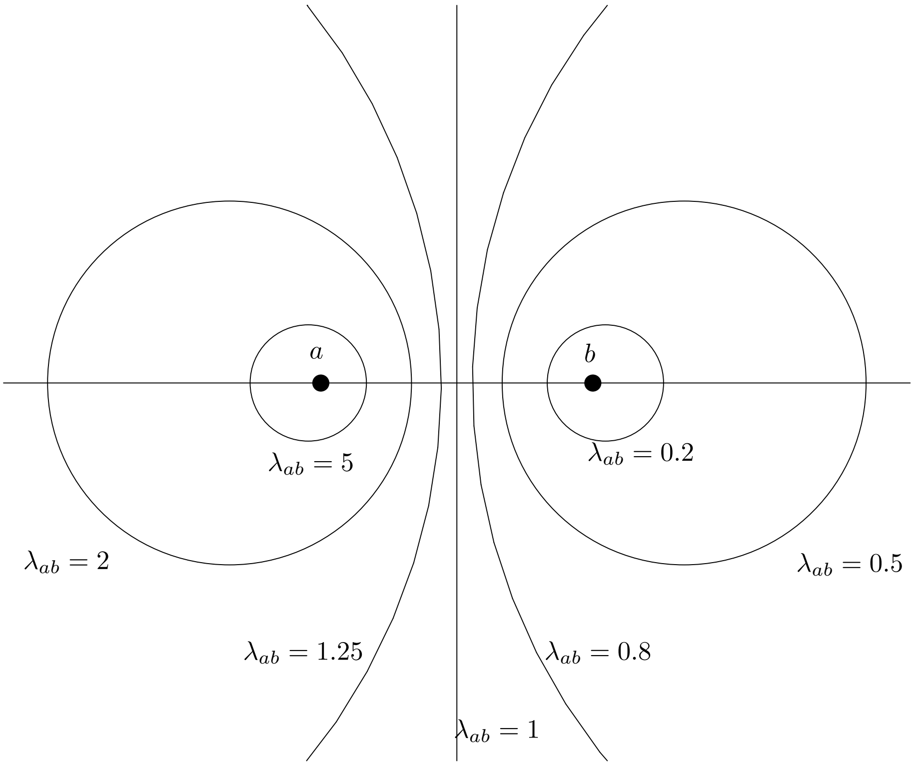

Figure 7: $\text{med}(a, b)$ for different values of $\lambda_{ab} = \lambda_a/\lambda_b$

$\lambda_{ab} \neq 1$ the centre m_{ab} lies on the line $x_a x_b$, but always outside the segment $[x_a, x_b]$, closest to that point among x_a, x_b with highest corresponding λ, and that it is this latter point that lies inside the circle. Figure 7 illustrates these properties.

Assuming $\lambda_{ab} < 1$, the points of $\text{med}(a, b)$ may be described parametrically as $x = (t, h)$ with

$$\frac{1}{1+\lambda_{ab}} \leq t \leq \frac{1}{1-\lambda_{ab}}$$

and

$$h^2 = (t - \frac{1}{1+\lambda_{ab}})(\frac{1}{1-\lambda_{ab}} - t)$$

while the inflated distances to x_a and x_b are given by

$$d_a(x) = d_b(x) = \sqrt{\frac{2t-1}{1-\lambda_{ab}^2}}$$

which is an increasing function of t. For $\lambda_{ab} > 1$ the first inequalities above should simply be inverted, and the inflated distance then decreases with t.

In any case inflated distance is always minimised on $\text{med}(a, b)$ at the point z_{ab} for $t = \frac{1}{1+\lambda_{ab}}$ and grows continuously along both upper and lower half-circles in symmetric way towards its (common) maximum at Z_{ab} reached when $t = \frac{1}{1-\lambda_{ab}}$.

Lemma 11 *Let a, b and c be three non-collinear points in the plane with strictly positive inflation factors λ_a, λ_b and λ_c. Then at most one point equidistant from a, b and c exists lying within the triangle formed by them.*

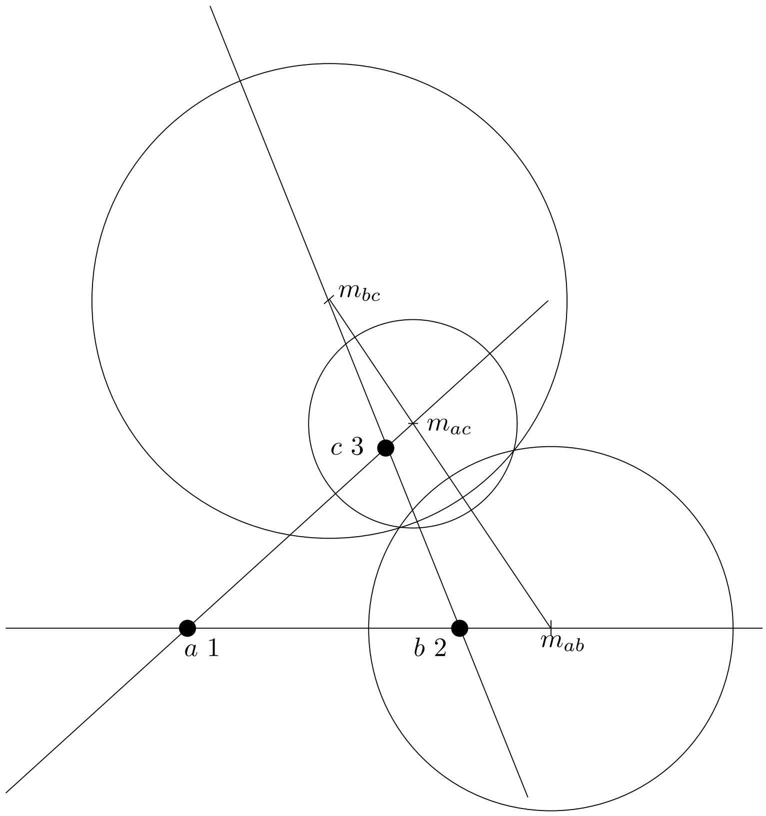

Figure 8: med(a, b), med(a, c) and med(b, c)

Proof.
The case of equal inflation factors is trivial, so we assume the inflation factors to be different.

The non-collinearity of points a, b, c implies that $\text{med}(a, b), \text{med}(a, c)$ and $\text{med}(b, c)$ are different circles. Any point common to at least two of the mediatrices med(a, b), med(b, c) and med(c, a) is a point equidistant from a, b and c, hence is common to all three of them.

Two (or three) different circles have at most two points in common. So we may assume further there are exactly two equidistant points. Three circles can only have two points in common if they have a common symmetry axis and this axis separates these two points. Any symmetry axis of a circle contains the circle's centre. It follows that the centres m_{ab}, m_{bc} and m_{ca} lie on one line; in fact they are the points of intersection of this symmetry axis with each of the boundary lines of the triangle (this is illustrated by Figure 8). Since we know that the centre of any mediatrix lies outside the segment joining the two defining points, it follows easily that the triangle lies fully at one side of the symmetry axis. And since this axis separates the two equidistant points the triangle may contain at most one of them. □

With these preparatory results we have obtained

Theorem 12 *Let $T = \{a, b, c\} \subset A$. Let x_T be the optimal solution of (Q_T), and denote by $act(T)$ the set of objectives of (Q_T) active at x_T, i.e.,*

$$act(T) = \left\{ e^* \in T : \, d_{e^*}(x_T) = \max_{e \in T} d_e(x_T) \right\}$$

1. *If act*$(T) = \{a\}$*, then* x_T *is the point of* S *closest (with respect to the Euclidean distance) to* x_a
2. *If act*$(T) = \{a, b\}$*, then* x_T *is the point of* S *on* $\text{med}(a, b)$ *minimising* $d_a(x)$*, if any, and is found*
   - *either on the line segment joining* x_a *and* x_b*, if this (unique) point* z_{ab} *of* $\text{med}(a, b)$ *lies in* S
   - *or by intersecting* $\text{med}(a, b)$ *with* S*'s boundary and selecting among the intersection points (if any) the one(s) (at most two of such) closest to* x_a *(or equivalently to* x_b*)*
3. *If act*$(T) = \{a, b, c\}$*, then* x_T *is the intersection of the three mediatrices corresponding to each choice of two points among them and the triangle with vertices the points in* T *and lies in* S*.*

It follows that the list of candidate efficient solutions is of length at most $n + 2n(n-1)/2 + n(n-1)(n-2)/6 = n(n+1)(n+2)/6$, where n is the cardinality of A. It is therefore an easy task to construct this full list of length $O(n^3)$, and to evaluate each candidate, taking $O(n)$ time for evaluating the captured weight each time, which leads to an overall $O(n^4)$ method.

In fact this same task may be done more efficiently by mixing the evaluation of each candidate solution with the generation process, leading to a much better algorithm of complexity $O(n^3 \log n)$. For the technical details of this method the interested reader is referred to Plastria (1996).

Now we illustrate the method just described by means of a small-size example.

Example 13
Consider a set $A = \{a_1, \ldots, a_{10}\}$ of consumers, located at the points plotted as empty circles in Figure 9, with coordinates x_a and weights ω_a given in Table 2.

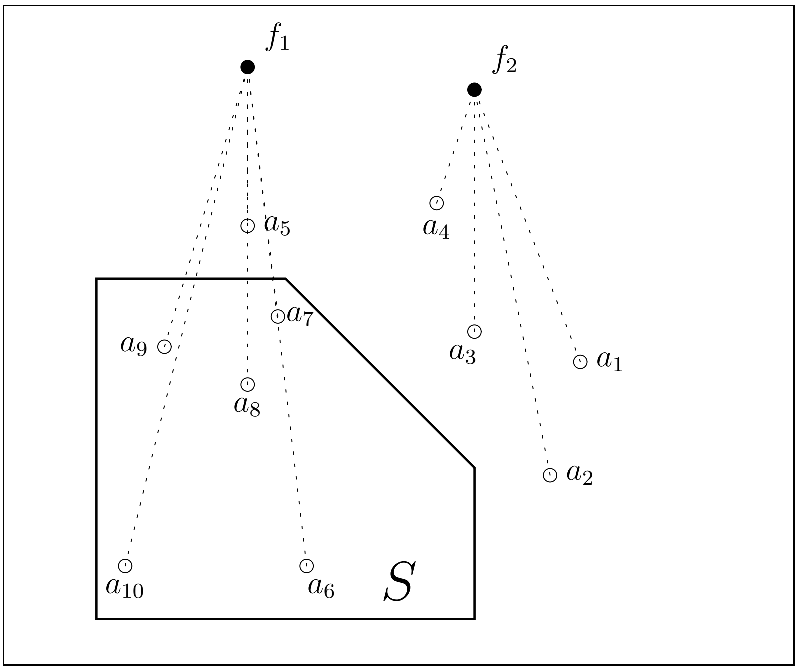

Figure 9: Consumers and existing facilities

A set of two facilities, $CF = \{f_1, f_2\}$, plotted as solid circles in Figure 9, already compete in the region. Their coordinates x_f and qualities α_f are given in Table 1.

f	x_f	α_f
f_1	$(20, 73)$	1250
f_2	$(50, 70)$	1000

Table 1: Existing facilities

We assume that for each consumer a the attraction to any facility has the form (25) with each $k_a = 1$, $\alpha_0 \approx 0$, $\alpha_0 = 0.000001$, say, and $p = 2$ for each a, i.e.

$$\mathcal{A}_a(\alpha, x) = \frac{\alpha}{\|x - x_a\|^2}$$

Before the advent of the new facility, consumer a is captured by the existing facility yielding highest attraction. This is determined by calculating the decisive attractions μ_a following (11):

$$\mu_a = \max\left\{ \frac{\alpha_f}{\|x_a - x_f\|^2} \mid f \in CF \right\}$$

which are listed together with the currently attracting facility $f(a)$ in the last two columns of Table 2. See also Figure 9 for the current patronisings.

a	x_a	ω_a	μ_a	$f(a)$
a_1	$(64, 34)$	600	0.6702	f_2
a_2	$(60, 19)$	100	0.3702	f_2
a_3	$(50, 38)$	100	0.9766	f_2
a_4	$(45, 55)$	100	4.0000	f_2
a_5	$(20, 52)$	400	2.8345	f_1
a_6	$(27.8, 7)$	300	0.2830	f_1
a_7	$(24, 40)$	100	1.1312	f_1
a_8	$(20, 31)$	100	0.7086	f_1
a_9	$(9, 36)$	100	0.8389	f_1
a_{10}	$(3.8, 7)$	600	0.2707	f_1

Table 2: Attraction parameters

A new facility must be located within the region S, — represented in Figure 9 by the convex polygon with endpoints $(0, 0), (50, 0), (50, 20), (25, 45), (0, 45)$ — to maximize a profit indicator function Π of type (8), whose parameters σ and γ will be specified later.

With this information, we already know, by Theorems 4 and 12, that an optimal solution (α^*, x^*) can be found within a finite set of candidate points, the locations and qualities of which are obtained by intersecting discs in the plane. This set of candidate locations is plotted in Figure 10 as small solid circles. One can see that 10 candidate points with a single active objective are present (the closest-point projections of each customer-site on S — coinciding with the site when in S), the other candidate points along S's boundary where two objectives are active, and several candidate points within $S \setminus A$ where three objectives are active.

However, most of the elements in this set are not real candidates, since they are not efficient for problem (19). After eliminating dominated elements, (see Carrizosa -Plastria 1995, 1998b for details), one obtains the reduced list of 12 candidates (α_i, x_i), given in Table 3 and shown in Figure 11 as small solid circles. Figure 12 shows a plot of attractiveness α_i versus market share $CW(x_i)$ for these efficient solutions.

Assume now that further assumptions on Π are made. For instance, suppose Π corresponds a sales-minus-cost model (9) in which sales income are assumed to be proportional to the weight

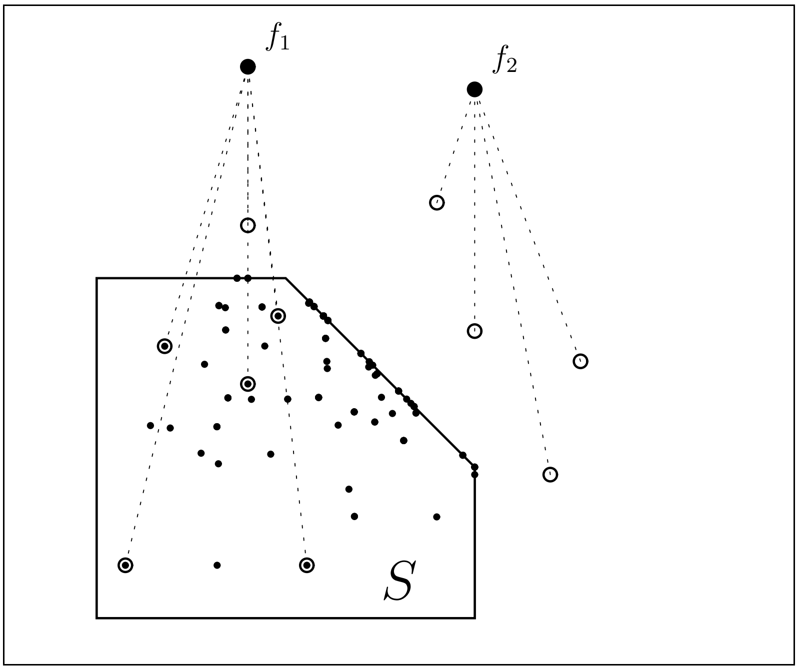

Figure 10: Location of candidates

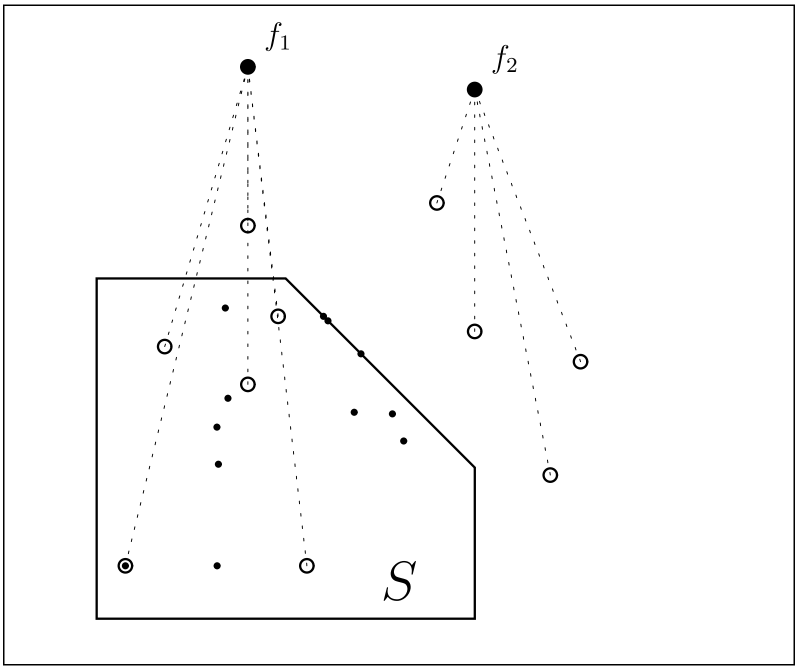

Figure 11: Location of efficient solutions

captured, with proportionality constant σ, and operating costs are proportional to the quality of the facility, with proportionality constant γ. In this case, the profit indicator Π takes the simpler form

$$\Pi(\alpha, x) = \sigma \cdot CW(\alpha, x) - \gamma \cdot \alpha, \tag{26}$$

for given constants $\sigma, \gamma > 0$.

i	x_i	α_i	$CW(\alpha_i, x_i)$
1	$(3.8000, 7.0000)$	0.0000	600
2	$(15.9339, 7.0000)$	39.8488	900
3	$(16.1018, 20.4373)$	89.8289	1000
4	$(15.9074, 25.3450)$	135.2698	1100
5	$(17.3649, 29.1604)$	182.7161	1200
6	$(34.0663, 27.3086)$	359.5603	1300
7	$(17.0163, 41.1000)$	361.9952	1600
8	$(40.6091, 23.5091)$	440.4785	1800
9	$(39.1179, 27.0960)$	446.9055	1900
10	$(34.9578, 35.0422)$	566.0434	2000
11	$(30.5932, 39.4068)$	767.5907	2400
12	$(30.0000, 40.0000)$	1800.0000	2500

Table 3: Efficient elements

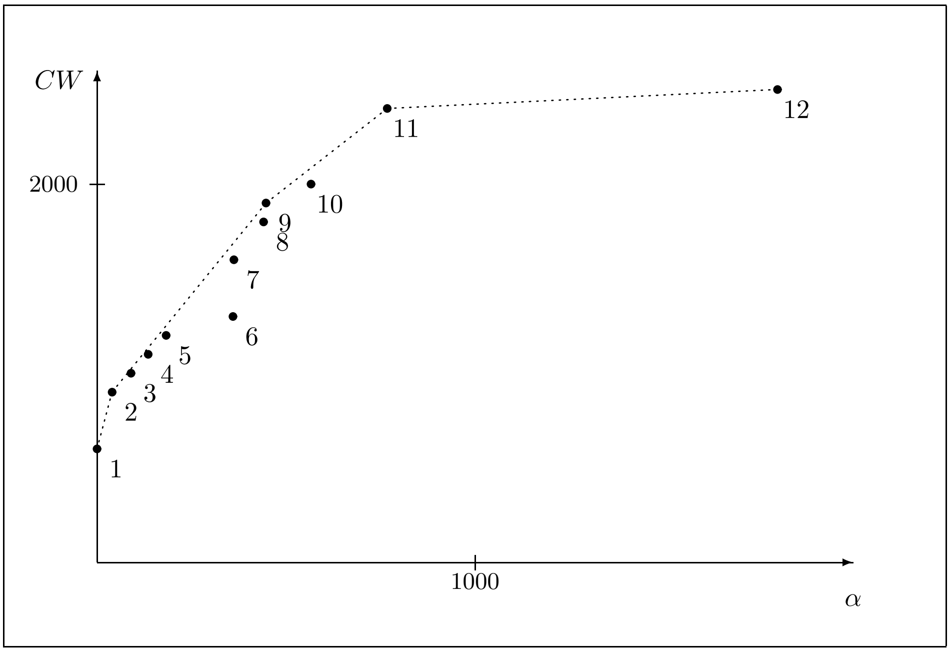

Figure 12: Plot of efficient solutions in value-space

Finding a profit maximising solution under model (26) amounts to evaluate the corresponding Π at the candidate solutions (α_i, x_i), $i = 1, 2, \ldots, 12$ of Table 3. For instance, for the choice $\sigma = 42$, $\gamma = 100$, a profit maximising solution consists of locating at $x_9 = (39.1179, 27.0960)$ a facility with quality $\alpha_9 = 446.9055$, as plotted in Figure 13. See also Table 6 in Section 4.4.

The fact that the candidate list is finite allows us to go much further: assuming model (26), for *unknown* parameters σ, γ, a full parametric analysis is possible by pairwise comparison of each solution in the candidate set. Indeed, observing that

$$\Pi(\alpha, x) = \left(\frac{\sigma}{\gamma} CW(\alpha, x) - \alpha\right) \cdot \gamma,$$

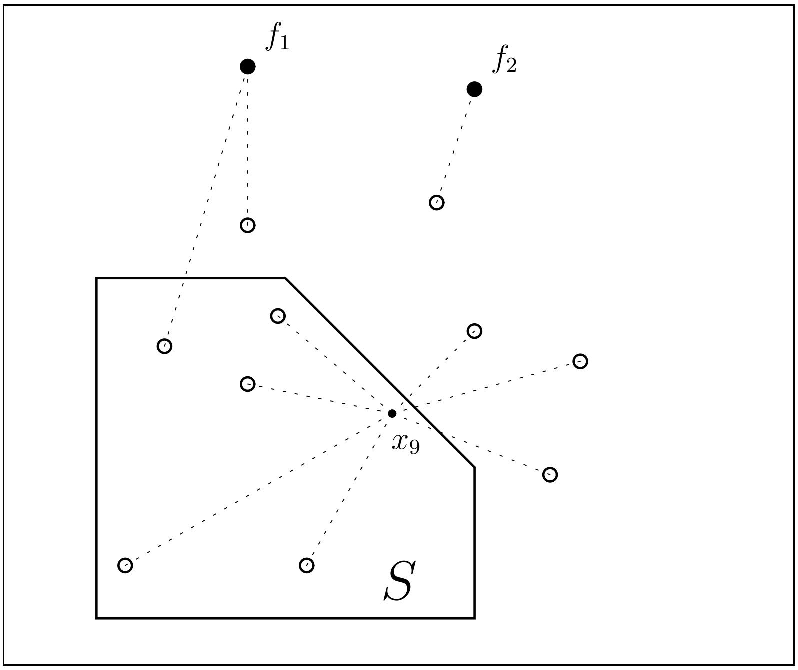

Figure 13: Optimal location for (26) with $\sigma = 42$, $\gamma = 100$

we see that maximising $\Pi(\alpha, x)$ is equivalent to maximising

$$\Pi_\tau(\alpha, x) = \tau CW(\alpha, x) - \alpha.$$

Comparing the solutions in Table 3 we obtain, for each (α_i, x_i), the interval $[\underline{\tau_i}, \overline{\tau_i}]$ such that (α_i, x_i) is a profit-maximising solution under model (26) if and only if τ belongs to $[\underline{\tau_i}, \overline{\tau_i}]$. The results are summarized in Table 4, where it is readily seen that not all the efficient solutions of Table 3 can be optimal under model (26): only the extreme efficient solutions, vertices of the upper left boundary of the convex hull of the efficient value pairs, shown by a dotted line in Figure 12, appear. This is due to the fact that profit is linear in α and CW.

i	x_i	α_i	$CW(\alpha_i, x_i)$	$\underline{\tau_i}$	$\overline{\tau_i}$
1	$(3.8000, 7.0000)$	0.0000	600	0.0000	0.1328
2	$(15.9339, 7.0000)$	39.8488	900	0.1328	0.4071
9	$(39.1179, 27.0960)$	446.9055	1900	0.4071	0.6414
11	$(30.5932, 39.4068)$	767.5907	2400	0.6414	10.3241
12	$(30.0000, 40.0000)$	1800.0000	2500	10.3241	∞

Table 4: Full parametric analysis under model (26)

As mentioned before, the applicability of our methodology is not at all restricted to the sales-minus-costs model (26), and a similar analysis can be carried out for other choices of the profit indicator function Π. For instance, assume now that Π is of the type sales divided by operating costs, where operating costs consist of a fixed cost γ_0 and a cost proportional to quality, i.e.,

$$\Pi(\alpha, x) = CW(\alpha, x) / (\gamma_0 + \gamma \cdot \alpha,) \tag{27}$$

Again by inspecting the efficient solutions (α_i, x_i), we can find, for each (α_i, x_i) the interval $[\underline{\tau_i}, \overline{\tau_i}]$ such that (α_i, x_i) is a profit-maximising solution under model (27) if and only if $\frac{\gamma_0}{\gamma}$ belongs to $[\underline{\tau_i}, \overline{\tau_i}]$. See the results in Table 5, where we again only find the extreme efficient solutions, now due to the fact that profit is a linear fractional function of quality and covered weight.

i	x_i	α_i	$CW(\alpha_i, x_i)$	$\underline{\tau_i}$	$\overline{\tau_i}$
1	$(3.8000, 7.0000)$	0.0000	600	0.0000	79.6976
2	$(15.9339, 7.0000)$	39.8488	900	79.6976	326.5023
9	$(39.1179, 27.0960)$	446.9056	1900	326.5023	771.6985
11	$(30.5932, 39.4068)$	767.5908	2400	771.6985	24010.2417
12	$(30.0000, 40.0000)$	1800.0005	2500	24010.2417	∞

Table 5: Full parametric analysis under model (27)

Remark 14 The geometric approach described above has as cornerstone the fact that mediatrices are circles. This property extends to slightly different attraction functions such as

$$\mathcal{A}_a^1(\alpha, d) = \frac{k_a \alpha}{h_a + d^2}$$

or the particular case of (2) given by

$$\mathcal{A}_a^2(\alpha, d) = \max\{0, \alpha k_a - h_a d^2\}.$$

Indeed, one easily obtains

$$\mathcal{B}_a^1(d) = \max\{\alpha_0, \frac{\mu_a}{k_a}(h_a + d^2)\}$$

and

$$\mathcal{B}_a^2(d) = \max\{\alpha_0, \frac{\mu_a + d^2}{k_a}\}$$

for $\mu_a > 0$ or $\mathcal{B}_a^2(d) = \alpha_0$ for $\mu_a = 0$, see Example 3.2.

In both cases mediatrices are of circular shape (see Carrizosa and Plastria, 1998a), and may be handled in a way similar to the method explained in this section.

4.3 Step attractions and Euclidean distances

Both the minimum disutility attraction (3) and the step attraction (5) lead to a binary $\mathcal{B}_a$ of the form (17),

$$\mathcal{B}_a(d) = \begin{cases} \underline{\alpha}_a, & \text{if } d \le \overline{d}_a \\ +\infty, & \text{else,} \end{cases}$$

Hence, the function $\mathcal{D}_a$ will have under Euclidean distances the form

$$\mathcal{D}_a(x) = \begin{cases} \underline{\alpha}_a, & \text{if } \|x - x_a\| \le \overline{d}_a \\ +\infty, & \text{else,} \end{cases}$$

for some threshold value $\overline{d}_a$.

This shows how different shapes for the attraction functions $\mathcal{A}_a$ can lead to the same function $\mathcal{D}_a$, thus also to the same subproblems (P_T) to be solved and the same set of candidate points as output of our general algorithm.

Let us now have a closer look at the optimisation problems (P_T) obtained in this particular case. Since $\mathcal{B}_a$ is no longer strictly increasing (it is in fact constant when finite!) no analog to Lemma 10 holds, and the list $\mathcal{L}$ obtained by directly using the general algorithm described in

Section 4.1 would be infinite (and useless!). Nevertheless, the structure of this problem can be used to develop simple procedures to skip this technical handicap.

Indeed, if we denote by D_T the (possibly empty) intersection of the disks centered at x_a, x_b, x_c with radii $\overline{d}_a, \overline{d}_b, \overline{d}_c$, it follows that the set of optimal solutions for (P_T) is D_T, whereas the optimal value is $\alpha_T := \max\{\underline{\alpha}_a, \underline{\alpha}_b, \underline{\alpha}_c\}$.

This implies the following

Theorem 15 *Let (α^*, x^*) be an efficient solution for (19) with $CW(\alpha^*, x^*) > 0$. Then there exists some $a \in A$ such that*

1. $\alpha^* = \underline{\alpha}_a$
2. $\|x^* - x_a\| \leq \overline{d}_a$

Moreover, the set $C(x^*)$ of points

$$\left\{x \in S : \|x - x_b\| \leq \overline{d}_b \text{ for all } b \in A \text{ such that } \|x^* - x_b\| \leq \overline{d}_b\right\}$$

is such that any pair (α^*, x), with $x \in C(x^*)$, is also efficient for (19), with same quality and coverage than (α^*, x^*).

Consider for each $a \in A$ a maxcovering problem as discussed by T. Drezner (1994) with customerset reduced to $A(\alpha_a) = \{b \in A : \alpha_b \leq \alpha_a\}$, keeping the original radii and weights. Then, finding one optimal solution for each of these maxcovering problems leads to a list $\mathcal{L}^*$ such that, for any efficient solution (α, x), there is some (α^*, x^*) in $\mathcal{L}^*$ which is equivalent (same coverage, same quality) to (α, x).

One may then find the optimal profit quality α^* by inspecting all pairs in $\mathcal{L}^*$ and then apply Theorem 15 using this α^* in order to obtain all optimal profit locations.

4.4 Other planar distance measures

The question whether non-Euclidean distances in the plane might be handled is a quite subtle one.

The basic theory carries through since (inflated) distance functions remain convex, yielding the same basic properties of the candidate efficient solutions, see Carrizosa and Plastria (1995). However, it is less evident whether the corresponding candidate points may easily be calculated.

First it will not be easy to detect that all candidates have indeed been generated because (some of) the problems (P_T) may have several (usually isolated) solutions. Assuming each $\mathcal{B}_a$ is strictly increasing and round norms, e.g. ℓ_p distances $(1 < p < \infty)$, Lemma 10 remains valid, so the process reduces to finding *the* $O(n^3)$ optimal solutions of the (quasi)convex small-scale problems (P_T).

However, the geometrical procedure described in Section 4.2 does not seem to extend to this more general context. Indeed, although determining closest points to a polygon in a non-Euclidean metric is not too hard, the notion of Appolonius circle does not work anymore; these should be replaced by equidistance sets, which often do not have analytical equations, meaning that intersecting them with S's boundary will not be easy, nor finding points of equal distance to a triplet of consumers.

In general one will have to rely on iterative techniques for determination of each type of point. Since $O(n^3)$ such iterative algorithms should be carried out, it seems evident that the computational burden will substantially increase with the number n of consumers.

This was attempted for different ℓ_r norms using the Optimization toolbox of Matlab to solve each of the subproblems (Q_T). Table 6 shows the results obtained for the additive model (26), with $\sigma = 42, \gamma = 100$, for different values of r.

For simple distance measures like the rectangular ℓ_1, or any other block norm, these difficulties are relatively easy to overcome and geometric procedures can again be used. A new difficulty arises, however, when considering such norms because they are not round, hence the subproblems (P_T) might have an infinite number of solutions and in each such case another subproblem of LP type has to be solved to obtain S_T^*. See Carrizosa and Plastria (1998b) for further details.

r	$x(r)$	$\alpha(r)$	$CW(\alpha(r), x(r))$	Π
1.60	(16.0556, 7.0000)	38.0152	900	33998.4794
1.65	(16.0359, 7.0000)	38.3388	900	33966.1219
1.70	(16.0177, 7.0000)	38.6292	900	33937.0758
1.75	(38.4713, 27.2559)	457.7094	1900	34029.0640
1.80	(38.6153, 27.2240)	455.4035	1900	34259.6471
1.85	(38.7515, 27.1921)	453.1701	1900	34482.9870
1.90	(38.8805, 27.1601)	451.0095	1900	34699.0519
1.95	(39.0024, 27.1281)	448.9215	1900	34907.8543
2.00	(39.1179, 27.0960)	446.9055	1900	35109.4438

Table 6: Optimal solutions under (27) for varying distance measure ℓ_r

5 Alternate tie resolution rules

Throughout this paper we have assumed novelty orientation as tie breaking rule. The other extreme tie resolution rule, as compared to novelty orientation, states that in case of an attraction tie with the new facility the existing facility will be favored. This rule may be called *conservative behaviour* by the customers.

T. Drezner's (1995) rule takes the middle path in between novelty orientation and conservatism: in case of attraction tie the demand is split equally between new and existing facility. When demand points are considered as population centres and attraction is viewed as behaviour of individual customers, this assumption means that customers are novelty oriented with probability of 0.5 and otherwise conservative. It is clear that we may generalise this to a *θ-mixed behaviour*, θ being a vector of the form $(\theta_a)_{a\in A}$, $0 \le \theta_a \le 1\, \forall a \in A$, stating that in case of a tie for consumer a, a fraction θ_a of his/her demand will patronise the new facility, the remaining fraction $1 - \theta_a$ patronising the existing facilities.

Note that 0-mixed behaviour is the same as conservatism, while novelty orientation corresponds with θ-mixed behaviour, θ being then a vector with all its coordinates equal to 1.

The following result states that in fact, novelty orientation is the unique $\theta-$mixed behaviour which is guaranteed to always yield well behaved problems.

For any (α, x) let us define the set of active demand points

$$act(\alpha, x) \equiv \{a \in A \mid \mathcal{B}_a(\mathrm{dist}_a(x)) = \alpha\}.$$

The total attracted weight in case of θ-mixed behaviour may then be written as

$$CW_\theta(\alpha, x) \equiv CW(\alpha, x) - \sum_{a\in act(\alpha,x)} (1 - \theta_a) w_a$$

The profit for θ-mixed behaviour is now given as

$$\Pi_\theta(\alpha, x) = \pi(\ \sigma(CW_\theta(\alpha, x))\ ,\ \gamma(\alpha)\ ).$$

Theorem 16 *Let the profit-indicator function $\pi(t,\cdot)$ be continuous for any fixed t and the cost function γ also be continuous. Under θ-mixed behaviour, if a profit-maximising pair (α^*, x^*) exists with $\alpha^* > \alpha_0$, then at least one consumer $a \in Capt(\alpha, x)$ (i.e., $\mathcal{B}_a(\mathrm{dist}_a(x^*)) \le \alpha^*$) is novelty-oriented, i.e. $\theta_a = 1$.*

In particular if all θ_a are equal to some fixed θ_0 and a profit-maximising pair (α^*, x^*) exists with $\alpha^* > \alpha_0$, then $\theta_0 = 1$, i.e. we have full novelty orientation.

Proof.
Let (α^*, x^*) be a profit-maximising solution with $\alpha^* > \alpha_0$. We first show that $act(\alpha^*, x^*)$ is

non-empty. Indeed, if $act(\alpha^*, x^*) = \emptyset$, then any consumer $a \in A$ is either strictly covered ($\mathcal{B}_a(\mathrm{dist}_a(x^*)) < \alpha^*$) or strictly uncovered, ($\mathcal{B}_a(\mathrm{dist}_a(x^*) > \alpha^*$).

Defining $\bar{\alpha}$ as the lowest possible quality yielding the same market share,

$$\bar{\alpha} = \max\{\alpha_0, \max\{\mathcal{B}_a(\mathrm{dist}_a(x^*)) : \mathcal{B}_a(\mathrm{dist}_a(x^*)) < \alpha^*\}\} < \alpha^*,$$

we then have that

$$\begin{aligned} CW(\bar{\alpha}, x^*) &= CW(\alpha^*, x^*) \\ \gamma(\bar{\alpha}) &< \gamma(\alpha^*), \end{aligned}$$

thus, by the strictly decreasing character of $\pi(t, \cdot)$,

$$\Pi(\bar{\alpha}, x^*) > \Pi(\alpha^*, x^*),$$

contradicting the optimality of (α^*, x^*). Hence,

$$act(\alpha^*, x^*) \neq \emptyset, \tag{28}$$

as asserted.

We are now in position to show that there exists some (partially) captured $a \in A$ such that $\theta_a = 1$. By contradiction, assume that

$$\theta_a < 1 \ \ \forall a \in A \text{ such that } \mathcal{B}_a(\mathrm{dist}_a(x^*)) \leq \alpha. \tag{29}$$

Define α_1 as

$$\alpha_1 = \max\{\alpha_0, \min\{\mathcal{B}_a(\mathrm{dist}_a(x^*)) : \mathcal{B}_a(\mathrm{dist}_a(x^*)) > \alpha^*, \, a \in A\}\},$$

or $\alpha_1 = +\infty$ if the latter set is empty.

In particular, $\alpha_1 > \alpha^*$. Hence, for any $\alpha \in]\alpha^*, \alpha_1[$

$$\begin{aligned} CW_\theta(\alpha, x^*) &= CW(\alpha, x^*) \\ &= CW(\alpha^*, x^*) \\ &> CW_\theta(\alpha^*, x^*) \end{aligned}$$

In particular,

$$\pi(CW_\theta(\alpha, x^*), \gamma(\alpha)) = \pi(CW_\theta(\alpha^*, x^*), \gamma(\alpha)),$$

thus, by the continuity of γ and $\pi(t, \cdot)$,

$$\begin{aligned} \lim_{\alpha \downarrow \alpha^*} \pi(CW_\theta(\alpha, x^*), \gamma(\alpha)) &= \lim_{\alpha \downarrow \alpha^*} \pi(CW(\alpha^*, x^*), \gamma(\alpha)) \\ &= \pi(CW(\alpha^*, x^*), \gamma(\alpha^*)) \\ &> \pi(CW_\theta(\alpha^*, x^*), \gamma(\alpha^*)) \end{aligned}$$

Hence, for α (slightly) greater than α^*, one would obtain, at the same location, a strictly greater profit, contradicting the optimality of (α^*, x^*).

The second claim is then straightforward. □

In spite of this negative result for $\theta \neq 1$ it turns out that the efficient solutions derived in Section 3 for $\theta = 1$ nevertheless yield interesting sites and information. Locating at such a point indeed guarantees that any other location sufficiently nearby may be beaten by choosing a quality slightly above the one of the corresponding $\theta = 1$ efficient solution.

References

[1] Avriel, M., Diewert, W.E., Schaible, S., and Zhang, I., (1988), *Generalized concavity*, Plenum, New York.

[2] Carrizosa, E., and Plastria, F. (1995), On minquantile and maxcovering optimisation, *Mathematical Programming* 71, 101-112.

[3] Carrizosa, E., and Plastria, F. (1998a), Locating an undesirable facility by generalized cutting planes, *Mathematics of Operations Research* 23, 3, 680-694.

[4] Carrizosa, E., and Plastria, F. (1998b), Polynomial algorithms for parametric minquantile and maxcovering planar location problems with locational constraints, *TOP* 6, 179-194.

[5] Drezner, T. (1994), Locating a single new facility among existing unequally attractive facilities, *Journal of Regional Science* 34, 2, 237-252.

[6] Drezner, T. (1995), Competitive facility location in the plane, in *Facility Location. A survey of applications and methods*, Z. Drezner (Ed.), Springer, Berlin, 285-300.

[7] Drezner, Z. (1981), On a modified one-center problem, *Management Science* 27, 848-851.

[8] Eiselt, H.A., and Laporte, G., (1988a), Location of a new facility on a linear market in the presence of weights, *Asia-Pacific Journal of Operational Research* 5, 160-165.

[9] Eiselt, H.A., and Laporte, G., (1988b), Trading areas of facilities with different sizes, *RAIRO Recherche Operationnelle/Operations Research* 22,1, 33-44.

[10] Eiselt, H.A., Laporte, G., and Thisse,J.–F. (1993), Competitive location models: a framework and bibliography, *Transportation Science*, 27, 44-54.

[11] Francis,R.L., Lowe,T.J., (1992), On worst case aggregation analysis for network location problems, *Annals of Operations Research*, 40, 229-246.

[12] Goodchild,M.F., (1979), The aggregation problem in location-allocation, *Geographical Analysis*, 11, 240-255.

[13] Hansen, P., Peeters, D. and Thisse, J.-F., (1995), The profit-maximizing Weber Problem, *Location Science* 3, 2, 67-85.

[14] Hearn,D.W., and Vijay, J., (1982), Efficient algorithms for the (weighted) minimum circle problem, *Operations Research* 30, 777-795.

[15] Mehrez, A., and Stulman, A. (1982), The maximal covering location problem with facility placement on the entire plane, *Journal of Regional Science* 22, 361-365.

[16] Michelot, C., (1993), The mathematics of continuous location, *Studies in Locational Analysis*, 5, 59–83.

[17] Okabe, A., Boots, B., and Sugihara, K., (1992), *Spatial tesselations, Concepts and applications of Voronoi diagrams*, John Wiley & Sons, Chichester.

[18] Pelegrín, B., Michelot, C. and Plastria, F., (1985), On the uniqueness of optimal solutions in continuous location theory, *European Journal of Operational Research*, 20, 327-331.

[19] Plastria, F., (1988), The minimisation of lower subdifferentiable functions under nonlinear constraints, an all feasible cutting plane method, *Journal of Optimisation Theory and Applications*, 57, 463-484.

[20] Plastria, F. (1995a). Continuous Location Problems, in Z. Drezner (Ed.), *Facility Location: A Survey of Applications and Methods*, Springer-Verlag, 1995, 225–262.

[21] Plastria, F. (1996), Profit maximising location in the plane of a single competitive facility with deterministic gravity-type consumer behaviour, Report BEIF/93, Vrije Universiteit Brussel, Brussels, Belgium.

[22] Plastria, F. (1997), Profit maximising single competitive facility location in the plane, *Studies in Locational Analysis*, 11, 115-126.

[23] Plastria, F. (2001), Static competitive facility location: an overview of optimisation approaches, *European Journal of Operational Research*, 129, 461-470.

[24] Roberts, J.H., and Lilien, G.L., (1993), Explanatory and Predictive Models of Consumer Behavior, in *Marketing, Handbooks in Operations Research and Management Science* (Vol. 5), (J. Eliashberg and G.L. Lilien, Eds.), North-Holland, Amsterdam.

[25] Steuer, R. (1986), *Multiple criteria optimization*, Wiley, New York.

[26] Vaughan, R., (1987), *Urban Spatial Traffic Patterns*, Pion Limited, London.