

# Multivariate Nonnegative Quadratic Mappings\*

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## Abstract

In this paper we study several issues related to the characterization of specific classes of multivariate quadratic mappings that are nonnegative over a given domain, with nonnegativity defined by a pre-specified conic order. In particular, we consider the set (cone) of nonnegative quadratic mappings defined with respect to the positive semidefinite matrix cone, and study when it can be represented by linear matrix inequalities. We also discuss the applications of the results in robust optimization, especially the robust quadratic matrix inequalities and the robust linear programming models. In the latter application the implementational errors of the solution is taken into account, and the problem is formulated as a semidefinite program.

**Keywords:** Linear Matrix Inequalities, Convex Cone, Robust Optimization, Bi-Quadratic Functions.

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# 1 Introduction

Let  $\mathcal{C} \subset \mathbb{R}^n$  be a closed and pointed convex cone. We can define a natural notion of conic ordering: for vectors  $x, y \in \mathbb{R}^n$ , we say  $x \succeq_{\mathcal{C}} y$  if and only if  $x - y \in \mathcal{C}$ . Thus,  $x \in \mathbb{R}^n$  is nonnegative if and only if  $x \in \mathcal{C}$ . In this paper, we will be primarily interested the conic ordering induced by the cone of positive semidefinite matrices, which is a very popular subject of study thanks to the recently developed high performance interior methods for conic optimization.

In general, given a closed and pointed convex cone  $\mathcal{C}$ , we wish to derive efficiently verifiable conditions under which a multivariate nonlinear mapping is nonnegative over a given domain (typically a unit ball), where nonnegativity is defined with respect to  $\mathcal{C}$ . In [13], Sturm and Zhang studied the problem of representing all nonnegative (defined with respect to the cone of nonnegative reals  $\mathbb{R}_+$ ) quadratic functions over a given domain. They showed that it is possible to characterize the set of nonnegative quadratic functions over some specific domains e.g. the intersection of an ellipsoid and a half-space. Moreover, the characterization is a necessary and sufficient condition in the form of Linear Matrix Inequalities (abbreviated as LMI hereafter) which is easy to verify. This type of easily computable necessary and sufficient conditions are particularly useful in systems theory and robust optimization where the problem data themselves may contain certain design variables to be optimized. In particular, using these LMI conditions, many robust control or minimax type of robust optimization problems can be reformulated as Semidefinite Programming (SDP) problems which can be efficiently solved using modern interior point methods.

The problems to be studied in this paper belong to the same category. In particular, we show that it is possible to characterize, by LMIs, when a certain type of Nonlinear Matrix Inequalities holds over a domain. The first case of this type is Quadratic Matrix Inequalities (QMI), where the quadratic matrix function is assumed to take a specific form. We prove that it is possible to give an LMI description, in terms of the problem data (i.e., the coefficients of the QMIs) for the quadratic matrix function to be positive semidefinite for all variables satisfying either a spectral or Frobenius norm bound. In fact, our methodology works for general quadratic matrix functions as well. What we derive is an equivalent condition in the dual conic space. However, the membership verification problem of this dual condition is NP-hard in general. There are several special cases in which the membership verification boils down to checking a system of LMIs, thus verifiable in polynomial-time. The first such case is when the variable is one-dimensional (the dimension of the matrix-valued mapping is arbitrary). Alternatively, if the dimension of the matrix mapping is 2 by 2 (the variable can be in any dimension), then we prove that the quadratic matrix inequality can again be characterized by LMIs of the problem data. We also show that our results can be applied to robust optimization. Specifically, we show that the robust linear programming models, where the implementational errors of the solution is taken into account, can be formulated as semidefinite programming problems.

This paper is organized as follows. In Section 2, we introduce the general conic framework and problem formulation. In Section 3, we present several results concerning the representation of matrix-valued quadratic matrix functions which are nonnegative over a domain. The discussion is continued in Section 4 for the general matrix valued mappings. A characterization for the nonnegativity of the mapping, in terms of the input data, over a general domain, is presented in the same section. This characterization is further shown to reduce to an LMI system in several special cases, when the underlying variable is one-dimensional, or 2-dimensional in the homogeneous case. Similarly, and in fact equivalently, we obtain LMI characterizations when the underlying variable is  $n$ -dimensional, but the mapping is  $2 \times 2$  matrix valued. Particular attention is given to the case where the domain is an  $n$ -dimensional unit ball. In Section 5 we discuss the applications of our results in robust optimization.

The notations we use are fairly standard. Vectors are in lower case letters, and matrices are in capital case letters. The transpose is expressed by  $^T$ . The set of  $n$  by  $n$  symmetric matrices is denoted by  $\mathcal{S}^n$ ; the set of  $n$  by  $n$  positive (semi)definite matrices is denoted by  $(\mathcal{S}_+^n) \mathcal{S}_{++}^n$ . For two given matrices  $A$  and  $B$ , we use ' $A \succ B$ ' (' $A \succeq B$ ') to indicate that  $A - B$  is positive (semi)-definite, ' $A \otimes B$ ' to indicate the Kronecker product between  $A$  and  $B$ , and  $A \bullet B := \sum_{i,j} A_{ij}B_{ij} = \text{tr} AB^T$  to indicate the matrix inner-product. For a given matrix  $A$ ,  $\|A\|_F$  stands for its Frobenius norm, and  $\|A\|_2$  stands for its spectrum norm. By 'cone  $\{x \mid x \in S\}$ ' ('span  $\{x \mid x \in S\}$ ') we mean the convex cone (respectively linear subspace) generated by the set  $S$ . The acronym 'SOC' stands for the second order cone  $\{(t, x) \in \mathbb{R}^n \mid t \geq \|x\|\}$ , and ' $\|\cdot\|$ ' represents the Euclidean norm. Given a Euclidean space  $\mathcal{L}$  with an inner-product  $X \bullet Y$  and a cone  $\mathcal{K} \subseteq \mathcal{L}$ , the dual cone  $\mathcal{K}^*$  is defined as

$$\mathcal{K}^* = \{Y \in \mathcal{L} \mid X \bullet Y \geq 0 \text{ for all } X \in \mathcal{K}\}.$$

Since the choice of  $\mathcal{L}$  can be ambiguous, we call  $\mathcal{K}^*$  the dual cone of  $\mathcal{K}$  in  $\mathcal{L}$ . Often,  $\mathcal{L}$  is chosen as  $\text{span}(\mathcal{K})$ .

## 2 Cones of Nonnegative Mappings

One fundamental problem in optimization is to check the membership with respect to a given cone. Any polynomial-time procedure for the membership problem will lead to a polynomial-time algorithm for optimizing a linear function over the cone intersected with some affine subspace; see [9]. Checking the membership for the dual cone is equivalent to asking whether a linear function is nonnegative over the whole cone itself. In Sturm and Zhang [13], a problem of this nature is investigated in detail. In particular, the authors studied the structure of all quadratic functions that are nonnegative over a certain domain  $D$ . It turns out that when  $D$  is either the level set of a quadratic function, or is the contour of a convex quadratic function, or is the intersection of the level set of a convex quadratic function with a half-space, then the cone generated by all nonnegative quadratic functions over this

domain can be described using *Linear Matrix Inequalities* (LMI). A consequence of this result is that the robust quadratic inequality over  $D$  can be converted equivalently to a single LMI.

If we consider a general vector-valued mapping, then questions such as the one posed in [13] can be generally formulated as

Determine a finite convex representation for the cone

$$\mathcal{K} = \{f : \mathfrak{R}^n \rightarrow \mathfrak{R}^m \mid f \in \mathcal{F}, f(D) \subseteq \mathcal{C}\}$$

where  $\mathcal{F}$  is a certain vector space of functions,  $D \subseteq \mathfrak{R}^n$  is a given domain, and  $\mathcal{C} \subseteq \mathfrak{R}^m$  is a given closed convex cone.

Solutions to the problems of this type are essential ingredients in robust optimization [3], since they allow conversion of semi-infinite constraints into finite convex ones. To appreciate the difficulty of these problems, let us quote a useful result from [3] as follows.

**Proposition 2.1.** *Let  $\mathcal{F}$  be the set of all affine linear mappings,  $D$  be a unit sphere,  $\mathcal{C}$  be the cone of positive semidefinite matrices. Then, it is NP-Complete to decide the membership problem for  $\mathcal{K}$ . More explicitly, for given symmetric matrices  $A_0, A_1, \dots, A_n$  of size  $m \times m$ , it is NP-Complete to test whether the following implication holds*

$$\sum_{i=1}^n x_i^2 \leq 1 \implies A_0 + \sum_{i=1}^n x_i A_i \succeq 0.$$

However, there are positive results as well. It is known [5, 11] that if  $\mathcal{F}$  is the set of polynomials of order no more than  $d$ ,  $D = \mathfrak{R}^1$ , and  $\mathcal{C}$  is the cone of positive semidefinite matrices, then there is a polynomial reduction of  $\mathcal{K}$  to an LMI. In other words,  $\mathcal{K}$  can be described by a reasonably sized LMI. In the next section, we will show that if  $\mathcal{F}$  is a certain quadratic matrix function set, and  $D$  is a unit ball defined either by the spectrum norm or the Frobenius norm, then  $\mathcal{K}$  can still be described by reasonably sized LMIs. Before we discuss specific results, we need to introduce some definitions.

Let  $D \subseteq \mathfrak{R}^n$  be a given domain. Then, its homogenization is given as

$$\mathcal{H}(D) = \text{cl} \left\{ \left[ \begin{array}{c} t \\ x \end{array} \right] \mid x/t \in D \right\} \subseteq \mathfrak{R}^{1+n}.$$

We consider the cone of co-positive matrices over  $D$  to be

$$\mathcal{C}_+(D) = \{Z \in \mathcal{S}^n \mid x^T Z x \geq 0 \text{ for all } x \in D\}. \quad (1)$$

Let  $D_1 \subseteq \mathfrak{R}^n$  and  $D_2 \subseteq \mathfrak{R}^m$  be two domains. The bi-linear positive cone is defined as

$$\mathcal{B}_+(D_1, D_2) = \{Z \in \mathfrak{R}^{n \times m} \mid x^T Z y \geq 0 \text{ for all } x \in D_1, y \in D_2\}.$$

Obviously, the description of  $\mathcal{C}_+$  or  $\mathcal{B}_+$  are the same as that of  $\mathcal{K}$ , when  $\mathcal{F}$  is taken as the set of quadratic forms, and  $\mathcal{C}$  is simply  $\mathfrak{R}_+$ .

If we have a general nonhomogeneous quadratic function  $q(x) = c + 2b^T x + x^T A x$ , then we introduce

$$M(q(\cdot)) = \begin{bmatrix} c & b^T \\ b & A \end{bmatrix}.$$

Consider

$$\mathcal{FC}_+(D) = \{M(q(\cdot)) \mid q(x) \geq 0 \text{ for all } x \in D\}.$$

It can be shown [13] that

$$\mathcal{FC}_+(D) = \mathcal{C}_+(\mathcal{H}(D)).$$

This implies that we need only to concentrate on the homogeneous form.

The following lemma plays a key role in our analysis.

**Lemma 2.2.** *Let  $\mathcal{K}$ ,  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be closed cones. It holds that*

$$\mathcal{C}_+^*(\mathcal{K}) = \text{cone}\{xx^T \mid x \in \mathcal{K}\}$$

and

$$\mathcal{B}_+^*(\mathcal{K}_1, \mathcal{K}_2) = \text{cone}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\}.$$

**Proof.** Let us only consider the second assertion. It can be shown ([13], Lemma 1) that

$$\text{cone}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\} \text{ is closed.}$$

Using the bi-polar theorem, it therefore suffices to prove that

$$\mathcal{B}_+(\mathcal{K}_1, \mathcal{K}_2) = (\text{cone}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\})^*. \quad (2)$$

It is clear that

$$\mathcal{B}_+(\mathcal{K}_1, \mathcal{K}_2) \subseteq (\text{conv}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\})^*.$$

We now wish to establish the other containing relation. Suppose, by contradiction, that there is

$$Z \in (\text{conv}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\})^* \setminus \mathcal{B}_+(\mathcal{K}_1, \mathcal{K}_2).$$

Then, since  $Z \notin \mathcal{B}_+(\mathcal{K}_1, \mathcal{K}_2)$ , by definition there exist  $u \in \mathcal{K}_1$  and  $v \in \mathcal{K}_2$  such that  $u^T Z v < 0$ . We arrive now at a contradiction, namely

$$0 > u^T Z v = Z \bullet (uv^T) \geq 0$$

where the latter inequality holds since  $Z \in (\text{conv}\{xy^T \mid x \in \mathcal{K}_1, y \in \mathcal{K}_2\})^*$ . For a proof of the first statement of the lemma, see Proposition 1 in [13]. **Q.E.D.**

We note that although we are primarily interested in  $\mathcal{C}_+(\mathcal{K})$  or  $\mathcal{B}_+(\mathcal{K}_1, \mathcal{K}_2)$ , it can be advantageous to work with their dual counterparts first, and then again dualize to get the original cone. For instance, in [13], Sturm and Zhang used this technique to show that

$$\mathcal{C}_+^*(\text{SOC}(1+n)) = \left\{ \begin{bmatrix} z_{11} & z^T \\ z & Z \end{bmatrix} \succeq 0 \mid z_{11} \geq \text{Tr}(Z) \right\}, \quad (3)$$

which is an explicit LMI system. Relation (3) is dual to the S-lemma [15], see Proposition 3.1 below.

### 3 Robust Quadratic Matrix Inequalities

Suppose that we consider an ordinary inequality, say  $f(x) \geq 0$ , where  $x$  can be viewed as a parameter, which is uncertain. Assume that this uncertain parameter  $x$  can attain any value within a set  $D$ . We call the inequality  $f(x) \geq 0$  to be robust if  $f(x) \geq 0$  for all  $x \in D$ .

In this regard, the S-lemma of Yakubovich [15] plays a key role in robust analysis, where  $f$  is quadratic and  $D$  is given as either the level set or the contour of a quadratic function. Actually, there are several variants of the S-lemma of Yakubovich, of which we list two. For proofs, see e.g. [13].

**Proposition 3.1 (S-lemma, level set).** *Let  $f : \Re^n \rightarrow \Re$  and  $g : \Re^n \rightarrow \Re$  be quadratic functions with  $g(\bar{x}) > 0$  for some  $\bar{x}$ . It holds that*

$$f(x) \geq 0 \text{ for all } x : g(x) \geq 0$$

*if and only if there exists  $t \geq 0$  such that*

$$f(x) - tg(x) \geq 0 \text{ for all } x \in \Re^n.$$

**Proposition 3.2 (S-lemma, contour).** *Let  $f : \Re^n \rightarrow \Re$  and  $g : \Re^n \rightarrow \Re$  be quadratic forms with  $g(x^{(1)}) < 0$  and  $g(x^{(2)}) > 0$  for some  $x^{(1)}$  and  $x^{(2)}$ . It holds that*

$$f(x) \geq 0 \text{ for all } x : g(x) = 0$$

*if and only if there exists  $t \in \Re$  such that*

$$f(x) + tg(x) \geq 0 \text{ for all } x \in \Re^n.$$

In this section, we derive extensions of the S-lemma to the matrix case, viz. the robust QMI.

Our first extension of Proposition 3.1 concerns the following robust QMI:

$$(S_1) : \quad C + X^T B + B^T X + X^T A X \succeq 0 \quad \text{for all } X \text{ with } I - X^T D X \succeq 0.$$

We show that this robust QMI holds if and only if the data matrices  $(A, B, C, D)$  satisfy a certain LMI relation.

**Theorem 3.3.** *The robust QMI  $(S_1)$  is equivalent to*

$$\left[ \begin{array}{cc} C & B^T \\ B & A \end{array} \right] \in \left\{ Z \mid Z - t \begin{bmatrix} I & 0 \\ 0 & -D \end{bmatrix} \succeq 0, t \geq 0 \right\}. \quad (4)$$

**Proof.** We first show that the robust QMI  $(S_1)$  is equivalent to the robust quadratic inequality  $(S_2)$  below:

$$(S_2): \quad \xi^T C \xi + 2\eta^T B \xi + \eta^T A \eta \geq 0, \quad \text{for all } \xi, \eta \text{ with } \xi^T \xi - \eta^T D \eta \geq 0.$$

To see that  $(S_2)$  implies  $(S_1)$ , we fix an  $X$  satisfying

$$I - X^T D X \succeq 0.$$

Then, by letting  $\xi$  be an arbitrary vector, and  $\eta := X\xi$ , we see that

$$\xi^T \xi - \eta^T D \eta \geq 0,$$

which, in light of  $(S_2)$ , implies

$$\xi^T C \xi + 2\eta^T B \xi + \eta^T A \eta \geq 0,$$

or, equivalently,

$$\xi^T (C + X^T B + B^T X + X^T A X) \xi \geq 0.$$

This shows that

$$C + X^T B + B^T X + X^T A X \succeq 0.$$

Next we shall show that  $(S_1)$  implies  $(S_2)$ . Suppose that  $(S_1)$  holds and let  $\xi$  and  $\eta$  be such that

$$\xi^T \xi - \eta^T D \eta \geq 0. \quad (5)$$

Consider first the case that  $\xi = 0$  and let  $X(u) = \eta u^T / u^T u$  for  $u \neq 0$ . Due to (5) we have

$$X(u)^T D X(u) = \frac{\eta^T D \eta}{(u^T u)^2} u u^T \preceq 0 \prec I \text{ for all } u \neq 0.$$

It thus follows from  $(S_1)$  that

$$0 \leq u^T (C + X(u)^T B + B^T X(u) + X(u)^T A X(u)) u = \eta^T A \eta + o(\|u\|),$$

and hence  $\eta^T A \eta \geq 0$ . This establishes  $(S_2)$  for the case that  $\xi = 0$ . If  $\xi \neq 0$  we let  $X = \eta \xi^T / \xi^T \xi$ . Due to (5) we have

$$X^T D X = \frac{\eta^T D \eta}{(\xi^T \xi)^2} \xi \xi^T \preceq \frac{1}{\xi^T \xi} \xi \xi^T \preceq I.$$

Then, by  $(S_1)$  we have

$$C + X^T B + B^T X + X^T A X \succeq 0.$$

Pre- and post-multiplying on both sides of the above matrix inequality by  $\xi^T$  and  $\xi$  respectively, we get

$$\xi^T C \xi + 2\eta^T B \xi + \eta^T A \eta \geq 0.$$

This establishes the equivalence between  $(S_1)$  and  $(S_2)$ . Now, applying Proposition 3.1 on  $(S_2)$ , Theorem 3.3 follows. **Q.E.D.**

Theorem 3.3 may be applied with  $D = I$  (or a multiple of the identity matrix) to yield a robust QMI where the uncertainty set is a level set of the spectral radius. At first sight, this is a more conservative robustness than one based on the Frobenius norm, since  $\|X\|_2 \leq \|X\|_F$ , with a strict inequality if the rank of  $X$  is more than one. Nevertheless, these uncertainty sets turn out to be equivalent for the form of QMIs treated in this section. More precisely, we have the following:

**Proposition 3.4.** *If  $D \succeq 0$ , then  $(S_1)$  is equivalent to the following robust QMI:*

$$(S_3) : \quad C + X^T B + B^T X + X^T A X \succeq 0, \quad \text{for all } X \text{ with } \text{Tr}(D(XX^T)) \leq 1.$$

**Proof.** Observe first that if  $X$  is such that  $1 \geq \text{Tr}(D(XX^T)) = \text{Tr}(X^T D X)$  with  $D \succeq 0$  then also  $I - X^T D X \succeq 0$ . Therefore,  $(S_1)$  implies  $(S_3)$ .

Now we wish to show the converse. Suppose that  $(S_3)$  holds, and let  $\xi$  and  $\eta$  be such that

$$\xi^T \xi - \eta^T D \eta \geq 0.$$

Then by letting  $X = \eta \xi^T / \xi^T \xi$  we have  $\text{Tr}(D(XX^T)) = \eta^T D \eta / \xi^T \xi \leq 1$ . It thus follows from  $(S_3)$  that

$$C + X^T B + B^T X + X^T A X \succeq 0.$$

By pre- and post-multiplying both sides by  $\xi^T$  and  $\xi$  we further get

$$\xi^T C \xi + 2\eta^T B \xi + \eta^T A \eta \geq 0,$$

establishing  $(S_2)$ . By Theorem 3.3,  $(S_3)$  is also implies  $(S_1)$ . **Q.E.D.**

As a consequence of the above results, we have derived an LMI description (4) for the data  $(A, B, C, D)$  when the following quadratic matrix function inequality holds:

$$C + X^T B + B^T X + X^T A X \succeq 0 \text{ for all } I - X^T D X \succeq 0.$$

If  $D \succeq 0$ , then the same LMI description (4) applies to the nonnegativity condition

$$C + X^T B + B^T X + X^T A X \succeq 0 \text{ for all } \text{Tr}(D(XX^T)) \leq 1.$$



Below we shall further extend the results in Theorem 3.3 to a setting where a matrix quadratic fraction is present.

Consider the data matrices  $(A, B, C, D, F, G, H)$  satisfying the following robust fractional QMI

$$I - X^T D X \succeq 0 \implies \begin{cases} H \succeq 0, \\ C + X^T B + B^T X + X^T A X \succeq 0, \\ H - (F + G X)(C + X^T B + B^T X + X^T A X)^+(F + G X)^T \succeq 0, \end{cases} \quad (6)$$

where  $M^+$  stands for the pseudo inverse of  $M \succeq 0$ . We remark that  $(A, B, C, D)$  satisfies  $(S_1)$  if and only if  $(A, B, C, D, 0, 0, 0)$  satisfies (6).

**Theorem 3.5.** *The data matrices  $(A, B, C, D, F, G, H)$  satisfy the robust fractional QMI (6) if and only if there is  $t \geq 0$  such that*

$$\begin{bmatrix} H & F & G \\ F^T & C & B^T \\ G^T & B & A \end{bmatrix} - t \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -D \end{bmatrix} \succeq 0.$$

**Proof.** Consider the QMI

$$\begin{bmatrix} H & F + G X \\ (F + G X)^T & C + X^T B + B^T X + X^T A X \end{bmatrix} \succeq 0 \text{ for all } I - X^T D X \succeq 0. \quad (7)$$

By taking Schur-complements, it is clear that (6) and (7) are equivalent. Unfortunately, the above QMI is not in the form of  $(S_1)$ ; Theorem 3.3 is therefore not applicable. Nevertheless, we can use a similar argument as in the proof of Theorem 3.3.

We shall show that the QMI (7) is equivalent to the robust quadratic inequality (8) below:

$$\xi^T H \xi + 2\xi^T F \eta + 2\xi^T G \gamma + \eta^T C \eta + \gamma^T B \eta + \eta^T B^T \gamma + \gamma^T A \gamma \geq 0, \quad \text{for all } \eta^T \eta - \gamma^T D \gamma \geq 0. \quad (8)$$

Suppose first that (8) holds, and fix an  $X$  satisfying  $I - X^T D X \succeq 0$ . Let  $\xi$  and  $\eta$  be arbitrary vectors, and let  $\gamma := X \eta$ . By construction, we have  $\eta^T \eta - \gamma^T D \gamma \geq 0$ , so that (8) implies

$$\begin{aligned} 0 &\leq \xi^T H \xi + 2\xi^T F \eta + 2\xi^T G \gamma + \eta^T C \eta + \gamma^T B \eta + \eta^T B^T \gamma + \gamma^T A \gamma \\ &= \begin{bmatrix} \xi \\ \eta \end{bmatrix}^T \begin{bmatrix} H & F + G X \\ (F + G X)^T & C + X^T B + B^T X + X^T A X \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \end{aligned}$$

establishing (7). Conversely, suppose that (7) holds, and let  $\xi$ ,  $\eta$  and  $\gamma$  be such that

$$\eta^T \eta - \gamma^T D \gamma \geq 0 \quad (9)$$

Consider first the case that  $\eta = 0$  and let  $X(u) = \gamma u^T / u^T u$  for  $u \neq 0$ . Due to (9) we have

$$X(u)^T D X(u) = \frac{\gamma^T D \gamma}{(u^T u)^2} u u^T \preceq 0 \prec I \text{ for all } u \neq 0.$$

It thus follows from (7) that

$$\begin{aligned} 0 &\leq \begin{bmatrix} \xi \\ u \end{bmatrix}^T \begin{bmatrix} H & F + GX(u) \\ (F + GX(u))^T & C + X(u)^T B + B^T X(u) + X(u)^T A X(u) \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} \\ &= \begin{bmatrix} \xi \\ \gamma \end{bmatrix}^T \begin{bmatrix} H & G \\ G^T & A \end{bmatrix} \begin{bmatrix} \xi \\ \gamma \end{bmatrix} + o(\|u\|). \end{aligned}$$

This establishes (8) for the case that  $\eta = 0$ . If  $\eta \neq 0$  we let  $X = \gamma\eta/\eta^T\eta$ . Due to (9) we have  $X^T D X \preceq I$ . Then, by (7) we have

$$\begin{aligned} 0 &\leq \begin{bmatrix} \xi \\ \eta \end{bmatrix}^T \begin{bmatrix} H & F + GX(u) \\ (F + GX(u))^T & C + X(u)^T B + B^T X(u) + X(u)^T A X(u) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \\ &= \xi^T H \xi + 2\xi^T F \eta + 2\xi^T G \gamma + \eta^T C \eta + \gamma^T B \eta + \eta^T B^T \gamma + \gamma^T A \gamma, \end{aligned}$$

establishing (8). We have proved the equivalence between (6) and (8). The theorem now follows by applying Proposition 3.1 to (8). **Q.E.D.**

Analogous to Proposition 3.4, we have the following equivalence result.

**Proposition 3.6.** *If  $D \succeq 0$ , then (6) is equivalent to the following robust fractional QMI:*

$$\text{Tr}(DXX^T) \leq 1 \implies \begin{cases} H \succeq 0, \\ C + X^T B + B^T X + X^T A X \succeq 0, \\ H - (F + GX)(C + X^T B + B^T X + X^T A X)^+(F + GX)^T \succeq 0. \end{cases}$$

It is interesting to note a related, but somewhat surprising result which we formulate in the following theorem.

**Theorem 3.7.** *The data matrices  $(A, B, C, F, G, H)$  satisfy*

$$\begin{bmatrix} H & F + GX \\ (F + GX)^T & C + X^T B + B^T X + X^T A X \end{bmatrix} \succeq 0, \quad \text{for all } X^T X = I \quad (10)$$

*if and only if*

$$\begin{bmatrix} H & F & G \\ F^T & C & B^T \\ G^T & B & A \end{bmatrix} - t \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} \succeq 0, \quad \text{for some } t \in \mathfrak{R}.$$

**Proof.** Just as in the proof of Theorem 3.5, the robust QMI (10) is equivalent to the following robust quadratic inequality:

$$\xi^T H \xi + 2\xi^T F \eta + 2\xi^T G \gamma + \eta^T C \eta + \gamma^T B \eta + \eta^T B^T \gamma + \gamma^T A \gamma \geq 0, \quad \text{for all } \eta^T \eta - \gamma^T \gamma = 0.$$

Applying Proposition 3.2 to the above relation, the theorem follows. **Q.E.D.**

Theorem 3.7 allows us to model the robust QMI over the orthonormal matrix constraints as a linear matrix inequality.

Matrix orthogonality constrained quadratic optimization problems were studied in [1, 14], where it was shown that if the objective function is homogeneous, either purely linear or quadratic, then by adding some seemingly redundant constraints one achieves strong duality with its Lagrangian dual problem.

## 4 General Robust Quadratic Matrix Inequalities

Section 3 shows how we can transform some special type of robust QMIs into a linear matrix inequality. In this section, we consider general robust quadratic matrix inequalities.

Remark that the matrix inequality  $Z \succeq 0$  is equivalent to the fact that  $x^T Z x \geq 0$  for all  $x \in \mathfrak{R}^n$ . Thus, the linear matrix inequality itself is nothing but a special type of robust quadratic inequality. The same is true for the co-positive matrix cone (1). From this viewpoint, we may formulate the general robust quadratic matrix inequalities as an ordinary robust inequality involving polynomials of order no more than 4.

Consider a domain  $D \subseteq \mathfrak{R}^n$  and a domain  $\Delta \subseteq \mathfrak{R}^m$ . In the same spirit as (1), let us define

$$\mathcal{C}_+(D, \Delta) := \left\{ Z \in \mathcal{L}_{n,m} \left| \sum_{i=1}^n \sum_{j=1}^n x_i x_j y^T Z_{ij} y \geq 0 \text{ for all } x \in D, y \in \Delta \right. \right\}, \quad (11)$$

where  $\mathcal{L}_{n,m}$  represents the  $mn(m+1)(n+1)/4$ -dimensional linear space of bi-quadratic forms. More precisely,  $\mathcal{L}_{n,m}$  is defined as follows:

$$\mathcal{L}_{n,m} := \left\{ \left[ \begin{array}{cccc} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{array} \right] \in \mathcal{S}^{n \times m} \left| G_{ij}^T = G_{ij} \in \mathcal{S}^m \text{ for all } i, j = 1, 2, \dots, n \right. \right\}.$$

Notice that  $\mathcal{C}_+(D) = \mathcal{C}_+(D, \mathfrak{R}_+) = \mathcal{C}_+(D, \mathfrak{R})$ .

Certainly,  $\mathcal{C}_+(\Delta)$  is a well-defined closed convex cone. It is easy to see that  $\mathcal{C}_+(D, \Delta)$  can be equivalently viewed as robust quadratic matrix inequality over  $D$  in the conic order defined by  $\mathcal{C}_+(\Delta)$ , i.e.

$$\mathcal{C}_+(D, \Delta) = \left\{ Z \in \mathcal{L}_{n,m} \left| \sum_{i=1}^n \sum_{j=1}^n x_i x_j Z_{ij} \in \mathcal{C}_+(\Delta) \text{ for all } x \in D \right. \right\}.$$

Given a quadratic function  $q : \mathfrak{R}^n \rightarrow \mathfrak{S}^m$ ,

$$q(x) = C + 2 \sum_{j=1}^n x_j B_j + \sum_{i=1}^n \sum_{j=1}^n x_i x_j A_{ij}, \quad (12)$$

we let  $M(q(\cdot)) \in \mathcal{L}_{n+1,m}$  denote the matrix representation of  $q(\cdot)$ , i.e.

$$M(q(\cdot)) = \begin{bmatrix} C & B_1 & \cdots & B_n \\ B_1 & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ B_n & A_{n1} & \cdots & A_{nn} \end{bmatrix}.$$

The cone of  $\mathcal{C}_+(\Delta)$ -nonnegative quadratic functions over  $D$  is now conveniently defined as

$$\mathcal{FC}_+(D, \Delta) = \{M(q(\cdot)) \mid q(x) \in \mathcal{C}_+(\Delta) \text{ for all } x \in D\}.$$

Clearly,  $\mathcal{FC}_+(D) = \mathcal{FC}_+(D, \mathfrak{R})$ . Furthermore, it can be shown [13] that

$$\mathcal{FC}_+(D, \Delta) = \mathcal{C}_+(\mathcal{H}(D), \Delta).$$

This implies that we need only to concentrate on the homogeneous form.

Similar to Lemma 2.2, we have the following representation.

**Lemma 4.1.** *Let  $D \subseteq \mathfrak{R}^n$  and  $\Delta \subseteq \mathfrak{R}^m$ . In the linear space  $\mathcal{L}_{n,m}$  it holds that*

$$\mathcal{C}_+^*(D, \Delta) = \text{cone} \{(xx^T) \otimes (yy^T) \mid x \in D, y \in \Delta\} \quad (13)$$

$$= \text{cone} \{(xx^T) \otimes Y \mid x \in D, Y \in \mathcal{C}_+(\Delta)^*\} \quad (14)$$

$$= \text{cone} \{X \otimes Y \mid X \in \mathcal{C}_+(D)^*, Y \in \mathcal{C}_+(\Delta)^*\}. \quad (15)$$

**Proof.** It can be shown ([13, Lemma 1]) that

$$\text{cone} \{(xx^T) \otimes (yy^T) \mid x \in D, y \in \Delta\} \text{ is closed.}$$

Using also the bi-polar theorem, an equivalent statement of (13) is therefore

$$\mathcal{C}_+(D, \Delta) = \text{cone} \{(xx^T) \otimes (yy^T) \mid x \in D, y \in \Delta\}^*. \quad (16)$$

If  $Z \in \mathcal{C}_+^*(D, \Delta)$  then for all  $x \in D$  and  $y \in \Delta$  we have

$$0 \leq y^T \left( \sum_{i=1}^n \sum_{j=1}^n x_i x_j Z_{ij} \right) y = (x \otimes y)^T Z (x \otimes y) = Z \bullet ((xx^T) \otimes (yy^T)).$$

This shows that

$$\mathcal{C}_+^*(D, \Delta) \subseteq \text{cone} \{(xx^T) \otimes (yy^T) \mid x \in D, y \in \Delta\}.$$

In order to establish the converse relation, suppose by contradiction that there exists

$$Z \in \text{cone} \{(xx^T) \otimes (yy^T) \mid x \in D, y \in \Delta\}^* \setminus \mathcal{C}_+(D, \Delta). \quad (17)$$

Since  $Z \notin \mathcal{C}_+(D, \Delta)$  there must exist  $x \in D$  and  $y \in \Delta$  such that

$$0 > y^T \left( \sum_{i=1}^n \sum_{j=1}^n x_i x_j Z_{ij} \right) y = Z \bullet ((xx^T) \otimes (yy^T)) \geq 0,$$

where the latter inequality follows from (17). This impossible inequality completes the proof of (16) and hence (13). The equivalence between (13) and (14) and (15) follows from Lemma 2.2. **Q.E.D.**

It can be seen that

$$\mathcal{L}_{n,m} = \text{span} \{X \otimes Y \mid X \in \mathcal{S}^n, Y \in \mathcal{S}^m\}.$$

To verify this relation, we first notice that the right hand side linear subspace is contained in  $\mathcal{L}_{n,m}$  since each matrix of the form  $X \otimes Y$  is in  $\mathcal{L}_{n,m}$ . Then we check that the dimension of the two linear subspaces are actually equal. This establishes the above equality.

There is an one-to-one correspondence between  $\mathcal{L}_{n,m}$  and  $\mathcal{L}_{m,n}$  by means of a permutation operator. In particular, we implicitly define the permutation matrix  $N_{m,n}$  by

$$N_{m,n} \text{vec}(X) = \text{vec}(X^T) \text{ for all } X \in \mathfrak{R}^{m \times n}. \quad (18)$$

We are now in a position to list some standard results on the Kronecker product.

**Proposition 4.2.** *Let  $A \in \mathfrak{R}^{p \times m}$ ,  $B \in \mathfrak{R}^{m \times n}$  and  $C \in \mathfrak{R}^{n \times q}$ . Then*

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B), \quad (19)$$

$$N_{m,n}^{-1} = N_{m,n}^T = N_{n,m}, \quad (20)$$

$$N_{p,q}(C^T \otimes A)N_{n,m} = A \otimes C^T. \quad (21)$$

**Proof.** We prove only (21), since the other two results are straightforward. We have

$$\begin{aligned} N_{p,q}(C^T \otimes A)N_{n,m} \text{vec}(B^T) &\stackrel{(18)}{=} N_{p,q}(C^T \otimes A) \text{vec}(B) \\ &\stackrel{(19)}{=} N_{p,q} \text{vec}(ABC) \\ &\stackrel{(18)}{=} \text{vec}(C^T B^T A^T) \\ &\stackrel{(19)}{=} (A \otimes C^T) \text{vec}(B^T) \end{aligned}$$

for arbitrary  $B$ . Hence (21). **Q.E.D.**

Notice that in particular from (21) that

$$N_{m,n}(X \otimes Y)N_{n,m} = Y \otimes X \text{ for all } X \in \mathcal{S}^n, Y \in \mathcal{S}^m,$$

so that

$$\mathcal{L}_{m,n} = \{N_{m,n}ZN_{n,m} \mid Z \in \mathcal{L}_{n,m}\}.$$

**Theorem 4.3.** *Let  $D \subseteq \mathfrak{R}^n$  and  $\Delta \subseteq \mathfrak{R}^m$ . Consider the cones  $\mathcal{C}_+(D, \Delta)$ ,  $\mathcal{C}_+(\Delta, D)$  and their duals in  $\mathcal{L}_{n,m}$  and  $\mathcal{L}_{m,n}$  respectively. It holds that*

$$\mathcal{C}_+(\Delta, D) = \{N_{m,n}XN_{n,m} \in \mathcal{L}_{m,n} \mid X \in \mathcal{C}_+(D, \Delta)\}, \quad (22)$$

and

$$\mathcal{C}_+^*(\Delta, D) = \{N_{m,n}ZN_{n,m} \in \mathcal{L}_{m,n} \mid Z \in \mathcal{C}_+^*(D, \Delta)\}. \quad (23)$$

**Proof.** Relation (23) follows from applying (21) to Lemma 4.1. Relation (22) follows by dualization. **Q.E.D.**

We shall first consider the cone  $\mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}^m)$  residing in  $\mathcal{L}_{n,m}$ . The following theorem provides an LMI characterization if either  $n = 2$  or  $m = 2$ .

**Theorem 4.4.** *Consider the cone  $\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$  and its dual in  $\mathcal{L}_{2,m}$ . It holds that*

$$\begin{aligned} \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m) &= \mathcal{FC}_+(\mathfrak{R}, \mathfrak{R}^m) = (\mathcal{L}_{2,m} \cap \mathcal{S}_+^{2m})^* \\ &= \left\{ \begin{bmatrix} A & B \\ B & C \end{bmatrix} \in \mathcal{L}_{n,m} \mid \begin{bmatrix} A & B + \tilde{B} \\ B - \tilde{B} & C \end{bmatrix} \in \mathcal{S}_+^{2m} \text{ for some } \tilde{B} = -\tilde{B}^T \right\}. \end{aligned}$$

The cone

$$\mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}^2) = \mathcal{FC}_+(\mathfrak{R}^{n-1}, \mathfrak{R}^2) = (\mathcal{L}_{n,2} \cap \mathcal{S}_+^{2n})^*$$

has a similar LMI characterization due to Theorem 4.3.

**Proof.** The relation  $\mathcal{FC}_+(\mathfrak{R}, \mathfrak{R}^m) = (\mathcal{L}_{2,m} \cap \mathcal{S}_+^{2m})^*$  is a special case of Theorem 2 in Genin et al. [6] on matrix polynomials. The second part of the lemma follows from Theorem 4.3. For completeness, we provide a direct proof below.

By Lemma 4.1, we have

$$\mathcal{C}_+^*(\mathfrak{R}^2, \mathfrak{R}^m) \subseteq \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m}.$$

Now we proceed to showing the converse containing relation. For this purpose, we take any

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m},$$

and prove that  $G \in \mathcal{C}_+^*(\mathfrak{R}^2, \mathfrak{R}^m)$ . We will use the obvious invariance relation

$$\mathcal{C}_+^*(\mathfrak{R}^2, \mathfrak{R}^m) = \left\{ \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} Z \begin{bmatrix} P^T & 0 \\ 0 & P^T \end{bmatrix} \mid Z \in \mathcal{C}_+^*(\mathfrak{R}^2, \mathfrak{R}^m) \right\},$$

where  $P$  is any nonsingular real matrix.

Let  $G(\epsilon) = G + \epsilon I \succ 0$ , where  $\epsilon > 0$  is an arbitrarily small (but fixed) positive number. Since  $G_{22}(\epsilon) \succ 0$  and  $G_{12}(\epsilon) = G_{12}$  is symmetric, there exists a nonsingular matrix  $P_\epsilon$  such that

$$P_\epsilon G_{22}(\epsilon) P_\epsilon^T = I \text{ and } P_\epsilon G_{12}(\epsilon) P_\epsilon^T = \Lambda_\epsilon$$

where  $\Lambda_\epsilon = \text{diag}(\lambda_1(\epsilon), \dots, \lambda_m(\epsilon))$  is a diagonal matrix. In fact,  $P_\epsilon = G_{22}(\epsilon)^{-1/2} Q_\epsilon$  for some orthogonal matrix  $Q_\epsilon$ . To show  $G(\epsilon) \in \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$ , we only need to prove

$$\begin{bmatrix} P_\epsilon & 0 \\ 0 & P_\epsilon \end{bmatrix} G(\epsilon) \begin{bmatrix} P_\epsilon^T & 0 \\ 0 & P_\epsilon^T \end{bmatrix} = \begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix} \in \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m).$$

By the well known Schur complement lemma, we have

$$P_\epsilon G_{11}(\epsilon) P_\epsilon^T - \Lambda_\epsilon^2 \succ 0.$$

Therefore, we obtain the following representation:

$$\begin{aligned} \begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix} &= \begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T - \Lambda_\epsilon^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Lambda_\epsilon^2 & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes (P_\epsilon G_{11}(\epsilon) P_\epsilon^T - \Lambda_\epsilon^2) + \sum_{i=1}^m \begin{bmatrix} \lambda_i^2 & \lambda_i \\ \lambda_i & 1 \end{bmatrix} \otimes (e_i e_i^T), \end{aligned} \quad (24)$$

where  $e_i \in \mathfrak{R}^m$  is the  $i$ -th column of the  $m \times m$  identity matrix. By Lemma 4.1, the above representation shows that the matrix

$$\begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix}$$

lies in  $\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$ . Consequently,  $G(\epsilon) \in \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$ . Since  $\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$  is a closed cone, we have  $G = \lim_{\epsilon \rightarrow 0} G(\epsilon) \in \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$ . This proves the first part of the theorem. The characterization of the primal cone  $\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$  follows by dualization, viz.

$$\begin{aligned} \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m) &= \text{cl}(\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)) = \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m) \\ &= \text{cl}(\mathcal{L}_{n,m}^\perp + \mathcal{S}_+^{2m}) \cap \mathcal{L}_{n,m} \\ &= \left\{ \begin{bmatrix} A & B \\ B & C \end{bmatrix} \in \mathcal{L}_{n,m} \mid \begin{bmatrix} A & B + \tilde{B} \\ B - \tilde{B} & C \end{bmatrix} \in \mathcal{S}_+^{2m} \text{ for some } \tilde{B} = -\tilde{B}^T \right\}. \end{aligned}$$

**Q.E.D.**

We remark that  $\mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^m)$  (or equivalently  $\mathcal{C}_+(\mathfrak{R}^m, \mathfrak{R}^2)$ ) is not self-dual. For instance, we have

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^2) \setminus \mathcal{C}_+(\mathfrak{R}^2, \mathfrak{R}^2).$$

The membership problem of  $\mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}^m)$  for general  $n$  and  $m$  is a hard problem; see Corollary 4.10 later in this paper.

We study now the mixed co-positive/positive semi-definite bi-quadratic forms, i.e.  $\mathcal{C}_+(\mathfrak{R}_+^n, \mathfrak{R}^m)$  and  $\mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}_+^m)$ . For  $m = 2$ , we arrive at a special case of nonnegative polynomial matrices on the positive real half-line (see [11] for the scalar case).

**Theorem 4.5.** *There holds*

$$\mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m) = \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{L}_{2,m} \cap \mathcal{S}_+^{2m} \mid G_{12} \succeq 0 \right\}. \quad (25)$$

Consequently, the primal cone  $\mathcal{C}_+(\mathfrak{R}_+^2, \mathfrak{R}^m)$  can be characterized as

$$\mathcal{C}_+(\mathfrak{R}_+^2, \mathfrak{R}^m) = \mathcal{FC}_+(\mathfrak{R}_+, \mathfrak{R}^m) = \left\{ \begin{bmatrix} C & B \\ B & A \end{bmatrix} \in \mathcal{L}_{2,m} \mid \begin{bmatrix} C & B \\ B & A \end{bmatrix} - \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \succeq 0, E + E^T \succeq 0 \exists E \right\}.$$

**Proof.** First, it follows from Lemma 4.1 that

$$\mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m) \subseteq \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m} \mid G_{12} \succeq 0 \right\}.$$

It remains to argue the inclusion in the reverse direction. To this end, let

$$G \in \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m} \mid G_{12} \succeq 0 \right\}$$

be arbitrary. We follow the same proof technique for Theorem 4.4, and we use  $G(\epsilon)$ ,  $P_\epsilon$  and  $\Lambda_\epsilon$  defined there. The only difference is that  $\Lambda_\epsilon \succeq 0$ , due to the fact that  $G_{12} \succeq 0$ . Relation (24) states that

$$\begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes (P_\epsilon G_{11}(\epsilon) P_\epsilon^T - \Lambda_\epsilon^2) + \sum_{i=1}^m \begin{bmatrix} \lambda_i^2 & \lambda_i \\ \lambda_i & 1 \end{bmatrix} \otimes (e_i e_i^T).$$

Obviously,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_i \\ 1 \end{bmatrix} \in \mathfrak{R}_+^2$ . By Lemma 4.1, the above representation thus shows that the matrix

$$\begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix}$$

lies in  $\mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ . Consequently,  $G(\epsilon) \in \mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ , and by continuity  $G \in \mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ .

The remaining claim about the characterization of the primal cone  $\mathcal{C}_+(\mathfrak{R}_+^2, \mathfrak{R}^m)$  can be easily verified by taking the dual on both sides of (25). **Q.E.D.**

As a special case of the above theorem, we see that  $\mathcal{C}_+(\mathfrak{R}_+^2) = \mathcal{S}_+^2 + \mathfrak{R}_+^{2 \times 2}$ , which is a well known characterization of the  $2 \times 2$  co-positive cone. However, for  $n > 2$  one merely has  $\mathcal{S}_+^n + \mathfrak{R}_+^{n \times n} \subset \mathcal{C}_+(\mathfrak{R}_+^n)$ .



In fact, the membership problem of the co-positive cone in co-NP-Complete [10]. Hence also the membership problem of  $\mathcal{C}_+(\mathfrak{R}_+^m, \mathfrak{R}^m)$  is co-NP-Complete.

Consider the case of  $D = [0, 1]$ . This is a special case of nonnegative polynomial matrices on an interval (see [11] for the scalar case).

**Theorem 4.6.** *Let  $D = [0, 1]$ . Then, we have*

$$\mathcal{C}_+^*(\mathcal{H}([0, 1]), \mathfrak{R}^m) = \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{L}_{2,m} \cap \mathcal{S}_+^{2m} \mid G_{12} - G_{22} \succeq 0 \right\}. \quad (26)$$

As a result, the primal cone  $\mathcal{C}_+(\mathcal{H}([0, 1]), \mathfrak{R}^m) = \mathcal{FC}_+([0, 1], \mathfrak{R}^m)$  can be characterized as

$$\mathcal{FC}_+([0, 1], \mathfrak{R}^m) = \left\{ \begin{bmatrix} C & B \\ B & A \end{bmatrix} \in \mathcal{L}_{2,m} \mid \begin{bmatrix} C & B - E \\ B - E^T & A + E + E^T \end{bmatrix} \succeq 0, E + E^T \succeq 0, \exists E \right\}.$$

**Proof.** First, it follows from Lemma 4.1 that

$$\mathcal{C}_+^*(\mathcal{H}([0, 1]), \mathfrak{R}^m) \subseteq \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m} \mid G_{12} \succeq G_{22} \right\}.$$

(Of course, one also has  $G_{11} \succeq G_{12}$ , but this relation is implied by  $G \succeq 0$  and  $G_{12} \succeq G_{22}$ .)

It remains to argue the inclusion in the reverse direction. To this end, let

$$G \in \left\{ \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \in \mathcal{S}_+^{2m} \cap \mathcal{L}_{2,m} \mid G_{12} \succeq G_{22} \right\}$$

be arbitrary. We follow the same proof technique for Theorem 4.4, and we use  $G(\epsilon)$ ,  $P_\epsilon$  and  $\Lambda_\epsilon$  defined there. The only difference is that  $\Lambda_\epsilon \succeq I$ , due to the fact that  $G_{12} \succeq G_{22}$ . Relation (24) states that

$$\begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes (P_\epsilon G_{11}(\epsilon) P_\epsilon^T - \Lambda_\epsilon^2) + \sum_{i=1}^m \begin{bmatrix} \lambda_i^2 & \lambda_i \\ \lambda_i & 1 \end{bmatrix} \otimes (e_i e_i^T).$$

Obviously,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} \lambda_i \\ 1 \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ 1/\lambda_i \end{bmatrix} \in \mathcal{H}([0, 1])$ , because  $1/\lambda_i \in [0, 1]$  for all  $i$ . By Lemma 4.1, the above representation thus shows that the matrix

$$\begin{bmatrix} P_\epsilon G_{11}(\epsilon) P_\epsilon^T & \Lambda_\epsilon \\ \Lambda_\epsilon & I \end{bmatrix}$$

lies in  $\mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ . Consequently,  $G(\epsilon) \in \mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ , and by continuity  $G \in \mathcal{C}_+^*(\mathfrak{R}_+^2, \mathfrak{R}^m)$ .

The characterization of the primal cone  $\mathcal{FC}_+([0, 1], \mathfrak{R}^m)$  can be easily established by taking the dual on both sides of (26). **Q.E.D.**

Quadratic programming over a box  $[0, 1]^n$  is well-known to be NP-Complete for general  $n$ , see [10]. Hence, also the membership problems of  $\mathcal{FC}_+([0, 1]^n)$  and  $\mathcal{FC}_+([0, 1]^n, \mathfrak{R}^m)$  with general  $n$  are co-NP-Complete.

Recall from (3) that

$$\mathcal{C}_+(\text{SOC}(n)) = \mathcal{FC}_+(\{x \in \mathfrak{R}^n \mid x^T x \leq 1\}) = \{Z \in \mathcal{S}_+^n \mid J \bullet Z \geq 0\}^*,$$

where  $J := 2e_1 e_1^T - I$ . Using Lemma 4.1, we have

$$\begin{aligned} \mathcal{C}_+(\text{SOC}(n), \mathfrak{R}^m) &= \text{cone} \{X \otimes Y \mid X \in \mathcal{C}_+(\text{SOC}(n))^*, Y \in \mathcal{C}_+(\mathfrak{R}^m)^*\} \\ &= \text{cone} \{X \otimes Y \mid X \in \mathcal{S}_+^n, Y \in \mathcal{S}_+^m, J \bullet X \geq 0\}. \end{aligned} \quad (27)$$

Furthermore, we know from Theorem 4.3 that this cone is isomorphic to (i.e. in one-to-one correspondence with)

$$\mathcal{C}_+(\mathfrak{R}^m, \text{SOC}(n)) = \text{cone} \{X \otimes Y \mid X \in \mathcal{S}_+^m, Y \in \mathcal{S}_+^n, J \bullet Y \geq 0\}. \quad (28)$$

From this relation, it is clear that

$$\mathcal{C}_+(\mathfrak{R}^2, \text{SOC}(m)) \subseteq \left\{ \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \in \mathcal{L}_{2,m} \cap \mathcal{S}_+^{2m} \mid \begin{bmatrix} J \bullet Z_{11} & J \bullet Z_{12} \\ J \bullet Z_{21} & J \bullet Z_{22} \end{bmatrix} \succeq 0 \right\}. \quad (29)$$

A natural conjecture is that (29) might be an equality. Unfortunately, this conjecture turns out to be false.

**Counter-Example:** Let  $p = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$  and

$$Z_{11} = pp^T + 2e_3 e_3^T, \quad Z_{12} = Z_{21} = pp^T, \quad Z_{22} = pp^T + 6e_1 e_1^T,$$

that is,

$$Z = \begin{bmatrix} p \\ p \end{bmatrix} \begin{bmatrix} p \\ p \end{bmatrix}^T + 2 \begin{bmatrix} e_3 \\ 0 \end{bmatrix} \begin{bmatrix} e_3 \\ 0 \end{bmatrix}^T + 6 \begin{bmatrix} 0 \\ e_1 \end{bmatrix} \begin{bmatrix} 0 \\ e_1 \end{bmatrix}^T,$$

where  $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$  and  $e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ . Clearly,  $Z \in \mathcal{L}_{2,3}$  and  $Z \succeq 0$ . Moreover, we find that

$$\begin{bmatrix} J \bullet Z_{11} & J \bullet Z_{12} \\ J \bullet Z_{21} & J \bullet Z_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}^T. \quad (30)$$

So  $Z$  lies in the cone defined by the right hand side of (29). We claim that  $Z \notin \mathcal{C}_+(\mathfrak{R}^2, \text{SOC}(3))$ . Suppose to the contrary that  $Z \in \mathcal{C}_+(\mathfrak{R}^2, \text{SOC}(3))$ . Then  $Z = \sum_{i=1}^k (x_i x_i^T) \otimes (y_i y_i^T)$ , where  $x_i \in \mathfrak{R}^2$ ,  $y_i \in \text{SOC}(3)$ . Notice that

$$\begin{bmatrix} J \bullet Z_{11} & J \bullet Z_{12} \\ J \bullet Z_{21} & J \bullet Z_{22} \end{bmatrix} = \sum_{i=1}^k (J \bullet (y_i y_i^T)) x_i x_i^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}^T,$$

where the last step follows from (30). Since  $y_i \in \text{SOC}(3)$ , it follows  $J \bullet (y_i y_i^T) \geq 0$ , for all  $i$ . Therefore, the above relation implies that each  $x_i$  must be a constant multiple of the vector  $\begin{bmatrix} 1 & 3 \end{bmatrix}^T$ . By a renormalization if necessary, we can assume  $x_i = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$  for all  $i$ . As a result, we obtain

$$Z = \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}^T \right) \otimes Y = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \otimes Y,$$

where  $Y = \sum_i y_i y_i^T$ . This implies that  $Z_{22} = 3Z_{12} = 9Z_{11}$ . This clearly contradicts with the definitions of  $Z_{11}$ ,  $Z_{12}$  and  $Z_{22}$ . We therefore have proved that  $Z \notin \mathcal{C}_+(\mathfrak{R}^2, \text{SOC}(3))$ .

It remains an open question as to whether the cone  $\mathcal{C}_+(\mathfrak{R}^2, \text{SOC}(m))$  is representable by linear matrix inequalities. Below, we shall characterize the cone

$$\left\{ (A, C) \in \mathcal{L}_{n,m} \times \mathcal{S}^m \mid \begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix} \in \mathcal{C}_+(\text{SOC}(n), \Delta) \right\}, \quad (31)$$

for given  $\Delta \subseteq \mathfrak{R}^m$ . In other words, we consider quadratic functions  $q : \{x \in \mathfrak{R}^n \mid x^T x \leq 1\} \rightarrow \mathcal{C}_+(\Delta)$  where the  $B_j$ 's in (12) are all zero. Our result is the following.

**Theorem 4.7.** *Let  $r > 0$  be a given scalar quantity and let  $\emptyset \neq \Delta \subseteq \mathfrak{R}^m$  be a given domain. It holds that  $A \in \mathcal{L}_{n,m}$ ,  $C \in \mathcal{S}^m$  satisfy*

$$y^T C y + \sum_{i=1}^n \sum_{j=1}^n x_i x_j y^T A_{ij} y \geq 0 \text{ for all } x^T x \leq r, y \in \Delta \quad (32)$$

if and only if

$$C \in \mathcal{C}_+(\Delta), rA + I \otimes C \in \mathcal{C}_+(\mathfrak{R}^n, \Delta). \quad (33)$$

**Proof.** We shall use the Rayleigh-Ritz characterization of the smallest eigenvalue of a symmetric matrix  $Z = Z^T$ . The smallest eigenvalue, denoted  $\lambda_{\min}(Z)$ , is characterized as follows:

$$\lambda_{\min}(Z) = \min\{u^T Z u \mid u^T u = 1\}. \quad (34)$$

Suppose now that (32) holds. Setting  $x = 0$  we obtain that  $C \in \mathcal{C}_+(\Delta)$ . It also follows immediately from (32) that for arbitrary  $y \in \Delta$ ,

$$\begin{aligned} 0 &\leq y^T C y + \min_x \left\{ \sum_{i=1}^n \sum_{j=1}^n x_i x_j y^T A_{ij} y \mid x^T x = r \right\} \\ &= y^T C y + r \min_{\xi} \{ \xi^T (I \otimes y)^T A (I \otimes y) \xi \mid \xi^T \xi = 1 \} \\ &= y^T C y + r \lambda_{\min}((I \otimes y)^T A (I \otimes y)) \end{aligned}$$

where we used (34). It follows that

$$r(I \otimes y)^T A(I \otimes y) \succeq -(y^T C y)I \text{ for all } y \in \Delta.$$

Pre- and post-multiplying both sides with an arbitrary  $\xi \in \Re^n$ , we obtain that

$$\begin{aligned} 0 &\leq r(\xi \otimes y)^T A(\xi \otimes y) + (y^T C y)\xi^T \xi \\ &= (\xi \otimes y)^T (rA + I \otimes C)(\xi \otimes y) \\ &= (rA + I \otimes C) \bullet ((\xi \xi^T) \otimes (y y^T)) \end{aligned}$$

for all  $\xi \in \Re^n$ ,  $y \in \Delta$ . From Lemma 4.1, this in turn is equivalent to

$$rA + I \otimes C \in \mathcal{C}_+(\Re^n, \Delta).$$

We have shown that (32) implies (33). Conversely, suppose that  $(A, C)$  satisfies (33), and let  $x \in \Re^n$ ,  $y \in \Delta$ ,  $x^T x \leq r$  be arbitrary. We have

$$y^T C y + \sum_{i=1}^n \sum_{j=1}^n x_i x_j y^T A_{ij} y = \frac{(x \otimes y)^T (rA + I \otimes C)(x \otimes y) + (r - x^T x)y^T C y}{r} \geq 0,$$

where the inequality follows immediately from (33) and the nonnegativity of  $r - x^T x$ . **Q.E.D.**

By the same argument, the following theorem is readily proven.

**Theorem 4.8.** *Let  $\emptyset \neq \Delta \subseteq \Re^m$  and let  $r > 0$  be a given scalar quantity. It holds that  $A \in \mathcal{L}_{n,m}$ ,  $C \in \mathcal{S}^m$  satisfy*

$$y^T C y + \sum_{i=1}^n \sum_{j=1}^n x_i x_j y^T A_{ij} y \geq 0 \text{ for all } x^T x = r, y \in \Delta$$

*if and only if*

$$rA + I \otimes C \in \mathcal{C}_+(\Re^n, \Delta).$$

Recall from Theorems 4.4–4.6 that  $\mathcal{C}_+(\Re^n, \Re^2)$ ,  $\mathcal{C}_+(\Re^n, \Re_+^2)$  and  $\mathcal{FC}_+(\Re^n, [0, 1])$  are efficiently LMI representable. Theorems 4.7–4.8 therefore provide an efficient LMI characterization for the class of  $2 \times 2$  robust multivariate QMIs whose entries are co-centered (e.g., centered at the origin) over the unit ball. Stated more clearly, we have obtained an efficient LMI representation for the following robust QMI:

$$\begin{bmatrix} x^T C x + c & x^T B x + b \\ x^T B x + b & x^T A x + a \end{bmatrix} \in \mathcal{FC}_+(\Delta), \text{ for all } x \in D,$$

where  $D$  is either  $\{x \in \Re^n \mid x^T x \leq r\}$  or  $\{x \in \Re^n \mid x^T x = r\}$ , and  $\Delta$  is either  $\Re$ ,  $\Re_+$  or  $[0, 1]$ . In particular, we have the following equivalences:

1. For symmetric matrices  $A$ ,  $B$  and  $C$ , the robust QMI

$$\begin{bmatrix} x^T C x + c & x^T B x + b \\ x^T B x + b & x^T A x + a \end{bmatrix} \succeq 0, \text{ for all } \|x\|^2 \leq r$$

holds if and only if

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix} \in \mathcal{L}_{2,m}, \quad \begin{bmatrix} rC + cI & rB + bI - E \\ rB + bI + E & rA + aI \end{bmatrix} \succeq 0, \quad \begin{bmatrix} c & b \\ b & a \end{bmatrix} \succeq 0,$$

for some  $E$  with  $E + E^T = 0$ .

2. For symmetric matrices  $A$ ,  $B$  and  $C$ , the condition

$$y^T \begin{bmatrix} x^T C x + c & x^T B x + b \\ x^T B x + b & x^T A x + a \end{bmatrix} y \geq 0, \text{ for all } \|x\|^2 \leq r \text{ and for all } y \in \mathfrak{R}_+^2$$

holds if and only if

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix} \in \mathcal{L}_{2,m}, \quad \begin{bmatrix} rC + cI & rB + bI - E \\ rB + bI - E^T & rA + aI \end{bmatrix} \succeq 0, \quad \begin{bmatrix} c & b - e \\ b - e & a \end{bmatrix} \succeq 0,$$

for some  $e \geq 0$  and some  $E$  with  $E + E^T \succeq 0$ .

3. For symmetric matrices  $A$ ,  $B$  and  $C$ , the condition

$$y^T \begin{bmatrix} x^T C x + c & x^T B x + b \\ x^T B x + b & x^T A x + a \end{bmatrix} y \geq 0, \text{ for all } \|x\|^2 \leq r \text{ and for all } y \in \mathfrak{R}_+^2 \text{ with } y_1 \geq y_2$$

holds if and only if

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix} \in \mathcal{L}_{2,m}, \quad \begin{bmatrix} rC + cI & rB + bI - E \\ rB + bI - E^T & rA + aI + E + E^T \end{bmatrix} \succeq 0, \quad \begin{bmatrix} c & b - e \\ b - e & a + 2e \end{bmatrix} \succeq 0,$$

for some  $e \geq 0$  and some  $E$  with  $E + E^T \succeq 0$ .

Similar equivalence relations hold for the case where ‘ $\|x\|^2 \leq r$ ’ is replaced by ‘ $\|x\|^2 = r$ ’. In this case, we only need to remove from the above equivalence relations the nonnegative parameter  $e$  and the respective conditions on the  $2 \times 2$  matrix involving  $a, b, c, d$ .

It remains an open question whether one can obtain an LMI description for the general  $2 \times 2$  robust QMIs over the unit ball without the co-centeredness condition.

For  $\Delta = \mathfrak{R}^m$  with general  $m$ , however, checking the membership problem (31) is a hard problem:

**Theorem 4.9.** For general  $n$  and  $m$ , the ( $\epsilon$ -approximate) membership problem

$$\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix} \in \mathcal{C}_+(\text{SOC}(n), \mathfrak{R}^m) \quad (35)$$

with data  $(A, C) \in \mathcal{L}_{n,m} \times \mathcal{S}^m$  is co-NP-Complete.

**Proof.** We choose to use the well-known NP-Complete partition problem for the purpose of reduction:

Decide whether or not one can partition a given set of integers  $a_1, \dots, a_n$  such that the two subsets will have the same subset sum.

The above decision problem can be further reduced to the following decision problem

Given  $a \in \mathbb{Z}^n$  (the  $n$ -dimensional integer lattice) and a scalar  $t \geq 0$ , decide whether or not  $p(x; t) \geq 0$  for all  $\|x\|_2^2 = n$ , where  $p(x; t) = (t + (a^T x)^2)^2 - n^2 + \sum_{i=1}^n x_i^4$ .

To see why it is so, we notice that for  $t \geq 0$  and  $x$  with  $\|x\|^2 = n$  that

$$p(x; t) = (t + (a^T x)^2)^2 - n^2 + \sum_{i=1}^n x_i^4 \geq t^2 - n^2 + \sum_{i=1}^n x_i^4 \geq t^2 - n^2 + \frac{(\sum_{i=1}^n x_i^2)^2}{n} = t^2 - n(n-1)$$

where the second inequality is based on the Cauchy-Schwarz inequality.

The lower bound is attained, i.e.  $p(x; t) = t^2 - n(n-1)$ , if and only if  $x_i^2 = 1$  for all  $i = 1, \dots, n$ , and  $a^T x = 0$ , which is equivalent to the existence of a partition. Thus, a partition does *not* exist if and only if for  $t = \sqrt{n(n-1)} - 1$  there holds  $p(x; t) \geq 0$  for all  $\|x\|_2^2 = n$ .

Next we notice that

$$\|x\|_2^4 = \left(\sum_{i=1}^n x_i^2\right)^2 = \sum_{i=1}^n x_i^4 + \sum_{i \neq j} x_i^2 x_j^2,$$

so that

$$p(x; t) = (t + (a^T x)^2)^2 + (\|x\|_2^4 - n^2) - \sum_{i \neq j} x_i^2 x_j^2,$$

where the second term vanishes if  $\|x\|_2^2 = n$ . It follows that  $p(x; t) \geq 0$  for all  $\|x\|_2^2 = n$  if and only if

$$t + (a^T x)^2 \geq \|\{x_i x_j\}_{i \neq j}\|_2 \text{ for all } x^T x = n. \quad (36)$$

The above robust quadratic SOC-constraint can be transformed into an equivalent robust QMI in the familiar way, viz.

$$L(t + (a^T x)^2, \{x_i x_j\}_{i \neq j}) \in \mathcal{S}_+^{1+n(n-1)} \text{ for all } x^T x = n, \quad (37)$$

where  $L(\cdot, \cdot)$  denotes the so-called arrow-hat (or Jordan product representation) matrix

$$L(s, y) = \begin{bmatrix} s & y^T \\ y & sI \end{bmatrix}.$$

We have reduced the partitioning problem to the robust QMI (37), which is of the form (35). **Q.E.D.**

**Corollary 4.10.** *The membership problem  $X \in \mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}^m)$  is co-NP-Complete.*

**Proof.** Due to Theorem 4.8, the co-NP-Complete problem in Theorem 4.9 can be reduced to the membership problem for  $\mathcal{C}_+(\mathfrak{R}^n, \mathfrak{R}^m)$ , which must therefore also be co-NP-Complete. **Q.E.D.**

To close the section, we remark that it is NP-hard in general to check whether a fourth order polynomial is nonnegative over the unit sphere (or over the whole space). The same partition problem as in the proof of Theorem 4.9 can also be used to reduce to the unconstrained minimization of the following fourth order polynomial

$$\sum_{i=1}^n (x_i^2 - 1)^2 + (a^T x)^2.$$

In particular, a partition exists if and only if the polynomial attains zero. Now we pose as an open question the complexity of deciding whether a third order polynomial is nonnegative over the unit sphere. If this can be done in polynomial-time, then the next question will be: Can we describe the set of the coefficients of such nonnegative third order polynomials (over the unit sphere) by (Linear) Matrix Inequalities?

## 5 Applications in Robust Linear Programming

Robust optimization models in mathematical programming have received much attention recently; see, e.g. [2, 3, 7]. In this section we will discuss some of these models using the techniques developed in the previous sections.

Consider the following formulation of a robust linear program:

$$\begin{aligned} & \text{minimize} && \max_{\|\Delta x\| \leq \delta, \|\Delta c\| \leq \epsilon_0} (c + \Delta c)^T (x + \Delta x) \\ & \text{subject to} && (a_i + \Delta a_i)^T (x + \Delta x) \geq (b_i + \Delta b_i), \\ & && \text{for all } \|(\Delta a_i, \Delta b_i)\| \leq \epsilon_i, \quad i = 1, 2, \dots, m, \quad \|\Delta x\| \leq \delta. \end{aligned} \tag{38}$$

Here two types of perturbation are considered. First, the problem data  $(\{a_i\}, \{b_i\}, c)$  might be affected by unpredictable perturbation (e.g., measurement error). Second, the optimal solution  $x^{opt}$  is subject to implementation errors caused by the finite precision arithmetic of digital hardware. That is, we

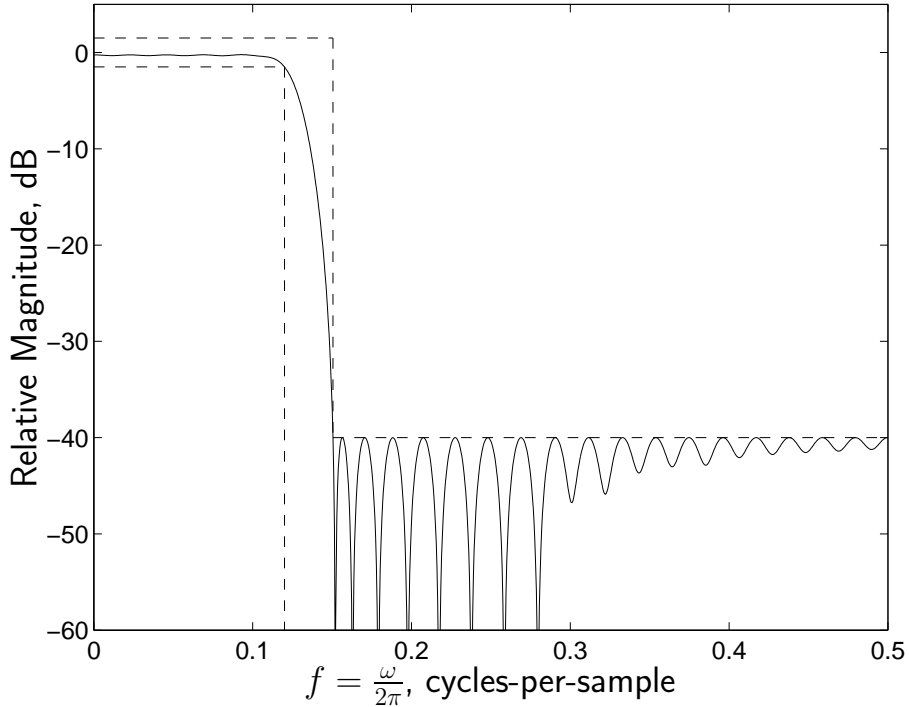


Figure 1: An example of spectral mask constraint

have  $x^{actual} := x^{opt} + \Delta x$ , where  $x^{actual}$  is the actually implemented solution. To ensure  $x^{actual}$  remains feasible and delivers a performance comparable to that of  $x^{opt}$ , we need to make sure  $x^{opt}$  is robust against both types of perturbations. This is essentially the motivation of the above robust linear programming model. Notice that our model is more general than the ones proposed by Ben-Tal and Nemirovskii [3] in that the latter only considers perturbation error in the data ( $\{a_i\}, \{b_i\}, c$ ).

The above model of robust linear programming arises naturally from the design of a linear phase FIR (Finite Impulse Response) filter for digital signal processing. In particular, for a linear phase FIR filter  $h = (h_1, \dots, h_n) \in \Re^n$ , the frequency response is

$$H(e^{j\omega}) = e^{-jn\omega}(h_1 + h_2 \cos \omega + \dots + h_n \cos(n\omega)) = e^{-jn\omega}(\mathbf{cos} \omega)^T h,$$

where  $\mathbf{cos} \omega = (1, \cos \omega, \dots, \cos(n\omega))^T$ . The FIR filter usually must satisfy a given spectral envelope constraint (typically specified by design requirement or industry standards), see Figure 1 for an example.

This gives rise to

$$L(e^{-j\omega}) \leq (\mathbf{cos} \omega)^T h \leq U(e^{-j\omega}), \quad \forall \omega \in [0, \pi]. \quad (39)$$

To find a discrete  $h$  (say, 4-bit integer) satisfying (39) is NP-hard. Ignoring discrete structure of  $h$ , we can find a  $h$  satisfying (39) in polynomial time [4]. However, rounding such solution to the nearest



discrete  $h$  may degrade performance significantly. Our design strategy is then to first discretize the frequency  $[0, \pi]$ , then find a solution robust to discretization and rounding errors. This leads to the following notion of robustly feasible solution:

$$L(e^{-j\omega_i}) \leq (\mathbf{cos} \omega_i + \Delta_i)^T (h + \Delta h) \leq U(e^{-j\omega_i}), \quad \text{for all } \|\Delta_i\| \leq \epsilon, \|\Delta h\| \leq \delta,$$

where  $\Delta_i$  accounts for discretization error, while  $\Delta h$  models the rounding errors.

We now reformulate the robust linear program (38) as a semidefinite program. We say the solution  $x$  is *robustly feasible* if, for all  $i = 1, 2, \dots, m$ ,

$$(a_i + \Delta a_i)^T (x + \Delta x) \geq (b_i + \Delta b_i), \quad \text{for all } \|(\Delta a_i, \Delta b_i)\| \leq \epsilon_i, \|\Delta x\| \leq \delta, \quad i = 1, 2, \dots, m.$$

It can be shown [3] that  $x$  is robustly feasible if and only if

$$a_i^T x - b_i - \epsilon_i \sqrt{\|x + \Delta x\|^2 + 1} \geq 0, \quad \forall \|\Delta x\| \leq \delta, \quad i = 1, 2, \dots, m. \quad (40)$$

Constraint (40) can be formulated as

$$\begin{bmatrix} I & \sqrt{\epsilon_i} \begin{bmatrix} x + \Delta x \\ 1 \end{bmatrix} \\ \sqrt{\epsilon_i} \begin{bmatrix} (x + \Delta x)^T & 1 \end{bmatrix} & a_i^T (x + \Delta x) - b_i \end{bmatrix} \succeq 0, \quad \forall \|\Delta x\| \leq \delta, \quad i = 1, 2, \dots, m. \quad (41)$$

Now the objective function can also be modelled by introducing an additional variable  $t$  to be minimized, and at the same time set as a constraint  $t - (c + \Delta c)^T (x + \Delta x) \geq 0$ , for all  $\|\Delta c\| \leq \epsilon_0$  and  $\|\Delta x\| \leq \delta$ . Then the objective can be modelled by  $t - c^T (x + \Delta x) \geq \epsilon_0 \|x + \Delta x\|$ , for all  $\|\Delta x\| \leq \delta$ , which is equivalent to

$$\begin{bmatrix} I & \sqrt{\epsilon_0} (x + \Delta x) \\ \sqrt{\epsilon_0} (x + \Delta x)^T & t - c^T (x + \Delta x) \end{bmatrix} \succeq 0, \quad \forall \|\Delta x\| \leq \delta. \quad (42)$$

Now we apply the results in Section 3 to get an explicit form in terms of LMIs with respect to  $(A, b, c)$  and  $x$ . In particular, we apply Theorem 3.5. It is clear that if we let  $H = I$ ,  $F = \sqrt{\epsilon_i} \begin{bmatrix} x \\ 1 \end{bmatrix}$ ,

$G = \sqrt{\epsilon_i} \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $C = a_i^T x - b$ ,  $B = \frac{1}{2} a_i$  and  $A = 0$ , then Theorem 3.5 gives us an equivalent condition for (41) as follows: there exists a  $\mu_i \geq 0$  such that

$$\begin{bmatrix} I & \sqrt{\epsilon_i} \begin{bmatrix} x \\ 1 \end{bmatrix} & \sqrt{\epsilon_i} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ \sqrt{\epsilon_i} \begin{bmatrix} x^T \\ 1 \end{bmatrix} & a_i^T x - b_i & \frac{1}{2} a_i^T \\ \sqrt{\epsilon_i} \begin{bmatrix} I \\ 0 \end{bmatrix} & \frac{1}{2} a_i & 0 \end{bmatrix} - \mu_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & -I \end{bmatrix} \succeq 0. \quad (43)$$

Similarly, (42) holds for all  $\|\Delta x\| \leq \delta$  if and only if there is a  $\mu_0 \geq 0$  such that

$$\begin{bmatrix} I & \sqrt{\epsilon_0}x & \sqrt{\epsilon_0}I \\ \sqrt{\epsilon_0}x^T & t - c^T x & -\frac{1}{2}c^T \\ \sqrt{\epsilon_0}I & -\frac{1}{2}c & 0 \end{bmatrix} - \mu_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & -I \end{bmatrix} \succeq 0. \quad (44)$$

Therefore, the robust linear programming model becomes a semidefinite program: minimize  $t$ , subject to (43) and (44).

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