Approximation Bounds for Quadratic Maximization with Semidefinite Programming Relaxation

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December 2002

Abstract

In this paper, we consider a class of quadratic maximization problems. One important instance in that class is the famous quadratic maximization formulation of the max-cut problem studied by Goemans and Williamson [6]. Since the problem is NP-hard in general, following Goemans and Williamson, we apply the approximation method based on the semidefinite programming (SDP) relaxation. For a subclass of the problems, including the ones studied by Helmberg [9] and Zhang [23], we show that the SDP relaxation approach yields an approximation solution with the worst-case performance ratio at least $\alpha = 0.87856\cdots$. This is a generalization of the results obtained in [6, 9, 23]. In fact, the estimated worst-case performance ratio is dependent on the data of the problem with α being a uniform lower bound. In light of this new bound, we study the original max-cut problem and show that the actual worst-case performance ratio of the SDP relaxation approach (with the triangle inequalities added) is at least $\alpha + \delta_d$, where $\delta_d > 0$ is a constant depending on the problem dimension and data. Karloff [10] showed that for any positive $\epsilon > 0$ there is an instance of the max-cut problem such that the SDP relaxation (with triangle inequalities) bound is worse than $\alpha + \epsilon$. Hence the improvement is in this sense best possible for the Goemans and Williamson type approach to the max-cut problem.

Key words: quadratic maximization, semidefinite programming relaxation, approximation algorithm, performance ratio.

AMS subject classification: 49M27, 90C20, 90C22.

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1. Introduction

One of the recent ground-breaking works in approximation algorithms for combinatorial optimization is the seminal paper by Goemans and Williamson [6] (1995), in which a semidefinite programming relaxation is proposed for the quadratic maximization formulation of the classical NP-hard max-cut problem, followed by a novel randomized rounding scheme. Goemans and Williamson managed to show that this procedure gives an approximate solution for the max-cut problem with the worst-case performance ratio guaranteed to be no less than $0.87856\cdots$, denoted by α in this paper. Soon it turned out that the bound α is essentially tight for the Goemans-Williamson approach. In particular, Karloff [10] shows that for any positive $\epsilon > 0$ there exists an instance of the max-cut problem such that the optimal value of the corresponding SDP relaxation is large than $1/(\alpha+\epsilon)$ times the optimal max-cut value. Up to date, no known polynomial-time approximation algorithm solves the max-cut problem with a guaranteed worst-case performance ratio substantially better than α. On the negative side, Håstad [8] shows that it is NP-hard to approximate the max-cut problem with a guaranteed worst case bound more than $16/17+\epsilon = 0.94117\cdots+\epsilon$, where $\epsilon > 0$ is any fixed positive constant.

The technique introduced by Goemans and Williamson has found applications in many other combinatorial optimization problems. For surveys on the subject, see [5, 7, 9, 12, 16, 19]. The power of the method has also been extended to more general non-convex quadratic programming problems; see [9, 13, 14, 15, 18, 20, 21, 23]. As a consequence of their rank-1 matrix decomposition technique, Sturm and Zhang [17] show that the SDP relaxation can be exact for some interesting quadratic optimization problems. Tseng [18], and Ye and Zhang [22] further extend the results for quadratic optimization problems. Zhang [23] studies the quality of the SDP relaxation method under various assumptions on the structure of the quadratic maximization problem. In particular, it turns out that the worst-case performance ratio α holds whenever a certain off-diagonal nonpositivity (termed OD nonpositivity in the remainder of the paper) condition in the cost matrix is satisfied. Moreover, it is shown that the SDP relaxation is actually exact whenever the cost matrix is off-diagonally nonnegative (termed OD nonnegative). The last result was extended by Kim and Kojima [11], where they show that a simpler second-order cone relaxation is in fact already exact.

For the max-cut problem itself, Goemans and Williamson [6] show that their rounding algorithm behaves even better if the percentage of edges in the cut is relatively high, or more precisely, if the cut contains more than 85% of the total weight of the edges. Using outward rotations, Zwick [24] obtains an approximation algorithm for max-cut and the performance guarantee is strictly greater than α, unless the maximum cut of the input graph contains about 84.458% of the total weight of the edges of the graph. Feige and Langberg [3] present a procedure called RPR^2 (Random Projection followed by Randomized Rounding) for rounding the solution of semidefinite programs and improve the tradeoff curve (presented by Zwick) for max-cut. In general, however, the approximation ratio remains α.

In this paper we mainly present three new results for quadratic maximization. In Section 2, we discuss some special cases of quadratic maximization problems where the signs in the constraint matrix are structured. We show that the SDP relaxation approach has a worst-case bound α for these problem instances. The result can be considered as generalizations of the ones established in [9] and [23]. In Section 3 we show that if the constraint matrix is indeed OD nonpositive, then the SDP relaxation bound can be estimated using the SDP optimal solution itself, with α as a uniform lower bound of the new estimated bound. This analysis relies on the convexity of a related value function. Making further use of the convexity we prove in Section 4 that the true approximation ratio for the max-cut problem is $\alpha + \delta$ where $\delta > 0$ is in the order of $\Omega(\frac{1}{n^2 L})$ with L denoting the ratio between the maximum weight and the minimum weight, if we add the triangle inequalities in the SDP relaxation and assume that every weight is strictly positive. We remark here that recently Feige, Karpinski and Langberg [2] show that if the degree of the graph is at most Δ, then the SDP relaxation approach yields an approximation bound $\alpha + \Omega(\frac{1}{\Delta^4})$, a result with an accent similar to ours.

Throughout, $\Re^n$ denotes the space of n-dimensional Euclidean space; $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices; ^{T} denotes the transpose operation for a matrix. We denote $e_i \in \Re^n$ as the i-th unit vector, i.e. the vector whose i-th component is 1 and all other components are zeros. For any $x \in \Re^n$, we denote x_i as the i-th component of x. For $A \in \Re^{n\times n}$, A_{ij} denotes the (i,j)-th entry of A and $\mathrm{diag}(A)$ denotes an n-dimensional vector formed by the diagonal elements of A. For $A, B \in \mathcal{S}^n$, we denote $\langle A, B\rangle = \sum_{i,j} A_{ij}B_{ij}$ as the inner-product between A and B. We denote $A \succeq B$ (respectively $A \succ B$) by the fact that $A - B$ is positive semidefinite (respectively positive definite). For a given one-dimensional function f, we denote $f(A)$ to be $[f(A_{ij})]_{n\times n}$. In particular, for $a \in \Re^n$, we write a^2 to denote the n-dimensional vector which is component-wise square of a. For a given vector d, we use the capitalized letter D to denote the diagonal matrix which takes d as its diagonal elements.

2 Quadratic maximization and its approximation

Consider the follow quadratic maximization problem

$$\begin{array}{lll} (QP) & \text{maximize} & x^T Q x \\ & \text{subject to} & x^2 \in \mathcal{F}, \end{array}$$

where $\mathcal{F}$ is a closed convex subset of $\Re^n$, and Q is an arbitrary symmetric matrix.

A related nonlinear semidefinite programming formulation is given as follows:

$$\begin{array}{lll} (SP) & \text{maximize} & \frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle \\ & \text{subject to} & d \geq 0,\ d^2 \in \mathcal{F} \\ & & X \succeq 0,\ \mathrm{diag}(X) = e, \end{array}$$

where $\arcsin(X) := [\arcsin(X_{ij})]_{n\times n}$ and e denotes the all-one vector.

Let $v(QP)$ denote the optimal value of (QP) and $v(SP)$ denote the optimal value of (SP). It can be shown that these two optimal values coincide; see e.g. [23].

Theorem 2.1 *It holds that* $v(QP) = v(SP)$.

Consider a relaxed semidefinite maximization problem:

$$\begin{array}{lll} (R) & \text{maximize} & \langle Q, Z\rangle \\ & \text{subject to} & \operatorname{diag}(Z) \in \mathcal{F} \\ & & Z \succeq 0. \end{array}$$

It follows that

$$v(SP) \le v(R). \tag{2.1}$$

Because $\mathcal{F}$ is a closed convex set, (R) is a well-formulated convex optimization problem.

If $Q \succeq 0$, Nesterov [14] proved that

$$v(QP) \ge \frac{2}{\pi} v(R). \tag{2.2}$$

The solution of (R) functions as a good upper bound for (QP) whereas (QP) itself is in general an NP-hard problem. Based on the solution for (R), one obtains an approximative solution for (QP) with the worst-case performance ratio being $2/\pi = 0.63661\cdots$. Before proceeding let us introduce some terms used in [23].

Definition 2.2 *We call a symmetric matrix* $Q = [q_{ij}]_{n\times n}$ OD-nonnegative *if* $q_{ij} \ge 0$ *for all* $i, j = 1, \cdots, n$, *and* $i \neq j$. *Let the set of all OD-nonnegative matrices be denoted by* ODP.

Definition 2.3 *We call a symmetric matrix* $Q = [q_{ij}]_{n\times n}$ OD-nonpositive *if* $q_{ij} \le 0$ *for all* $i, j = 1, \cdots, n$, *and* $i \neq j$. *Let the set of all OD-nonpositive matrices be denoted by* ODN.

A slight extension of the OD-signed constrained matrices are the following classes of matrices.

Definition 2.4 *We call a symmetric matrix* $Q = [q_{ij}]_{n\times n}$ almost OD-nonnegative *if there exists a sign vector* $\sigma \in \{-1, +1\}^n$ *such that*

$$q_{ij}\sigma_i\sigma_j \ge 0$$

for all $i, j = 1, \cdots, n$, *and* $i \neq j$.

Definition 2.5 *We call a symmetric matrix* $Q = [q_{ij}]_{n\times n}$ almost OD-nonpositive *if there exists a sign vector* $\sigma \in \{-1,+1\}^n$ *such that*

$$q_{ij}\sigma_i\sigma_j \le 0$$

for all $i, j = 1, \cdots, n$*, and* $i \neq j$*.*

In [23], Zhang proved that $v(QP) = v(SP) = v(R)$ if Q is almost OD-nonnegative, and $v(QP) = v(SP) \ge \alpha v(R)$ if $Q \succeq 0$ is almost OD-nonpositive where $\alpha = 0.87856\cdots$. The last statement is an improvement on the bound (2.2) under the sign condition.

Along a similar line, Helmberg [9] observes that $v(QP) \ge \alpha v(R)$ holds if Q is in the matrix cone generated by the $n \times n$ rank-1 positive semidefinite matrices

$$(\delta_i e_i + \delta_j e_j)(\delta_i e_i + \delta_j e_j)^T \tag{2.3}$$

where δ_i, $\delta_j \in \{-1,+1\}$, $i \neq j$.

Clearly, Q being in the above mentioned matrix cone does not imply that it is OD-nonpositive, nor the other implication direction is true. In this section we shall prove a general result unifying both considerations.

Let us start by introducing a matrix operation as follows. Let $A = [a_{ij}]_{n\times n} \in \Re^{n\times n}$. Then, $\mathcal{O}(A) \in \Re^{n\times n}$ is defined as

$$(\mathcal{O}(A))_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ \ \ a_{ij} & \text{if } i = j. \end{cases}$$

In simple terms, $\mathcal{O}(A)$ changes the signs of the off-diagonal elements of A to the opposite ones, while the diagonal elements remain intact.

Following Definitions 2.2 and 2.3, let us further define

$$M^+ := \{A \mid A \in \text{ODP},\ \mathcal{O}(A) \succeq 0\}$$

and

$$M^- := \{A \succeq 0 \mid A \in \text{ODN}\}.$$

It is interesting to note the following fact.

Lemma 2.6 *It holds that* $M^+ \subseteq \mathcal{S}^n_+$*.*

Proof:

Let $A = [a_{ij}]_{n\times n} \in M^+$. This means that $a_{ij} \ge 0$ for $i \neq j$, and $\mathcal{O}(A) \succeq 0$.

Let x be an arbitrary n-dimensional real vector. We have

$$x^T Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij} = \sum_{i=1}^{n} x_i^2 a_{ii} + \sum_{i\neq j} x_i x_j a_{ij} \geq \sum_{i=1}^{n} |x_i|^2 a_{ii} - \sum_{i\neq j} |x_i||x_j| a_{ij} \geq 0$$

where the last step is due to the fact that $\mathcal{O}(A) \succeq 0$. Hence, it follows that $A \succeq 0$. Therefore, we have $M^+ \subseteq \mathcal{S}_+^n$ as asserted by the lemma. □

One immediate consequence of Lemma 2.6 is that if $A \in M^-$ then $\mathcal{O}(A) \succeq 0$.

Now we are in a position to study the relationships among (QP), (SP), and (R).

Let Z be a feasible solution to (R). Let

$$d = \sqrt{\operatorname{diag}(Z)} \quad \text{and} \quad X = D^+ZD^+ + \bar{D} \tag{2.4}$$

where D^+ stands for the pseudo-inverse of D, i.e. it is also diagonal and

$$(D)^+_{ii} = \begin{cases} (d_i)^{-1}, & \text{if } d_i > 0 \\ 0, & \text{if } d_i = 0, \end{cases}$$

and $\bar{D}$ denotes a binary diagonal matrix where $\bar{D}_{ii} = 1$ if $Z_{ii} = 0$ and $\bar{D}_{ii} = 0$ otherwise. It can be easily verified that $Z_{ij} = d_i d_j X_{ij}$ for all i and j. Conversely, if (D, X) is a feasible solution for (SP) then $Z = DXD$ is a feasible solution for (R).

We first prove the following lemma.

Lemma 2.7 *Suppose that the objective matrix Q of (QP) is in $M^+ \cup M^-$. Then, for any feasible solution (D, X) of (SP) and $Z = DXD$ which is a feasible solution of (R), it holds that*

$$\frac{2}{\pi}\langle Q, D \arcsin(X) D\rangle \geq \alpha \langle Q, Z\rangle.$$

Proof:

First we note the following inequality, which plays a crucial role in [6], i.e.

$$\frac{2}{\pi}\arcsin t \leq 1 - \alpha + \alpha t, \text{ for all } t \in [-1, 1]. \tag{2.5}$$

By setting $t := -t$, we have

$$\frac{2}{\pi}\arcsin t \geq \alpha - 1 + \alpha t, \text{ for all } t \in [-1, 1]. \tag{2.6}$$

Suppose that $Q \in M^+$. Then,

$$\begin{aligned}
\frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle &= \sum_{i,j} q_{ij}d_id_j(\frac{2}{\pi}\arcsin(X_{ij})) \\
&= \sum_{i\neq j} q_{ij}d_id_j(\frac{2}{\pi}\arcsin(X_{ij})) + \sum_{i=1}^{n} q_{ii}d_i^2 \\
&\geq \sum_{i\neq j} q_{ij}d_id_j(\alpha - 1 + \alpha X_{ij}) + \sum_{i=1}^{n} q_{ii}d_i^2 \qquad \text{(by (2.6))} \\
&= \alpha \sum_{i,j} q_{ij}d_id_jX_{ij} + (1-\alpha)\sum_{i,j} \mathcal{O}(Q)_{ij}d_id_j \\
&= \alpha\langle Q, Z\rangle + (1-\alpha)d^T(\mathcal{O}(Q))d \\
&\geq \alpha\langle Q, Z\rangle.
\end{aligned}$$

Similarly, suppose that $Q \in M^-$. Then

$$\begin{aligned}
\frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle &= \sum_{i,j} q_{ij}d_id_j(\frac{2}{\pi}\arcsin(X_{ij})) \\
&= \sum_{i\neq j} q_{ij}d_id_j(\frac{2}{\pi}\arcsin(X_{ij})) + \sum_{i=1}^{n} q_{ii}d_i^2 \\
&\geq \sum_{i\neq j} q_{ij}d_id_j(1 - \alpha + \alpha X_{ij}) + \sum_{i=1}^{n} q_{ii}d_i^2 \qquad \text{(by (2.5))} \\
&= \alpha \sum_{i,j} q_{ij}d_id_jX_{ij} + (1-\alpha)\sum_{i,j} q_{ij}d_id_j \\
&= \alpha\langle Q, Z\rangle + (1-\alpha)\sum_{i,j} q_{ij}d_id_j \\
&= \alpha\langle Q, Z\rangle + (1-\alpha)d^TQd \\
&\geq \alpha\langle Q, Z\rangle,
\end{aligned}$$

where the last step is due to the fact that $Q \succeq 0$. □

Because of the linearity of the inner product, the above result extends to matrices that are in the matrix cone generated by $M^+ \cup M^-$, denoted by cone $(M^+ \cup M^-)$.

Theorem 2.8 *Suppose that the objective matrix Q of (QP) is in* cone $(M^+ \cup M^-)$. *Then, for any feasible solution (D, X) of (SP) and $Z = DXD$ which is a feasible solution of (R), it holds that*

$$\frac{2}{\pi}\langle Q, D\arcsin(X)D\rangle \geq \alpha\langle Q, Z\rangle.$$

Consequently, we have

Theorem 2.9 *Suppose that the objective matrix Q of (QP) is in* cone $(M^+ \cup M^-)$. *Then it holds that*

$$v(QP) = v(SP) \geq \alpha v(R).$$

Proof: We note that the inequality guaranteed by Theorem 2.8 holds true for the optimal solution (D^*, X^*) of (SP) as well. The result thus follows. $\square$

Obviously, the matrix $(\delta_i e_i + \delta_j e_j)(\delta_i e_i + \delta_j e_j)^T$ is in cone $(M^+ \cup M^-)$. Hence, Theorem 2.9 is a generalization of the result in [9].

Since $M^- \subseteq$ cone $(M^+ \cup M^-)$, Theorem 2.9 also takes Theorem 3 in [23] as a special case.

To see that Theorem 2.9 genuinely generalizes the results in [9] and [23], consider

$$\begin{aligned} Q_1 &= \begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &\triangleq Q_2 + Q_3. \end{aligned}$$

It is easy to verify that $Q_2 \in M^-$ and $Q_3 \in M^+$. So Theorem 2.9 applies, when the objective matrix in (QP) is Q_1. However, Q_1 is neither almost OD-nonpositive nor it is a positive sum of the matrices as given in (2.3).

In the same vein as in [23] we can extend Theorem 2.9 a little bit further.

Theorem 2.10 *Let the objective matrix of (QP) be Q. Suppose that there is a sign vector $\sigma \in \{-1, +1\}^n$ such that*

$$\text{diag }(\sigma) Q \text{diag }(\sigma) \in \text{cone } (M^+ \cup M^-).$$

Then it holds that

$$v(QP) = v(SP) \geq \alpha v(R).$$

The following remains an interesting open question: How can one verify whether a given symmetric matrix Q belongs to cone $(M^+ \cup M^-)$ or not? If this can be done, then the next question would be: How to verify the condition assumed in Theorem 2.10?

3 OD nonpositive quadratic maximization

In this section, we concentrate on the OD nonpositive quadratic maximization problem. The goal is to get an estimation on the worst-case performance ratio in terms of the problem/solution data. In most cases, this ratio will turn out to be better that α, which plays the role as a uniform lower bound.

Similar as in (2.4), let Z^* be an optimal solution of (R), and $d^* = \sqrt{\mathrm{diag}(Z^*)}$.

For this fixed d^*, consider

$$\begin{array}{lll} (SP)_{d^*} & \text{maximize} & \frac{2}{\pi}\langle Q, D^* \arcsin(X) D^* \rangle \\ & \text{subject to} & X \succeq 0, \ \mathrm{diag}(X) = e, \end{array}$$

and its SDP relaxation

$$\begin{array}{lll} (R)_{d^*} & \text{maximize} & \langle Q, D^* X D^* \rangle \\ & \text{subject to} & X \succeq 0, \ \mathrm{diag}(X) = e. \end{array}$$

It is easy to see that the optimal value of $(R)_{d^*}$ equals the optimal value of (R), and the optimal value of $(SP)_{d^*}$ is no more than the optimal value of (SP).

Let X^* be an optimal solution of $(R)_{d^*}$. Obviously, $X = ee^T$ (the all-one matrix) is in any case a feasible solution for $(R)_{d^*}$. In case ee^T is indeed optimal for $(R)_{d^*}$, then it is also optimal for $(SP)_{d^*}$. Hence the relaxation is exact. For the interesting case we may assume that ee^T is not optimal for $(R)_{d^*}$, i.e.

$$\sum_{i,j} q_{ij} d_i^* d_j^* X_{ij}^* > \sum_{i,j} q_{ij} d_i^* d_j^*$$

or equivalently,

$$\sum_{i \neq j} q_{ij} d_i^* d_j^* X_{ij}^* > \sum_{i \neq j} q_{ij} d_i^* d_j^*.$$

Denote

$$W := \sum_{i \neq j} q_{ij} d_i^* d_j^* (1 - X_{ij}^*) < 0. \tag{3.7}$$

Since we are now concerned with the case $Q \in \mathrm{ODN}$, we have $q_{ij} d_i^* d_j^* (1 - X_{ij}^*) \le 0$ for all $i \neq j$. Let us further define

$$\lambda_{ij} := \frac{q_{ij} d_i^* d_j^* (1 - X_{ij}^*)}{W} \ge 0$$

for all $i \neq j$. Naturally we have $\sum_{i \neq j} \lambda_{ij} = 1$, by definition of W. In other words, λ_{ij}'s can be viewed as coefficients of a convex combination.

We further note that $\mathrm{diag}((X^*)^2) = e$ and $(X^*)^2 \succeq 0$ and so $(X^*)^2$ forms a feasible solution for $(R)_{d^*}$. Therefore, by optimality of X^* we have

$$\sum_{i \neq j} q_{ij} d_i^* d_j^* X_{ij}^* \ge \sum_{i \neq j} q_{ij} d_i^* d_j^* (X_{ij}^*)^2. \tag{3.8}$$

Now denote

$$A := \frac{\sum_{i \neq j} q_{ij} d_i^* d_j^* (1 - X_{ij}^*) X_{ij}^*}{W} = \sum_{i \neq j} \lambda_{ij} X_{ij}^*.$$

Thus,

$$-1 \leq A \leq 0 \tag{3.9}$$

where the second inequality is due to (3.8) and the first inequality is due to the fact that $X^* \succeq 0$ and so $|X_{ij}^*| \leq 1$ for all i, j.

Now let us consider a function that plays an important role in our analysis:

$$h(t) = \frac{1 - \frac{2}{\pi} \arcsin(t)}{1 - t}, \qquad t \in [-1, 1). \tag{3.10}$$

Lemma 3.1 *The function $h(t)$ defined in (3.10) is strongly convex in the interval [-1,1), and it attains its minimum at $t = -0.6892\cdots$ with the minimum value equal to $\alpha = 0.87856\cdots$.*

Proof:

Simple computation shows that

$$h'(t) = -\frac{2}{\pi} \frac{1}{(1 - t^2)^{1/2}(1 - t)} + \frac{1 - \frac{2}{\pi} \arcsin t}{(1 - t)^2}$$

and

$$h''(t) = -\frac{2}{\pi} \frac{1}{(1 - t^2)^{3/2}} \frac{t}{1 - t} - \frac{4}{\pi} \frac{1}{(1 - t^2)^{1/2}(1 - t)^2} + 2 \frac{1 - \frac{2}{\pi} \arcsin t}{(1 - t)^3}.$$

One further computes that

$$h''(t) \geq 0.4472 =: \mu, \tag{3.11}$$

for all $t \in [-1, 1)$. The second part of the lemma is proved by numerically solving the root of $h'(t)$. $\square$

To visualize the picture, see Figure 1.

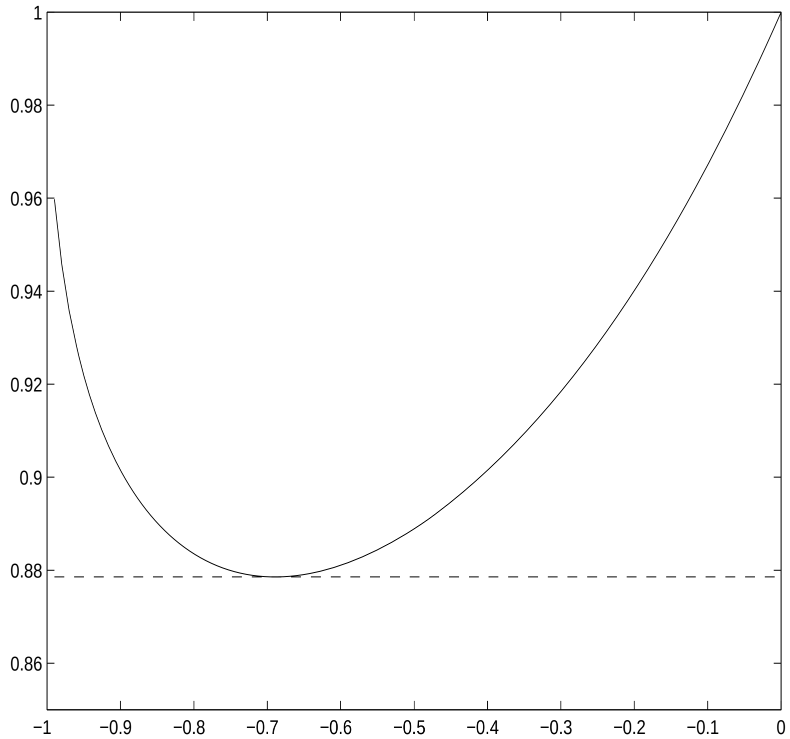

Figure 1: Function h

In Figure 1, the solid line corresponds to the function $h(t)$, and the dashed line corresponds to the minimum value α.

Theorem 3.2 *If $Q \in \text{ODP}$, then*

$$v(QP) = v(SP) \geq h(A)v(R).$$

Proof:

Let $\mathcal{I} = \{(i,j) \mid X^*_{ij} \neq 1\}$.

Obviously, $\lambda_{ij} = 0$ whenever $(i,j) \notin \mathcal{I}$.

By the convexity of h it follows that

$$\begin{aligned} h(A) &= h(\sum_{(i,j)\in\mathcal{I}} \lambda_{ij} X^*_{ij}) \\ &\leq \sum_{(i,j)\in\mathcal{I}} \lambda_{ij} h(X^*_{ij}) \\ &= \sum_{(i,j)\in\mathcal{I}} \frac{q_{ij} d^*_i d^*_j (1 - X^*_{ij})}{W} \times \frac{1 - \frac{2}{\pi}\arcsin(X^*_{ij})}{1 - X^*_{ij}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{W}\sum_{(i,j)\in\mathcal{I}} q_{ij}d_i^*d_j^*(1-\frac{2}{\pi}\arcsin(X_{ij}^*)) \\
&= \frac{1}{W}\sum_{i\neq j} q_{ij}d_i^*d_j^*(1-\frac{2}{\pi}\arcsin(X_{ij}^*)).
\end{aligned}
$$

Noting (3.7), the above inequality can be rewritten as

$$
\sum_{i\neq j} q_{ij}d_i^*d_j^*(\frac{2}{\pi}\arcsin(X_{ij}^*)) \geq \sum_{i\neq j} q_{ij}d_i^*d_j^*(1-h(A)+h(A)X_{ij}^*). \tag{3.12}
$$

Since X^* forms a feasible solution to $(SP)_{d^*}$, therefore we have

$$
\begin{aligned}
v(SP) &\geq v(SP)_{d^*} \\
&\geq \sum_{i,j} q_{ij}d_i^*d_j^*(\frac{2}{\pi}\arcsin(X^*)) \\
&= \sum_{i\neq j} q_{ij}d_i^*d_j^*(\frac{2}{\pi}\arcsin(X_{ij}^*)) + \sum_{i=1}^n q_{ii}(d_i^*)^2 \\
&\geq \sum_{i\neq j} q_{ij}d_i^*d_j^*(1-h(A)+h(A)X_{ij}^*) + \sum_{i=1}^n q_{ii}(d_i^*)^2 \qquad \text{(by (3.12))} \\
&= (1-h(A))\sum_{i,j} q_{ij}d_i^*d_j^* + h(A)\sum_{i,j} q_{ij}d_i^*d_j^*X_{ij}^* \\
&= (1-h(A))(d^*)^TQd^* + h(A)v(R) \\
&\geq h(A)v(R). \qquad \text{(by } Q\succeq 0)
\end{aligned}
$$

This concludes the proof. $\square$

Remark that Theorem 3.2 directly applies to the max-cut and the max-2Sat problems.

Combining Theorem 3.2 and Lemma 3.1, we have

Corollary 3.3 *If $Q \in$ ODP, then $v(QP) = v(SP) \geq \alpha v(R)$.*

4 Further analysis of the max-cut problem

In this section, we shall revisit the original max-cut problem in light of the strong convexity property of $h(\cdot)$ (Lemma 3.1).

Let us start by introducing the max-cut problem. Consider an undirected graph $G(V,E)$ with $|V|=n$ and nonnegative (but not all zeros) weights $w_{ij}=w_{ji}$ on each edge $(i,j)\in E$. The max-cut problem aims at finding the set of vertices S that maximizes the total weight of the edges in the *cut* $(S, \bar{S})$; that is, the total weight of the edges with one endpoint in S and the other endpoint in $\bar{S}$. In this paper, we assume that the graph is complete and that $w_{ij} > 0$ for all $(i,j) \in E$. The max-cut problem is a classical NP-hard problem. It is provable that it is even NP-hard to get an approximate algorithm with a worst-case performance ratio higher than 16/17; see [8].

The Goemans and Williamson approach to the max-cut problem begins with a mathematical programming reformulation of the problem:

$$\begin{array}{lll} (MC) & \text{maximize} & \frac{1}{2}\sum_{i,j} w_{ij}(1 - x_i x_j) \\ & \text{subject to} & x_i^2 = 1, \quad i = 1, \cdots, n, \end{array}$$

which can be viewed as one instance of (QP) with the cost matrix $Q = [q_{ij}]_{n \times n}$ given as

$$q_{ij} = \begin{cases} -w_{ij}, & \text{if } i \neq j \\ \displaystyle\sum_{k=1}^{n} w_{ik}, & \text{if } i = j, \end{cases}$$

and $\mathcal{F} = \{e\}$, which is a singleton. This Q is known as the Laplacian matrix of the graph. In the case that the weights are all nonnegative then $Q \succeq 0$.

The equivalent nonlinear semidefinite formulation is

$$\begin{array}{lll} (MCSP) & \text{maximize} & \frac{2}{\pi}\langle Q, \arcsin(X)\rangle \\ & \text{subject to} & X \succeq 0, \ \text{diag}(X) = e, \end{array}$$

and the convex semidefinite programming relaxation for $(MCSP)$ is (see e.g. [1])

$$\begin{array}{lll} (MCR) & \text{maximize} & \langle Q, X\rangle \\ & \text{subject to} & X \succeq 0, \ \text{diag}(X) = e. \end{array} \tag{4.13}$$

As a special case, Theorem 3.2 and Corollary 3.3 apply to problems (MC) and (MCR). In particular, the result in Corollary 3.3, i.e. $v(MC) \geq \alpha v(MCR)$, is the celebrated approximation bound of Goemans and Williamson [6].

Let X^* be an optimal solution of (MCR), i.e., $v(MCR) = \langle Q, X^*\rangle$.

Let us denote

$$\lambda_{ij} = w_{ij}(1 - X_{ij}^*) \geq 0$$

for all $i \neq j$, and

$$\lambda = \sum_{i,j} \lambda_{ij} = \langle Q, X^*\rangle > 0.$$

Furthermore, denote

$$A = \sum_{i \neq j} \frac{\lambda_{ij}}{\lambda} X_{ij}^*. \tag{4.14}$$

As in Section 3 (see (3.9)), we have $-1 \le A \le 0$.

Also along the same line as in Section 3, denote $\mathcal{I} = \{(i,j) \mid X^*_{ij} \neq 1,\ i,j = 1,2,\cdots,n\}$.

By the strong convexity of h as established in Lemma 3.1 we have

$$h(X^*_{ij}) \ge h(A) + h'(A)(X^*_{ij} - A) + \frac{\mu}{2}(X^*_{ij} - A)^2, \tag{4.15}$$

for all i,j, where $\mu = 0.4472$ (cf. (3.11)) is the lower bound on the second order derivative of h in the domain $[-1,1)$.

Multiplying $\frac{\lambda_{ij}}{\lambda}$ to (4.15) and sum up for all $(i,j) \in \mathcal{I}$, we have

$$\sum_{(i,j)\in\mathcal{I}} \frac{\lambda_{ij}}{\lambda} h(X^*_{ij}) \ge h(A) + \frac{\mu}{2} \sum_{(i,j)\in\mathcal{I}} \frac{\lambda_{ij}}{\lambda}(X^*_{ij} - A)^2 = h(A) + \frac{\mu}{2}\sum_{i,j}\frac{\lambda_{ij}}{\lambda}(X^*_{ij} - A)^2. \tag{4.16}$$

Now the left side of the above inequality is

$$\begin{aligned}
& \sum_{(i,j)\in\mathcal{I}} \frac{\lambda_{ij}}{\lambda} h(X^*_{ij}) \\
= \ & \sum_{i,j} w_{ij}(1 - \frac{2}{\pi}\text{arcsin} X^*_{ij})/\langle Q, X^*\rangle \\
= \ & \langle Q, \frac{2}{\pi}\text{arcsin}(X^*_{ij})\rangle/\langle Q, X^*\rangle.
\end{aligned}$$

In combination of (4.16) we have

$$\begin{aligned}
v(MC) &= v(MCSP) \\
&\ge \langle Q, \frac{2}{\pi}\text{arcsin}(X^*_{ij})\rangle \\
&\ge h(A)\langle Q, X^*\rangle + \frac{\mu}{2}\sum_{i,j}\lambda_{ij}(X^*_{ij} - A)^2 \\
&= h(A)v(MCR) + \frac{\mu}{2}\sum_{i,j} w_{ij}(1 - X^*_{ij})(X^*_{ij} - A)^2. \qquad (4.17)
\end{aligned}$$

Geomans and Williams [6] use the random hyperplane rounding technique to convert a solution of (MCR) to a feasible cut in the graph. In that sense, any ratio between $v(MC)$ and $v(MCR)$ will serve as a worst-case performance bound.

Whenever $A \ge -0.5$ or $A \le -0.8$, we shall have $h(A) \ge 0.8835$, hence a bound better that α based on (4.17). Let us now consider the case

$$A \in [-0.8, -0.5]. \tag{4.18}$$

In the subsequent discussion we assume that

$$w_{ij} > 0 \qquad \text{for all } i \neq j. \tag{4.19}$$

Let

$$I_\varepsilon = \{(i,j) \mid |X^*_{ij} - A| \leq \varepsilon\}$$

and

$$J_\varepsilon = \{(i,j) \mid X^*_{ij} \geq 1 - \varepsilon, i \neq j\}$$

with $\varepsilon = 0.03$.

Let us consider an enhanced (but still polynomially sized) version of the SDP relaxation for the max-cut problem. That is, we add the following so-called triangle inequalities into the constraint set of (4.13):

$$\begin{aligned} X_{ij} + X_{jk} + X_{ki} &\geq -1, & (4.20)\\ -X_{ij} - X_{jk} + X_{ki} &\geq -1, & (4.21)\\ -X_{ij} + X_{jk} - X_{ki} &\geq -1, & (4.22)\\ X_{ij} - X_{jk} - X_{ki} &\geq -1, & (4.23) \end{aligned}$$

for all possible triplets $1 \leq i < j < k \leq n$.

Lemma 4.1 *Let X^* be an optimal solution to the problem (MCR) with (4.20), (4.21), (4.22) and (4.23) added to the constraint set. Then, $I_\varepsilon \cap J_\varepsilon = \emptyset$ and $\{(i,j) \mid 1 \leq i \neq j \leq n\} \setminus (I_\varepsilon \cup J_\varepsilon) \neq \emptyset$.*

Proof:

That $I_\varepsilon \cap J_\varepsilon = \emptyset$ is immediately clear by the fact that $A \in [-0.8, -0.5]$ and $\varepsilon = 0.03$.

We wish to prove the second assertion by contradiction. Namely, for the sake of contradiction, we assume that $I_\varepsilon \cup J_\varepsilon = \{(i,j) \mid 1 \leq i \neq j \leq n\}$.

Let $m = \text{rank}(X^*)$ and let $X^* = [v_1, \cdots, v_n]^T [v_1, \cdots, v_n]$. Therefore, $v_i \in \Re^m$, $\|v_i\| = 1$, $i = 1, ..., n$, and $X^*_{ij} = v_i^T v_j$ for all $1 \leq i, j \leq n$. Let

$$\mathcal{V}_1 = \{v_i \mid v_i^T v_1 \geq 1 - \varepsilon, \; i = 1, ..., n\}$$

and

$$\mathcal{V}_2 = \{v_j \mid |v_j^T v_1 - A| \leq \varepsilon, \; j = 1, ..., n\}.$$

Obviously, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \{v_1, \cdots, v_n\}$ by the contradiction assumption.

We shall first prove the following three facts regarding $\mathcal{V}_1$ and $\mathcal{V}_2$.

Fact 1. For any $v_i,\ v_j \in \mathcal{V}_1$, it must follow that $v_i^T v_j \geq 1 - \varepsilon$.

This is because if the above statement is incorrect then we must have $|v_i^T v_j - A| \leq \varepsilon$ instead (note the contradiction assumption). However,

$$v_i^T v_j = X_{ij}^* \geq -1 + X_{i1}^* + X_{1j}^* \geq -1 + 2 \times (1 - \varepsilon) = 1 - 2\varepsilon$$

due to (4.21). This is impossible as

$$|v_i^T v_j - A| \geq 1 - 2\varepsilon - A > \varepsilon.$$

Fact 2. For any $v_i,\ v_j \in \mathcal{V}_2$, we have $v_i^T v_j \geq 1 - \varepsilon$.

To see this, we again prove by contradiction. Suppose there are two vectors $v_i,\ v_j \in \mathcal{V}_2$ such that $|v_i^T v_j - A| \leq \varepsilon$. Then the triplet $\{v_1, v_i, v_j\}$ would violate the constraint (4.20).

Fact 3. For any $v_i \in \mathcal{V}_1,\ v_j \in \mathcal{V}_2$, we have $|v_i^T v_j - A| \leq \varepsilon$.

Once again, this can be shown by contradiction. Suppose that the above is not true. Then we would have $v_i \in \mathcal{V}_1,\ v_j \in \mathcal{V}_2$, and $v_i^T v_j - 1 \geq -\varepsilon$. Then, the triplet $\{v_1, v_i, v_j\}$ would contradict the constraints (4.21)-(4.23).

Next we shall perform an operation on the vectors $v_1, \cdots, v_n$ described by the following steps. First of all, let

$$v_k^T v_l = \max\{v_i^T v_j \mid v_i \in \mathcal{V}_1 \text{ and } v_j \in \mathcal{V}_2\}.$$

Using **Fact 3** we have $-0.83 \leq v_k^T v_l \leq -0.47$.

By **Fact 1** and **Fact 2** we know that all the vectors in $\mathcal{V}_1$ reside in the second order cone

$$\text{SOC}(v_k) := \{d \mid \angle(d, v_k) \leq \arccos(1 - \varepsilon)\},$$

and all the vectors in $\mathcal{V}_2$ reside in the second order cone

$$\text{SOC}(v_l) := \{d \mid \angle(d, v_l) \leq \arccos(1 - \varepsilon)\}.$$

By **Fact 3**, we know that $\text{SOC}(v_k) \cap \text{SOC}(v_l) = \{0\}$.

Let $h = (v_k + v_l)/2$. Then,

$$\angle(v_k, h) = \angle(v_l, h) \leq \arccos(A - \varepsilon)/2 \leq \arccos(-0.83)/2 \leq 1.28.$$

Moreover, for any $d \in \text{SOC}(v_k)$ we have

$$\angle(d, h) \leq \angle(d, v_k) + \angle(v_k, h) \leq \arccos(1 - \varepsilon) + \arccos(-0.83)/2 \leq 1.53 < \pi/2.$$

By symmetry, we also have $\angle(d,h) \le 1.53 < \pi/2$ for all $d \in \mathrm{SOC}(v_l)$.

This implies that there exists a solid and pointed convex cone containing the sum of the two second order cones, namely there is a solid, pointed convex cone $\mathcal{K}$ such that $\mathcal{K} \supseteq \mathrm{SOC}(v_k) + \mathrm{SOC}(v_l)$. In particular, we may, for instance, let

$$\mathcal{K} := \mathrm{SOC}(h) := \{d \mid \angle(d,h) \le 1.53 < \pi/2\}.$$

Therefore $\mathrm{int}\ \mathcal{K}^* \neq \emptyset$.

Before proceeding let us consider a geometric property. Suppose that there is a given direction ι and a vector x. Suppose $\angle(\iota, x) > \pi/2$, i.e. $\iota^T x < 0$. Consider a movement from x to $x + \Delta x$, where Δx is a small displacement. Then,

$$\begin{aligned}
\cos\angle(\iota, x+\Delta x) &= \frac{\langle \iota, x+\Delta x\rangle}{\|\iota\|\cdot\|x+\Delta x\|} \\
&= \frac{\langle \iota, x\rangle + \langle \iota, \Delta x\rangle}{\|\iota\|\cdot\|x\|}\left[1 - \frac{x^T\Delta x}{\|x\|^2} + o(\|\Delta x\|)\right] \\
&= \cos\angle(\iota, x) + \langle \frac{\iota}{\|\iota\|} - \cos\angle(\iota,x)\frac{x}{\|x\|}, \frac{\Delta x}{\|x\|}\rangle + o(\|\Delta x\|).
\end{aligned}$$

Therefore the angle between ι and $x + \Delta x$ actually increases from the angle between ι and x for small displacement Δx if and only if

$$\langle \frac{\iota}{\|\iota\|} - \cos\angle(\iota,x)\frac{x}{\|x\|}, \Delta x\rangle < 0. \tag{4.24}$$

Now let us take any $\Delta x \in -\mathrm{int}\ \mathcal{K}^*$. So we have

$$\langle \theta u + \xi v, \Delta x\rangle < 0 \tag{4.25}$$

for all $\theta \ge 0$, $\xi \ge 0$, $u \in \mathrm{SOC}(v_k)$ and $v \in \mathrm{SOC}(v_l)$ with $0 \neq \theta u + \xi v \in \mathcal{K}$. This implies that if we shift the whole cone $\mathrm{SOC}(v_l)$ along the direction Δx, then for *each* fixed pair of vectors in $\mathrm{SOC}(v_k)$ and $\mathrm{SOC}(v_l)$, the (obtuse) angle between them actually increases, due to condition (4.24) and the actual choice of the moving direction (4.25), which satisfies condition (4.24).

Let us now perform such a shifting operation. First, we keep $\mathrm{SOC}(v_k)$ unchanged. In this case, all the vectors in $\mathcal{V}_1$ remain unchanged. Then, we shift the entire cone $\mathrm{SOC}(v_l)$ slightly along the direction Δx. In this case, all the vectors in $\mathcal{V}_2$ are also shifted along the same direction Δx, while their relative positions remain unchanged.

It is clear that if we do so, then

$$\begin{aligned}
&\angle(v_i, v_j) \text{ is invariant if } v_i, v_j \in \mathcal{V}_1, \\
&\angle(v_i, v_j) \text{ is invariant if } v_i, v_j \in \mathcal{V}_2, \\
&\angle(v_i, v_j) \text{ increases if } v_i \in \mathcal{V}_1 \text{ and } v_j \in \mathcal{V}_2,
\end{aligned}$$

where the last assertion is due to (4.24), (4.25) and the above observations.

Let $\hat{X}_{ij} = v_i^T v_j$ where v_i's, v_j's are the new vectors after the shifting operation. Since the shifting does not change the norm, so $\hat{X} \succeq 0$ and $\mathrm{diag}(\hat{X}) = e$, implying that $\hat{X}$ remains feasible for (MCR). Moreover, $\hat{X}_{ij}$ remains unchanged for either $i, j \in \mathcal{V}_1$ or $i, j \in \mathcal{V}_2$. However, $\hat{X}_{ij}$ decreases for $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$.

It can be easily verified that the corresponding objective value for $\hat{X}$ is strictly greater than $v(R)$. Moreover, $\hat{X}$ also satisfies the triangle inequalities due to the three facts established earlier. The contradiction proves the lemma. □

Let

$$L = \frac{\max_{i \neq j} w_{ij}}{\min_{i \neq j} w_{ij}} > 0. \tag{4.26}$$

Now, we apply Lemma 4.1 and use (4.17) to obtain

$$\begin{aligned}
v(MC) &\geq h(A)v(MCR) + \frac{\mu}{2}\sum_{i,j} w_{ij}(1 - X^*_{ij})(X^*_{ij} - A)^2 \\
&\geq h(A)v(MCR) + \frac{\mu}{2} \sum_{(i,j)\in\{(i,j)|1\leq i\neq j\leq n\}\setminus\{I_\varepsilon \cup J_\varepsilon\}} w_{ij}(1 - X^*_{ij})(X^*_{ij} - A)^2 \\
&\geq h(A)v(MCR) + \frac{\mu\varepsilon^3}{2}\min_{i\neq j} w_{ij} \\
&= h(A)v(MCR) + \frac{\mu\varepsilon^3}{2L}\max_{i\neq j} w_{ij} \\
&\geq h(A)v(MCR) + \frac{\mu\varepsilon^3}{4Ln^2}\sum_{i,j} w_{ij}(1 - X^*_{ij}) \\
&= (h(A) + \frac{\mu\varepsilon^3}{4Ln^2})v(MCR).
\end{aligned}$$

Finally we conclude the following theorem:

Theorem 4.2 *Consider the Goemans and Williamson algorithm for the max-cut problem where all the weights are positive. Suppose that the triangle inequalities (4.20), (4.21), (4.22), (4.23) are added in the SDP relaxation. Then, the worst-case performance ratio is bounded below by*

$$\alpha(A) = \begin{cases} h(A)\ (\geq 0.8835), & \text{if } A \in [-1, -0.8] \cup [-0.5, 0], \\ h(A) + \frac{\mu\varepsilon^3}{4Ln^2}\ (\geq \alpha + \frac{\mu\varepsilon^3}{4Ln^2}), & \text{otherwise}, \end{cases}$$

where $\mu > 0$ and $\varepsilon > 0$ are some universal constants, and A is as defined in (4.14) and L is as defined in (4.26).

Summarize: We show that the worst-case performance ratio for the Goemans and Williamson method is at least $\alpha + \Omega(\frac{1}{n^2 L})$ if we add the triangle inequalities, where L is the ratio between the maximum weight and the minimum weight and n is the number of nodes. Karloff [10] proves that for every given $\epsilon > 0$, there is an instance of max-cut such that the performance ratio of the Geomans and Williamson algorithm using the SDP relaxation with any valid linear constraints (such as the triangle inequalities) is lower than $\alpha + \epsilon$ (see also [4]). Feige, Karpinski and Langberg [2] show that for graphs with degree bounded by Δ, the performance ratio is $\alpha + \Omega(\frac{1}{\Delta^4})$. Certainly, whether the approximation ratio for max-cut can be improved substantially beyond α remains a challenging open problem (cf. [3, 4, 24]).

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