Stable Sets of Weak Tournaments

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1. Introduction :- An abiding problem in choice theory has been one of characterizing those choice functions which are obtained as a result of some kind of optimisation. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin [1985].
Miller [1977], [1980], introduces the concept of a tournament, which is an asymmetric and complete binary relation. Such binary relations arise very naturally in majority voting situations, where one candidate defeats another by a strict majority of votes. A consequence of majority voting and hence of the tournament it generates on the set of alternatives, is the well known Condorcet paradox: the tournament may fail to exhibit transitivity and thus no alternative qualifies as a best alternative. This paradoxical situation called for alternative solution concepts for tournaments, which were then axiomatically characterized in Moulin [1986]. Peris and Subiza [1999] refer to binary relations which are reflexive and complete as weak tournaments and replicate some of the results in Moulin [1986]. Weak tournaments can arise very naturally, when in an election comprising an even number of voters, two candidates secure an equal number of votes against each other. Alternatively, if the number of voters who prefer a certain candidate A over B, is equal to the number of voters who prefer B over A, then we arrive at a weak tournament in the sense of Peris and Subiza [1999].
Among the many different solutions, which have been prescribed for problems of choice, one of the most significant is the solution related to the (von Neumann-Morgenstern) stable set. Lucas [1994] surveys the very large literature dealing with this concept, particularly in the context of co-operative games. Given a weak tournament, a nonempty subset is said to be internally stable, if no alternative in the set is preferred to any other alternative in the set. It is said to be externally stable, if for every alternative that lies outside the set, one can always find an alternative within the set, which the decision maker prefers to the former.
The intuition behind the concept of a stable set is clear in the context of majority voting. Consider a situation where a committee is formed by majority voting. Internal stability would require, that there is no selected candidate, who is preferred by a majority to any other selected candidate. A violation of internal stability would lead to asymmetry and imbalance in the committee, where given a conflict of opinion between two candidates one of whom defeats the other by a majority, the losing candidate would always have to withdraw his opinion to the winning candidate, if democratic

representation were to be meaningful at all. On the other hand a violation of external stability would mean that a candidate who is not selected, is no less popular that any other candidate who is selected, and hence his/her exclusion from the committee cannot be justified on democratic principles alone.

The purpose of this paper is quite straightforward: to obtain conditions on weak tournaments, which guarantee that every non-empty subset of alternatives admits a stable set. It is shown in this paper, that if a weak tournament is acyclic then every non-empty subset of alternatives, admits a stable set. However, there are several instances where acyclicity is not necessary for every non-empty subset of alternatives to admit a stable set. For instance given any four-element set, which does admit any triplet, forming a strict preference cycle, it can be shown that every subset admits a stable set. Another example due to Kim and Roush [1980], is that of a set formed by the union of two disjoint sets of equal cardinality, where for every element in one set there is exactly one element in the other set which is strictly preferred to the former, while any two elements in the same set are perceived as being equally desirable. In this case, the weak tournament is not acyclic. However, both the disjoint sets we started out with turn out to be stable sets. We also show that, the set of best elements of any non-empty subset must necessarily be contained within each of its stable sets.

It turns out, that a stable set must always be the best element of a weak tournament defined on the set of its indifference sets, which satisfies the following two properties: (a) Any indifference set which contains another as a strict subset, must be preferred to the latter; (b) Any indifference set A which is not contained in another indifference set B and does not contain B either, is preferred to the latter indifference set, if and only if every alternative in A that is not contained in B, is preferred to every alternative in B. Further, any best element of this weak tournament defined on indifferent sets, turns out to be a stable set. Property (b) of the weak tournament described above, bears a striking resemblance to the one that is used in the study of college admissions problem, by Roth and Sotomayor (1990).

The most interesting result of this paper is that there exists a unique stable set for each non-empty subset of alternatives which coincides with its set of best elements, if and only if, the weak tournament is quasi-transitive. A somewhat weaker version of this result, which is also established in this paper, is that there exists a unique stable set for each non-empty subset of alternatives (: which may or may not coincide with its set of best elements), if and only if the weak tournament is acyclic.

2. The Model and preliminary definitions: Let X be a finite, non-empty set and given any non empty subset A of X, let [A] denote the collection of all non-empty subsets of A. Thus in particular, [X] denotes the set of all non-empty subsets of X. If $A \in [X]$, then #(A) denotes the number of elements in A.

Given a binary relation R on X, let $P(R) = \{(x, y) \in R \,/\, (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R \,/\, (y, x) \in R\}$. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R. Given a binary relation R on X and $A \in [X]$, let $R|A = R \cap (A \times A)$ and $R^{-1} = \{(x, y) / (y, x) \in R\}$.

A binary relation R on X is said to be (a) reflexive if $\forall\, x \in X : (x, x) \in R$; (b) complete if $\forall\, x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$;(c) quasi-transitive if

$\forall$ x, y,z $\in$ X :[(x, y) $\in$ P(R) & (y, z) $\in$ P(R)] implies [(x, z) $\in$ P(R)]. Let Π denote the set of all reflexive and complete binary relations. Following Peris and Subiza[1999], we refer to such binary relations as weak tournaments.

Given a binary relation R on X and A $\in$ [X], let G(A,R) ={ x $\in$ A/ $\forall$y $\in$ A :(x,y) $\in$ R} and let W(A,R) = G(A,R^{-1}). The set G(A,R) is variously referred to as the set of "best alternatives", or the "core" of A with respect to the weak tournament R. Given A $\in$ [X], let Δ (A) denote the diagonal of A i.e. Δ (A)$\equiv${(x,x)/x$\in$ A}.

Given R $\in\Pi$, let us say that G is well defined at R if $\forall$A$\in$ [X], G(A,R) is non-empty valued.

R$\in\Pi$ is said to be acyclic if $\forall$A$\in$ [X], G(A,R) is non-empty valued.

Let (A,R) $\in$ [X]$\times$ Π. A set B$\in$[A] is said to be a (von Neumann-Morgenstern) stable set for (A,R) if:(i)$\forall$x,y$\in$B: (x,y)$\in$I(R) (i.e. B satisfies internal stability);(ii) $\forall$x$\in$A\B, there exists y$\in$B:(y,x) $\in$P(R) (i.e. B satisfies external stability).Let Ψ(A,R)={ B$\in$[A]/B is a stable set for (A,R)}.

Given R $\in\Pi$, let us say that Ψ is well defined at R if $\forall$A$\in$ [X], Ψ(A,R) is non-empty.

Observation 1:-Let A$\in$ [X] and suppose #(A) $\leq$ 2. Then,$\forall$R$\in\Pi$:[G(A,R)$\neq\phi$ &Ψ(A,R)$\neq\phi$].

2. Non-emptiness of Ψ: The primary aim of this section is to study conditions under which Ψ is well defined. Further, in Theorem 3, we show that the set of best elements is always contained in a stable set.

Theorem 1 :- Let R belong to Π. If R is acyclic then Ψ is well defined at R.

Proof :- It is enough to show that Ψ(X,R) $\neq\phi$ under the assumption that R is acyclic, since the proof replicates for A$\in$[X]\ {X}. Since R is acyclic G(A,R) $\neq\phi$, whenever A$\in$[X]. If G(X,R) = X, then Ψ(X,R) = {X}. Hence suppose G(X,R) $\subset\subset$ X. Let Y^1 = X \ G(X,R). Let X^0 = X and X^1= X \ G(Y^1,R)}. Clearly, x$\in$G(Y^1,R) implies there exits y $\in$G(X,R), such that (y,x) $\in$P(R). Further, G(X,R) $\subset$ G(X^1,R). It is possible that there exists y $\in$G(X^1,R)\ G(X,R) and y$\in$G(Y^1,R), such that (y,x) $\in$P(R), though not necessarily so. Let Y^2 = X^1 \ (G(X^1,R) and X^2 = X^1 \ G(Y^2,R). Clearly, G(X^2,R) $\subset$ G(X^1,R). Having defined Y^k, X^k for k $\geq$ 1, let Y^{k+1} = X^k \ (G(X^k,R) and X^{k+1}= X^k \ (G(Y^{k+1},R). Since X is finite, there exists a smallest K such that either X^K \ (G(X^K,R) = ϕ. Let B = X^K. Clearly, x,y $\in$ B implies (x,y) $\in$I(R). Let y $\in$ X\ B. Thus, there exists 'k' such that y$\in Y^k$. Hence there exists x $\in$ G(X^{k-1},R) $\subset$B , such that (x,y) $\in$P(R). Thus, B$\in\Psi$(X,R). ♣

Thus if a weak tournament R is such that G(A,R) $\neq\phi$ whenever A $\in$[Y] where Y$\in$[X], then Ψ(A,R) $\neq\phi$, whenever A $\in$[Y].

Note: I am indebted to Jozsep Mala for the following observation on an earlier version of the paper.

It is not true that given (A,R) [X]× Π : G(A,R)≠ϕ implies Ψ(A,R) ≠ ϕ. Let X = {x,y,z,w}. Suppose (y,z), (z,w), (w,y) ∈P(R) and (x,z), (x,w), (x,y) ∈I(R). Then G(X,R) = {x} whilst Ψ(X,R) = ϕ.

Given a weak-tournament R on X, let a set A∈[X], with A = {a_1,…,a_p} be called a quasi-chain in X if for all i∈{1,…,p-1}: (a_i , a_{i+1}) ∈P(R). A is said to be a cycle if in addition, (a_p , a_1)∈P(R). It is said to be a dominated quasi-chain if there exists b∈X\ A such that (b, a_1) ∈P(R).

Observation: G(X,R) ≠ ϕ if and only there does not exist a collection X_1,…, X_k of quasi-chains in X which form a partition of X and such that each X_i is either a cycle or a dominated quasi-chain.

If G(X,R) = ϕ, and x∈X, is not part of a cycle that has already been constructed, then {x} itself can be considered as a quasi-chain and a member of the partition of X.

Further, G(X,R) is a singleton if and only there exists a collection X_1,…, X_k (k ≥ 2) of quasi-chains in X which form a partition of X and such that each X_i ,i=1,…,k-1 is either a cycle or a dominated quasi-chain and X_k is neither a cycle nor a dominated quasi-chain.

However it is not necessary that given R∈Π, it must be the case that R is acyclic for Ψ to be well defined at R.

Let X = {a_1 ,...,a_m }where m is a positive integer greater than or equal to four and let R∈Π. We say that X is a minimal m-cycle with respect to R if : (i) (a_i,a_{i+1}) ∈P(R) for all i∈{1,…,m-1}; (ii) (a_m,a_1)∈P(R); (iii) there does not exist any non-empty proper subset {b_1 ,...,b_n }of X such that [(a) (b_i,b_{i+1}) ∈P(R) for all i∈{1,…,n-1}; (b) (b_n ,b_1)∈P(R)].

Proposition 1 (Kim and Roush [1980]):- Let m be a positive integer greater than or equal to two. Let X = {a_1 ,...,a_{2m} } and suppose that { a_1 ,...,a_{2m} } is a minimal 2m - cycle with respect to R. Then, Ψ is well defined at R in spite of the fact that G(X,R)=ϕ.

Proof:- It is easy to see that Ψ(X,R) = {A, B} ,where A = { a_{2i} / i = 1,...,m} and B = { a_{2i-1} / i = 1,...,m}, although G(X,R) =ϕ. Further, by Theorem 1, Ψ(A,R)≠ ϕ whenever A∈ [X], since for A∈ [X] with A ≠ X, G(A,R) ≠ϕ. Thus, Ψ is well defined at R in spite of G(X,R)=ϕ.♣

Proposition 1 shows that G(A,R) can be a proper subset of a stable set. In fact in the above example G(X,R) is empty and Ψ(X,R) is non-empty.

The following two propositions provide additional sufficient conditions for a non-empty subset of alternatives to admit a stable set.

Proposition 2 :-Let R be an weak tournament and suppose that for some A ∈ [X] and x ∈A, Ψ(A\{x},R) ≠ ϕ. If G(A,R) ≠ ϕ.Then, Ψ(A,R) ≠ ϕ.

Proof :- Let x ∈ G(A,R). Since Ψ(A\{x}, R) ≠ ϕ, there exists B ∈ Ψ(A\{x}, R). Clearly, B ∪ {x} ∈ Ψ(A,R). ♣

<u>Proposition 3</u> :-Let R be an weak tournament and suppose that for some $A \in [X]$ and x $\in W(A, R)$, $\Psi(A\backslash\{x\},R) \neq \phi$. Then, $\Psi(A,R) \neq \phi$.

<u>Proof</u> :- Let $x \in W(A,R)$. Since $\Psi(A\backslash\{x\}, R) \neq \phi$, there exists $B \in \Psi(A\backslash\{x\}, R)$. If there exists $y \in B$ such that $(y,x) \in P(R)$, then $B \in \Psi(A,R)$. Otherwise $\forall$ $y \in B$, $(x,y) \in I(R)$ (: since $x \in W(A,R)$) and so $B \cup \{x\} \in \Psi(A,R)$. Thus $W(A,R) \neq \phi$ implies $\Psi(A,R) \neq \phi$. ♣

Let $\Pi(3) = \{R \in \Pi$/ there does not exist x , y, $z \in X$ with $(x,y),(y,z),(z,x) \in P(R)\}$. $\Pi(3)$ is a set consisting only of those weak tournaments which do not satisfy the requirements of the three element Condorcet Paradox.

<u>Example 1</u>:- Let $\{x,y,z\} \subset X$ and let R be a weak tournament on X, such that $(x,y),(y,z),(z,x) \in P(R)$ .Then $\Psi(\{x,y,z\},R)=\phi$. Thus Ψ is not well defined at R.

<u>Proposition 4</u> **:-** Let $R \in \Pi(3)$ and let $A \in [X]$ with # (A) ≤ 4. Then $\Psi(A,R) \neq \phi$.

<u>Proof</u> :- For # (A) equal to 1 or 2 there is nothing to prove and for # (A) equal to 3, R $\in \Pi(3)$ implies $G(A,R) \neq \phi$. Thus, by Theorem 1, $\Psi(A,R) \neq \phi$. Hence suppose # (A) = 4. If $G(A,R) \neq \phi$, then by Proposition 1, $\Psi(A,R) \neq \phi$. If $G(A,R) = \phi$ and $W(A,R) \neq \phi$, then let $y \in W(A,R)$. Thus $\#(A\backslash\{y\}) = 3$ and by the above $\Psi(A\backslash\{y\},R) \neq \phi$. Hence, by Proposition 3, $\Psi(A,R) \neq \phi$.
Finally suppose $G(A,R) = W(A,R) = \phi$. Let $A = \{x,y,z,w\}$ where all elements are distinct. Since $G(A,R) = \phi$, there exists $a \in A : (a, x) \in P(R)$. Without loss of generality suppose $(y, x) \in P(R)$. Since $W(A,R) = \phi$, there exists $a \in A : (x, a) \in P(R)$. Without loss of generality suppose $(x,z) \in P(R)$. Since $R \in \Pi(3)$, $(z,y) \notin P(R)$.Hence $(y,z) \in R$. Since $W(A,R)=\phi$, it must be the case that $(w,z) \notin R$. Thus $(z,w) \in P(R)$. If $(y,w) \in R$, then combined with $(y,x) \in P(R)$ and $(y,z) \in R$ we get $y \in G(A,R)$.Since $G(A,R) = \phi$ we must therefore have $(w,y) \in P(R)$. If $(w,x) \in P(R)$ then along with $(x,z) \in P(R)$ and $(z,w) \in P(R)$ we get a contradiction of the assumption that $R \in \Pi(3)$. Hence $(w,x) \notin P(R)$. If $(x,w) \in P(R)$ then along with $(w,y) \in P(R)$ and $(y,x) \in P(R)$ we get a contradiction of the assumption that $R \in \Pi(3)$. Thus $(w,x) \in I(R)$.If $(y,z) \in P(R)$ then along with $(z,w) \in P(R)$ and $(w,y) \in P(R)$ we get contradiction of the assumption that $R \in \Pi(3)$. Thus $(z,y) \in R$, which combined with $(y,z) \in R$ yields $(y,z) \in I(R)$. Thus $\Psi(A,R) = \{\{w,x\},\{y,z\}\}$. Thus $\Psi(A,R) \neq \phi$. ♣

However, the conclusion of Proposition 4 does not hold if # (A) > 4.

<u>Example 2</u> (Kim and Roush [1980]):- Let m be a positive integer greater than or equal to two. Let $X = \{ a_1 ,...,a_{2m+1} \}$ and suppose that $\{ a_1 ,...,a_{2m+1} \}$ is a minimal (2m+1) - cycle with respect to R. Suppose towards a contradiction $\Psi(X,R) \neq \phi$. Let $B \in \Psi(X,R)$. Suppose $a_1 \in B$. Then since $(a_{2m+1} , a_1) \in P(R)$, $a_{2m+1} \notin B$. Since $(a_i , a_{2m+1}) \in P(R)$ implies $i = 2m$, clearly $a_{2m} \in B$. By repeating the argument we arrive at the conclusion that $a_2 \in B$. But $a_1 \in B$, $a_2 \in B$ and $(a_1 , a_2) \in P(R)$ contradicts the assumption that B is a

stable set. Hence, $a_1 \notin B$. By symmetry of the elements in X, we get $B = \phi$ contradicting B is a stable set. Hence $\Psi(X,R) = \phi$. However R belongs to Π (3).

Theorem 2 :- Let $(A,R) \in [X] \times \Pi$ and suppose $B \in \Psi(A,R)$. Then $G(A,R) \subset B$.

Proof :- Let $B \in \Psi(A,R)$ and towards a contradiction suppose $x \in G(A,R) \setminus B$. Thus $x \in A \setminus B$. Since $B \in \Psi(A,R)$, there exists $y \in B \subset A: (y,x) \in P(R)$. This contradicts $x \in G(A,R)$ and proves the theorem.♣

Given a weak tournament R and $A \in [X]$, a non-empty subset B of A is said to be an indifference set of R in A, if for all $x, y \in B$, $(x, y) \in I(R)$. Clearly for all $x \in A$, $\{x\}$ is an indifference set of R in A. Let $I(A,R) = \{ B \in [A] / B$ is an indifference set of R in A$\}$. Thus, $I(A,R) \neq \phi$. Let $\Re(A,R)$ be a binary relation on $I(A,R)$ defined as follows: for all $B, C \in I(A,R)$: $(B,C) \in \Re(A,R)$ if either (i) $C \subset B$; or (ii) $R \cap (B \times (C \setminus B)) \neq \phi$ & $B \setminus C \neq \phi$. Let $P(\Re(A,R))$ denote the asymmetric part of R and $I(\Re(A,R))$ denote the symmetric part of R. Observe that $\Re(A,R)$ is a weak tournament on $I(A,R)$.

Proposition 5: Given $(A,R) \in [X] \times \Pi$: $[B \in \Psi(A,R)]$ if and only if $[B \in I(A,R)$ and $(B,C) \in \Re(A,R)$ whenever $C \in I(A,R)]$.

Proof: Let $B \in \Psi(A,R)$. Clearly, $B \in I(A,R)$. Let $C \in I(A,R)$. If $B \subset\subset C$, then B cannot belong to $\Psi(A,R)$. If $C \subset B$, then $(B,C) \in \Re(A,R)$. Suppose, $C \setminus B \neq \phi$ and $B \setminus C \neq \phi$. Then for every $x \in C \setminus B$, there exists $y \in B$, such that $(y,x) \in P(R)$. Thus, $(B,C) \in \Re(A,R)$.
Now suppose, $B \in I(A,R)$ and $(B,C) \in \Re(A,R)$ whenever $C \in I(A,R)$. Let $x \in A \setminus B$. If $B \cup \{x\} \in I(A,R)$, then $(B \cup \{x\}, B) \in P(\Re(A,R))$. If for all $y \in B$, it is the case that $(x, y) \in P(R)$, then $(\{x\}, B) \in P(\Re(A,R))$, contrary to hypothesis. Thus, there exists $y \in B$ such that $(y,x) \in P(R)$. Thus, $B \in \Psi(A,R)$. ♣

The following concept is a slight modification of one available in Kim and Roush [1980]:
Given $(A,R) \in [X] \times \Pi$ a set $B \in [A]$ is said to be a competitive solution for (A,R), if:
(i) for all $x,y \in B$, $(x,y) \in I(R)$;
(ii) $[x \in B, y \in A \setminus B, (y,x) \in P(R)]$ implies [there exists $z \in B$, such that $(z, y) \in P(R)]$.

It is easily verified that given $(A,R) \in [X] \times \Pi$, if B is a competitive solution for (A, R), then there exists $C \in \Psi(A,R)$ such that $B \subset C$.

3. Unique Stable Sets of Weak Tournaments : - The following result is of immense significance for quasi-transitive rational choice.

Theorem 3 :- Let $R \in \Pi$. Then the following statements are equivalent:
(1) R is quasi-transitive;

(2) [$\Psi(A,R) = \{G(A,R)\} \forall A \in [X]$].

<u>Proof</u> :- Let us first show that (1) implies (2). Suppose R is quasi-transitive. Clearly $\forall A \in [X]: G(A,R) \in \Psi(A,R)$. Let $B \in \Psi(A,R)$. Towards a contradiction suppose $x \in B \backslash G(A,R)$. Then there exists $y \in G(A,R)$ such that $(y,x) \in P(R)$. Since $B \in \Psi(A,R)$, $y \in A \backslash B$. Since $y \in G(A,R)$ there does not exist z in A (and hence in B) such that $(z,y) \in P(R)$. This contradicts $B \in \Psi(A,R)$. Thus $B \subset G(A,R)$.
Now suppose $x \in G(A,R) \backslash B$. Thus there does not exist $z \in B: (z,x) \in P(R)$. Thus $B \notin \Psi(A,R)$. Thus $G(A,R) \subset B$. Hence $B = G(A,R)$. Thus, (1) implies (2).
Now let us show that (2) implies (1). Suppose $\Psi(A, R) = \{G(A,R)\}$ for all $A \in [X]$. Towards a contraction suppose R is not quasi-transitive. Thus there exists $x, y, z \in X$: $(x, y) \in P(R)$, $(y,z) \in P(R)$ and $(x,z) \notin P(R)$. Since, $\Psi(\{x,y,z\}, R) = \{G(\{x,y,z\}, R)\}$, $G(\{x,y,z\}, R) \neq \phi$. Hence $(z,x) \notin P(R)$. Thus $(x,z) \in I(R)$. Thus $\Psi(\{x,y,z\}, R) = \{\{x,z\}\} \neq \{G(\{x,y,z\}, R)\} = \{x\}$. Thus R must be quasi-transitive. Thus, (2) implies (1). ♣

Interesting applications of this result occur in the literature on voting games. Given a weak tournament R, Gillies (1959) and Miller (1980), proposed two different dominance relations, both of which turn out to be quasi-transitive. Gillies (1959) proposed the following quasi-transitive weak tournament: for all $x,y \in X$, $(x,y) \in G^*$ if and only if [either (i) $(x,y) \in R$; or (ii) there exists $z \in X$, such that $(z,y) \in P(R)$ & $(x,z) \in R$]. Miller (1980) proposed the following the following quasi-transitive weak tournament: for all $x,y \in X$, $(x,y) \in M^*$ if and only if [either (i) $(x,y) \in R$; or (ii) there exists $z \in X$, such that $(z,x) \in P(R)$ & $(z,x) \in R$]. In view of Theorem 3, we may conclude that [$\Psi(A,G^*) = \{G(A,G^*)\} \forall A \in [X]$]and [$\Psi(A,M^*) = \{G(A,M^*)\} \forall A \in [X]$].

Note: It is possible for $\Psi(A,R)$ to be a singleton $\forall A \in [X]$, without R being quasi-transitive. Let $X = \{x,y,z\}$, such that $(x, y) \in P(R)$, $(y,z) \in P(R)$ and $(x,z) \in I(R)$. Clearly, R is not quasi-transitive. However, $\Psi(\{x,y,z\}, R) = \{\{x,z\}\}$, $\Psi(\{x,y\}, R) = \{\{x\}\}$, $\Psi(\{y,z\}, R) = \{\{y\}\}$ and $\Psi(\{x, z\}, R) = \{\{\{x,z\}\}$(: although both $\{x,z\}$ and $\{x\}$ are competitive solutions for (X, R)).

A necessary and sufficient condition for $\Psi(A,R)$ to be a singleton $\forall A \in [X]$ is that R is acyclic.

<u>Theorem 4</u> :- Let $R \in \Pi$. Then R is acyclic if and only if [$\Psi(A,R)$ is a singleton $\forall A \in [X]$].

<u>Proof</u> :- Suppose R is acyclic. By Theorem 1, Ψ is well defined at R. Let $A \in [X]$ and B, $C \in \Psi(A,R)$ with $B \neq C$. Suppose $C \subset B$. Let $x \in B \backslash C$. Since $C \in \Psi(A,R)$, there exists $y \in C$ such that $(y ,x) \in P(R)$. But, $C \subset B$ implies, $y \in B$. This contradicts the assumption that $B \in \Psi(A,R)$. Thus $C \not\subset B$. A similar argument shows that $B \not\subset C$. Since, B, C $\in \Psi(A,R)$, for all $x \in B \backslash C$, there exists $y \in C \backslash B$, such that $(y,x) \in P(R)$ and for all $x \in C \backslash B$, there exists $y \in B \backslash C$, such that $(y,x) \in P(R)$. Since X is finite, this yields a set $\{x_1$,

$x_2,\ldots, x_k\}$, where $(x_i,x_{i+1}) \in P(R)$ for i =1,…,k-1 and $(x_k,x_1)\in P(R)$. However, $G(\{x_1, x_2,\ldots, x_k\},R) = \phi$. This contradicts the acyclicity of R. Thus [$\Psi(A,R)$ is a singleton $\forall A\in[X]$].

The proof of the converse is by induction on the cardinality of X. For $\#X \leq 3$, [$\Psi(A,R)$ is a singleton $\forall A\in[X]$] if and only if R is acyclic has been observed earlier in Example 1, Theorem 3 and in the note preceding Theorem 4. Hence suppose the theorem is true for $\#X \leq k$, for some positive integer greater than or equal or equal to three. Let $\#X = k+1$. Let $x \in X$, and let Y be a non-empty subset of $X\setminus \{x\}$. Suppose [$\Psi(A,R)$ is a singleton $\forall A\in[X]$]. Thus, [$\Psi(A,R)$ is a singleton $\forall A\in[Y]$]. By the induction hypothesis the restriction of R to Y is acyclic. Towards a contradiction suppose $G(X,R) = \phi$. Let $X = \{ x = x_1, x_2,\ldots, x_{k+1}\}$, where $(x_i,x_{i+1}) \in P(R)$ for i=1,…,k and $(x_{k+1},x_1)\in P(R)$. By the induction hypothesis $B = G(X\setminus \{x\},R) \neq \phi$. Clearly, $B = \{x_2\}$. Thus, $(x_2, x_j)\in P(R)$, for $2 < j \leq k+1$. Thus, $G(\{x_1,x_2, x_{k+1}\},R) = \phi$, contradicting the induction hypothesis. Thus, R is acyclic. The theorem now follows by a standard induction argument. ♣

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References :

1. D.B. Gillies [1959]: " Solutions to general zero-sum games", in Tucker, A.W. and Luce, R.D. (eds), Contributions to the Theory of Games, Vol.4, Princeton Univ. Press, Princeton.
2. K. H. Kim and F. W. Roush [1980]: " Introduction to Mathematical Consensus Theory", Lecture Notes in Pure and Applied Mathematics, Vol. 59, Marcel Dekker, Inc.
3. W.F. Lucas [1992] : "Von Neumann-Morgenstern Stable Sets", Chapter 17 in R. Aumann and S.Hart (ed.) Handbook of Game Theory, Vol.1, Elsevier, Amsterdam.
4. N. Miller [1977]: " Graph – Theoretical Approaches to the Theory of Voting", American Journal of Political Science. Vol. 21, No.4, 769-803

5. N. Miller [1981]: “ A new solution set for tournaments and majority voting: further graph theoretical approaches to the theory of voting”, American Journal of Political Science, Vol. 24, 68-96.
6. H. Moulin [1985] : “Choice Functions Over A Finite Set : A Summary”, Social Choice Welfare. Vol. 2 : 147-160.
7. H. Moulin [1986] : “Choosing from a tournament”, Social Choice Welfare, Vol. 3: 271-291.
8. J.E. Peris and B. Subiza [1999]: “Condorcet choice correspondences for weak tournaments”, Social Choice Welfare.Vol.16: 217-231.
9. A. Roth and M. Sotomayor [1990] : “ Two- Sided Matching”, Econometric Society Monograph 18, Cambridge University Press.