THE LAX CONJECTURE IS TRUE

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Abstract

In 1958 Lax conjectured that hyperbolic polynomials in three variables are determinants of linear combinations of three symmetric matrices. This conjecture is equivalent to a recent observation of Helton and Vinnikov.

A homogeneous polynomial p on $\mathbf{R}^n$ is *hyperbolic* with respect to a vector $e \in \mathbf{R}^n$ if $p(e) \neq 0$ and, for all vectors $w \in \mathbf{R}^n$, the univariate polynomial $t \mapsto p(w - te)$ has all real roots. The corresponding *hyperbolicity cone* is the open convex cone (see [5])

$$\{w \in \mathbf{R}^n : p(w - te) = 0 \ \Rightarrow \ t > 0\}.$$

For example, the polynomial $w_1 w_2 \cdots w_n$ is hyperbolic with respect to the vector $(1, 1, \ldots, 1)$, with hyperbolicity cone the open positive orthant.

Hyperbolic polynomials and their hyperbolicity cones originally appeared in the partial differential equations literature [4]. They have attracted attention more recently as fundamental objects in modern convex optimization [6, 1].

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The following conjecture [8] proposes that all hyperbolic polynomials in three variables are easily described in terms of determinants of symmetric matrices. We denote the space of d-by-d real symmetric matrices by $\mathbf{S}^d$, with positive definite cone $\mathbf{S}^d_{++}$ and identity matrix I.

Conjecture 1 (Lax, 1958) *A polynomial p on $\mathbf{R}^3$ is hyperbolic of degree d with respect to the vector $e = (1,0,0)$ and satisfies $p(e) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that p is given by*

$$p(x,y,z) = \det(xI + yB + zC). \tag{2}$$

This conjecture would imply that the corresponding hyperbolicity cone is the set

$$\Big\{(x,y,z) : xI + yB + zC \in \mathbf{S}^d_{++}\Big\}.$$

More generally, we could quickly deduce that any three-dimensional hyperbolicity cone is a "semidefinite slice", namely a set of the form

$$\Big\{(x,y,z) : xA + yB + zC \in \mathbf{S}^d_{++}\Big\},$$

for some matrices $A, B, C \in \mathbf{S}^d$. In particular, such cones would be "semidefinite representable" in the sense of [9]. In passing, it is worth remarking that any *homogeneous cone* (an open convex pointed cone whose automorphism group acts transitively) is both a hyperbolicity cone [6] and a semidefinite slice [2] (see also [3]).

A polynomial on $\mathbf{R}^2$ is a *real zero polynomial* [7] if, for all vectors $(y,z) \in \mathbf{R}^2$, the univariate polynomial $t \mapsto q(t(y,z))$ has all real roots. Such polynomials are closely related to hyperbolic polynomials via the following elementary result.

Proposition 3 *If p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1,0,0)$, and $p(e) = 1$, then the polynomial on $\mathbf{R}^2$ defined by $q(y,z) = p(1,y,z)$ is a real zero polynomial of degree no more than d, and satisfying $q(0,0) = 1$.*

Conversely, if q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0,0) = 1$, then the polynomial on $\mathbf{R}^3$ defined by

$$p(x,y,z) = x^d q\Big(\frac{y}{x}, \frac{z}{x}\Big) \quad (x \neq 0) \tag{4}$$

is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$.

Proof To prove the first statement, note that for any point $(y, z) \in \mathbf{R}^2$ and complex μ, if $q(\mu(y, z)) = 0$ then $\mu \neq 0$ and $0 = p(1, \mu y, \mu z) = \mu^d p(\mu^{-1}, y, z)$, using the homogeneity of p. Hence, by the hyperbolic property, $-\mu^{-1}$ is real, and hence so is μ.

For the converse direction, clearly p is homogeneous of degree d and satisfies $p(e) = 1$. If $p(\mu, y, z) = 0$, then either $\mu = 0$ or $q(\mu^{-1}(y, z)) = 0$, in which case μ^{-1} and hence also μ must be real. ◇

Helton and Vinnikov [7, p. 10] observe the following result, based heavily on [10].

Theorem 5 *A polynomial q on $\mathbf{R}^2$ is a real zero polynomial of degree d and satisfies $q(0, 0) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that q is given by*

$$q(y, z) = \det(I + yB + zC). \tag{6}$$

We claim that this result is equivalent to the Lax conjecture.

To see this, suppose p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1, 0, 0)$, and $p(e) = 1$. Then by Proposition 3, the polynomial on $\mathbf{R}^2$ defined by $q(y, z) = p(1, y, z)$ is a real zero polynomial of degree $d' \leq d$, and satisfying $q(0, 0) = 1$. Hence by Theorem 5, equation (6) holds: we can assume $d' = d$ by replacing $B, C \in \mathbf{S}^{d'}$ with block diagonal matrices $\mathrm{Diag}(B, 0), \mathrm{Diag}(C, 0) \in \mathbf{S}^d$. Then, by homogeneity, for $x \neq 0$,

$$\begin{aligned} p(x, y, z) &= x^d p\Big(1, \frac{y}{x}, \frac{z}{x}\Big) = x^d q\Big(\frac{y}{x}, \frac{z}{x}\Big) \\ &= x^d \det\Big(I + \frac{y}{x}B + \frac{z}{x}C\Big) = \det(Ix + yB + zC). \end{aligned}$$

as required. The converse direction in the Lax conjecture is immediate.

Conversely, let us assume the Lax conjecture, and suppose q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0, 0) = 1$. (The converse direction in Theorem 5 is immediate.) Then by Proposition 3 the polynomial p defined by equation (4) is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$. According to the Lax conjecture, equation (2) holds, so

$$q(y, z) = p(1, y, z) = \det(I + yB + zC),$$

as required.

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