An interior point cutting plane method for convex feasibility problem with second-order cone inequalities

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Abstract

Convex feasibility problem in general is a problem of finding a point in a convex set contains a full dimensional ball and is contained in a compact convex set. We assume that the outer set is described by second-order cone inequalities and propose an analytic center cutting plane technique to solve this problem. We discuss primal and dual settings simultaneously. Two complexity results are reported; the complexity of restoration procedure and complexity of the overall algorithm. We prove that an approximate analytic center is updated after adding a second-order cone cut (SOCC) in *one* Newton step, and that the ACCPM with SOCC is a fully polynomial approximation scheme.

Keywords: cutting plane algorithm, nondifferentiable optimization, analytic center, second-order cone

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1 Introduction

Many large-scale optimization problems can be cast as nondifferentiable optimization (NDO) and solved more efficiently by NDO techniques. Amongst several techniques for solving NDO problems, cutting plane methods have many advantages when applied to large models. The general idea of cutting plane algorithms is to iteratively build a model, when it is too large. In other words, it assumes that a complete description of the model is not initially available and puts it together as needed. This is done by generating subgradient cuts at each test point.

Selecting a test point is a crucial step in the cutting plane techniques. One efficient tactic is by means of the analytic center, which leads to the analytic center cutting plane method (ACCPM) introduced by Sonnevend [23]. In this technique, first an outer approximation of the original problem is formed by means of subgradients of the objective function. Next a convex set composed of the subdifferential set and an upper bound of the optimal objective value is constructed. This set is known as the *set of localization*. At each iteration the analytic center of the set of localization is computed. This point serves as a test point. If the current test point is not in the solution set, a set of subgradients is added to the localization set and the analytic center is updated.

Depending on the original NDO problem, the subgradient cuts may be linear, quadratic, semidefinite or of the form of second-order cone. The ACCPM has been studied in the literature for some of these cases. Ye [26], Goffin, Haurie, and Vial [4], Atkinson and Vaidya [2], Nesterov [16], and Goffin, Lou, and Ye [5] studied the method in case of single linear cuts. Ye [27], and Goffin and Vial [6] studied the complexity of the ACCPM when the oracle returns a set of linear cuts. The method was extended to employ quadratic and nonlinear cuts by Lou and Sun [12], Luthi and Bueler [13] and Sharifi Mokhtarian and Goffin [21, 22].

Cutting plane techniques have been recently employed into nonpolyhedral models such as semidefinite programming (SDP). The first paper of this type was proposed by Helmberg and Rendl [8], in which the authors provide a spectral Bundle method for a class of SDP. Sun, Toh, and Zhao [24], Toh, Zhao, and Sun [25], and Chua, Toh, and Zhao [3] apply the ACCPM to convex semidefinite feasibility problem. Oskoorouchi [17] and Oskoorouchi and Goffin [18] modify the ACCPM by integrating semidefinite cuts (SDC), and apply this technique to the eigenvalue optimization and maxcut problem [19]. Krishnan [9] and Krishnan and Mitchell [10] propose an LP cutting plane and make use of polyhedral scheme for semidefinite programming. A survey paper of cutting plane methods for semidefinite programming by Krishnan and Mitchell [11] provide a summary of some of the aforementioned techniques and a nice comparison between them.

In this paper we extend the cutting plane techniques into another non-polyhedral model; namely, the second-order cone programming (SOCP). More precisely, we employ the second-order cone cut (SOCC) into the ACCPM. Ingredients of the ACCPM are modified to make the best use of this integration. SOCP is a convex optimization problem, and a general case of LP, QP, and QCQP. On the other hand it is well known that SOCP can be cast as a SDP. However, interior point algorithms designed for SOCP are computationally less expensive than those designed for SDP. This advantage of SOCP over SDP motivated the current study. Computation of SDCs could be very expensive in practice and that may drastically slow down the algorithm. One approach to overpass this difficulty is to replace SDCs by SOCCs. However, the theoretical issues of integrating ACCPM and SOCC should be fully elaborated. This is our intent in this paper.

We present the ACCPM in the context of convex feasibility problem (CFP). Let $\mathcal{F}^*$ be a convex set contains a full dimensional ball with ε radius and is contained in a compact convex set (set of localization). We employ the ACCPM to obtain a point in $\mathcal{F}^*$. We assume that there exists an oracle that at each iteration of the algorithm determines whether the current point is in $\mathcal{F}^*$ or returns a SOCC and updates the localization set by adding it to the center. We discuss how to obtain an interior point of the updated set of localization, as a *warm start* for recentering procedure after adding a central SOCC. We prove that analytic center of the set of localization is updated in *one* Newton step after adding a SOCC. Furthermore, we prove that the ACCPM with SOCC obtains a point in $\mathcal{F}^*$ after adding at most $O(m/\varepsilon^2\mu)$ SOCCs, where $\mu > 0$ is a condition number on SOCC.

The paper is organized as follows: In Section 2 we study the definition and most important properties of second-order cone. Section 3 presents localization sets and their corresponding potential functions in primal, dual and primal-dual settings. We also describe a Newton algorithm to compute an approximate analytic center and discuss its complexity. In Section 4 we present the framework of our algorithm, and a procedure for computing the recentering direction. We report the complexity of this procedure in this section, and finally in Section 5 we discuss the convergence analysis and the complexity of the ACCPM with SOCC.

2 Second-order cone: definitions and properties

In this section we study the most important properties of the second-order cone and a particular algebra associated with this cone. The material of this section is mostly based on a survey paper by Alizadeh and Goldfarb [1]. Throughout this paper we use the notations and definitions described in this section without specific reference. First some notations: we indicate vectors by lower case letters and matrices by uppercase letters. For $x \in \Re^p$ and $y \in \Re^q$, $(x; y)$ indicates a column vector in $\Re^{p+q}$. We also use bold fonts for block vector and matrices, e.g., $\mathbf{x} = (x_1; x_2; \ldots; x_k)$ is a k-block column vector composed of vectors x_1, x_2, ..., x_k. We refer to the space of symmetric matrices by $\mathcal{M}^n$, positive semidefinite matrices by $\mathcal{M}^n_+$, and positive definite matrices by $\mathcal{M}^n_{++}$. For matrices A and B we define $A \oplus B$ a block diagonal matrix composed of A and B:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

and $A \bullet B = tr A^T B$ the inner product of A and B.

Let us begin with the definition of the second-order cone. Let $\mathcal{S}_n$ be defined as follows:

$$\mathcal{S}_n = \{x \in \Re^n : x = (\xi_0; \bar{\xi}), \xi_0 \geq \|\bar{\xi}\|\}.$$

Then $\mathcal{S}_n$ is a closed, pointed and convex cone called the *second-order cone*. It is well-known that $\mathcal{S}_n$ induces a partial order on $\Re^n$: $x \succeq_{\mathcal{S}_n} s$ ($x \succ_{\mathcal{S}_n} s$) if $x - s \in \mathcal{S}_n$ ($x - s \in \mathcal{S}_n^\circ$), where $\mathcal{S}_n^\circ$ is the interior of $\mathcal{S}_n$, defined by

$$\mathcal{S}_n^\circ = \{x \in \mathcal{S} : \xi_0 > \|\bar{\xi}\|\}.$$

We denote $\mathcal{S}_n$ by $\mathcal{S}$ when there is no ambiguity. We also use the notation $\succeq$ and $\succ$ (without subscripts) for the Löwner partial order on the symmetric matrices. That is $A \succeq B$ ($A \succ B$) if $A - B \in \mathcal{M}^n_+$ ($A - B \in \mathcal{M}^n_{++}$).

For each vector $x = (\xi_0; \bar{\xi}) \in \Re^n$ we define a matrix representation

$$\mathbf{Mat}(x) = \begin{pmatrix} \xi_0 & \bar{\xi} \\ \bar{\xi} & \xi_0 I \end{pmatrix}.$$

It is easily verified that $\mathbf{Mat}(x)$ is a positive semidefinite (positive definite) matrix iff $x \in \mathcal{S}$ ($x \in \mathcal{S}^\circ$). A special case of Euclidean Jordan algebra can be defined on the second-order cone. Let $x = (\xi_0; \bar{\xi}) \in \Re^n$ and $s = (\sigma_0; \bar{\sigma}) \in \Re^n$. Define

$$x \circ s = \begin{pmatrix} x^T s \\ \xi_0 \sigma_1 + \sigma_0 \xi_1 \\ \vdots \\ \xi_0 \sigma_n + \sigma_0 \xi_n \end{pmatrix}$$

Then $(\Re^n, \circ)$ define an algebra on the second-order cone. Observe that

$$x \circ s = \mathbf{Mat}(x)s = \mathbf{Mat}(x)\mathbf{Mat}(s)e,$$

where $e = (1; \mathbf{0})$ is the unique identity element of the algebras, i.e.,

$$x \circ e = e \circ x = x.$$

It can easily be verified that operator "$\circ$" is commutative and distributive. That is $x \circ s = s \circ x$, and

$$x \circ (\alpha s + \beta t) = \alpha x \circ s + \beta x \circ t$$

and

$$(\alpha s + \beta t) \circ x = \alpha s \circ x + \beta t \circ x$$

for all $\alpha, \beta \in \Re$ and $x, s, t \in \Re^n$.

Similar to the cone of symmetric matrices one can define Spectral decomposition of $x \in \mathcal{S}$

$$x = \lambda_1 c_1 + \lambda_2 c_2, \tag{1}$$

where $c_1 = \frac{1}{2}(1, \bar{\xi}/\|\bar{\xi}\|)$, $c_2 = \frac{1}{2}(1, -\bar{\xi}/\|\bar{\xi}\|)$, and

$$\lambda_1 = \xi_0 + \|\bar{\xi}\| \text{ and } \lambda_2 = \xi_0 - \|\bar{\xi}\|.$$

λ_1 and λ_2 are eigenvalues of x that can alternatively be derived as the two roots of the characteristic polynomial of x:

$$p(\lambda, x) = \lambda^2 - 2\xi_0 \lambda + (\xi_0^2 - \|\bar{\xi}\|^2).$$

Observe that λ_1 and λ_2 are the largest and smallest eigenvalues of $\mathbf{Mat}(x)$ respectively, with corresponding eigenvectors c_1 and c_2. Furthermore ξ_0 is an eigenvalue of $\mathbf{Mat}(x)$ with multiplicity $n-2$.

We now define some important functions for algebra $(\Re^n, \circ)$. Trace and determinant functions are defined as follows:

$$\text{tr}(x) = \lambda_1 + \lambda_2 = 2\xi_0,$$

and

$$\det(x) = \lambda_1 \lambda_2 = \xi_0^2 - \|\bar{\xi}\|^2.$$

Frobenius and 2-norm functions are defined analogs to their counterpart in symmetric matrix algebra:

$$\|x\|_F = \sqrt{\lambda_1^2 + \lambda_2^2} = \sqrt{2}\|x\|,$$

and

$$\|x\|_2 = \max\{|\lambda_1|, |\lambda_2|\} = |\xi_0| + \|\bar{\xi}\|.$$

Vector x is called singular if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. In this case

$$x^{-1} = \lambda_1^{-1} c_1 + \lambda_2^{-1} c_2,$$

and finally if $x \in \mathcal{S}$, then the square root of x is uniquely defined via

$$x^{1/2} = \lambda_1^{1/2} c_1 + \lambda_2^{1/2} c_2.$$

Algebraic functions can similarly be defined in block sense. Let $\mathbf{x} = (x_1; \ldots, x_k)$ and $\mathbf{e} = (e_1; \ldots; e_k)$, then

$\|\mathbf{x}\|_F = \sum_{i=1}^k \|x_i\|_F$

$\|\mathbf{x}\|_2 = \max_i \|x_i\|_2$

$tr(\mathbf{x}) = 2\mathbf{e}^T\mathbf{x} = \sum_{i=1}^k tr(x_i)$

$\mathbf{x}^{-1} = (x_1^{-1}; \ldots; x_k^{-1})$

Associated with each vector s in $\Re^n$ there is a quadratic operator that maps any vector $x \in \Re^n$ to a vector composed of quadratic terms of s:

$$Q_s x = 2s \circ (s \circ x) - s^2 \circ x,$$

where $s^2 = s \circ s$. Since this operator plays an important role in our analysis we explain it in greater detail here. One can explicitly represent Q_s as

$$Q_s = 2\mathbf{Mat}^2(s) - \mathbf{Mat}(s^2). \tag{2}$$

By substituting $\mathbf{Mat}(s)$ and $\mathbf{Mat}(s^2)$ in (2) one can obtain

$$Q_s = \begin{pmatrix} \|s\|^2 & 2\sigma_0 \bar{\sigma}^T \\ 2\sigma_0 \bar{\sigma} & \det(s) I + 2\bar{\sigma}\bar{\sigma}^T \end{pmatrix}.$$

Note that the analogous operator to Q_s in symmetric matrix algebra is the one that maps any symmetric matrix X into SXS.

Eigenvalues of Q_s are

$$\lambda_1^2 = (\sigma_0 + \|\bar{\sigma}\|)^2$$

and

$$\lambda_2^2 = (\sigma_0 - \|\bar{\sigma}\|)^2,$$

each with multiplicity one and

$$\det(s) = \sigma_0^2 - \|\bar{\sigma}\|^2$$

with multiplicity $n-2$. Consequently, for a vector $s \in \Re^n$, Q_s is nonsingular iff s is nonsingular. Moreover, if $s \in \mathcal{S}$ then λ_1^2 and λ_2^2 are the maximum and minimum eigenvalues of Q_s respectively. We now present some important properties of the quadratic operator Q_s.

Lemma 1 *For $x \in \Re^n$ and nonsingular, $s \in \Re^n$, $\alpha \in \Re$ and integer p, we have*

1. $Q_x x^{-1} = x$ *and thus* $Q_x^{-1} x = x^{-1}$
2. $Q_s e = s^2$
3. $Q_{x^{-1}} = Q_x^{-1}$ *and more generally* $Q_{x^p} = Q_x^p$
4. $\nabla_x(\log\det(x)) = 2x^{-1}$ *and* $\nabla_x^2(\log\det(x)) = -2Q_x^{-1}$
5. $Q_{Q_s x} = Q_s Q_x Q_s$
6. $\det(Q_x s) = \det^2(x)\det(s) = \det(x^2)\det(s)$
7. $Q_x(\mathcal{S}) = \mathcal{S}$ *and* $Q_x(\mathcal{S}^\circ) = \mathcal{S}^\circ$.

Proof. See Alizadeh and Goldfarb [1], Theorem 3 and Theorem 4. ■

Lemma 2 *If $x \succeq_{\mathcal{S}} s$ and $x \succeq_{\mathcal{S}} 0$, then*

$$\det(x) \geq \det(s).$$

Proof. Since $x - s \succeq_{\mathcal{S}} 0$ and $x \succeq_{\mathcal{S}} 0$, then

$$Q_{x^{1/2}}(e - Q_{x^{-1/2}} s) \succeq_{\mathcal{S}} 0,$$

and from Part (7) of Lemma 1, $e - Q_{x^{-1/2}} s \succeq_{\mathcal{S}} 0$. Therefore $\det(Q_{x^{-1/2}} s) \leq 1$, and from Part (6) of Lemma 1, $\det(x^{-1})\det(s) \leq 1$. ■

Next lemma gives extensions of the well-known inequalities on the logarithmic function.

Lemma 3 *Let $s \in \mathcal{S}$, If $\|s\|_2 < 1$, then*

$$\log\det(e+s) \geq tr(s) - \frac{\|s\|_F^2}{2(1-\|s\|_2)}.$$

Moreover, if $\|s\|_F \leq 1$, then

$$\log\det(e+s) \geq tr(s) + \|s\|_F + \log(1-\|s\|_F),$$

Proof. Let $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_1 = \lambda_1(s)$ and $\lambda_2 = \lambda_2(s)$, then $\det(e+s) = (1+\lambda_1)(1+\lambda_2)$ and

$$\log\det(e+s) = \log(1+\lambda_1) + \log(1+\lambda_2).$$

If $\|s\|_2 < 1$ then

$$\log(1+\lambda_1) + \log(1+\lambda_2) \geq \lambda_1 + \lambda_2 - \frac{\|\lambda\|_2^2}{2(1-\|\lambda\|_\infty)}.$$

The first inequality follows. On the other hand, if $\|s\|_F \leq 1$, then (see [20], Page 439)

$$\log(1+\lambda_1) + \log(1+\lambda_2) \geq \lambda_1 + \lambda_2 + \|\lambda\|_2 + \log(1-\|\lambda\|_2),$$

which leads to the second inequality. ■

3 Analytic center

In this section we present optimality conditions of the analytic center of a compact convex set described by second-order cone inequalities. Since in this paper we frequently switch between primal, dual and primal-dual settings, we present the three characterizations here. Moreover, we introduce the potential functions and their corresponding feasible sets and show that the optimality conditions for the analytic center coincide in primal, dual and primal-dual cases.

We also present a computational algorithm based on a dual approach to compute an approximate analytic center when an interior point is available. It should be noted that computational algorithms based on the primal and primal-dual settings can be similarly derived. However, the dual algorithm better suits our case.

3.1 Optimality conditions

Let us start with the definitions of the feasible sets and potential functions. Let $A \in \Re^{m\times n}$ $(m \leq n)$ be a full-rank matrix and $c \in \Re^n$. Let

$$\mathcal{F}_P \stackrel{\text{def}}{=} \{x \in \mathcal{S}_n : Ax = 0\},$$

$$\mathcal{F}_D \stackrel{\text{def}}{=} \{y \in \Re^m : s = c - A^T y \succeq_{\mathcal{S}_n} 0\},$$

$$\mathcal{F}_{PD} \stackrel{\text{def}}{=} \mathcal{F}_P \times \mathcal{F}_D.$$

be convex compact sets. We refer to $\mathcal{F}_P$, $\mathcal{F}_D$, and $\mathcal{F}_{PD}$ as primal, dual and primal-dual feasible sets (localization sets) respectively. Throughout this paper we assume that

$$\mathcal{F}_P^\circ = \{x \in \mathcal{S}_n^\circ : Ax = 0\}$$

and

$$\mathcal{F}_D^\circ = \{y \in \Re^m : s = c - A^T y \succ_{\mathcal{S}_n} 0\}$$

are nonempty. We sometimes represent $\mathcal{F}_D$ only by the slack vector s,

$$\mathcal{F}_D = \{s \in \mathcal{S}_n : A^T y + s = c, \text{ for some } y \in \Re^m\},$$

Let $x \in \mathcal{F}_P^\circ$ and $s \in \mathcal{F}_D^\circ$, then the primal, dual and primal-dual potential functions are respectively defined via

$$\phi_P(x) \stackrel{\text{def}}{=} c^T x - \tfrac{1}{2} \log\det x,$$

$$\phi_D(s) \stackrel{\text{def}}{=} -\tfrac{1}{2} \log\det s, \text{ and}$$

$$\begin{aligned}
\phi_{PD}(x,s) &\stackrel{\text{def}}{=} \phi_P(x) + \phi_D(s) \\
&= x^T s - \frac{1}{2} \log\det Q_{x^{1/2}} s \\
&= x^T s - \frac{1}{2} \log\det Q_{x^{1/2}} s
\end{aligned}$$

The analytic center of $\mathcal{F}_P$ is a unique point in the interior of $\mathcal{F}_P$ that minimizes the primal potential function $\phi_P(x)$, i.e., the optimal solution of the following convex optimization problem:

$$\begin{array}{ll}
\min & c^T x - \frac{1}{2} \log\det x \\
s.t. & \\
& Ax = 0 \\
& x \in \mathcal{S}_n.
\end{array} \tag{3}$$

From the first order optimality conditions, $x^a \in \mathcal{S}_n^\circ$ is the analytic center of $\mathcal{F}_P$ iff there exist $y^a \in \Re^m$ and $s^a \in \mathcal{S}_n^\circ$ such that

$$Ax^a = 0, \;\; A^T y^a + s^a = c, \text{ and } x^a \circ s^a = e \tag{4}$$

Similarly, the analytic center of $\mathcal{F}_D$ is defined as the minimizer of the dual potential function $\phi_D(s)$ over the dual feasible region:

$$\begin{array}{ll} \min & -\frac{1}{2}\log\det s \\ s.t. & \\ & A^T y + s = c \\ & s \in \mathcal{S}_n, \end{array} \tag{5}$$

and the analytic center of the $\mathcal{F}_{PD}$ is defined as the minimizer of the primal-dual potential function $\phi_{PD}(x,s)$ over the primal-dual feasible region:

$$\min\{\phi_{PD}(x,s) : x \in \mathcal{F}_P, s \in \mathcal{F}_D\} \tag{6}$$

One can easily verify that the first order optimality conditions for Problems (5) and (6) coincide with (4). Since the three characterization of the analytic center lead to the same system of equations, with abusing notation we sometimes refer to either one of x^a, y^a, s^a, or (x^a, s^a) as the analytic center, without a specific reference to the set.

Note that the value of the primal-dual potential function at the analytic center is equal to 1, independent of n the dimension of $\mathcal{S}_n$. This is because for (x^a, y^a, s^a) obtained from (4), $x^a \circ x^a = e$. Therefore $(x^a)^T s^a = 1$ and

$$\log\det Q_{(x^a)^{1/2}} s^a = \log\det e = 0.$$

The analytic center in the block sense can similarly be defined. For $i = 1, ..., k$, let x_i, s_i, $c_i \in \Re^{p_i}$ and define $\mathbf{x} = (x_1; x_2; \ldots; x_k)$, $\mathbf{s} = (s_1; s_2; \ldots; s_k)$, $\mathbf{c} = (c_1; c_2; \ldots; c_k)$, and

$$\mathbf{A} \stackrel{\text{def}}{=} [A_1 \;\; A_2 \;\; \ldots \;\; A_k]$$

$$\mathcal{S} \stackrel{\text{def}}{=} \mathcal{S}_{p_1} \times \mathcal{S}_{p_2} \times \ldots \times \mathcal{S}_{p_k}.$$

Then the block forms of the primal, dual and primal-dual feasible sets are defined via

$$\mathcal{F}_P^k \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathcal{S} : \mathbf{A}\mathbf{x} = 0\},$$

$\mathcal{F}_D^k \stackrel{\text{def}}{=} \{y \in \Re^m : \mathbf{s} = \mathbf{c} - \mathbf{A}^T y \succeq_{\mathcal{S}} \mathbf{0}\}$, and

$\mathcal{F}_{PD}^k \stackrel{\text{def}}{=} \mathcal{F}_P^k \times \mathcal{F}_D^k$,

and the primal, dual and primal-dual potential functions in the block form are defined

$\phi_P(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{c}^T\mathbf{x} - \frac{1}{2}\log\det\mathbf{x}$,

$\phi_D(\mathbf{s}) \stackrel{\text{def}}{=} -\frac{1}{2}\log\det\mathbf{s}$, and

$\phi_{PD}(\mathbf{x},\mathbf{s}) \stackrel{\text{def}}{=} \phi_P(\mathbf{x}) + \phi_D(\mathbf{s}) = \mathbf{x}^T\mathbf{s} - \frac{1}{2}\log\det Q_{\mathbf{x}^{1/2}}\mathbf{s}$,

where $Q_{\mathbf{x}} = Q_{x_1} \oplus Q_{x_2} \oplus \ldots \oplus Q_{x_k}$. Now with the above definitions, the first order optimality conditions for the analytic center in block form read

$$\mathbf{A}\mathbf{x}^a = 0, \quad \mathbf{A}^T y^a + \mathbf{s}^a = \mathbf{c}, \quad \mathbf{x}^a \circ \mathbf{s}^a = \mathbf{e},$$

where $\mathbf{x} \circ \mathbf{s} = (x_1 \circ s_1; \ldots; x_k \circ s_k)$ and $\mathbf{e} = (e_1; \ldots; e_k)$. From the definitions of $\mathbf{x}$, $\mathbf{s}$ and $\mathbf{e}$, and since $\mathbf{x}^a \circ \mathbf{s}^a = \mathbf{e}$, the optimal value of the primal-dual potential function composed of k blocks is

$$\phi_{PD}(\mathbf{x}^a, \mathbf{s}^a) = k.$$

3.2 A computational algorithm

In this section we present a dual algorithm to compute the analytic center. Let us start with a definition:

Definition 4 *Let $\bar{x} \in \mathcal{S}_n^\circ$, $\bar{y} \in \Re^m$ and $\bar{s} \in \mathcal{S}_n^\circ$ satisfy*

$$A\bar{x} = 0, A^T\bar{y} + \bar{s} = c, \text{ and } \|Q_{\bar{x}^{1/2}}\bar{s} - e\|_F \le \theta < 1.$$

Then $(\bar{x}, \bar{y}, \bar{s})$ is called a θ-approximate analytic center.

Note that if $\|Q_{\bar{x}^{1/2}}\bar{s} - e\|_F = 0$, then $\lambda_1(Q_{\bar{x}^{1/2}}\bar{s}) = 1$ and $\lambda_2(Q_{\bar{x}^{1/2}}\bar{s}) = 1$, and thus $Q_{\bar{x}^{1/2}}\bar{s} = e$, and in view of Lemma 1, $\bar{x} \circ \bar{s} = e$. In other words, if $\theta = 0$, then θ-approximate analytic center is the exact analytic center.

We develop an algorithm for computation of an approximate analytic center of a compact convex set, composed of one block. The extension to the general case is trivial.

Let $s \in \mathcal{F}_D^\circ$ and d_s be a feasible direction for Problem (5), i.e., $A^T d_y + d_s = 0$. From Part 4 of Lemma 1

$$\begin{aligned}\phi_D(s+d_s) &= -\frac{1}{2}\log\det(s+d_s)\\ &= -\frac{1}{2}\log\det s - (s^{-1})^T d_s + \frac{1}{2} d_s^T Q_s^{-1} d_s.\end{aligned}$$

let $\Delta\phi_D(s) = \phi_d(s+d_s) - \phi_d(s)$, then

$$\Delta\phi_D(s) = d_y^T A s^{-1} + \frac{1}{2} dy^T A Q_s^{-1} A^T d_y,$$

and $\nabla_{d_y} \Delta\phi_D(s) \equiv 0$ implies that

$$d_y = -(AQ_s^{-1}A^T)^{-1} A s^{-1},$$

and

$$d_s = A^T (AQ_s^{-1}A^T)^{-1} A s^{-1}. \tag{7}$$

Starting from $s \in \mathcal{F}_D^\circ$, the above direction with a step size, reduces the value of potential function at each iteration. Let

$$x_s = s^{-1} - Q_s^{-1} d_s,$$

and define

$$p_s = Q_{s^{1/2}} x_s - e.$$

Note that $\|p_s\|_F$ measures the distance from the current point (x_s, s) to the analytic center. We first establish a lower bound on the potential reduction at each iteration:

Lemma 5 *Let for a given $s \in \mathcal{F}_D^\circ$, $\|p_s\|_F \geq 1$, and consider the dual directions with step size $\alpha/\|p_s\|$, for $\alpha < 1$*

$$\tilde{d}_y = -\frac{\alpha}{\|p_s\|_F}(AQ_s^{-1}A^T)^{-1} A s^{-1},$$

and $\tilde{d}_s = -A^T \tilde{d}_y$, and update $y^+ = y + \tilde{d}_y$ and $s^+ = s + \tilde{d}_s$. Then

$$\phi_D(s^+) - \phi_D(s) \leq -\delta,$$

where $\delta = \alpha - \frac{\alpha^2}{2(1-\alpha)}$.

Proof. First since

$$\begin{aligned} p_s &= Q_{s^{1/2}} x_s - e \\ &= Q_{s^{1/2}} s^{-1} - Q_{s^{1/2}} Q_s^{-1} d_s - e \\ &= Q_{s^{1/2}} Q_{s^{-1/2}} e - Q_{s^{1/2}} Q^2_{s^{-1/2}} d_s - e \\ &= e - Q_{s^{1/2}} Q_{s^{-1/2}} Q_{s^{-1/2}} d_s - e \\ &= -Q_{s^{-1/2}} d_s \end{aligned}$$

then

$$\|Q_{s^{-1/2}} \tilde{d}_s\|_F = \frac{\alpha}{\|p_s\|_F} \|Q_{s^{-1/2}} d_s\|_F = \alpha < 1$$

and from Lemma 3

$$\begin{aligned} -\log\det(Q_{s^{-1/2}} s^+) &\leq -tr(Q_{s^{-1/2}} s^+ - e) + \frac{\|Q_{s^{-1/2}} s^+ - e\|_F^2}{2(1 - \|Q_{s^{-1/2}} s^+ - e\|_2)} \\ &\leq -tr(Q_{s^{-1/2}} s^+ - e) + \frac{\alpha^2}{2(1-\alpha)}. \end{aligned}$$

On the other hand

$$-tr(Q_{s^{-1/2}} s^+ - e) = tr(Q_{s^{-1/2}} A^T \tilde{d}_y) = 2(AQ_{s^{-1/2}} e)^T \tilde{d}_y$$

Therefore

$$\begin{aligned} \phi_D(s^+) - \phi_D(s) &= -\log\det(Q_{s^{-1/2}} s^+) \\ &\leq 2(As^{-1})^T \tilde{d}_y + \frac{\alpha^2}{2(1-\alpha)}. \end{aligned} \qquad (8)$$

The tightest bound on the reduction of the potential function can be obtained by minimizing the right hand side of (8). That is

$$\begin{array}{ll} \min & (As^{-1})^T \tilde{d}_y \\ s.t. & \\ & \|Q_{s^{-1/2}} A^T \tilde{d}_y\|_F \leq \alpha, \quad \alpha < 1. \end{array}$$

From the first order optimality conditions, the optimal solution of the above problem reads

$$\tilde{d}_y = \frac{-\alpha}{\|p_s\|_F} (AQ_s^{-1} A^T)^{-1} A s^{-1}.$$

Since $\|p_s\|_F = \|Q_{s^{-1/2}} d_s\|_F$, then

$$\begin{aligned}(As^{-1})^T \tilde{d}_y &= \frac{-\alpha}{\|p_s\|_F} s^{-T} A^T (A Q_s^{-1} A^T)^{-1} A s^{-1} \\ &= \frac{-\alpha}{2\|p_s\|_F} \|p_s\|_F^2 \\ &\leq -\alpha/2.\end{aligned}$$

The proof follows from (8) now. ■

Note that for $1/3 \leq \alpha \leq 1/2$, the potential reduction δ is at least $1/4$ (and at most 0.268). However in practice, a line search method is employed to maximize the reduction and ensure dual feasibility.

Next lemma shows that once we are close enough to the analytic center, full Newton steps yields dual feasibility and converges quadratically.

Lemma 6 *If $\|p_s\|_F < 1$, then $s^+ = s + d_s \in \mathcal{S}_n^\circ$ and*

$$\|p_{s^+}\|_F \leq \|p_s\|_F^2. \tag{9}$$

Proof. Since

$$\begin{aligned}Q_{s^{1/2}}(e - p_s) &= Q_{s^{1/2}}(2e - Q_{s^{1/2}} x_s) \\ &= 2s - Q_s x_s \\ &= s + d_s,\end{aligned}$$

then $s^+ = Q_{s^{1/2}}(e - p_s)$. On the other hand, since $\|p(s)\|_F < 1$, then $\lambda_1(e - p_s) = 1 - \lambda_2(p_s) > 0$ and $\lambda_2(e - p_s) = 1 - \lambda_1(p_s) > 0$. That is $e - p_s \in \mathcal{S}_n^\circ$, and since $Q_{s^{1/2}}(\mathcal{S}_n^\circ) = \mathcal{S}_n^\circ$, then $s^+ \in \mathcal{S}_n^\circ$.

Now since x_s is the minimizer of the following least-square problem:

$$\begin{aligned}\min \quad & \|Q_{s^{1/2}} x - e\|_F \\ s.t. \quad & \\ & Ax = 0,\end{aligned}$$

then

$$\|p_{s^+}\|_F = \|Q_{(s^+)^{1/2}} x_{s^+} - e\|_F \leq \|Q_{(s^+)^{1/2}} x_s - e\|_F,$$

or

$$\|p_{s^+}\|_F \leq \|Q_{x^{1/2}} s^+ - e\|_F,$$

where $x = x_s$. By substituting s^+ in the above inequality one has

$$\|p_{s^+}\|_F \leq \|2Q_{x^{1/2}} s - Q_{x^{1/2}} Q_s x - e\|_F. \tag{10}$$

Now since $''\circ''$ is distributive

$$\begin{aligned}(Q_{x^{1/2}}s - e)^2 &= (Q_{x^{1/2}}s - e)\circ(Q_{x^{1/2}}s - e)\\ &= (Q_{x^{1/2}}s)^2 - 2Q_{x^{1/2}}s + e\end{aligned}$$

and in view of Lemma 1

$$(Q_{x^{1/2}}s)^2 = Q_{Q_{x^{1/2}}s}e = Q_{x^{1/2}}Q_sQ_{x^{1/2}}e = Q_{x^{1/2}}Q_sx.$$

Thus

$$(Q_{x^{1/2}}s - e)^2 = Q_{x^{1/2}}Q_sx - 2Q_{x^{1/2}}s + e.$$

The proof now follows from (10). ■

Computational algorithms based on primal and primal-dual settings can be developed similar to the dual algorithm. Primal algorithm starts with an interior point $x^0 \in \mathcal{F}_P^\circ$ and updates the primal direction d_x at each iteration. In this case one should be careful with the round-off error when updates x^+. In other words, at each iteration the updated primal direction d_x should be projected to the null space of A to ensure primal feasibility. We refer the reader to a survey paper by Goffin and Vial [7] for the analysis of primal algorithm in linear case and to Oskoorouchi [17] and Oskoorouchi and Goffin [19] for that of the semidefinite, and combination of linear and semidefinite cases.

The initial interior point plays a very important role in the performance of the Newton algorithm. In Lemma 5 we showed that the potential function is reduced by $\delta > 0$ at each iteration. This implies that after at most

$$O\left(\frac{\phi_D(s_0) - \phi_D(s^a)}{\delta}\right)$$

Newton steps, the algorithm stops with an approximate analytic center. Therefore the complexity of the algorithm depends on the proximity of the initial point to the analytic center.

4 The analytic center cutting plane method

In this section we present the framework of our algorithm and discuss the detail of the updating direction. The goal is to find a feasible solution in a convex set $\mathcal{F}^*$, which contains a full dimensional ball with $\varepsilon(<1)$ radius, and is contained in

$$\mathcal{F}_D = \{y \in \Re^m : A^Ty \preceq_{\mathcal{S}} c\};$$

a convex compact set described by second-order cone inequalities.

Let us begin with a definition.

Definition 7 *Let $\bar{y}$ be an approximate analytic center of the current set of localization $\mathcal{F}_D$, and $B \in \Re^{m \times p}$ be a full rank matrix, and $d \in \Re^p$ Then for all $y \notin \mathcal{F}^*$*

$$B^T y \preceq_{\mathcal{S}_p} d, \tag{11}$$

is called a second-order cone cut (SOCC) in $\mathcal{S}_p$. If $d = B^T \bar{y}$ then the cut passes through the center $\bar{y}$ and is called a central second-order cone cut. If $d \succ_{\mathcal{S}_p} B^T \bar{y}$ ($d \prec_{\mathcal{S}_p} B^T \bar{y}$), then (11) is called shallow (deep) cut.

In our analysis in this paper we work with the central cuts. Therefore, we assume that $d = B^T \bar{y}$ when we refer to (11) as the second-order cone cut.

First we make three assumptions:

Assumption 1 *The initial set of localization is the unit ball. That is*

$$\mathcal{F}_D^0 = \{y \in \Re^m : \|y\| \leq 1\}.$$

If $A_0 = (0 \quad -I_m) \in \Re^{m \times m+1}$, and $c_0 = e$, then $\mathcal{F}_D^0$ can be written in the standard form

$$\mathcal{F}_D^0 = \{y \in \Re^m : A_0^T y \preceq_{\mathcal{S}_{m+1}} c_0\}.$$

Notice that Assumption 1 is only a scaling assumption and does not reduce the generality of our analysis.

Assumption 2 *The set of localization $\mathcal{F}_D$ is composed of second-order cone cutting planes generated by an oracle, i.e., at each iteration, an oracle determines if the current point is in the solution set $\mathcal{F}^*$ or returns SOCC $A^T y \preceq_{\mathcal{S}_p} c$ that contains $\mathcal{F}^*$, and*

$$\max\{\|a_i\|, i = 1, 2, ..., p\} = 1,$$

where a_is are the columns of matrix A.

Assumption 2 implies that at the kth iteration of the algorithm, the set of localization takes the block form

$$\mathcal{F}_D^k = \{y \in \mathcal{F}_D^0 : \mathbf{A}^T y \preceq_{\mathcal{S}} \mathbf{c}\}$$

Assumption 3 *Let* $A_i^T y \preceq_{\mathcal{S}_{p_i}} c_i$ *be a SOCC. Define*

$$\mu_i = \max\{\det(A_i^T u) : A_i^T u \succeq_{\mathcal{S}_{p_i}} 0, \ \ \|u\| \leq 1\}.$$

Then

$$\mu = \min\{\mu_i, i = 1, ..., k\} > 0.$$

Assumption 3 is necessary for establishing a lower bound on the optimal value of the potential function.

The framework of the ACCPM can be presented as follows:

Algorithm 1 *Let* $\mathcal{F}_D^0 = \{y : \|y\| \leq 1\}$ *be the initial set of localization, and* $k = 0$

Step 1. *Compute* $\bar{y}^k$*, an approximate analytic center of* $\mathcal{F}_D^k$*.*

Step 2. *Call the oracle: if* $\bar{y}^k \in \mathcal{F}^*$*, stop. Otherwise add the cutting plane*

$$A_{k+1}^T y \preceq_{\mathcal{S}_{p_{k+1}}} A_{k+1}^T \bar{y}^k \tag{12}$$

to the current set of localization, and update

$$\mathcal{F}_D^{k+1} = \{y \in \mathcal{F}_D^k : A_{k+1}^T(\bar{y}^k - y) \succeq_{\mathcal{S}_{p_{k+1}}} 0\} \tag{13}$$

Step 3. *Set* $k = k + 1$ *and go to Step 1.*

Algorithm 1 starts from the unit ball and at each iteration adds a central SOCC to the set of localization, and updates the analytic center. We need an interior point of the updated set of localization in order to use the algorithm described in Section 3 to compute an approximate center. On the other hand, since the complexity of the algorithm depends on the initial point, one should select an interior point which is as close as possible to the next analytic center. Such a point is known as a *warm start* in the literature (see Mitchell and Todd [14] and Mitchell [15]). To obtain a warm start after adding a central cut, one can start from the current analytic center which is on the boundary of $\mathcal{F}_D^{k+1}$ and move towards minimizing the log det of the new slack vector. Let us describe this procedure now.

Dropping the index, let $(\bar{x}, \bar{y}, \bar{s})$ be a θ-approximate analytic center of

$$\mathcal{F}_D^k = \{y \in \mathcal{F}_D^0 : \mathbf{s} = \mathbf{c} - \mathbf{A}^T y \succeq_{\mathcal{S}} \mathbf{0}\}$$

and let the oracle returns the SOCC (12) and updates the set of localization as in (13). Observe that $\bar{y}$ is on the boundary of $A_{k+1}^T(\bar{y}-y) \succeq_{\mathcal{S}_{p_{k+1}}} 0$ and therefore it is not an interior point of $\mathcal{F}_D^{k+1}$. Corresponding to the updated dual set of localization, one can obtain the updated primal set of localization

$$\mathcal{F}_P^{k+1} = \{\mathbf{x} \in \mathcal{S}, x_{k+1} \in \mathcal{S}_{p_{k+1}} : \mathbf{A}\mathbf{x} + A_{k+1}x_{k+1} = 0\}.$$

Let $d_y = y - \bar{y}^k$ and $d_{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}}^k$. Consider the following optimization problems:

$$\begin{array}{llcll}
\multicolumn{2}{l}{\underline{\textbf{Primal direction}}} & & \multicolumn{2}{l}{\underline{\textbf{Dual direction}}} \\
\min & -\frac{1}{2}\log\det x_{k+1} & & \min & -\frac{1}{2}\log\det s_{k+1} \\
s.t. & & & s.t. & \\
& \mathbf{A}d_{\mathbf{x}} + A_{k+1}x_{k+1} = 0 & \textit{and} & & s_{k+1} = -A_{k+1}^T\hat{d}_y \\
& \|Q_{\bar{\mathbf{s}}^{1/2}}d_{\mathbf{x}}\|_F \le 1 & & & \|Q_{\bar{\mathbf{s}}^{-1/2}}\mathbf{A}^T\hat{d}_y\|_F \le 1 \\
& x_{k+1} \in \mathcal{S}_{p_{k+1}} & & & s_{k+1} \in \mathcal{S}_{p_{k+1}}.
\end{array} \tag{14}$$

The optimal solution of Problems in (14) give the primal and dual directions to obtain a warm start for computing the analytic center of the updated set. Now this lemma.

Lemma 8 *Let $U = A_{k+1}^T(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1}A_{k+1}$ be a $p_{k+1} \times p_{k+1}$ symmetric matrix. Then*

$$\hat{x}_{k+1} = \arg\min\{x_{k+1}^T U x_{k+1} - \frac{1}{2}\log\det x_{k+1} : x_{k+1} \in \mathcal{S}_{p_{k+1}}\}, \tag{15}$$

and

$$\hat{d}_{\mathbf{x}} = -Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1}A_{k+1}\hat{x}_{k+1},$$

are the optimal solutions of primal problem of (14) and thus the primal updating direction; and

$$\hat{s}_{k+1} = \arg\min\{s_{k+1}^T U^{-1} s_{k+1} - \frac{1}{2}\log\det s_{k+1} : s_{k+1} \in \mathcal{S}_{p_{k+1}}\}, \tag{16}$$

and

$$\hat{d}_y = -\frac{1}{2}(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1}A_{k+1}\hat{s}_{k+1}^{-1},$$

are the optimal solutions of the dual problem of (14) and thus the dual updating direction.

Proof. We first derive the KKT optimality conditions for the primal direction. The norm constraint can be written as

$$2\hat{d_{\mathbf{x}}}^T Q_{\bar{\mathbf{s}}} \hat{d_{\mathbf{x}}} \leq 1.$$

Thus $\hat{d_{\mathbf{x}}}$ and $\hat{x}_{k+1}$ are optimal iff there exist $u \in \Re^m$ and $u_0 > 0$ such that

$$\begin{aligned} -\hat{x}_{k+1}^{-1} + A_{k+1}^T u &= 0 && (17)\\ \mathbf{A}^T u + u_0 Q_{\bar{\mathbf{s}}} \hat{d_{\mathbf{x}}} &= \mathbf{0} && (18)\\ \|Q_{\bar{\mathbf{s}}^{1/2}} \hat{d_{\mathbf{x}}}\|_F - 1 &= 0 && (19)\\ \mathbf{A}\hat{d_{\mathbf{x}}} + A_{k+1}\hat{x}_{k+1} &= 0. && (20) \end{aligned}$$

By multiplying $\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}$ to (18) from the left hand side, one has

$$(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)u + u_0 \mathbf{A}\hat{d_{\mathbf{x}}} = 0,$$

and in view of (20)

$$u = u_0(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1} A_{k+1}\hat{x}_{k+1}.$$

On one hand from (18)

$$\hat{d_{\mathbf{x}}} = -Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1} A_{k+1}\hat{x}_{k+1},$$

and on the other hand by substituting u to (17), we have

$$\hat{x}_{k+1}^{-1} = u_0 U \hat{x}_{k+1}. \qquad (21)$$

Now from (19)

$$2x_{k+1}^T U x_{k+1} = 1$$

or $x_{k+1}^T x_{k+1}^{-1} = u_0$. Since $x_{k+1}^{-1} \circ x_{k+1} = e$, then $u_0 = 2$. Therefore Equation (21) is the optimality condition of Problem (15).

To derive the dual direction, let $u_0 > 0$ be the unique multiplier corresponding to the norm constraint in dual problem. Then $\hat{d}_y$ and $\hat{s}_{k+1}$ are optimal iff

$$A_{k+1}\hat{s}_{k+1}^{-1} + u_0(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)\hat{d}_y = 0,$$

where $\hat{s}_{k+1} = -A_{k+1}^T \hat{d}_y$. Therefore

$$\hat{d}_y = \frac{-1}{u_0}(\mathbf{A}Q_{\bar{\mathbf{s}}^{-1}}\mathbf{A}^T)^{-1} A_{k+1} s_{k+1}^{-1},$$

and

$$s_{k+1} = \frac{1}{u_0} U s_{k+1}^{-1}. \tag{22}$$

Similar to the primal case, for optimal $\hat{d}_y$ one has

$$2\hat{d}_y^{\,T}(\mathbf{A} Q_{\bar{\mathbf{s}}^{-1}} \mathbf{A}^T)\hat{d}_y = 1. \tag{23}$$

Substituting $\hat{d}_y$ in (23), we have $u_0 = 2$. ■

Note that the objective functions of Problems (15) and (16) are self-concordant functions, and therefore $\hat{x}_{k+1}$ and $\hat{s}_{k+1}$ can be efficiently obtained by the Newton method and a line search. In the next lemma we show that the optimal updating directions obtained from Lemma 8 lead to interior points of primal and dual localization sets.

Lemma 9 *Let* $\mathbf{x}^+ = (\bar{\mathbf{x}} + \alpha \hat{d}_{\mathbf{x}}; \alpha x_{k+1})$ *and* $\mathbf{s}^+ = (\bar{\mathbf{s}} + \alpha d_{\mathbf{s}}; \alpha s_{k+1})$ *for* $\alpha < 1 - \theta$. *Then* $(\mathbf{x}^+, \mathbf{s}^+)$ *is an interior point of* $\mathcal{F}_{PD}^{k+1}$.

Proof. The dual feasibility is trivial. Let us prove the primal feasibility. First observe that since $\bar{\mathbf{x}} \in \mathcal{F}_P^k$ and $\hat{d}_{\mathbf{x}}$ and $\hat{x}_{k+1}$ are optimal for the primal problem in (14), then clearly $\mathbf{A}(\bar{\mathbf{x}} + \alpha \hat{d}_{\mathbf{x}}) + A_{k+1}(\alpha \hat{x}_{k+1}) = 0$ and $\hat{x}_{k+1} \in \mathcal{S}^{\circ}_{p_{k+1}}$. We prove that $\bar{x}_i + \alpha \hat{d}_{x_i} \succ_{\mathcal{S}_{p_i}} 0$, for $i = 1, ..., k$. In what follows we indicate $\bar{x}_i$ by x, $\bar{s}_i$ by s, $\hat{d}_x$ by d_x, $\bar{\mathbf{s}}$ by $\mathbf{s}$, and $\hat{d}_{\mathbf{x}}$ by $d_{\mathbf{x}}$.

$$\begin{aligned}
\|Q_{x^{-1/2}} d_x\|_F^2 &= 2\|Q_{x^{-1/2}} d_x\|^2 \\
&= 2\|Q_{s^{-1/2}} Q_{x^{-1/2}} Q_{s^{1/2}} d_x\|^2 \\
&= 2(Q_{s^{1/2}} d_x)^T Q_{x^{-1/2}} Q_{s^{-1}} Q_{x^{-1/2}} (Q_{s^{1/2}} d_x) \\
&= 2 Q_{x^{-1/2}} Q_{s^{-1}} Q_{x^{-1/2}} \bullet (Q_{s^{1/2}} d_x)(Q_{s^{1/2}} d_x)^T \\
&= 2 Q^{-1}_{Q_{x^{1/2}} s} \bullet (Q_{s^{1/2}} d_x)(Q_{s^{1/2}} d_x)^T \\
&\leq 2 \sum \lambda_j (Q^{-1}_{Q_{x^{1/2}} s}) \lambda_j ((Q_{s^{1/2}} d_x)(Q_{s^{1/2}} d_x)^T) \\
&\leq 2 \lambda_1 (Q^{-1}_{Q_{x^{1/2}} s}) \mathrm{tr}(Q_{s^{1/2}} d_x)(Q_{s^{1/2}} d_x)^T \\
&= 2 \lambda_1 (Q^{-1}_{Q_{x^{1/2}} s}) \|Q_{s^{1/2}} d_x\|^2 \\
&= \lambda_1 (Q^{-1}_{Q_{x^{1/2}} s}) \|Q_{s^{1/2}} d_x\|_F^2,
\end{aligned} \tag{24}$$

and since $\|Q_{\mathbf{s}^{1/2}} d_{\mathbf{x}}\|_F = 1$, then $\|Q_{s^{1/2}} d_x\|_F \leq 1$. Thus

$$\|Q_{x^{-1/2}} d_x\|_F^2 \leq \lambda_1 (Q^{-1}_{Q_{x^{1/2}} s}). \tag{25}$$

Now since $\lambda_1(Q_x) = \lambda_1^2(x)$, then

$$\lambda_1(Q_{Q_{x^{1/2}}s}) = \lambda_1^2(Q_{x^{1/2}}s),$$

and since $s \in \mathcal{S}_{p_i}$ and $x^{1/2}$ is nonsingular, then $Q_{x^{1/2}}s \in \mathcal{S}_{p_i}$ and therefore

$$\lambda_1(Q^{-1}_{Q_{x^{1/2}}s}) = \lambda_2^{-2}(Q_{x^{1/2}}s). \tag{26}$$

On the other hand since (x, s) is a θ-approximate analytic center, then

$$\|Q_{x^{1/2}}s - e_i\|_F \leq \theta,$$

which implies that $|\lambda_2(Q_{x^{1/2}}s) - 1| \leq \theta$, or

$$\lambda_2^{-1}(Q_{x^{1/2}}s) \leq \frac{1}{1-\theta}. \tag{27}$$

Now in view of (25), (26), and (27) one has

$$\|Q_{x^{-1/2}}d_x\|_F \leq \frac{1}{1-\theta}, \tag{28}$$

and since $\alpha < 1 - \theta$

$$e_i + \alpha Q_{x^{-1/2}}d_x \succ_{\mathcal{S}_{p_i}} 0.$$

Therefore $x + \alpha d_x = Q_{x^{1/2}}(e_i + \alpha Q_{x^{-1/2}}d_x) \succ_{\mathcal{S}_{p_i}} 0.$ ■

In the next two lemmas we establish upper bounds on the primal and dual potential functions at the updated points.

Lemma 10 *Let $\bar{\mathbf{x}}$ be a θ-approximate analytic center of $\mathcal{F}_P^k$ and $\mathbf{x}^+$ be the interior point of $\mathcal{F}_P^{k+1}$ defined in Lemma 9. Then*

$$\phi_P(\mathbf{x}^+) - \phi_P(\bar{\mathbf{x}}) \leq \alpha(\theta - \frac{1}{2}) - \frac{1}{2}\log(1-\alpha) - \log\alpha - \frac{1}{2}\log\det\hat{x}_{k+1}$$

Proof. The primal potential function at the updated point is

$$\begin{aligned}\phi_P(\mathbf{x}^+) &= \mathbf{c}^T(\bar{\mathbf{x}} + \alpha\hat{d}_{\mathbf{x}}) - \frac{1}{2}\log\det(\bar{\mathbf{x}} + \alpha\hat{d}_{\mathbf{x}}) + \\ &\quad \alpha(A_{k+1}^T\bar{y})^T\hat{x}_{k+1} - \frac{1}{2}\log\det\alpha\hat{x}_{k+1}.\end{aligned}$$

First from Part (6) of Lemma 1

$$\log\det(\bar{\mathbf{x}} + \alpha\hat{d}_{\mathbf{x}}) = \log\det\bar{\mathbf{x}} + \log\det(\mathbf{e} + \alpha Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}).$$

Now since $\|\alpha Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}\|_F \leq 1$, from Lemma 3

$$\begin{aligned}\log\det(\bar{\mathbf{x}} + \alpha\hat{d}_{\mathbf{x}}) \quad &\geq \quad \log\det\bar{\mathbf{x}} + \alpha\mathrm{tr}(Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}) + \\ &\quad\ \alpha\|Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}\|_F + \log(1 - \alpha\|Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}\|_F),\end{aligned}$$

and noting that $t + \log(1-t)$ is a nonincreasing function

$$\log\det(\bar{\mathbf{x}} + \alpha\hat{d}_{\mathbf{x}}) \geq \log\det\bar{\mathbf{x}} + \alpha\mathrm{tr}(Q_{\bar{\mathbf{x}}^{-1/2}}\hat{d}_{\mathbf{x}}) + \alpha + \log(1-\alpha). \tag{29}$$

Also since $A_{k+1}\hat{x}_{k+1} = -\mathbf{A}\hat{d}_{\mathbf{x}}$

$$\mathbf{c}^T\hat{\hat{d}}_{\mathbf{x}} + \bar{y}^T A_{k+1}\hat{x}_{k+1} = (\mathbf{c} - \mathbf{A}^T\bar{y})^T\hat{\hat{d}}_{\mathbf{x}} = \bar{\mathbf{s}}^T\hat{\hat{d}}_{\mathbf{x}}. \tag{30}$$

Incorporating (29) and (30) into $\phi_P(\mathbf{x}^+)$ one has

$$\begin{aligned}\phi_P(\mathbf{x}^+) - \phi_P(\bar{\mathbf{x}}) \quad &= \quad \alpha\bar{\mathbf{s}}^T\hat{\hat{d}}_{\mathbf{x}} - \frac{\alpha}{2}\mathrm{tr}(Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}}) \\ &\quad -\frac{1}{2}(\alpha + \log(1-\alpha)) - \frac{1}{2}\log\det\alpha\hat{x}_{k+1}.\end{aligned}$$

The lemma now follows from $\mathrm{tr}(Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}}) = 2\mathbf{e}^T Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}}$ and

$$\begin{aligned}\bar{\mathbf{s}}^T\hat{\hat{d}}_{\mathbf{x}} - \mathbf{e}^T Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}} \quad &\leq \quad \left|(Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e})^T Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}}\right| \\ &\leq \quad \|(Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e})\|_F \left\|Q_{\bar{\mathbf{x}}^{-1/2}}\hat{\hat{d}}_{\mathbf{x}} y\right\|_F \\ &\leq \quad \theta.\end{aligned}$$

■

Lemma 11 *Let* $\bar{\mathbf{s}}$ *be a* θ*-approximate analytic center of* $\mathcal{F}_D^k$ *and* $\mathbf{s}^+$ *be the interior point of* $\mathcal{F}_D^{k+1}$ *defined in Lemma 9. Then*

$$\phi_D(\mathbf{s}^+) - \phi_D(\bar{\mathbf{s}}) \leq \alpha(\theta - \frac{1}{2}) - \frac{1}{2}\log(1-\alpha) - \log\alpha - \frac{1}{2}\log\det\hat{s}_{k+1}$$

Proof. Following the same line of proof as in Lemma 10 we have

$$\phi_D(\mathbf{s}^+) - \phi_D(\bar{\mathbf{s}}) \leq -\frac{\alpha}{2}\mathrm{tr}(Q_{\bar{\mathbf{s}}^{-1/2}}d_{\mathbf{s}}) - \frac{\alpha}{2} - \frac{1}{2}\log(1-\alpha) - \frac{1}{2}\log\det\alpha\hat{s}_{k+1}. \tag{31}$$

Let $\bar{\mathbf{x}}$ be an approximate analytic center of $\mathcal{F}_P^k$. Since

$$\begin{aligned}\mathrm{tr}(Q_{\bar{\mathbf{x}}^{1/2}}d_{\mathbf{s}}) \quad &= \quad -2\mathbf{e}^T Q_{\bar{\mathbf{x}}^{1/2}}\mathbf{A}^T\hat{d}_y \\ &= \quad -2\hat{d}_y{}^T\mathbf{A}Q_{\bar{\mathbf{x}}^{1/2}}\mathbf{e} \\ &= \quad -2\hat{d}_y{}^T\mathbf{A}\bar{\mathbf{x}} \\ &= \quad 0,\end{aligned}$$

then

$$
\begin{aligned}
\mathrm{tr}(Q_{\bar{\mathbf{s}}^{-1/2}} d_{\mathbf{s}}) &= \mathrm{tr}(Q_{\bar{\mathbf{s}}^{-1/2}} - Q_{\bar{\mathbf{x}}^{1/2}}) d_{\mathbf{s}} \\
&= 2 d_{\mathbf{s}}^T (Q_{\bar{\mathbf{s}}^{-1/2}} - Q_{\bar{\mathbf{x}}^{1/2}}) \mathbf{e} \\
&= 2 d_{\mathbf{s}}^T (Q_{\bar{\mathbf{s}}^{-1/2}} \mathbf{e} - Q_{\bar{\mathbf{x}}^{1/2}} \mathbf{e}) \\
&= 2 d_{\mathbf{s}}^T Q_{\bar{\mathbf{s}}^{-1/2}} (\mathbf{e} - Q_{\bar{\mathbf{s}}^{1/2}} \bar{\mathbf{x}}).
\end{aligned}
$$

Thus

$$
\frac{1}{2} \left| \alpha \mathrm{tr}\, (Q_{\bar{\mathbf{s}}^{-1/2}} d_{\mathbf{s}}) \right| \leq \alpha \| Q_{\bar{\mathbf{s}}^{-1/2}} d_{\mathbf{s}} \|_F \| Q_{\bar{\mathbf{s}}^{1/2}} \bar{\mathbf{x}} - \mathbf{e} \|_F \leq \alpha \theta.
$$

The proof now follows from the above inequality and (31). ■

In the next theorem we show that the number of Newton steps needed to recover the analytic center is bounded by $O(1)$.

Theorem 12 *Let $(\bar{\mathbf{x}}, \bar{\mathbf{s}})$ be a θ-approximate analytic center of $\mathcal{F}_{PD}^k$, and $(\mathbf{x}^+, \mathbf{s}^+)$ be an interior point of the updated set of localization $\mathcal{F}_{PD}^{k+1}$ derived in Lemma 9. Then the number of Newton steps needed to compute an approximate analytic center of $\mathcal{F}_{PD}^{k+1}$ is bounded by $O(1)$ when starting from $(\mathbf{x}^+, \mathbf{s}^+)$.*

Proof. Adding up the two inequalities in Lemma 10 and Lemma 11 gives

$$
\begin{aligned}
&\phi_{PD}(\mathbf{x}^+, \mathbf{s}^+) - \phi_{PD}(\bar{\mathbf{x}}, \bar{\mathbf{s}}) \leq \\
&\qquad 2\alpha\theta - \alpha - \log(1-\alpha) - 2\log\alpha - \log\det Q_{\hat{x}_{k+1}^{1/2}} \hat{s}_{k+1}.
\end{aligned}
$$

Recall that $\hat{x}_{k+1}$ and $\hat{s}_{k+1}$ are optimal for Problem (15) and Problem (16) and from the optimality conditions (21) and (22), $\hat{s}_{k+1} = U \hat{x}_{k+1}$. Therefore

$$
\hat{x}_{k+1}^{-1} = 2 \hat{s}_{k+1},
$$

or

$$
\hat{x}_{k+1} \circ \hat{s}_{k+1} = \frac{1}{2} e.
$$

Thus

$$
\phi_{PD}(\mathbf{x}^+, \mathbf{s}^+) \leq \phi_{PD}(\bar{\mathbf{x}}, \bar{\mathbf{s}}) + 2\alpha\theta - \alpha - \log(1-\alpha) - 2\log\alpha + 2\log 2. \quad (32)
$$

Let us first bound $\phi_{PD}(\bar{\mathbf{x}}, \bar{\mathbf{s}})$. From Lemma 3, since $(\bar{\mathbf{x}}, \bar{\mathbf{s}})$ is a θ approximate center

$$
\log\det(Q_{\bar{\mathbf{x}}^{1/2}} \bar{\mathbf{s}}) \geq \mathrm{tr}(Q_{\bar{\mathbf{x}}^{1/2}} \bar{\mathbf{s}} - \mathbf{e}) - \frac{\| Q_{\bar{\mathbf{x}}^{1/2}} \bar{\mathbf{s}} - \mathbf{e} \|_F^2}{2(1 - \| Q_{\bar{\mathbf{x}}^{1/2}} \bar{\mathbf{s}} - \mathbf{e} \|_2)}.
$$

But $\mathbf{e} = (e_1; e_2; \ldots; e_k)$ and $\mathbf{e}^T\mathbf{e} = k$. Thus

$$\operatorname{tr}(Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e}) = 2\mathbf{e}^T(Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e}) = 2(\bar{\mathbf{x}}^T\bar{\mathbf{s}} - k),$$

and therefore

$$\begin{aligned}
\phi_{PD}(\bar{\mathbf{x}}, \bar{\mathbf{s}}) &= \bar{\mathbf{x}}^T\bar{\mathbf{s}} - \frac{1}{2}\log\det(Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}}) \\
&\leq \bar{\mathbf{x}}^T\bar{\mathbf{s}} - (\bar{\mathbf{x}}^T\bar{\mathbf{s}} - k) + \frac{\|Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e}\|_F^2}{4(1 - \|Q_{\bar{\mathbf{x}}^{1/2}}\bar{\mathbf{s}} - \mathbf{e}\|_2)} \\
&\leq k + \frac{\theta^2}{4(1-\theta)} \qquad (33)
\end{aligned}$$

Thus (32) reads

$$\phi_{PD}(\mathbf{x}^+, \mathbf{s}^+) - k \leq \frac{\theta^2}{4(1-\theta)} + 2\alpha\theta - \alpha - \log(1-\alpha) - 2\log\alpha + 2\log 2.$$

Now let $((\mathbf{x}^a)^{k+1}, (\mathbf{s}^a)^{k+1})$ be the exact analytic center of $\mathcal{F}_{PD}^{k+1}$, then since $\phi_{PD}((\mathbf{x}^a)^{k+1}, (\mathbf{s}^a)^{k+1}) = k+1$, we have

$$\phi_{PD}(\mathbf{x}^+, \mathbf{s}^+) - \phi_{PD}\left((\mathbf{x}^a)^{k+1}, (\mathbf{s}^a)^{k+1}\right) \leq \rho(\theta, \alpha),$$

where

$$\rho(\theta, \alpha) = \frac{\theta^2}{4(1-\theta)} + 2\alpha\theta - \alpha - \log(1-\alpha) - 2\log\alpha + 2\log 2 - 1$$

is a constant for fixed values of θ and $\alpha < 1 - \theta$. On the other hand from Lemma 5, Newton method reduces the potential function by a constant amount δ at each iteration. Therefore after at most

$$\left\lceil \frac{\rho(\theta, \alpha)}{\delta} \right\rceil = O(1),$$

iterations, the algorithm stops with an approximate center of $\mathcal{F}_{PD}^{k+1}$. ∎

Selecting arbitrary values for $\theta < 1$ and $\alpha < 1 - \theta$, say $\theta = 0.25$ and $\alpha = 0.70$, one has

$$\rho(\theta, \alpha) < 0.11.$$

As mentioned in Section 3.2, $\delta \geq 0.25$. Therefore

$$\left\lceil \frac{\rho(\theta, \alpha)}{\delta} \right\rceil \leq \lceil 0.44 \rceil = 1$$

This means that an approximate analytic center for $\mathcal{F}_{PD}^{k+1}$ can be obtained in at most one Newton step when starting from $(\mathbf{x}^+, \mathbf{s}^+)$.

5 Convergence analysis

In this section we establish a bound on the number of SOCC needed by the algorithm before it obtains a solution in $\mathcal{F}^*$. Let us first introduce some notations for optimal value of potential functions at the analytic center:

$$opt(\mathcal{F}_P^k) = \phi_P(\mathbf{x}^a)$$

$$opt(\mathcal{F}_D^k) = \phi_D(\mathbf{s}^a)$$

$$opt(\mathcal{F}_{PD}^k) = \phi_{PD}(\mathbf{x}^a, \mathbf{s}^a)$$

We bound the dual potential function at the kth iteration. Recall that

$$\mathcal{F}_D^{k+1} = \{y \in \Re^m : \mathbf{s} = \mathbf{c} - \mathbf{A}^T y \succeq_{\mathcal{S}} 0, \text{ and } A_{k+1}^T(\bar{y}^k - y) \succeq_{\mathcal{S}_{p_k+1}} 0\},$$

where $\mathbf{A} = [A_1 \;\; A_2 \ldots A_k]$.

Lemma 13 *Let*

$$\gamma = (a_1^T(\mathbf{A}Q_{\mathbf{s}^{-1}}\mathbf{A}^T)^{-1}a_1)^{1/2},$$

where a_1 is the first column of matrix A_{k+1}. Then

$$opt(\mathcal{F}_D^{k+1}) \geq opt(\mathcal{F}_D^k) - \log\gamma,$$

Proof. First observe that from Inequality (33)

$$\phi_{PD}(\bar{\mathbf{x}}, \bar{\mathbf{s}}) - opt(\mathcal{F}_{PD}^k) \leq \frac{\theta^2}{4(1-\theta)},$$

which implies that

$$\Big(\phi_P(\bar{\mathbf{x}}) - opt(\mathcal{F}_P^k)\Big) + \Big(\phi_D(\bar{\mathbf{s}}) - opt(\mathcal{F}_D^k)\Big) \leq \frac{\theta^2}{4(1-\theta)}$$

or

$$\phi_P(\bar{\mathbf{x}}) - opt(\mathcal{F}_P^k) \leq \frac{\theta^2}{4(1-\theta)}. \tag{34}$$

On the other hand

$$opt(\mathcal{F}_D^{k+1}) = k + 1 - opt(\mathcal{F}_P^{k+1})$$

and from Lemma 10

$$opt(\mathcal{F}_P^{k+1}) \leq \phi_P(\mathbf{x}^+) \leq \phi_P(\bar{\mathbf{x}}) - \frac{1}{2}\log\det\hat{x}_{k+1} + \eta(\theta, \alpha),$$

where $\eta(\theta,\alpha) = \alpha(\theta - \frac{1}{2}) - \frac{1}{2}\log(1-\alpha) - \log\alpha$. Thus

$$opt(\mathcal{F}_D^{k+1}) \geq (k+1) - \phi_P(\bar{\mathbf{x}}) + \frac{1}{2}\log\det\hat{x}_{k+1} - \eta(\theta,\alpha).$$

Now in view of (34)

$$opt(\mathcal{F}_D^{k+1}) \geq (k+1) - opt(\mathcal{F}_P^k) + \frac{1}{2}\log\det\hat{x}_{k+1} - \eta(\theta,\alpha) - \frac{\theta^2}{4(1-\theta)}$$

or

$$opt(\mathcal{F}_D^{k+1}) \geq opt(\mathcal{F}_D^k) + \frac{1}{2}\log\det\hat{x}_{k+1} + 1 - \eta(\theta,\alpha) - \frac{\theta^2}{4(1-\theta)}.$$

For arbitrary values $\theta = 0.25$ and $\alpha = 0.70$ we have

$$opt(\mathcal{F}_D^{k+1}) \geq opt(\mathcal{F}_D^k) + \frac{1}{2}\log\det\hat{x}_{k+1} + \log 2. \tag{35}$$

Now recall that $\hat{x}_{k+1}$ satisfies (15), and therefore $\hat{x}_{k+1}^T U \hat{x}_{k+1} = 1/2$. Let

$$x = \frac{1}{\sqrt{2}}\gamma^{-1}e,$$

then $x \in \mathcal{S}_{p_{k+1}}$ and

$$\begin{aligned} x^T U x &= \frac{1}{2\gamma^2}e^T U e \\ &= \frac{1}{2\gamma^2}e^T A_{k+1}^T (\mathbf{A}Q_{\mathbf{s}}^{-1}\mathbf{A}^T)^{-1} A_{k+1} e \\ &= \frac{1}{2\gamma^2}a_1^T (\mathbf{A}Q_{\mathbf{s}}^{-1}\mathbf{A}^T)^{-1} a_1 \\ &= \frac{1}{2}. \end{aligned}$$

Therefore $\log\det\hat{x}_{k+1} \geq \log\det x = -2\log\gamma - \log 2$. Lemma follows from (35) now. ■

Next lemma establishes a preliminary bound on k, the number of SOCC, in terms of γ_i.

Lemma 14 *Let $A_1, A_2, \ldots, A_k$ be k SOCCs with a condition number $\mu > 0$, and for each $i = 1, 2, ..., k$*

$$\gamma_i = ((a_1^i)^T (\mathbf{A}Q_{\mathbf{s}}^{-1}\mathbf{A}^T)^{-1} a_1^i)^{1/2}, \tag{36}$$

where a_1^i is the first column of matrix A_i. Then

$$\sum_{i=1}^{k} \log \gamma_i \geq \frac{k}{2} \log \varepsilon^2 \mu,$$

where ε is the radius of the full dimensional ball contained in $\mathcal{F}^*$.

Proof. From Lemma 13

$$\begin{aligned} opt(\mathcal{F}_D^k) &\geq opt(\mathcal{F}_D^{k-1}) - \log \gamma_1 \\ &\vdots \\ &\geq opt(\mathcal{F}_D^0) - \sum_{i=1}^{k} \log \gamma_i. \end{aligned} \tag{37}$$

Since $\mathcal{F}_D^0$ is the unit ball $\|y\| \leq 1$ with the analytic center $y^a = 0$, then

$$opt(\mathcal{F}_D^0) = \log(1 - \|y^a\|^2) = 0.$$

Now let y^c be the center of the full dimensional ball contained in $\mathcal{F}^*$. Since for every $y \in \mathcal{F}_D^k$, $\mathbf{c} - \mathbf{A}^T y \succeq_{\mathcal{S}} 0$, then for u with $\|u\| \leq 1$, one has

$$\mathbf{c} - \mathbf{A}^T (y^c + \varepsilon u) \succeq_{\mathcal{S}} 0,$$

or $\mathbf{c} - \mathbf{A}^T y^c \succeq_{\mathcal{S}} \varepsilon \mathbf{A}^T u$. Now from Lemma 2

$$\det(\mathbf{c} - \mathbf{A}^T y^c) \geq \det(\varepsilon \mathbf{A}^T u).$$

In view of Assumption 3, since $\mathbf{A}$ contains k blocks

$$\det(\mathbf{c} - \mathbf{A}^T y^c) \geq (\varepsilon^2 \mu)^k. \tag{38}$$

On the other hand,

$$opt(\mathcal{F}_D^k) \leq -\frac{1}{2} \log \det(\mathbf{c} - \mathbf{A}^T y^c). \tag{39}$$

The proof follows from (37)- (39). ∎

We need to bound $\sum \gamma_i$. First this lemma:

Lemma 15 *Let $\mathbf{A} = [A_1 \;\; A_2 \;\; \ldots \;\; A_k]$, and a_1^i be the first column of matrix A_i. Then*

$$\mathbf{A} Q_{\mathbf{s}}^{-1} \mathbf{A}^T \succeq I + \frac{1}{4} \sum_{i=1}^{k} a_1^i (a_1^i)^T. \tag{40}$$

Proof. We drop the index i. First observe that for any vector u

$$u^T u - \frac{1}{\lambda_1(Q_s)} u^T Q_s u \geq 0.$$

Let $u = Q_{s^{-1/2}} A^T z$ for an arbitrary vector z. Then clearly

$$AQ_s^{-1}A^T \succeq \frac{1}{\lambda_1(Q_s)} AA^T. \tag{41}$$

Now let $s = (\sigma_0; \bar{\sigma})$. From the definition of Q_s

$$Q_s^{-1} = \frac{1}{\det^2(s)} \begin{pmatrix} \|s\|^2 & -2\sigma_0 \bar{\sigma}^T \\ -2\sigma_0 \bar{\sigma} & \det(s) I_m + 2\bar{\sigma}\bar{\sigma}^T \end{pmatrix}.$$

From Assumption 1, for the initial set $\mathcal{F}_D^0$, we have $A_0 = (0 \quad -I_m)$, and $c_0 = e$. Thus $s_0 = (1; y)$ and

$$Q_{s_0}^{-1} = \frac{1}{(1 - \|y\|^2)^2} \begin{pmatrix} 1 + \|y\|^2 & -2y^T \\ -2y & (1 - \|y\|^2) I_m + 2yy^T \end{pmatrix}.$$

Thus

$$\begin{aligned} A_0 Q_{s_0}^{-1} A_0^T &= \frac{1}{(1 - \|y\|^2)^2} \left[(1 - \|y\|^2) I_m + 2yy^T \right] \\ &= \frac{1}{1 - \|y\|^2} I_m + \frac{2}{(1 - \|y\|^2)^2} yy^T \end{aligned}$$

and clearly

$$A_0 Q_{s_0^{-1}} A_0^T \succeq I_m. \tag{42}$$

Now since

$$\mathbf{A} Q_{\mathbf{s}}^{-1} \mathbf{A}^T = A_0 Q_{s_0}^{-1} A_0^T + \sum_{i=1}^{k} A_i Q_{s_i}^{-1} A_i^T,$$

in view of (41) and (42)

$$\mathbf{A} Q_{\mathbf{s}^{-1}} \mathbf{A}^T \succeq I_m + \sum_{i=1}^{k} \frac{1}{\lambda_1(Q_{s_i})} A_i A_i^T \tag{43}$$

On the other hand for $s_i = c_i - A_i^T y \in \mathcal{S}_{p_i}$

$$\lambda_1(Q_{s_i}) = \lambda_1^2(s_i) = (\sigma_0^i + \|\bar{\sigma}^i\|)^2 \leq (2\sigma_0^i)^2$$

and from Assumption 2

$$\lambda_1(Q_{s_i}) \leq 4. \tag{44}$$

Inequality (40) now follows from (43) and (44) ■

We are now ready to establish a bound on the number of second order cone cutting planes.

Theorem 16 *Let $\mathcal{F}^*$ contains a full dimensional ball with ε radius and is contained in $\mathcal{F}_D^0$. Let Algorithm 1 is employed to generate a series of the nested convex sets $\mathcal{F}_D^k \subset \mathcal{F}_D^{k-1} \subset \ldots \subset \mathcal{F}_D^0$ by means of the second-order cone cuts $A_i^T y \preceq_{\mathcal{S}_{p_i}} c_i$, for $i = 1, 2, ..., k$. Then the algorithm finds a solution in $\mathcal{F}^*$ when*

$$k \geq O(\frac{m}{\varepsilon^2 \mu}).$$

Proof. Let $G^0 = I$ and define

$$G^{k+1} = G^k + \frac{1}{4} a_1^k (a_1^k)^T, \tag{45}$$

where a_1^k is the first column of matrix A_k. Observe that

$$G^{k+1} = I + \frac{1}{4} \sum_{i=1}^{k} a_1^i (a_1^i)^T, \tag{46}$$

and therefore $G^k \succeq I$. Taking logdet from both sides of (45)

$$\log\det(G^{k+1}) = \log(1 + \frac{r_k^2}{4}) + \log\det(G^k), \tag{47}$$

where $r_k^2 = (a_1^k)^T (G^k)^{-1} a_1^k$. Since $G^k \succeq I$ and $\|a_1^k\| \leq 1$, therefore $r_k^2 \leq 1$ and consequently

$$\log(1 + \frac{r_k^2}{4}) \geq \frac{r_k^2}{4} - \frac{(r_k^2/4)^2}{2(1 - r_k^2/4)} \geq \frac{r_k^2}{5}.$$

Thus since $\log\det(G^0) = 0$, from (47)

$$\log\det(G^{k+1}) \geq \frac{1}{5} \sum_{i=1}^{k} r_i^2. \tag{48}$$

Now from Lemma 15 and Inequality (46)

$$\mathbf{A} Q_{\mathbf{s}^{-1}} \mathbf{A}^T \succeq G^k,$$

and therefore $r_i \geq \gamma_i$, where γ_i is defined as in (36). Thus

$$\log\det(G^{k+1}) \geq \frac{1}{5}\sum_{i=1}^{k}\gamma_i^2. \tag{49}$$

On the other hand

$$\log\det(G^{k+1}) \leq m\log\left(\frac{\operatorname{tr}(G^{k+1})}{m}\right).$$

But from (46) $\operatorname{tr}(G^{k+1}) = m + k/4$. Thus

$$\log\det(G^{k+1}) \leq m\log(1+\frac{k}{4m}). \tag{50}$$

Combining (49) and (50) one has

$$\sum_{i=1}^{k}\gamma_i^2 \leq 5m\log(1+\frac{k}{4m})$$

Now since $\prod_{i=1}^k \gamma_i \leq (\frac{\sum_{i=1}^k \gamma_i}{k})^k$, then

$$\frac{1}{k}\sum_{i=1}^{k}\log\gamma_i^2 \leq \log\left(\frac{\sum_{i=1}^k\gamma_i^2}{k}\right) \leq \log\left(\frac{5m}{k}\log(1+\frac{k}{4m})\right)$$

Now in view of Lemma 14

$$\begin{aligned}
\frac{k}{2}\log\varepsilon^2\mu &\leq \frac{1}{2}\sum_{i=1}^{k}\log\gamma_i^2 \\
&\leq \frac{k}{2}\log\left(\frac{5m}{k}\log(1+\frac{k}{4m})\right),
\end{aligned}$$

or

$$\varepsilon^2\mu \leq \frac{5m}{k}\log(1+\frac{k}{4m}).$$

The algorithm stops when this inequality is violated. ■

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