Global Optimization of Homogeneous Polynomials on the Simplex and on the Sphere

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Abstract We obtain rigorous estimates for linear and semidefinite relaxations of global optimization problems on the simplex and on the sphere

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1 Introduction

Recently, there has been a considerable activity in the area of global polynomial optimization based on the ideas of linear, second order and semidefinite programming relaxations. In the present paper we consider a particular but an important case of the global optimization of homogeneous polynomials on the sphere and on the simplex. Our results should be considered as natural generalizations of the work of [9] and [7], where the case of the global optimization of a quadratic form on the simplex has been addressed. Our situation can be analyzed, in principle, with the help of the technique developed in [8]. The major advantage of our approach, however, is that we obtain rigorous estimates of proximity of optimal values of relaxations to global optimal values of original problems. The technique which we are using is mostly due to B. Reznick and some of his coauthors (see e.g. [3],[4],[5]). This technique has been developed with the purpose of understanding the structure of cones of nonnegative polynomials of several variables and, to the best of our knowledge, has never been used in optimization theory before. We believe that it will play an important role in the computation of global optimal solutions to polynomial optimization problems.

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2 Minimization of Homogeneous Polynomials of even degree on the Sphere

Let $T_{n-1} = \{x \in R: \ x = (x_1, ...x_n)^T, x_i \geq 0, i = 1, 2, ...n, \ x_1 + x_2 + ... + x_n = 1\}$,

$$S^{n-1} = \{x \in R^n : x_1^2 + x_2^2 + ... + x_n^2 = 1\}$$

Observe that the map $\varphi : R^n \to R^n$

$$\varphi(x_1, ..., x_n) = (x_1^2, x_2^2, ..., x_n^2)$$

maps S^{n-1} onto T_{n-1}. Thus, if p is a homogeneous polynomial of degree d on R^n, then $q = p \circ \varphi$ is a homogenous polynomial of degree $2d$ and, moreover,

$$\min\{q(x) : x \in S^{n-1}\} = \min\{p(x) : x \in T_{n-1}\}$$

Denote by $H_d(R^n)$ the vector space of homogenous polynomials of degree d on R^n.
First, we consider the problem

$$p(x) \to min. \tag{1}$$

$$x \in S^{n-1} \tag{2}$$

for some $p \in H_{2d}(R^n)$. The remark above enable one to reduce the problem of minimization (maximization) of any homogeneous polynomial on T_{n-1} to (1), (2). It will be convenient to introduce the following scalar product on $H_d(R^n)$.

Let $f \in H_d(R^n)$

$$f(x_1, ...x_n) = \sum_{(i_1,...,i_n)\in\tau(n,d)} c_d(i_1, ..., i_n)a(f; i_1, ..., i_n)x_1^{i_1} x_2^{i_2} ...x_n^{i_n}.$$

where

$$\tau(n, d) = \{(i_1, ..., i_n) : i_1 + ... + i_n = d, i_j \ are \ nonnegative \ integers, j = 1, 2, ..., n\},$$

$$c_d(i_1, ..., i_n) = \frac{d!}{i_1!...i_n!}.$$

Then

$$\langle f, g\rangle_d = \sum_{(i_1,...,i_n)\in\tau(n,d)} c_d(i_1, ..., i_n)a(f; i_1, ..., i_n)a(g; i_1, ..., i_n)$$

$$f, g \in H_d(R^n).$$

We will use the following two properties of this scalar product. Given $\alpha \in R^n$, let

$$g(x) = \sum_{i=1}^{n} (\alpha_i x_i)^d \in H_d(R^n).$$

Then

$$\langle f, g\rangle_d = f(\alpha), \forall f \in H_d(R^n). \tag{3}$$

Define by $f(D)$ the differential operator:

$$f(D) = \sum_{(i_1,\dots i_n)\in\tau(n,d)} c_d(i_1,\dots i_n)a(f;i_1,\dots i_n)\frac{\partial^d}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}.$$

If $p \in H_e(R^n), h \in H_{e+d}(R^n)$, then

$$d!\langle f, p(D)h\rangle_d = (d+e)!\langle pf, h\rangle_{d+e}. \tag{4}$$

Following the general construction in [2], consider a bilinear map:

$$B : H_d(R^n) \times H_d(R^n) \to H_{2d}(R^n)$$

$$B(u_1, u_2) = u_1 u_2.$$

According to [2], B induces a linear map

$$\Lambda : Hom(H_d(R^n)) \to H_{2d}(R^n).$$

Here $Hom(H_d(R^n))$ is the vector space of a linear maps of $H_d(R^n)$ into itself.
More precisely,

$$\Lambda(u_1 \otimes u_2) = u_1 u_2\ ,\ \ u_1, u_2 \in H_d(R^n),$$

where $u_1 \otimes u_2 \in Hom(H_d(R^n))$,

$$u_1 \otimes u_2(u_3) =< u_2, u_3 >_d u_1\ ,\ \ u_3 \in H_d(R^n).$$

Consider the vector subspace $S_d \subset Hom(H_d(R^n))$ of symmetric linear maps of $H_d(R^n)$ into itself : $A \in S_d$ if and only if $\langle Au_1, u_2\rangle_d =< u_1, Au_2 >_d\ ,\ \ \forall u_1, u_2 \in H_d(R^n)$. We can naturally define the cone S_d^+ :

$$S_d^+ = \{A \in S_d :< Au, u >_d \geq 0,\ \forall u \in H_d(R^n)\}$$

Then, according to [2] :

$$\Lambda(S_d^+) = cone\{B(u, u) : u \in H_d(R^n)\}.$$

In other words, the image of S_d^+ under Λ coincides with the "cone of squares" in $H_{2d}(R^n)$. We can define the dual $M : H_{2d}(R^n) \to Hom(H_{2d}(R^n))$ of Λ as follows:

$$Tr(M(v)A) = \langle v, \Lambda(A)\rangle_{2d}, \tag{5}$$

$$\forall v \in H_{2d}(R^n), A \in Hom(H_d(R^n).$$

In particular,

$$\langle v, u^2\rangle_{2d} = \langle u, M(v)u\rangle_d,$$

$\forall v \in H_{2d}(R^n), \forall u \in H_d(R^n)$. It then easily follows that the dual $[\Lambda(S_d^+)]^*$ of the cone of squares $\Lambda(S_d^+)$ has the form:

$$[\Lambda(S_d^+)]^* = \{v \in H_{2d}(R^n) : M(v) \in S_d^+\}. \tag{6}$$

Let us denote by K_{2d} the "cone of squares" in $H_{2d}(R^n)$ and by K_{2d}^* its dual. To address the original problem (1), (2), consider the following sequence of optimization problems, parametrized by the integer parameter $l \geq 0$.

$$\mu \to min \tag{7}$$

$$\|x\|^{2l} p + \mu \|x\|^{2(d+l)} \in K_{2(d+l)}. \tag{8}$$

Here $\|x\| = (x_1^2 + ... + x_n^2)^{1/2}$ and , hence, $\|x\|^{2l} \in H_{2l}(R^n)$. The problem (7), (8) is in the so-called conic form and, hence, its conic dual can be calculated.

Let us briefly recall a general formalism of the conic duality. Let $(V, \langle, \rangle)$ be an Euclidean finite-dimensional vector space; K be a closed, convex, pointed cone with a nonempty interior; $A : R^m \to V$ be a linear map; $c \in R^m, b \in V$. Consider the problem:

$$\sum_{i=1}^{m} c_i \mu_i \to \min,$$

$$A\mu - b \in K.$$

Here $\mu = (\mu_1, \ldots \mu_m)^T, c = (c_1, \ldots c_m)^T \in R^m$. Denote by $A^T : V \to R^m$ the linear map uniquely defined by:

$$\mu^T (A^T v) = \langle A\mu, v \rangle, \tag{9}$$

$\mu \in R^m, v \in V$. Consider , also, the dual problem:

$$\langle b, v \rangle \to \max,$$

$$A^T v = c, v \in K^*.$$

Here

$$K^* = \{w \in V : \langle w, v \rangle \geq 0, \forall v \in K\}.$$

Let us calculate the dual problem in our situation. We have: $V = H_{2(d+l)}(R^n), \langle, \rangle = \langle, \rangle_{2(d+l)}, m = 1$;

$$A\mu = \mu \parallel x \parallel^{2(d+l)} .$$

We have by (9):

$$\mu A^T v = \langle \mu \parallel x \parallel^{2(d+l)}, v \rangle_{2(d+l)},$$

$v \in H_{2(d+l)}(R^n)$. Hence,

$$A^T v = \langle \parallel x \parallel^{2(d+l)}, v \rangle.$$

Correspondingly, the dual problem takes the form:

$$-\langle \parallel x \parallel^{2l} p, u \rangle_{2(d+l)} \to \max, \tag{10}$$

$$u \in K_{2(l+d)}^*, \tag{11}$$

$$\langle u, \parallel x \parallel^{2(d+l)} \rangle_{2(d+l)} = 1. \tag{12}$$

Since $\parallel x \parallel^{2l} \in int(K_{2l}), \forall l$, (see [3], Theorem 8.15)we conclude that the feasible set of (10)-(12) is compact and nonempty. Moreover, $int(K_{2l}^*)$ is nonempty for any $l > 0$. Hence, the problem (10)-(12)) is strictly feasible. This implies (see e.g. [1]) that optimal solutions to both (7)-(8) and (10)-(12) exist and, moreover, there is no duality gap. In other words, if μ_l and u_l are optimal solutions to (7),(8) and (10)-(12), respectively, then

$$\mu_l = -\langle \| x \|^{2l} p, u_l \rangle_{2(d+l)} = -\xi_l. \tag{13}$$

Remark 1 *Observe that according to (6) condition (11) is equivalent to:*

$$M(u) \in S^+_{l+d}. \tag{14}$$

Let

$$p_{min} = min\{p(x) : x \in S^{n-1}\},$$
$$p_{max} = max\{p(x) : x \in S^{n-1}\}.$$

Lemma 1 *Let μ be feasible for (7), (8) then*

$$\mu \geq -p_{min}.$$

Proof Let $x \in S^{n-1}$, i.e. $\|x\| = 1$. According to (8) $p(x) + \mu \geq 0$. Hence, $\mu \geq -p(x)$. Since p attains its minimum on S^{n-1}, the result follows. ⋄

Lemma 2 *The feasible set of the problem (7), (8)with parameter l is a subset of the feasible set of the problem (7), (8) with parameter $l + 1$*

Proof If μ satisfies (8), then $\|x\|^{2l}p + \mu\|x\|^{2(d+l)} = \sum_{j=1}^k h_j^2$ for some $h_j \in H_{d+l}(R^n)$. Then $\|x\|^{2(l+1)}p + \mu\|x\|^{2(d+l+1)} = (\sum x_i^2)(\sum_{j=1}^k h_j^2) \in K_{2(d+l+1)}$, i.e. μ is feasible for (7), (8) with the parameter $l + 1$. ⋄

Corollary 1 *Let*

$$\mu_l = \min\{\mu : \|x\|^{2l}p + \mu\|x\|^{2(d+l)} \in K_{2(d+l)}\}.$$

Then

$$\mu_l \geq \mu_{l+1} \geq -p_{min}$$

for any $l \geq 0$.

In particular,

$$\mu_* = \lim \mu_l, l \to \infty,$$

exists and $\mu_* \geq -p_{min}$.

Proposition 1 *We have: $\mu_* = -p_{min}$.*

Proof By Corollary $\mu_* \geq -p_{min}$. Let $\mu_* > -p_{min}$. Consider a polynomial

$$q(x) = p(x) + (-p_{min} + \epsilon)\|x\|^{2d} \in H_{2d}(R^n),$$

where $\epsilon > 0$ is chosen in such a way that $\mu_* - \epsilon > -p_{min}$. Since $\min\{q(x) : x \in S^{n-1}\} = \epsilon > 0$, by [4] there exists $l > 0$ such that

$$\|x\|^{2l}q \in K_{2(l+d)}.$$

Hence $-p_{min}+\epsilon$ is feasible for (7), (8) with parameter l. Hence, $-p_{min}+\epsilon \geq \mu_*$. Contradicts our choice of ϵ.$\diamond$

Denote by $p_{\nu,l}$ the polynomial:

$$p_{\nu,l}(x) = \|x\|^{2l} p + \nu \|x\|^{2(l+d)} \in H_{2(d+l)}(R^n).$$

We obviously, can rewrite (10)-(12) in the form:

$$< u, p_{\nu,l} >_{2(l+d)} - \nu \rightarrow min,$$

$$u \in K^*_{2(l+d)}$$

$$\langle u, \|x\|^{2(d+l)} \rangle_{2(d+l)} = 1$$

If $p_{\nu,l} \in K_{2(d+l)}$, then

$$\xi_l = \langle u_l, p_{\nu,l} \rangle_{2(d+l)} - \nu \geq -\nu. \tag{15}$$

On the other hand, due to the duality theorem (see [6]):

$$\xi_l = -\mu_l \leq p_{min}$$

Theorem 1 *Let*

$$\alpha = \alpha(l, n, d) = (l + \frac{n}{2} + d) 4 \ln 2$$

$$\beta = \beta(n, d) = n2d(2d - 1).$$

Suppose that $\alpha(l, n, d) > \beta(n, d)$, *i.e.*

$$l > \frac{2nd(2d-1)}{4 \ln 2} - \frac{n}{2} - d. \tag{16}$$

Then

$$\frac{\beta(p_{max} - p_{min})}{\alpha - \beta} \geq p_{min} - \xi_l \geq 0. \tag{17}$$

In particular,

$$|p_{min} - \xi_l| \leq \frac{\beta(p_{max} - p_{min})}{\alpha - \beta}.$$

Proof We need to establish the left inequality in (17).
It is obvious that

$$\max\{p_{\nu,l} : x \in S^{2(d+l)-1}\} = p_{max} + \nu$$

$$\min\{p_{\nu,l} : x \in S^{2(d+l)-1}\} = p_{min} + \nu$$

Thus according to the main result of [4] $p_{\nu,l} \in K_{2(d+l)}$, provided $p_{min} + \nu > 0$ and

$$l \geq \frac{n2d(2d-1)(p_{max} + \nu)}{4 \ln 2 (p_{min} + \nu)} - \frac{n}{2} - d. \tag{18}$$

Take

$$\nu^* = \frac{\beta(n,d)p_{max} - \alpha(l,n,d)p_{min}}{\alpha(l,n,d) - \beta(n,d)}$$

.

Then, if (16) holds, we obtain that (18) holds as an equality. Now by (15):

$$\xi_l \geq -\nu^*$$

which is the left inequality in (17) (it is essential that $\alpha(l,n,d) > \beta(n,d)$). $\diamond$

3 Hilbert's Identities

The following result is proved e.g. in [10].

Proposition 2 *For $m \geq 0$, let*

$$c_m = \frac{m!}{2^m(\frac{m}{2})!} \;\; if\ m\ is\ even$$

$$c_m = 0 \;\; if\ m\ is\ odd$$

Let $\beta_1, ..., \beta_r$ be distinct real roots of Hermite polynomial $H_r(x)$. Let further $\rho_1, ..., \rho_r$ be a unique solution of the following system of linear equation.

$$\sum_{j=1}^{r} \beta_j^k x_j = c_k, \;\; k = 0, 1, ..., r-1 \;\; (r \geq 1)$$

Then $\rho_i > 0$ and

$$c_{2s}(x_1^2 + ... + x_n^2)^s = \sum_{j_1=1}^{r} ... \sum_{j_n=1}^{r} \rho_{j_1}...\rho_{j_n}(\beta_{j_1}x_1 + ... + \beta_{j_n}x_n)^{2s},$$

provided $2s < r$.

This result (known as a version of Hilbert's identities) provides an explicit representation of the form:

$$\|x\|^{2s} = \sum_{i=1}^{N(s)} < a_i^{(s)}, x >^{2s} \tag{19}$$

with $a_i^{(s)} \in R^n$.

Using this representation and properties (3),(4)of the scalar product $\langle , \rangle_d$ on $H_d(R^n)$ (see e.g. [3]), we can rewrite (10) - (12) in the form:

$$\sum_{i=1}^{N(l)} (p(D)u)(a_i^{(l)}) \to \min, \tag{20}$$

$$\sum_{i=1}^{N(l+d)} u(a_i^{(l+d)}) = 1, \tag{21}$$

$$M(u) \in S^+_{l+d}. \tag{22}$$

Here $p(D)$ is the differential operator obtained by substituting $\frac{\partial}{\partial x_i}$ instead of x_i.

The form (20) - (22) seems to be much more convenient for practical purposes than the original form (10) -(12). We refer to [3] for further discussion of representations (19) with choices of $N(s)$ as small as possible and their connections with spherical designs.

4 Minimization of Homogeneous Polynomials on the Simplex

Let $p \in H_d(R^n)$. Consider the following optimization problem:

$$p(x) \to \min,$$

$$x \in T_{n-1}.$$

To formulate our main result we need to introduce some notation.

For a positive number t, a nonnegative integer m and a single variable x, define

$$(x)^m_t = x(x-t)...(x-(m-1)t) = \prod_{i=0}^{m-1}(x-it).$$

Given a polynomial

$$f(x) = \sum_{(i_1,...,i_n)} b(i_1,...,i_n)x_1^{i_1}...x_n^{i_n},$$

define

$$f_t(x) = \sum_{(i_1,...,i_n)} b(i_1,...,i_n)(x_1)^{i_1}_t...(x_n)^{i_n}_t. \tag{23}$$

If $f \in H_m(R^n)$, we define

$$L(f) = \max\{\frac{|b(i_1,...,i_n)|}{c_m(i_1,...,i_n)} : (i_1,...,i_n) \in \tau(n,m)\}.$$

Here we use notation introduced in Section 2.

Let, further,

$$\omega_l(d) = \prod_{i=1}^{d-1}(1-\frac{i}{l+d}),$$

i.e

$$\omega_l(d) = (1)^d_{\frac{1}{l+d}}.$$

Suppose that

$$\delta_l(P) = \min\{p_{\frac{1}{l+d}}(\frac{\gamma_1}{l+d},...,\frac{\gamma_n}{l+d}) : (\gamma_1,...\gamma_n) \in \tau(n,d+l)\}.$$

With this notation, we can formulate the main result of this section.

Theorem 2 *For any positive l, we have :*

$$0 \le p_{min} - \frac{\delta_l(p)}{\omega_l(d)} \le (L(p) - p_{min})(\frac{1}{\omega_l(d)} - 1) \tag{24}$$

.

Remark 2 *Observe that $\omega_l(d) \to 1$ when $l \to \infty$. Hence, $p_{min} \approx \delta_l(p)/\omega_l(d)$ for large l. However, (24) provides an exact estimate for the proximity of p_{min} and $\frac{\delta_l(p)}{\omega_l(d)}$.*

To prove Theorem 2, we need several auxilliary statements taken from [5].

Proposition 3 *Let*

$$p(x) = \sum_{(i_1,...,i_n)\in\tau(n,d)} b_{i_1...i_n} x_1^{i_1}...x_n^{i_n}$$

Then

$$(x_1 + x_2 + ... + x_n)^l p(x) = \sum_{(\gamma_1,...,\gamma_n)\in\tau(n,d+l)} A^{(l)}_{\gamma_1,...,\gamma_n} x_1^{\gamma_1}...x_n^{\gamma_n},$$

$$A^{(l)}_{(\gamma_1,...,\gamma_n)} = \frac{l!(l+d)^d}{\gamma_1!...\gamma_n!} p_{(l+d)^{-1}}(\frac{\gamma_1}{l+d}, ..., \frac{\gamma_n}{l+d})$$

Proposition 4 *Let $(\gamma_1, ..., \gamma_n) \in \tau(n, l+d)$. Then*

$$p_{\frac{1}{l+d}}(\frac{\gamma_1}{l+d}, ..., \frac{\gamma_n}{l+d}) \ge p_{min} - L(p)(1 - \omega_l(d)).$$

For a proof see e.g. [5].

Proof of theorem 2 Given $\epsilon > 0$, consider the polynomial

$$g_e(x) = (x_1 + x_2 + ... + x_n)^l p(x) - (p_{min} + \epsilon)(x_1 + x_2 + ... + x_n)^{d+l}$$

It is clear that $\min\{g_e(x) : x \in T_{n-1}\} = -\epsilon < 0$. By Proposition 3

$$g_e(x) = \sum_{(\gamma_1,...,\gamma_n)\in\tau(n,d+l)} (A^{(l)}_{\gamma_1,...,\gamma_n} - (p_{min} + \epsilon)\frac{(d+l)!}{\gamma_1!...\gamma_n!}) x_1^{\gamma_1}...x_n^{\gamma_n}.$$

Hence,

$$\min\{(A^{(l)}_{\gamma_1,...,\gamma_n} - (p_{min} + \epsilon)\frac{(d+l)!}{\gamma_1!...\gamma_n!}) : (\gamma_1, ..., \gamma_n) \in \tau(n, d+l)\} < 0.$$

Otherwise g_e is nonnegative on T_{n-1}. But by Proposition 3

$$A^{(l)}_{\gamma_1,...,\gamma_n} - (p_{min} + \epsilon)\frac{(d+l)!}{\gamma_1!...\gamma_n!} = \frac{(l+d)^d l!}{\gamma_1!...\gamma_n!}[p_t(y_1, ..., y_n) - (p_{min} + \epsilon)\frac{(l+1)...(d+l)}{(l+d)^d}]$$

$$= \frac{(l+d)^d l!}{\gamma_1!...\gamma_n!}[p_t(y_1, ..., y_n) - \omega_l(d)(p_{min} + \epsilon)].$$

Here $t = (l+d)^{-1}, y_i = \frac{\gamma_i}{l+d}$. Thus, $\delta_l(p) < (p_{min} + \epsilon)\omega_l(d)$. Since it is true for any $\epsilon > 0$, we conclude that $\delta_l(p) \leq p_{min}\omega_l(d)$. On the other hand, by Proposition 4

$$p_t(\frac{\gamma_1}{l+d}, ..., \frac{\gamma_n}{l+d}) \geq p_{min} - L(p)(1 - \omega_l(d))$$

for any $(\gamma_1, ..., \gamma_n) \in \tau(n, l+d)$. Hence,

$$p_{min}\omega_l(d) \geq \delta_l(p) \geq p_{min} - L(p)(1 - \omega_l(d)). \tag{25}$$

From (25):

$$0 \leq p_{min} - \frac{\delta_l(p)}{\omega_l(d)} \leq p_{min} - \frac{p_{min}}{\omega_l(d)} + L(p)(\frac{1}{\omega_l(d)} - 1)$$
$$= (L(p) - p_{min})(\frac{1}{\omega_l(d)} - 1).\diamond$$

Remark 3 *Let $f \in H_m(R^n)$ be defined in (23). Introduce*

$$L_{max}(f) = \max\{\frac{b(i_1, ..., i_n)}{c_m(i_1, ..., i_n)} : (i_1, ..., i_n) \in \tau(n, m)\},$$

$$L_{min}(f) = \min\{\frac{b(i_1, ..., i_n)}{c_m(i_1, ..., i_n)} : (i_1, ..., i_n) \in \tau(n, m)\},$$

One can easily see that

$$0 \leq L(p) - p_{min} \leq \max\{L_{max}(p) - L_{min}(p), -2L_{min}(p)\}.$$

This, along with Theorem 2, gives an estimate of the proximity of p_{min} and $\frac{\delta_l}{\omega_l(d)}$.

Remark 4 *Let $d = 2$. Then*

$$\omega_l(d) = 1 - \frac{1}{l+2} = \frac{l+1}{l+2}.$$

Hence, (23) takes the form:

$$0 \leq p_{min} - \frac{\delta_l(p)}{\omega_l(2)} \leq \frac{L(p) - p_{min}}{l+1}.$$

This is very similar to the result in [7]. The difference is that instead of $L(p)$ they have p_{max}.

Remark 5 *Using elementary inequality*

$$\prod_{j=1}^{n}(1 - w_j) \geq 1 - \sum_{j=1}^{m} w_j,$$

provided $0 \leq w_j \leq 1$, we see that

$$w_l \geq 1 - \frac{d(d-1)}{2(l+d)}.$$

Thus, if $d(d-1) < 2(l+d)$, we have:

$$0 \leq p_{min} - \frac{\delta_l(p)}{\omega_l(d)} \leq (L(p) - p_{min})\frac{d(d-1)}{2(l+d) - d(d-1)}$$

.

Remark 6 *Let*

$$p = \sum_{i=0}^{d} p_i$$

be an arbitrary polynomial of degree d ($degp_i = i$). *Consider*

$$q(x) = \sum_{i=0}^{d} p_i(x)(x_1 + x_2 + \ldots x_n)^{d-i}.$$

It is clear that

$$q(x) = p(x), \forall x \in T_{n-1}$$

and $q \in H_d(R^n)$. *Thus, the problem of the minimization of an arbitrary polynomial on* T_{n-1} *can be reduced to the homogeneous case considered above. Since any continuous function on* T_{n-1} *can be, in principle, uniformly approximated by polynomials, it is possible to use the technique developed above for the global optimization of a continuous function on the simplex.*

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