DUALITY AND A FARKAS LEMMA FOR INTEGER PROGRAMS

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Abstract. We consider the integer program $\max\{c'x \mid Ax = b, x \in \mathbb{N}^n\}$. A formal parallel between linear programming and continuous integration on one side, and discrete summation on the other side, shows that a natural duality for integer programs can be derived from the $\mathbb{Z}$-transform and Brion and Vergne' s counting formula. Along the same lines, we also provide a discrete Farkas lemma and show that the existence of a nonnegative integral solution $x \in \mathbb{N}^n$ to $Ax = b$ can be tested via a linear program.

1. Introduction

In this paper we are interested in a comparison between linear and integer programming, and particularly in a *duality* perspective. So far, and to the best of our knowledge, the duality results available for integer programs are obtained via the use of *subadditive* functions as in e.g. Wolsey [21], and the smaller class of *Chvátal* and *Gomory* functions as in e.g. Blair and Jeroslow [6] (see also Schrijver [19, pp. 346-353]). For more details the interested reader is referred to [1, 6, 19, 21] and the many references therein. However, as subadditive, Chvátal and Gomory functions are only defined implicitly from their properties, the resulting dual problems defined in [6] or [21], are conceptual in nature and Gomory functions are rather used to generate valid inequalities for the primal problem.

We claim that another natural *duality* for integer programs can be derived from the $\mathbb{Z}$-transform (or generating function) associated with the *counting* version (defined below) of the integer program. Results for counting problems, notably by Barvinok [4], Barvinok and Pommersheim [5], Khovanskii and Pukhlikov [12], and in particular, Brion and Vergne' s counting formula [7], will prove specially useful.

For this purpose, we will consider the four related problems $\mathbb{P}, \mathbb{P}_d, \mathbb{I}$ and $\mathbb{I}_d$ displayed in the diagram below, in which the integer program $\mathbb{P}_d$ appears

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in the upper right corner.

$$
\begin{array}{ccc}
\textbf{Continuous Optimization} & & \textbf{Discrete Optimization} \\
- & | & - \\
\begin{array}{rl} \mathbb{P}: & f(b,c) := \max\, c'x \\ & \text{s.t.} \left[\begin{array}{rcl} Ax & = & b \\ x & \in & \mathbb{R}^n_+ \end{array}\right. \end{array} & \longleftrightarrow & \begin{array}{rl} \mathbb{P}_d: & f_d(b,c) := \max\, c'x \\ & \text{s.t.} \left[\begin{array}{rcl} Ax & = & b \\ x & \in & \mathbb{N}^n \end{array}\right. \end{array} \\
\updownarrow & - & \updownarrow \\
\textbf{Integration} & & \textbf{Summation} \\
- & | & - \\
\begin{array}{rl} \mathbb{I}: & \widehat{f}(b,c) := \displaystyle\int_{\Omega(b)} \mathrm{e}^{c'x} ds \\ & \Omega(b) := \left[\begin{array}{rcl} Ax & = & b \\ x & \in & \mathbb{R}^n_+ \end{array}\right. \end{array} & \longleftrightarrow & \begin{array}{rl} \mathbb{I}_d: & \widehat{f}_d(b,c) := \displaystyle\sum_{x\in\Omega(b)} \mathrm{e}^{c'x} \\ & \Omega(b) := \left[\begin{array}{rcl} Ax & = & b \\ x & \in & \mathbb{N}^n. \end{array}\right. \end{array}
\end{array}
$$

Problem $\mathbb{I}$ (in which ds denotes the Lebesgue measure on the affine variety $\{x \in \mathbb{R}^n \,|\, Ax = b\}$ that contains the convex polyhedron $\Omega(b)$) is the *integration* version of the linear program $\mathbb{P}$, whereas Problem $\mathbb{I}_d$ is the *counting* version of the (discrete) integer program $\mathbb{P}_d$.

Why do these four problems should help in analyzing $\mathbb{P}_d$? Because firstly, $\mathbb{P}$ and $\mathbb{I}$, as well as $\mathbb{P}_d$ and $\mathbb{I}_d$, are simply related, and in the same manner. Next, as we will see, the nice and complete duality results available for $\mathbb{P}, \mathbb{I}$ and $\mathbb{I}_d$, extend in a natural way to $\mathbb{P}_d$.

1.1. **Preliminaries.** In fact, $\mathbb{I}$ and $\mathbb{I}_d$ are the respective formal analogues in the algebra $(+,\times)$, of $\mathbb{P}$ and $\mathbb{P}_d$ in the algebra $(\oplus,\times)$, where in the latter, the addition $a \oplus b$ stands for $\max(a,b)$; indeed, the "max" in $\mathbb{P}$ and $\mathbb{P}_d$ can be seen as an *idempotent* integral (or, *Maslov integral*) in this algebra (see e.g. Litvinov et al. [17]). For a nice parallel between results in probability ($(+,\times)$ algebra) and optimization ($(\max,+)$ algebra), the reader is referred to Bacelli et al. [3, §9].

Moreover, $\mathbb{P}$ and $\mathbb{I}$, as well as $\mathbb{P}_d$ and $\mathbb{I}_d$, are simply related via

$$
\mathrm{e}^{f(b,c)} = \lim_{r\to\infty} \widehat{f}(b,rc)^{1/r}; \quad \mathrm{e}^{f_d(b,c)} = \lim_{r\to\infty} \widehat{f}_d(b,rc)^{1/r}. \tag{1.1}
$$

Equivalently, by continuity of the logarithm

$$
f(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}(b,rc); \quad f_d(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \widehat{f}_d(b,rc), \tag{1.2}
$$

a relationship that will be useful later.

Next, concerning *duality*, the standard *Legendre-Fenchel* transform which yields the usual dual LP of $\mathbb{P}$,

$$
\mathbb{P}^* \to \min_{\lambda\in\mathbb{R}^m} \{b'\lambda \,|\, A'\lambda \geq c\}, \tag{1.3}
$$

has a natural analogue for integration, the *Laplace transform*, and thus, the *inverse Laplace transform* problem (that we call $\mathbb{I}^*$) is the formal analogue of $\mathbb{P}^*$ and provides a nice duality for integration (although not usually presented in these terms). Finally, the $\mathbb{Z}$-*transform* is the obvious analogue for *summation* of the Laplace transform for integration. We will see that in the light of recent results in *counting problems*, it permits to establish a nice duality for $\mathbb{I}_d$ of the same vein as the duality for (continuous) integration and by (1.2), it also provides a powerful tool to analyze the integer program $\mathbb{P}_d$.

1.2. **Summary of content.** (a) We first review the duality principles that are available for $\mathbb{P}$, $\mathbb{I}$ and $\mathbb{I}_d$ and underline the parallels and connections between them. In particular, a fundamental difference between the continuous and discrete cases is that, in the former, the data appear as *coefficients* of the dual variables whereas in the latter, the *same* data appear as *exponents* of the dual variables. Consequently, the (discrete) $\mathbb{Z}$-transform has many more *poles* than the Laplace transform. While the Laplace transform has only *real* poles, the $\mathbb{Z}$-transform has additional *complex* poles associated with each real pole, which induces some *periodic* behavior, a well known phenomenon in Number theory where the $\mathbb{Z}$-transform (or *generating function*) is a standard tool (see e.g. Iosevich [11], Mitrinović et al [18]). So, if the procedure of inverting the Laplace transform or the $\mathbb{Z}$-transform (i.e. solving the dual problems $\mathbb{I}^*$ and $\mathbb{I}^*_d$) is basically of the same nature, a complex integral, it is significantly more complicated in the discrete case, due to the presence of these additional complex poles.

(b) Then we use results from (a) to analyze the discrete optimization problem $\mathbb{P}_d$. Central in the analysis is Brion and Vergne' s inverse formula [7] for counting problems. In particular, we provide a closed form expression of the optimal value $f_d(b, c)$ which highlights the special role played by the so-called *reduced-costs* of the linear program $\mathbb{P}$ and the *complex poles* of the $\mathbb{Z}$-transform associated with each basis of the linear program $\mathbb{P}$. We also show that each basis B of the linear program $\mathbb{P}$ provides exactly $\det(B)$ complex *dual* vectors in $\mathbb{C}^m$, the complex (periodic) analogues for $\mathbb{P}_d$ of the unique dual vector in $\mathbb{R}^m$ for $\mathbb{P}$, associated with the basis B. As in linear programming (but in a more complicated way), the optimal value $f_d(b, c)$ of $\mathbb{P}_d$ can be found by inspection of (certain sums of) reduced costs associated with each vertex of $\Omega(b)$.

(c) We also provide a *discrete* Farkas Lemma for the existence of nonnegative integral solutions $x \in \mathbb{N}^n$ to $Ax = b$. Its form also confirms the special role of the $\mathbb{Z}$-transform described earlier. Moreover, it permits to check the existence of a nonnegative integral solution by solving a related linear program.

2. Duality for the continuous problems $\mathbb{P}$ and $\mathbb{I}$

With $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, let $\Omega(b) \subset \mathbb{R}^n$ be the convex polyhedron

$$\Omega(b) := \{x \in \mathbb{R}^n \mid Ax = b; \quad x \geq 0\}, \tag{2.1}$$

and consider the standard linear program (LP)

$$\mathbb{P}: \quad f(b, c) := \max\{c'x \mid Ax = b; \quad x \geq 0\} \tag{2.2}$$

with $c \in \mathbb{R}^n$, and its associated *integration* version

$$\mathbb{I}: \quad \widehat{f}(b,c) := \int_{\Omega(b)} \mathrm{e}^{c'x}\, ds \tag{2.3}$$

where ds is the Lebesgue measure on the affine variety $\{x \in \mathbb{R}^n \mid Ax = b\}$ that contains the convex polyhedron $\Omega(b)$.

For a vector c and a matrix A we denote by c' and A' their respective transpose. We also use both notations $c'x$ and $\langle c, x\rangle$ for the usual scalar product of two vector c and x. We assume that both $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ have rational entries.

2.1. **Duality for $\mathbb{P}$.** It is well-known that the standard duality for (2.2) is obtained from the *Legendre-Fenchel* transform $F(.,c) : \mathbb{R}^m \to \mathbb{R}$ of the value function $f(b,c)$ w.r.t. b, i.e., here (as $y \mapsto f(y,c)$ is concave)

$$\lambda \mapsto F(\lambda, c) := \inf_{y\in\mathbb{R}^m} \langle \lambda, y\rangle - f(y,c), \tag{2.4}$$

which yields the usual dual LP problem

$$\mathbb{P}^* \to \quad \inf_{\lambda\in\mathbb{R}^m} \langle \lambda, b\rangle - F(\lambda,c) = \min_{\lambda\in\mathbb{R}^m} \{b'\lambda \mid A'\lambda \geq c\}. \tag{2.5}$$

2.2. **Duality for integration.** Similarly, the analogue for integration of the Fenchel transform is the two-sided *Laplace* transform $\widehat{F}(.,c) : \mathbb{C}^m \to \mathbb{C}$ of $\widehat{f}(b,c)$, given by

$$\lambda \mapsto \widehat{F}(\lambda, c) := \int_{\mathbb{R}^m} \mathrm{e}^{-\langle\lambda,y\rangle} \widehat{f}(y,c)\, dy. \tag{2.6}$$

It turns out that developing (2.6) yields

$$\widehat{F}(\lambda,c) = \prod_{k=1}^{n} \frac{1}{(A'\lambda - c)_k} \quad [\text{whenever } \Re(A'\lambda - c) > 0]. \tag{2.7}$$

(See e.g. [7, p. 798] or [13]). Thus, $\widehat{F}(\lambda,c)$ is well-defined provided

$$\Re(A'\lambda - c) > 0, \tag{2.8}$$

and $\widehat{f}(b,c)$ can be computed by solving the *inverse* Laplace transform problem, that we call the (integration) *dual* problem $\mathbb{I}^*$ of (2.9), that is,

$$\begin{aligned} \mathbb{I}^* \to \widehat{f}(b,c) &:= \frac{1}{(2i\pi)^m} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{\langle b,\lambda\rangle} \widehat{F}(\lambda,c)\, d\lambda \\ &= \frac{1}{(2i\pi)^m} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\mathrm{e}^{\langle b,\lambda\rangle}}{\prod_{k=1}^{n} (A'\lambda - c)_k}\, d\lambda \end{aligned} \tag{2.9}$$

where $\gamma \in \mathbb{R}^m$ is fixed and satisfies $A'\gamma - c > 0$. Incidently, observe that the domain of definition (2.8) of $\widehat{F}(.,c)$ is precisely the interior of the feasible set of the dual problem $\mathbb{P}^*$ in (2.5). We will comment more on this and the link with the logarithmic barrier function for linear programming (see §2.5 below).

We may indeed call $\mathbb{I}^*$ a dual problem of $\mathbb{I}$ as it is defined on the space $\mathbb{C}^m$ of variables $\{\lambda_k\}$ associated with the nontrivial constraints $Ax = b$; notice that we also retrieve the standard "ingredients" of the dual optimization problem $\mathbb{P}^*$, namely, $b'\lambda$ and $A'\lambda - c$.

2.3. **Comparing $\mathbb{P}, \mathbb{P}^*$ and $\mathbb{I}, \mathbb{I}^*$.** One may compute directly $\widehat{f}(b, c)$ by using Cauchy residue techniques. That is, one computes the integral (2.9) by successive one-dimensional complex integrals with respect to (w.r.t.) one variable λ_k at a time (e.g. starting with $\lambda_1, \lambda_2, \ldots$) and by repeated application of Cauchy 's Residue Theorem [8]. This is possible because the integrand is a rational fraction, and after application of Cauchy 's Residue Theorem at step k w.r.t. λ_k, the ouput is still a rational fraction of the remaining variables $\lambda_{k+1}, \ldots, \lambda_m$. For more details the reader is referred to Lasserre and Zeron [13]. It is not difficult to see that the whole procedure is a summation of partial results, each of them corresponding to a (multi-pole) vector $\widehat{\lambda} \in \mathbb{R}^m$ that anhilates m terms of n products in the denominator of the integrand.

This is formalized in the nice formula of Brion and Vergne [7, Proposition 3.3 p. 820] that we describe below. For the interested reader, there are several other nice closed form formula for $\widehat{f}(b, c)$ notably by Barvinok [4], Barvinok and Pommersheim [5], and Khovanskii and Pukhlikov [12].

2.4. **The continuous Brion and Vergne' s formula.** The material in this section is taken from [7]. To explain the closed form formula of Brion and Vergne we need some notation.

Write the matrix $A \in \mathbb{R}^{m\times n}$ as $A = [A_1|\ldots|A_n]$ where $A_j \in \mathbb{R}^m$ denotes the j-th column of A for all $j = 1, \ldots, n$. With $\Delta := (A_1, \ldots, A_n)$ let $C(\Delta) \subset \mathbb{R}^m$ be the closed convex cone generated by Δ. Let $\Lambda \subseteq \mathbb{Z}^m$ be a lattice.

A subset σ of $\{1, \ldots, n\}$ is called a *basis* of Δ if the sequence $\{A_j\}_{j\in\sigma}$ is a basis of $\mathbb{R}^m$, and the set of bases of Δ is denoted by $\mathcal{B}(\Delta)$. For $\sigma \in \mathcal{B}(\Delta)$ let $C(\sigma)$ be the cone generated by $\{A_j\}_{j\in\sigma}$. With any $y \in C(\Delta)$ associate the intersection of all cones $C(\sigma)$ which contain y. It defines a subdivision of $C(\Delta)$ into polyhedral cones. The interiors of the maximal cones in this subdivision are called *chambers* in Brion and Vergne [7]. For every $y \in \gamma$, the convex polyhedron $\Omega(y)$ in (2.1) is *simple*. Next, for a chamber γ (whose closure is denoted by $\overline{\gamma}$), let $\mathcal{B}(\Delta, \gamma)$ be the set of bases σ such that γ is contained in $C(\sigma)$, and let $\mu(\sigma)$ denote the volume of the convex polytope $\{\sum_{j\in\sigma} t_j A_j \,|\, 0 \le t_j \le 1\}$ (normalized so that $\mathrm{vol}(\mathbb{R}^m/\Lambda) = 1$). Observe that for $b \in \overline{\gamma}$ and $\sigma \in \mathcal{B}(\Delta, \gamma)$ we have $b = \sum_{j\in\sigma} x_j(\sigma) A_j$ for some $x_j(\sigma) \ge 0$. Therefore, the vector $x(\sigma) \in \mathbb{R}^n_+$ with $x_j(\sigma) = 0$ whenever $j \notin \sigma$, is a *vertex* of the polytope $\Omega(b)$. In the LP terminology, the bases $\sigma \in \mathcal{B}(\Delta, \gamma)$ correspond to the *feasible bases* of the linear program $\mathbb{P}$. Denote by V the subspace $\{x \in \mathbb{R}^n \mid Ax = 0\}$. Finally, given $\sigma \in \mathcal{B}(\Delta)$, let $\pi^\sigma \in \mathbb{R}^m$ be the row vector that solves $\pi^\sigma A_j = c_j$ for all $j \in \sigma$. A vector $c \in \mathbb{R}^n$ is said to be *regular* if $c_j - \pi^\sigma A_j \neq 0$ for all $\sigma \in \mathcal{B}(\Delta)$ and all $j \notin \sigma$.

Let $c \in \mathbb{R}^n$ be regular with $-c$ in the interior of the dual cone $(\mathbb{R}^n_+ \cap V)^*$ (which is the case if $A'u > c$ for some $u \in \mathbb{R}^m$). Then, with $\Lambda = \mathbb{Z}^m$, Brion and Vergne' s formula [7, Proposition 3.3, p. 820] states that

$$(2.10) \qquad \widehat{f}(b,c) \;=\; \sum_{\sigma \in \mathcal{B}(\Delta,\gamma)} \frac{\mathrm{e}^{\langle c, x(\sigma)\rangle}}{\mu(\sigma) \prod_{k \notin \sigma} (-c_k + \pi^\sigma A_k)} \qquad \forall\, b \in \overline{\gamma}.$$

Notice that in the Linear Programming terminology, $c_k - \pi^\sigma A_k$ is nothing less than the so-called *reduced cost* of the variable x_k, with respect to the basis $\{A_j\}_{j \in \sigma}$. Equivalently, we can rewrite (2.10) as

$$(2.11) \qquad \widehat{f}(b,c) \;=\; \sum_{x(\sigma):\ \text{vertex of } \Omega(b)} \frac{\mathrm{e}^{\langle c, x(\sigma)\rangle}}{\mu(\sigma) \prod_{k \notin \sigma} (-c_k + \pi^\sigma A_k)}.$$

Thus, $\widehat{f}(b,c)$ is a weighted *summation* over the vertices of $\Omega(b)$ whereas $f(b,c)$ is a *maximization* over the vertices (or a summation with $\oplus \equiv \max$).

So, if c is replaced with rc and $x(\sigma^*)$ denotes the vertex of $\Omega(b)$ at which $c'x$ is maximized, we obtain

$$\widehat{f}(b,rc)^{1/r} \;=\; \mathrm{e}^{\langle c, x(\sigma^*)\rangle} \left[\sum_{x(\sigma):\ \text{vertex of } \Omega(b)} \frac{\mathrm{e}^{r\langle c, x(\sigma) - x(\sigma^*)\rangle}}{r^{n-m} \mu(\sigma) \prod_{k \notin \sigma} (-c_k + \pi^\sigma A_k)} \right]^{1/r},$$

from which it easily follows that

$$\lim_{r \to \infty} \ln \widehat{f}(b,rc)^{1/r} \;=\; \langle c, x(\sigma^*)\rangle \;=\; \max_{x \in \Omega(b)} \langle c, x\rangle \;=\; f(b,c),$$

as indicated in (1.2).

2.5. **The logarithmic barrier function.** It is also worth noticing that

$$\begin{aligned} \widehat{f}(b,rc) &= \frac{1}{(2i\pi)^m} \int_{\gamma_r - i\infty}^{\gamma_r + i\infty} \frac{\mathrm{e}^{\langle b,\lambda\rangle}}{\prod_{k=1}^{n} (A'\lambda - rc)_k}\, d\lambda \\ &= \frac{1}{(2i\pi)^m} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{r^{m-n} \mathrm{e}^{\langle rb,\lambda\rangle}}{\prod_{k=1}^{n} (A'\lambda - c)_k}\, d\lambda \end{aligned}$$

with $\gamma_r = r\gamma$ and we can see that (up to the constant $(m-n)\ln r$) the logarithm of the integrand is nothing less than the well-known *logarithmic barrier function*

$$\lambda \;\mapsto\; \phi_\mu(\lambda, b) \;=\; \mu^{-1} \langle b, \lambda\rangle - \sum_{j=1}^{n} \ln\,(A'\lambda - c)_j,$$

with parameter $\mu := 1/r$, of the dual problem $\mathbb{P}^*$ (see e.g. Den Hertog [9]). This should not be a surprise as a self-concordant barrier function $\phi_K(x)$ of a cone $K \subset \mathbb{R}^n$ is given by the logarithm of Laplace transform $\int_{K^*} \mathrm{e}^{-\langle x,s\rangle} ds$ of its dual cone K^* (see e.g. Güler [10], Truong and Tunçel [20]).

Thus, when $r \to \infty$, minimizing the exponential logarithmic barrier function on its domain in $\mathbb{R}^m$ yields the same result as taking its residues.

2.6. **Summary.** The parallel bewteen $\mathbb{P}, \mathbb{P}^*$ and $\mathbb{I}, \mathbb{I}^*$ is summarized below.

Fenchel-duality:	**Laplace-duality**
$f(b,c) := \max_{Ax=b;\, x\ge 0} c'x$	$\widehat{f}(b,c) := \int_{Ax=b;\, x\ge 0} \mathrm{e}^{c'x}\, ds$
$F(\lambda,c) := \inf_{y\in\mathbb{R}^m} \{\lambda' y - f(y,c)\}$	$\widehat{F}(\lambda,c) := \int_{\mathbb{R}^m} \mathrm{e}^{-\lambda' y}\, \widehat{f}(y,c)\, dy = \dfrac{1}{\prod_{k=1}^{n} (A'\lambda - c)_k}$
with : $A'\lambda - c \ge 0$	with : $\Re(A'\lambda - c) > 0$
$f(b,c) = \min_{\lambda\in\mathbb{R}^m} \{\lambda' b - F(\lambda,c)\} = \min_{\lambda\in\mathbb{R}^m} \{b'\lambda \mid A'\lambda \ge c\}$	$\widehat{f}(b,c) = \dfrac{1}{(2i\pi)^m} \int_\Gamma \mathrm{e}^{\lambda' b} \widehat{F}(\lambda,c)\, d\lambda = \dfrac{1}{(2i\pi)^m} \int_\Gamma \dfrac{\mathrm{e}^{\lambda' b}}{\prod_{k=1}^{n} (A'\lambda - c)_k} d\lambda$
Simplex algorithm$\to$ vertices of $\Omega(b)$. $\to \max c'x$ over vertices.	Cauchy' s Residue$\to$ poles of $\widehat{F}(\lambda,c)$. $\to \sum \mathrm{e}^{c'x}$ over vertices.

3. Duality for the discrete problems $\mathbb{I}_d$ and $\mathbb{P}_d$

In the respective *discrete* analogues $\mathbb{P}_d$ and $\mathbb{I}_d$ of (2.2) and (2.3) one replaces the positive cone $\mathbb{R}^n_+$ by $\mathbb{N}^n$ (or, $\mathbb{R}^n_+ \cap \mathbb{Z}^n$), that is (2.2) becomes the integer program

$$\mathbb{P}_d : \quad f_d(b,c) := \max \{c'x \mid Ax = b; \quad x \in \mathbb{N}^n\} \tag{3.1}$$

whereas (2.3) becomes a summation over $\mathbb{N}^n \cap \Omega(b)$, i.e.,

$$\mathbb{I}_d : \quad \widehat{f}_d(b,c) := \sum \{\mathrm{e}^{c'x} \mid Ax = b; \quad x \in \mathbb{N}^n\}. \tag{3.2}$$

We here assume that $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, which implies in particular that the *lattice* $\Lambda := A(\mathbb{Z}^n)$ is a sublattice of $\mathbb{Z}^m$ ($\Lambda \subseteq \mathbb{Z}^m$). Note that b in (3.1) and (3.2) is necessarily in Λ.

In this section we are concerned with what we call the "dual" problem $\mathbb{I}_d^*$ of $\mathbb{I}_d$, the discrete analogue of the dual $\mathbb{I}^*$ of $\mathbb{I}$, and its link with the discrete optimization problem $\mathbb{P}_d$.

3.1. **The $\mathbb{Z}$-transform.** The natural discrete analogue of the Laplace transform is the so-called $\mathbb{Z}$-transform. Therefore, with $\widehat{f}_d(b,c)$ we associate its (two-sided) $\mathbb{Z}$-transform $\widehat{F}_d(.,c) : \mathbb{C}^m \to \mathbb{C}$ defined by

$$z \mapsto \widehat{F}_d(z,c) := \sum_{y\in\mathbb{Z}^m} z^{-y} \widehat{f}_d(y,c), \tag{3.3}$$

where the notation z^y with $y \in \mathbb{Z}^m$ stands for $z_1^{y_1} \cdots z_m^{y_m}$. Applying this definition yields

$$\begin{aligned} \widehat{F}_d(z,c) &= \sum_{y\in\mathbb{Z}^m} z^{-y} \widehat{f}_d(y,c) \\ &= \sum_{y\in\mathbb{Z}^m} z^{-y} \left[\sum_{x\in\mathbb{N}^n;\, Ax=y} \mathrm{e}^{c'x} \right] = \sum_{x\in\mathbb{N}^n} \mathrm{e}^{c'x} \left[\sum_{y=Ax} z_1^{-y_1} \cdots z_m^{-y_m} \right] \\ &= \sum_{x\in\mathbb{N}^n} \mathrm{e}^{c'x} z_1^{-(Ax)_1} \cdots z_m^{-(Ax)_m} \\ &= \prod_{k=1}^n \frac{1}{(1-\mathrm{e}^{c_k} z_1^{-A_{1k}} z_2^{-A_{2k}} \cdots z_m^{-A_{mk}})} = \prod_{k=1}^n \frac{1}{(1-\mathrm{e}^{c_k} z^{-A_k})}, \end{aligned} \tag{3.4}$$

wich is well-defined provided

$$|z_1^{A_{1k}} \cdots z_m^{A_{mk}}| \; (= |z^{A_k}|) \; > \mathrm{e}^{c_k} \quad \forall k = 1, \dots, n. \tag{3.5}$$

Observe that the domain of definition (3.5) of $\widehat{F}_d(.,c)$ is the *exponential* version of (2.8) for $\widehat{F}(.,c)$. Indeed, taking the real part of the logarithm in (3.5) yields (2.8).

3.2. **The dual problem $\mathbb{I}_d^*$.** Therefore, the value $\widehat{f}_d(b,c)$ is obtained by solving the *inverse* $\mathbb{Z}$-transform problem $\mathbb{I}_d^*$ (that we call the *dual* of $\mathbb{I}_d$)

$$\widehat{f}_d(b,c) = \frac{1}{(2i\pi)^m} \int_{|z_1|=\gamma_1} \cdots \int_{|z_m|=\gamma_m} \widehat{F}_d(z)\, z^{b-e_m}\, dz_m \cdots dz_1, \tag{3.6}$$

where e_m is the unit vector of $\mathbb{R}^m$ and $\gamma \in \mathbb{R}^m$ is a (fixed) vector that satisfies $\gamma_1^{A_{1k}} \gamma_2^{A_{2k}} \cdots \gamma_m^{A_{mk}} > \mathrm{e}^{c_k}$ for all $k = 1, \dots, n$. We may indeed call $\mathbb{I}_d^*$ the *dual problem* of $\mathbb{I}_d$ as it is defined on the space $\mathbb{Z}^m$ of dual variables z_k associated with the nontrivial constraints $Ax = b$ of the primal problem $\mathbb{I}_d$.

And we have the parallel

Continuous Laplace-duality	**Discrete $\mathbb{Z}$-duality**
$\widehat{f}(b,c) := \int_{Ax=b;\, x\in\mathbb{R}^n_+} \mathrm{e}^{c'x}\, ds$	$\widehat{f}_d(b,c) := \sum_{Ax=b;\, x\in\mathbb{N}^n} \mathrm{e}^{c'x}$
$\widehat{F}(\lambda,c) := \int_{\mathbb{R}^m} \mathrm{e}^{-\lambda' y}\, \widehat{f}(y,c)dy$	$\widehat{F}_d(z,c) := \sum_{y\in\mathbb{Z}^m} z^{-y}\, \widehat{f}_d(y,c)$
$= \prod_{k=1}^n \frac{1}{(A'\lambda - c)_k}$	$= \prod_{k=1}^n \frac{1}{1-\mathrm{e}^{c_k} z^{-A_k}}$
with $\Re(A'\lambda - c) > 0$.	with $\vert z^{A_k}\vert > \mathrm{e}^{c_k}, \quad k=1,\ldots,n.$

3.3. **Comparing $\mathbb{I}^*$ and $\mathbb{I}^*_d$.** Observe that the dual problem $\mathbb{I}^*_d$ in (3.6) is of the same nature as $\mathbb{I}^*$ in (2.9) because both reduce to compute a complex integral whose integrand is a rational function. In particular, as $\mathbb{I}^*$, the problem $\mathbb{I}^*_d$ can be solved by Cauchy residue techniques (see e.g. [14]).

However, there is an important difference between $\mathbb{I}^*$ and $\mathbb{I}^*_d$. While the data $\{A_{jk}\}$ appear in $\mathbb{I}^*$ as *coefficients* of the dual variables λ_k in $\widehat{F}(\lambda,c)$, the now appear as *exponents* of the dual variables z_k in $\widehat{F}_d(z,c)$. And, an immediate consequence of this fact is that the rational function $\widehat{F}_d(.,c)$ has many more poles than $\widehat{F}(.,c)$ (by considering one variable at a time), and in particular, many of them are *complex*, whereas $\widehat{F}(.,c)$ has only *real* poles. As a result, the integration of $\widehat{F}_d(z,c)$ is more complicated than that of $\widehat{F}(\lambda,c)$, which is reflected in the *discrete* (or, *periodic*) Brion and Vergne' s formula described below. However, we will see that the poles of $\widehat{F}_d(z,c)$ are simply related to those of $\widehat{F}(\lambda,c)$.

3.4. **The "discrete" Brion and Vergne' s formula.** Brion and Vergne [7] consider the *generating function* $H : \mathbb{C}^m \to \mathbb{C}$ defined by :

$$\lambda \mapsto H(\lambda,c) := \sum_{y\in\mathbb{Z}^m} \widehat{f}_d(y,c)\mathrm{e}^{-\langle\lambda,y\rangle},$$

which, after the change of variable $z_i = \mathrm{e}^{\lambda_i}$ for all $i=1,\ldots,m$, reduces to $\widehat{F}_d(z,c)$ in (3.6).

They obtain the following nice formula (3.7) below. Namely, and with same notation used in §2.4, let $c \in \mathbb{R}^n$ be regular with $-c$ in the interior of $(\mathbb{R}^n_+ \cap V)^*$, and let γ be a chamber. Then for all $b \in \Lambda\cap\overline{\gamma}$ (recall $\Lambda = A(\mathbb{Z}^n)$),

$$\widehat{f}_d(b,c) = \sum_{\sigma\in\mathcal{B}(\Delta,\gamma)} \frac{\mathrm{e}^{c'x(\sigma)}}{\mu(\sigma)} U_\sigma(b,c) \tag{3.7}$$

for some coefficients $U_\sigma(b,c)\in\mathbb{R}$ whose detailed expression, can be found in [7, Theorem 3.4, p. 821]. In particular, due to the occurrence of *complex poles* in $\widehat{F}(z,c)$, the term $U_\sigma(b,c)$ in (3.7) is the *periodic* analogue of $(\prod_{k\notin\sigma}(c_k-\pi_\sigma A_k))^{-1}$ in (2.11).

Again, as for $\widehat{f}(b,c)$, (3.7) can be re-written

$$\widehat{f}_d(b,c) \;=\; \sum_{x(\sigma):\ \text{vertex of } \Omega(b)} \frac{\mathrm{e}^{c'x(\sigma)}}{\mu(\sigma)} U_\sigma(b,c), \tag{3.8}$$

to compare with (2.11). To be more precise, by inspection of Brion and Vergne 's formula in [7, p. 821] in our present context, one may see that

$$U_\sigma(b,c) \;=\; \sum_{g\in G(\sigma)} \frac{\mathrm{e}^{2i\pi b}(g)}{V_\sigma(g,c)}, \tag{3.9}$$

where $G(\sigma) := (\oplus_{j\in\sigma}\mathbb{Z}A_j)^*/\Lambda^*$ (where * denotes the dual lattice); it is a finite abelian group of order $\mu(\sigma)$, and with (finitely many) characters $\mathrm{e}^{2i\pi b}$ for all $b\in\Lambda$; in particular, writing $A_k=\sum_{j\in\sigma}u_{jk}A_j$ for all $k\notin\sigma$,

$$\mathrm{e}^{2i\pi A_k}(g) \;=\; \mathrm{e}^{2i\pi\sum_{j\in\sigma}u_{jk}g_j} \quad k\notin\sigma.$$

Moreover,

$$V_\sigma(g,c) \;=\; \prod_{k\notin\sigma}\left(1-\mathrm{e}^{-2i\pi A_k}(g)\mathrm{e}^{c_k-\pi^\sigma A_k}\right), \tag{3.10}$$

with A_k,π^σ as in (2.10) (and π^σ is *rational*). Again note the importance of the *reduced-cost* $c_k-\pi^\sigma A_k$ in the expression of $\widehat{F}_d(z,c)$.

3.5. **The discrete optimization problem $\mathbb{P}_d$.** We now are in position to see how $\mathbb{I}_d^*$ provides some nice information about the optimal value $f_d(b,c)$ of the discrete optimization problem $\mathbb{P}_d$.

Theorem 3.1. *Let $A\in\mathbb{Z}^{m\times n}, b\in\mathbb{Z}^m$ and let $c\in\mathbb{Z}^n$ be regular with $-c$ in the interior of $(\mathbb{R}^n_+\cap V)^*$. Let $b\in\overline{\gamma}\cap A(\mathbb{Z}^n)$ and let $q\in\mathbb{N}$ be the least common multiple (l.c.m.) of $\{\mu(\sigma)\}_{\sigma\in\mathcal{B}(\Delta,\gamma)}$.*

If $Ax=b$ has no solution $x\in\mathbb{N}^n$ then $f_d(b,c)=-\infty$, else assume that

$$\max_{x(\sigma):\ \text{vertex of } \Omega(b)}\left[c'x(\sigma)+\lim_{r\to\infty}\frac{1}{r}\ln U_\sigma(b,rc)\right],$$

is attained at a unique vertex $x(\sigma)$ of $\Omega(b)$. Then

$$\begin{aligned} f_d(b,c) &= \max_{x(\sigma):\ \text{vertex of } \Omega(b)}\left[c'x(\sigma)+\lim_{r\to\infty}\frac{1}{r}\ln U_\sigma(b,rc)\right] \\ &= \max_{x(\sigma):\ \text{vertex of } \Omega(b)}\left[c'x(\sigma)+\frac{1}{q}(\deg(P_{\sigma b})-\deg(Q_{\sigma b}))\right] \end{aligned} \tag{3.11}$$

for some real-valued univariate polynomials $P_{\sigma b},Q_{\sigma b}$.

Moreover, the term $\lim_{r\to\infty} \ln U_\sigma(b, rc)/r$ *or* $(\deg(P_{\sigma b}) - \deg(Q_{\sigma b}))/q$ *in (3.11), is a sum of certain reduced costs* $c_k - \pi^\sigma A_k$ *(with* $k \notin \sigma$*).*

For a proof see §6.1.

Remark 3.2. Of course, (3.11) is not easy to obtain but it shows that the optimal value $f_d(b, c)$ of $\mathbb{P}_d$ is strongly related to the various complex poles of $\widehat{F}_d(z, c)$. It is also interesting to note the crucial role played by the reduced costs $(c_k - \pi^\sigma A_k)$ in linear programming. Indeed, from the proof of Theorem 3.1 the optimal value $f_d(b, c)$ is the value of $c'x$ at some vertex $x(\sigma)$ plus a sum of certain reduced costs (see (6.5) and the form of the coefficients $\alpha_j(\sigma, c)$). Thus, as for the LP problem $\mathbb{P}$, the optimal value $f_d(b, c)$ of $\mathbb{P}_d$ can be found by inspection of (certain sums of) reduced costs associated with each vertex of $\Omega(b)$.

We next derive an asymptotic result that relates the respective optimal values $f_d(b, c)$ and $f(b, c)$ of $\mathbb{P}_d$ and $\mathbb{P}$.

Corollary 3.3. *Let* $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ *and let* $c \in \mathbb{R}^n$ *be regular with* $-c$ *in the interior of* $(\mathbb{R}^n_+ \cap V)^*$*. Let* $b \in \gamma \cap \Lambda$ *and let* $x^* \in \Omega(b)$ *be an optimal vertex of* $\mathbb{P}$*, that is,* $f(b, c) = c'x^* = c'x(\sigma^*)$ *for* $\sigma^* \in \mathcal{B}(\Delta, \gamma)$*, the unique optimal basis of* $\mathbb{P}$*. Then for* $t \in \mathbb{N}$ *sufficiently large,*

$$f_d(tb, c) - f(tb, c) = \lim_{r\to\infty} \left[\frac{1}{r} \ln U_{\sigma^*}(tb, rc)\right]. \tag{3.12}$$

In particular, for $t \in \mathbb{N}$ *sufficently large, the function* $t \mapsto f(tb, c) - f_d(tb, c)$ *is periodic (constant) with period* $\mu(\sigma^*)$*.*

For a proof see §6.2. Thus, when $b \in \gamma \cap \Lambda$ is sufficiently large, say $b = tb_0$ with $b_0 \in \Lambda$ and $t \in \mathbb{N}$, the "max" in (3.11) is attained at the unique optimal basis σ^* of the LP (2.2) (see details in §6.2).

From Remark 3.2 it also follows that for sufficiently large $t \in \mathbb{N}$, the optimal value $f_d(tb, c)$ is equal to $f(tb, c)$ plus a certain sum of *reduced costs* $c_k - \pi^{\sigma^*} A_k$ (with $k \notin \sigma^*$) with respect to the optimal basis σ^*.

3.6. A dual comparison of $\mathbb{P}$ and $\mathbb{P}_d$.

We now provide an alternative formulation of Brion and Vergne 's discrete formula (3.8), which explicitly relates dual variables of $\mathbb{P}$ and $\mathbb{P}_d$. Recall that a *feasible basis* of the linear program $\mathbb{P}$ is a basis $\sigma \in \mathcal{B}(\Delta)$ for which $A_\sigma^{-1} b \geq 0$. Thus, let $\sigma \in \mathcal{B}(\Delta)$ be a feasible basis of the linear program $\mathbb{P}$, and consider the system of m equations in $\mathbb{C}^m$:

$$z_1^{A_{1j}} \cdots z_m^{A_{mj}} = e^{c_j} \qquad j \in \sigma. \tag{3.13}$$

Recall that A_σ is the nonsingular matrix $[A_{j_1}| \cdots |A_{j_m}]$, with $j_k \in \sigma$ for all $k = 1, \ldots, m$. The above system (3.13) has $\rho(\sigma)$ $(= \det(A_\sigma))$ solutions $\{z(k)\}_{k=1}^{\rho(\sigma)}$, written

$$z(k) = e^\lambda e^{2i\pi\theta(k)} \qquad k = 1, \ldots, \rho(\sigma) \tag{3.14}$$

for $\rho(\sigma)$ vectors $\{\theta(k)\}$ in $\mathbb{C}^m$.

Indeed, writing $z = \mathrm{e}^{\lambda}\mathrm{e}^{2i\pi\theta}$ (i.e., the vector $\{e^{\lambda_j}\mathrm{e}^{2i\pi\theta_j}\}_{j=1}^m$ in $\mathbb{C}^m$), and passing to the logarithm in (3.13), yields

$$A'_\sigma\lambda + 2i\pi\, A'_\sigma\theta \,=\, c_\sigma \tag{3.15}$$

where $c_\sigma \in \mathbb{R}^m$ is the vector $\{c_j\}_{j\in\sigma}$. Thus, $\lambda \in \mathbb{R}^m$ is the unique solution of $A'_\sigma\lambda = c_\sigma$ and θ satisfies

$$A'_\sigma\theta \,\in \mathbb{Z}^m. \tag{3.16}$$

Equivalently, θ belongs to $(\oplus_{j\in\sigma}A_j\mathbb{Z})^*$, the dual lattice of $\oplus_{j\in\sigma}A_j\mathbb{Z}$.

Thus, there is a one-to-one correspondence between the $\rho(\sigma)$ solutions $\{\theta(k)\}$ and the finite group $G'(\sigma) = (\oplus_{j\in\sigma}A_j\mathbb{Z})^*/\mathbb{Z}^m$, and $G(\sigma)$ is a subgroup of $G'(\sigma)$. Thus, with $G(\sigma) = \{g_1, \ldots, g_s\}$ and $s := \mu(\sigma)$, we can write $(A'_\sigma)^{-1}g_k = \theta_{g_k} = \theta(k)$, so that for every character $\mathrm{e}^{2i\pi y}$ of $G(\sigma)$, $y \in \Lambda$, we have

$$\mathrm{e}^{2i\pi y}(g) \,=\, \mathrm{e}^{2i\pi y'\theta_g} \qquad y \in \Lambda,\; g \in G(\sigma). \tag{3.17}$$

and

$$\mathrm{e}^{2i\pi A_j}(g) \,=\, \mathrm{e}^{2i\pi A'_j\theta_g} \,=\, 1 \qquad j \in \sigma. \tag{3.18}$$

So, for every $\sigma \in \mathcal{B}(\Delta)$, denote by $\{z_g\}_{g\in G(\sigma)}$ these $\mu(\sigma)$ solutions of (3.14), that is,

$$z_g \,=\, \mathrm{e}^{\lambda}\mathrm{e}^{2i\pi\theta_g} \,\in\, \mathbb{C}^m, \qquad g \in G(\sigma), \tag{3.19}$$

with $\lambda = (A'_\sigma)^{-1}c_\sigma$, and where $\mathrm{e}^{\lambda} \in \mathbb{R}^m$ is the vector $\{\mathrm{e}^{\lambda_i}\}_{i=1}^m$.

So, in the linear program $\mathbb{P}$ we have a dual vector $\lambda \in \mathbb{R}^m$ associated with each basis σ. In the integer program $\mathbb{P}$, with each (same) basis σ are now associated $\mu(\sigma)$ "dual" (complex) vectors $\lambda + 2i\pi\theta_g$, $g \in G(\sigma)$. Hence, with a basis σ in linear programming, the "dual variables" in integer programming are obtained from (a), the corrresponding dual variables $\lambda \in \mathbb{R}^m$ in linear programming, and (b), a periodic correction term $2i\pi\theta_g \in \mathbb{C}^m$, $g \in G(\sigma)$.

We next introduce what we call the *vertex residue function*.

Definition 3.4. Let $b \in \Lambda$ and let $c \in \mathbb{R}^n$ be regular. Let $\sigma \in \mathcal{B}(\Delta)$ be a feasible basis of the linear program $\mathbb{P}$ and for every $r \in \mathbb{N}$, let $\{z_{gr}\}_{g\in G(\sigma)}$ be as in (3.19), with rc in lieu of c, that is,

$$z_{gr} \,=\, \mathrm{e}^{r\lambda}\mathrm{e}^{2i\pi\theta_g} \in \mathbb{C}^m; \qquad g \in G(\sigma), \quad (\text{with } \lambda = (A'_\sigma)^{-1}c_\sigma).$$

The vertex residue function associated with a basis σ of the linear program $\mathbb{P}$, is the function $R_\sigma(z_g, .) : \mathbb{N} \to \mathbb{R}$ defined by :

$$r \,\mapsto\, R_\sigma(z_g, r) \,:=\, \frac{1}{\mu(\sigma)} \sum_{g\in G(\sigma)} \frac{z_{gr}^b}{\prod_{j\notin\sigma}(1 - z_{gr}^{-A_k}\mathrm{e}^{rc_k})}, \tag{3.20}$$

which is well-defined because when c is regular, $|z_{gr}|^{A_k} \neq \mathrm{e}^{rc_k}$ for all $k \notin \sigma$.

The name *vertex residue* is now clear because in the integration (3.6), $R_\sigma(z_g, r)$ is to be interpreted as a generalized Cauchy residue, with respect to the $\mu(\sigma)$ "poles" $\{z_{gr}\}$ of the generating function $\widehat{F}_d(z, rc)$.

Recall that from Corollary 3.3 , when $b \in \gamma \cap \Lambda$ is sufficiently large, say $b = tb_0$ with $b_0 \in \Lambda$ and some large $t \in \mathbb{N}$, the "max" in (3.11) is attained at the unique optimal basis σ^* of the linear program $\mathbb{P}$.

Proposition 3.5. *Let c be regular with $-c \in (\mathbb{R}^n_+ \cap V)^*$, and let $b \in \gamma \cap \Lambda$ be sufficiently large so that the max in (3.11) is attained at the unique optimal basis σ^* of the linear program $\mathbb{P}$. Let $\{z_g\}_{g\in G(\sigma^*)}$ be as in (3.19) with $\sigma = \sigma^*$.*

Then the optimal value of $\mathbb{P}_d$ satisfies

$$\begin{aligned} f_d(b, c) &= \lim_{r\to\infty} \frac{1}{r} \ln \left[\frac{1}{\mu(\sigma^*)} \sum_{g\in G(\sigma^*)} \frac{z_{gr}^b}{\prod_{k\notin\sigma^*}(1 - z_{gr}^{-A_k} e^{rc_k})} \right] \\ &= \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(z_g, r), \end{aligned} \tag{3.21}$$

and the optimal value of $\mathbb{P}$ satisfies

$$\begin{aligned} f(b, c) &= \lim_{r\to\infty} \frac{1}{r} \ln \left[\frac{1}{\mu(\sigma^*)} \sum_{g\in G(\sigma^*)} \frac{|z_{gr}|^b}{\prod_{k\notin\sigma^*}(1 - |z_{gr}|^{-A_k} e^{rc_k})} \right] \\ &= \lim_{r\to\infty} \frac{1}{r} \ln\, R_{\sigma^*}(|z_g|, r). \end{aligned} \tag{3.22}$$

For a proof see §6.3.

Proposition 3.5 shows that there is indeed a strong relationship between the integer program $\mathbb{P}_d$ and its continuous analogue, the linear program $\mathbb{P}$. Both optimal values obey exactly the same formula (3.21), but for the continuous version, the complex vector $z_g \in \mathbb{C}^m$ is replaced with the vector $|z_g| = e^{\lambda^*} \in \mathbb{R}^m$ of its component moduli, where $\lambda^* \in \mathbb{R}^m$ is the optimal solution of the LP dual of $\mathbb{P}$. In summary, when $c \in \mathbb{R}^n$ is regular and $b \in \gamma \cap \Lambda$ is sufficiently large, we have the correspondence

Linear program $\mathbb{P}$	**Integer program** $\mathbb{P}_d$
unique optimal basis σ^*	unique optimal basis σ^*
1 optimal dual vector $\lambda^* \in \mathbb{R}^m$	$\mu(\sigma^*)$ dual vectors $z_g \in \mathbb{C}^m$, $g \in G(\sigma^*)$
	$\ln z_g = \lambda^* + 2i\pi\theta_g$
$f(b, c) = \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(\lvert z_g\rvert, r)$	$f_d(b, c) = \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(z_g, r)$

4. A discrete Farkas lemma

In this section we are interested in a discrete analogue of the continuous Farkas lemma. That is, with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$, consider the issue of existence of a *nonnegative integral* solution $x \in \mathbb{N}^n$ to the system of linear equations $Ax = b$.

The (continuous) Farkas Lemma, which states that given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$,

$$\{x \in \mathbb{R}^n \,|\, Ax = b,\ x \geq 0\} \neq \emptyset \;\Leftrightarrow\; \left[A'\lambda \geq 0\right] \Rightarrow b'\lambda \geq 0, \tag{4.1}$$

has no discrete analogue in an *explicit* form. For instance, the *Gomory functions* used in Blair and Jeroslow [6] (see also Schrijver [19, Corollary 23.4b]) are implicitly and iteratively defined, and not directly in terms of the data A, b. On the other hand, for various characterizations of feasibility of linear diophantine equations $Ax = b$, $x \in \mathbb{Z}^n$, the interested reader is referred to Schrijver [19, §4].

Before proceeding to the general case $A \in \mathbb{Z}^{m\times n}$, we first consider the case $A \in \mathbb{N}^{m\times n}$ where A (and thus b) has only nonnegative entries.

4.1. The case $A \in \mathbb{N}^{m\times n}$.

In this section we assume that $A \in \mathbb{N}^{m\times n}$ and thus, necessarily $b \in \mathbb{N}^m$, otherwise $\{x \in \mathbb{N}^n \,|\, Ax = b\} = \emptyset$.

Theorem 4.1. *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$. Then the following two propositions (i) and (ii) are equivalent :*

(i) *The linear system $Ax = b$ has a solution $x \in \mathbb{N}^n$.*

(ii) *The real-valued polynomial $z \mapsto z^b - 1 := z_1^{b_1} \cdots z_m^{b_m} - 1$ can be written*

$$z^b - 1 \,=\, \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1), \tag{4.2}$$

for some real-valued polynomials $Q_j \in \mathbb{R}[z_1, \ldots, z_m]$, $j = 1, \ldots, n$, all of them with nonnegative coefficients.

In addition, the degree of the Q_j's in (4.2) is bounded by

$$b^* \,:=\, \sum_{j=1}^{m} b_j - \min_k \sum_{j=1}^{m} A_{jk}. \tag{4.3}$$

For a proof see §6.4. Hence Theorem 4.1 reduces the issue of existence of a solution $x \in \mathbb{N}^n$ to a particular *ideal membership problem*, that is, $Ax = b$ has a solution $x \in \mathbb{N}^n$ if and only if the polynomial $z^b - 1$ belongs to the *binomial ideal* $I = \langle z^{A_j} - 1\rangle_{j=1,\ldots,n} \subset \mathbb{R}[z_1, \ldots, z_m]$ for some weights Q_j with *nonnegative* coefficients.

Interestingly, consider the ideal $J \subset \mathbb{R}[z_1, \ldots, z_m, y_1, \ldots, y_n]$ generated by the binomials $z^{A_j} - y_j$, $j = 1, \ldots, n$, and let G be a Gröbner basis of J. Using the algebraic approach described in Adams and Loustaunau [2, §2.8], it is known that $Ax = b$ has a solution $x \in \mathbb{N}^n$ if and only if the monomial z^b is reduced (with respect to G) to some monomial y^α, in which case, $\alpha \in \mathbb{N}^n$ is a feasible solution. Observe that in this case, we do not know in advance $\alpha \in \mathbb{N}^n$ (we look for it!) to test whether $z^b - y^\alpha \in J$. One has to apply Buchberger's algorithm to (i) find a reduced Gröbner basis G of J, and (ii) reduce z^b with respect to G and check whether the final result is a monomial y^α. Moreover, in the latter approach one uses polynomials in $n+m$ (primal) variables y and (dual) variables z, in contrast with the (only) m dual variables z in Theorem 4.1.

Remark 4.2. (a) With b^* as in (4.3) denote by $s(b^*) := \binom{m+b^*}{b^*}$ the dimension of the vector space of polynomials of degree b^* in m variables. In view of Theorem 4.1, and given $b \in \mathbb{N}^m$, checking the existence of a solution $x \in \mathbb{N}^n$ to $Ax = b$ reduces to checking whether or not there exists a nonnegative solution y to a system of linear equations with :

- $n \times s(b^*)$ variables, the nonnegative coefficients of the Q_j's.
- $s(b^* + \max_k \sum_{j=1}^{n} A_{jk})$ equations to identify the terms of same power in both sides of (4.2).

This in turn reduces to solving a LP problem with $ns(b^*)$ variables and $s(b^* + \max_k \sum_j A_{jk})$ equality constraints. Observe that in view of (4.2), this LP has a matrix of constraints with only 0 and ± 1 coefficients.

(b) From the proof of Theorem 4.1 in §6.4, it easily follows that one may even enforce the weights Q_j in (4.2) to be polynomials in $\mathbb{Z}[z_1, \ldots, z_m]$ (instead of $\mathbb{R}[z_1, \ldots, z_m]$) with nonnegative coefficients. However, (a) shows that the strength of Theorem 4.1 is precisely to allow $Q_j \in \mathbb{R}[z_1, \ldots, z_m]$ as it permits to check feasibility by solving a (continuous) linear program. By enforcing $Q_j \in \mathbb{Z}[z_1, \ldots, z_m]$ one would end up with an integer linear system of size larger than that of the original problem.

4.2. **The general case.** In this section we consider the general case $A \in \mathbb{Z}^{m \times n}$ so that A may have negative entries, and we assume that the convex polyhedron $\Omega := \{x \in \mathbb{R}^n_+ \mid Ax = b\}$ is *compact.*

The above arguments cannot be repeated because of the occurence of negative powers. However, let $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}$ be such that

$$\widehat{A}_{jk} := A_{jk} + \alpha_k \geq 0, \quad k = 1, \ldots, n; \quad \widehat{b}_j := b_j + \beta \geq 0, \tag{4.4}$$

for all $j = 1, \ldots, m$. Moreover, as Ω is compact, we have that

$$\max_{x \in \mathbb{N}^n} \{\sum_{j=1}^{n} \alpha_j x_j \mid Ax = b\} \leq \max_{x \in \mathbb{R}^n_+} \{\sum_{j=1}^{n} \alpha_j x_j \mid Ax = b\} =: \rho^*(\alpha) < \infty. \tag{4.5}$$

Observe that given $\alpha \in \mathbb{N}^n$, the scalar $\rho^*(\alpha)$ is easily calculated by solving a LP problem. Choose $\mathbb{N} \ni \beta \geq \rho^*(\alpha)$, and let $\widehat{A} \in \mathbb{N}^{m \times n}, \widehat{b} \in \mathbb{N}^m$ be as in (4.4). Then the existence of solutions $x \in \mathbb{N}^n$ to $Ax = b$ is equivalent to the existence of solutions $(x,u) \in \mathbb{N}^n \times \mathbb{N}$ to the system of linear equations

$$(4.6) \qquad \mathbb{Q} \left\{ \begin{array}{rcrcl} \widehat{A}x & + & ue_m & = & \widehat{b} \\ \displaystyle\sum_{j=1}^n \alpha_j x_j & + & u & = & \beta \end{array} \right.$$

Indeed, if $Ax = b$ with $x \in \mathbb{N}^n$ then

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j - e_m \sum_{j=1}^n \alpha_j x_j \;=\; b + e_m\beta - e_m\beta,$$

or equivalently,

$$\widehat{A}x + \left(\beta - \sum_{j=1}^n \alpha_j x_j\right) e_m \;=\; \widehat{b},$$

and thus, as $\beta \geq \rho^*(\alpha) \geq \sum_{j=1}^n \alpha_j x_j$ (cf. (4.5)), we see that (x,u) with $\beta - \sum_{j=1}^n \alpha_j x_j =: u \in \mathbb{N}$ is a solution of (4.6). Conversely, let $(x,u) \in \mathbb{N}^n \times \mathbb{N}$ be a solution of (4.6). Using the definitions of $\widehat{A}$ and $\widehat{b}$, it then follows immediately that

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j + ue_m = b + \beta e_m; \quad \sum_{j=1}^n \alpha_j x_j + u \;=\; \beta,$$

so that $Ax = b$. The system of linear equations (4.6) can be put in the form

$$(4.7) \qquad B \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \widehat{b} \\ \beta \end{bmatrix} \text{ with } B := \begin{bmatrix} \widehat{A} & | & e_m \\ - & & - \\ \alpha' & | & 1 \end{bmatrix},$$

and as B has only entries in $\mathbb{N}$, we are back to the case analyzed in §4.1.

Corollary 4.3. *Let $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ and assume that $\Omega := \{x \in \mathbb{R}^n_+ \,|\, Ax = b\}$ is compact. Let $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}$ be as in (4.4) with $\beta \geq \rho^*(\alpha)$ (cf. (4.5)). Then the following two propositions (i) and (ii) are equivalent :*

(i) *The system of linear equations $Ax = b$ has a solution $x \in \mathbb{N}^n$.*

(ii) *The real-valued polynomial $z \mapsto z^b(zy)^\beta - 1 \in \mathbb{R}[z_1,\ldots,z_m,y]$ can be written*

$$(4.8) \qquad z^b(zy)^\beta - 1 \;=\; Q_0(z,y)(zy-1) + \sum_{j=1}^n Q_j(z,y)(z^{A_j}(zy)^{\alpha_j} - 1)$$

for some real-valued polynomials $\{Q_j\}_{j=0}^n$ in $\mathbb{R}[z_1,\ldots,z_m,y]$, all with non-negative coefficients.

The degree of the Q_j's in (4.8) is bounded by

$$(m+1)\beta + \sum_{j=1}^m b_j - \min\left[m+1, \min_{k=1,\ldots,n}\left[(m+1)\alpha_k + \sum_{j=1}^m A_{jk}\right]\right].$$

Proof. Let $\widehat{A} \in \mathbb{N}^{m\times n}, \widehat{b} \in \mathbb{N}^m$, $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}$ be as in (4.4) with $\beta \geq \rho^*(\alpha)$. Then apply Theorem 4.1 to the equivalent form (4.7) of the system $\mathbb{Q}$ in (4.6), where B and $(\widehat{b}, \beta)$ have only entries in $\mathbb{N}$, and use the definition of $\widehat{A}, \widehat{b}$. □

Indeed Theorems 4.1, and Corollary 4.3 have the flavor of a Farkas Lemma as it is stated with the transpose A' of A and involves the dual variables z_k associated with the constraints $Ax = b$. In addition, and as expected, it implies the continuous Farkas lemma because if $\{x \in \mathbb{N}^n \mid Ax = b\} \neq \emptyset$, then from (4.8), and with $z := \mathrm{e}^{\lambda}$, and $y := (z_1 \cdots z_m)^{-1}$,

$$\mathrm{e}^{b'\lambda} - 1 \;=\; \sum_{j=1}^{m} Q_j(e^{\lambda_1}, \dots e^{\lambda_m}, e^{-\sum_i \lambda_i})(\mathrm{e}^{(A'\lambda)_j} - 1). \tag{4.9}$$

Therefore, $A'\lambda \geq 0 \Rightarrow \mathrm{e}^{(A'\lambda)_j} - 1 \geq 0$ for all $j = 1, \dots, n$, and as the Q_j have nonnegative coefficients, we have $e^{b'\lambda} - 1 \geq 0$ which in turn implies $b'\lambda \geq 0$.

Equivalently, evaluating the partial derivatives of both sides of (4.9) with respect to λ_j, at the point $\lambda = 0$, yields $b_j = \sum_{k=1}^n A_{jk} x_k$ for all $j = 1, \dots, n$, with $x_k := Q_k(1, \dots, 1) \geq 0$. Thus, $Ax = b$ for some $x \in \mathbb{R}^n_+$.

5. Conclusion

We have proposed what we think is a natural duality framework for the integer program $\mathbb{P}_d$. It essentially relies on the $\mathbb{Z}$-transform of the associated counting problem $\mathbb{I}_d$, for which the important Brion and Vergne' s inverse formula appears to be an important tool for analyzing $\mathbb{P}_d$. In particular, it shows that the usual *reduced-costs* in linear programming, combined with periodicities phenomena associated with the complex poles of $\widehat{F}_d(z, c)$, also play an essential role for $\mathbb{P}_d$. Moreover, to the standard dual vector $\lambda \in \mathbb{R}^m$ associated with each basis B of the linear program $\mathbb{P}$, correspond $\det(B)$ dual vectors $z \in \mathbb{C}^m$ for the discrete problem $\mathbb{P}_d$. Moreover, for b sufficiently large, the optimal value of $\mathbb{P}_d$ is a function of these dual vectors associated with the optimal basis of the linear program $\mathbb{P}$. A topic of further research is to establish an *explicit* dual optimization problem $\mathbb{P}_d^*$ over these dual variables. We hope that the above results will stimulate further research in this direction.

6. Proofs

A proof in French of Theorem 3.1 can be found in Lasserre [15], whereas the English proof in [16] is reproduced below.

6.1. **Proof of Theorem 3.1.**

Proof. Use (1.1) and (3.8) to obtain

$$\begin{aligned}
\mathrm{e}^{f_d(b,c)} &= \lim_{r\to\infty}\left[\sum_{x(\sigma):\ \text{vertex of }\Omega(b)} \frac{\mathrm{e}^{rc'x(\sigma)}}{\mu(\sigma)} U_\sigma(b,rc)\right]^{1/r}\\
&= \lim_{r\to\infty}\left[\sum_{x(\sigma):\ \text{vertex of }\Omega(b)} \frac{\mathrm{e}^{rc'x(\sigma)}}{\mu(\sigma)} \sum_{g\in G(\sigma)} \frac{\mathrm{e}^{2i\pi b}(g)}{V_\sigma(g,rc)}\right]^{1/r}\\
&= \lim_{r\to\infty}\left[\sum_{x(\sigma):\ \text{vertex of }\Omega(b)} H_\sigma(b,rc)\right]^{1/r} \qquad (6.1)
\end{aligned}$$

Next, from the expression of $V_\sigma(b,c)$ in (3.10), and with rc in lieu of c, we see that $V_\sigma(g,rc)$ is a function of $y := e^r$, which in turn implies that $H_\sigma(b,rc)$ is also a function of e^r, of the form

$$H_\sigma(b,rc) \;=\; (\mathrm{e}^r)^{c'x(\sigma)} \sum_{g\in G(\sigma)} \frac{\mathrm{e}^{2i\pi b}(g)}{\sum_j \left(\delta_j(\sigma,g,A)\times (e^r)^{\alpha_j(\sigma,c)}\right)}, \qquad (6.2)$$

for finitely many coefficients $\{\delta_j(\sigma,g,A),\alpha_j(\sigma,c)\}$. Note that the coefficients $\alpha_j(\sigma,c)$ are sums of some reduced costs $c_k-\pi^\sigma A_k$ (with $k\notin\sigma$). In addition, the (complex) coefficients $\{\delta_j(\sigma,g,A)\}$ do not depend on b.

Let $y := \mathrm{e}^{r/q}$ with q the l.c.m. of $\{\mu(\sigma)\}_{\sigma\in\mathcal{B}(\Delta,\gamma)}$. As $q(c_k-\pi^\sigma A_k)\in\mathbb{Z}$ for all $k\notin\sigma$,

$$H_\sigma(b,rc) \;=\; y^{qc'x(\sigma)}\times\frac{P_{\sigma b}(y)}{Q_{\sigma b}(y)}, \qquad (6.3)$$

for some polynomials $P_{\sigma b},Q_{\sigma b}\in\mathbb{R}[y]$. In view of (6.2), the degree of $P_{\sigma b}$ and $Q_{\sigma b}$, which depends on b but *not* on the magnitude of b, is uniformly bounded in b.

Therefore, as $r\to\infty$,

$$H_\sigma(b,rc) \;\approx\; (\mathrm{e}^{r/q})^{qc'x(\sigma)+\deg(P_{\sigma b})-\deg(Q_{\sigma b})}, \qquad (6.4)$$

so that the limit in (6.1), which is given by $\max_\sigma \mathrm{e}^{c'x(\sigma)}\lim_{r\to\infty}U_\sigma(b,rc)^{1/r}$ (as we have assumed unicity of the maximizer σ), is also

$$\max_{x(\sigma):\ \text{vertex of }\Omega(b)} \mathrm{e}^{c'x(\sigma)+(\deg(P_{\sigma b})-\deg(Q_{\sigma b}))/q}.$$

Therefore, $f_d(b,c)=-\infty$ if $Ax=b$ has no solution $x\in\mathbb{N}^n$ and

$$f_d(b,c) \;=\; \max_{x(\sigma):\ \text{vertex of }\Omega(b)}\left[c'x(\sigma)+\frac{1}{q}(\deg(P_{\sigma b})-\deg(Q_{\sigma b}))\right] \qquad (6.5)$$

otherwise, from which (3.11) follows easily. □

6.2. **Proof of Corollary 3.3.**

Proof. Let $t \in \mathbb{N}$ and note that $f(tb, rc) = trf(b, c) = trc'x^* = trc'x(\sigma^*)$. As in the proof of Theorem 3.1, and with tb in lieu of b, we have

$$\widehat{f}_d(tb, rc)^{1/r} \;=\; \mathrm{e}^{tc'x^*} \left[\frac{U_{\sigma^*}(tb, rc)}{\mu(\sigma^*)} + \sum_{\text{vertex}\, x(\sigma) \neq x^*} \left(\frac{\mathrm{e}^{rc'x(\sigma)}}{\mathrm{e}^{rc'x(\sigma^*)}} \right)^t \frac{U_\sigma(tb, rc)}{\mu(\sigma)} \right]^{1/r}$$

and from (6.2)-(6.3), setting $\delta_\sigma := c'x^* - c'x(\sigma) > 0$ and $y := \mathrm{e}^{r/q}$,

$$\widehat{f}_d(tb, rc)^{1/r} \;=\; \mathrm{e}^{tc'x^*} \left[\frac{U_{\sigma^*}(tb, rc)}{\mu(\sigma^*)} + \sum_{\text{vertex}\, x(\sigma) \neq x^*} y^{-tq\delta_\sigma} \frac{P_{\sigma tb}(y)}{Q_{\sigma tb}(y)} \right]^{1/r}.$$

Observe that $c'x(\sigma^*) - c'x(\sigma) > 0$ whenever $\sigma \neq \sigma^*$ because $\Omega(y)$ is simple if $y \in \gamma$, and c is regular. Indeed, as x^* is an optimal vertex of the LP problem $\mathbb{P}$, the reduced costs $c_k - \pi^{\sigma^*} A_k$ ($k \notin \sigma^*$) with respect to the optimal basis σ^* are all nonpositive, and in fact, strictly negative because c is regular (see §2.4). Therefore, the term

$$\sum_{\text{vertex}\, x(\sigma) \neq x^*} y^{-tq\delta_\sigma} \frac{P_{\sigma tb}(y)}{Q_{\sigma tb}(y)}$$

is negligible for t sufficiently large, when compared with $U_{\sigma^*}(tb, rc)$. This is because the degrees of $P_{\sigma tb}$ and $Q_{\sigma tb}$ depend on tb but *not* on the magnitude of tb (see (6.2)-(6.3)), and they are uniformly bounded in tb. Hence, taking limit as $r \to \infty$ yields

$$\mathrm{e}^{f_d(tb,c)} \;=\; \lim_{r\to\infty} \left[\frac{\mathrm{e}^{rtc'x(\sigma^*)}}{\mu(\sigma^*)} U_{\sigma^*}(tb, rc) \right]^{1/r} \;=\; \mathrm{e}^{tc'x(\sigma^*)} \lim_{r\to\infty} U_{\sigma^*}(tb, rc)^{1/r},$$

from which (3.12) follows easily.

Finally, the periodicity is coming from the term $\mathrm{e}^{2i\pi tb}(g)$ in (3.9) for $g \in G(\sigma^*)$. The period is then the order of $G(\sigma^*)$. □

6.3. **Proof of Proposition 3.5.**

Proof. Let $U_{\sigma^*}(b, c)$ be as in (3.9)-(3.10). It is immediate to see that $\pi^{\sigma^*} = (\lambda^*)'$ and so

$$\mathrm{e}^{-\pi^{\sigma^*} A_k} \mathrm{e}^{-2i\pi A_k}(g) \;=\; \mathrm{e}^{-A_k'\lambda^*} \mathrm{e}^{-2i\pi A_k'\theta_g} \;=\; z_g^{-A_k}, \quad g \in G(\sigma^*).$$

Next, using $c'x(\sigma^*) = b'\lambda^*$,

$$\mathrm{e}^{c'x(\sigma^*)} \mathrm{e}^{2i\pi b}(g) \;=\; \mathrm{e}^{b'\lambda^*} \mathrm{e}^{2i\pi b'\theta_g} \;=\; z_g^b \qquad g \in G(\sigma^*).$$

Therefore,

$$\begin{aligned} \frac{1}{\mu(\sigma^*)} \mathrm{e}^{c'x(\sigma)} U_{\sigma^*}(b, c) \;&=\; \frac{1}{\mu(\sigma^*)} \sum_{g \in G(\sigma^*)} \frac{z_g^b}{(1 - z_g^{-A_k} \mathrm{e}^{c_k})} \\ &=\; R_{\sigma^*}(z_g, 1), \end{aligned}$$

and (3.21) follows from (3.11) because, with rc in lieu of c, z_g becomes $z_{gr} = \mathrm{e}^{r\lambda^*}\mathrm{e}^{2i\pi\theta_g}$ (only the modulus changes).

Next, as only the modulus of z_g is involved in (3.22), we have $|z_{gr}| = \mathrm{e}^{r\lambda^*}$ for all $g \in G(\sigma^*)$, so that

$$\frac{1}{\mu(\sigma^*)} \sum_{g\in G(\sigma^*)} \frac{|z_{gr}|^b}{\prod_{k\notin\sigma^*}(1-|z_{gr}|^{-A_k}\mathrm{e}^{rc_k})} = \frac{\mathrm{e}^{rb'\lambda^*}}{\prod_{k\notin\sigma^*}(1-\mathrm{e}^{r(c_k-A'_k\lambda^*)})},$$

and, as $r\to\infty$,

$$\frac{\mathrm{e}^{rb'\lambda^*}}{\prod_{k\notin\sigma^*}(1-\mathrm{e}^{r(c_k-A'_k\lambda^*)})} \approx \mathrm{e}^{rb'\lambda^*},$$

because $(c_k - A'_k\lambda^*) < 0$ for all $k \notin \sigma^*$. Therefore,

$$\lim_{r\to\infty} \frac{1}{r} \ln \left[\frac{\mathrm{e}^{rb'\lambda^*}}{\prod_{k\notin\sigma^*}(1-\mathrm{e}^{r(c_k-A'_k\lambda^*)})}\right] = b'\lambda^* = f(b,c),$$

the desired result. □

6.4. **Proof of Theorem 4.1.**

Proof. (ii) ⇒ (i). Assume that $z^b - 1$ can be written as in (4.2) for some polynomials $\{Q_j\}$ with nonnegative coefficients $\{Q_{j\alpha}\}$, that is,

$$Q_j(z) = \sum_{\alpha\in\mathbb{N}^m} Q_{j\alpha} z^\alpha = \sum_{\alpha\in\mathbb{N}^m} Q_{j\alpha} z_1^{\alpha_1}\cdots z_m^{\alpha_m},$$

for finitely many nonzero (and nonnegative) coefficients $\{Q_{j\alpha}\}$. Using notation of §3, the function $\widehat{f}_d(b,0)$, which (as $c = 0$) counts the nonnegative integral solutions $x \in \mathbb{N}^n$ to the equation $Ax = b$, is given by

$$\widehat{f}_d(b,0) = \frac{1}{(2\pi i)^m} \int_{|z_1|=\gamma_1} \cdots \int_{|z_m|=\gamma_m} \frac{z^{b-e_m}}{\prod_{j=1}^n (1-z^{-A_k})}\, dz$$

where $\gamma \in \mathbb{R}^m$ satisfies $A'\gamma > 0$ (see (3.4) and (3.6)).

Writing z^{b-e_m} as $z^{-e_m}(z^b - 1 + 1)$ we obtain

$$\widehat{f}_d(b,0) = B_1 + B_2,$$

with

$$B_1 = \frac{1}{(2\pi i)^m} \int_{|z_1|=\gamma_1} \cdots \int_{|z_m|=\gamma_m} \frac{z^{-e_m}}{\prod_{j=1}^n (1-z^{-A_k})}\, dz,$$

and

$$\begin{aligned}
B_2 &:= \frac{1}{(2\pi i)^m}\int_{|z_1|=\gamma_1}\cdots\int_{|z_m|=\gamma_m}\frac{z^{-e_m}(z^b-1)}{\prod_{j=1}^n(1-z^{-A_k})}\,dz\\
&= \sum_{j=1}^n \frac{1}{(2\pi i)^m}\int_{|z_1|=\gamma_1}\cdots\int_{|z_m|=\gamma_m}\frac{z^{A_j-e_m}Q_j(z)}{\prod_{k\neq j}(1-z^{-A_k})}\,dz\\
&= \sum_{j=1}^n\sum_{\alpha\in\mathbb{N}^m}\frac{Q_{j\alpha}}{(2\pi i)^m}\int_{|z_1|=\gamma_1}\cdots\int_{|z_m|=\gamma_m}\frac{z^{A_j+\alpha-e_m}}{\prod_{k\neq j}(1-z^{-A_k})}\,dz.
\end{aligned}$$

From (3.6) (with $b := 0$) we recognize in B_1 the number of solutions $x \in \mathbb{N}^n$ to the linear system $Ax = 0$, so that $B_1 = 1$. Next, again from (3.6) (now with $b := A_j + \alpha$), each term

$$C_{j\alpha} := \frac{Q_{j\alpha}}{(2\pi i)^m}\int_{|z_1|=\gamma_1}\cdots\int_{|z_m|=\gamma_m}\frac{z^{A_j+\alpha-e_m}}{\prod_{k\neq j}(1-z^{-A_k})}\,dz.,$$

is equal to

$$Q_{j\alpha}\times \text{ the number of integral solutions } x\in\mathbb{N}^{n-1}$$

of the linear system $\widehat{A}^{(j)}x = A_j + \alpha$, where $\widehat{A}^{(j)}$ is the matrix in $\mathbb{N}^{m\times(n-1)}$ obtained from A by deleting its j-th column A_j. As by hypothesis each $Q_{j\alpha}$ is nonnegative, it follows that

$$B_2 = \sum_{j=1}^n\sum_{\alpha\in\mathbb{N}^m} C_{j\alpha} \geq 0,$$

so that $\widehat{f_d}(b,0) = B_1 + B_2 \geq 1$. In other words, the sytem $Ax = b$ has at least one solution $x \in \mathbb{N}^n$.

(i) $\Rightarrow$ (ii). Let $x \in \mathbb{N}^n$ be a solution of $Ax = b$, and write

$$z^b - 1 = z^{A_1x_1} - 1 + z^{A_1x_1}(z^{A_2x_2} - 1) + \cdots + z^{\sum_{j=1}^{n-1}A_jx_j}(z^{A_nx_n} - 1),$$

and

$$z^{A_jx_j} - 1 = (z^{A_j} - 1)\left[1 + z^{A_j} + \cdots + z^{A_j(x_j-1)}\right] \quad j = 1,\ldots,n,$$

to obtain (4.2) with

$$z \mapsto Q_j(z) := z^{\sum_{k=1}^{j-1}A_kx_k}\left[1 + z^{A_j} + \cdots + z^{A_j(x_j-1)}\right], \quad j = 1,\ldots,n.$$

We immediately see that each Q_j has all its coefficients nonnegative (and even in $\{0, 1\}$).

Finally, the bound on the degree follows immediately from the proof of the (i) $\Rightarrow$ (ii). □

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