CALCULATION OF UNIVERSAL BARRIER FUNCTIONS FOR CONES GENERATED BY CHEBYSHEV SYSTEMS OVER FINITE SETS *

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Abstract. We explicitly calculate universal barrier functions for cones generated by (weakly) Chebyshev systems over finite sets. We show that universal barrier functions corresponding to Chebyshev systems on intervals are obtained as limits of universal barrier functions of their discretizations. The results are heavily rely upon classical work of M. Krein, A. Nudelman and I.J. Schoenberg .

Key words. interior-point methods , Chebyshev systems,semi-infinite programming

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1. Introduction. In [1] we calculated the universal barrier function for a broad class of cones generated by Chebyshev systems on intervals of the real line and the circle. In general, given a closed , convex, pointed cone K in $\mathbf{R^n}$, the universal barrier function (up to a multiplication by a positive constant) has the form [5]:

$$\Phi(x) = \ln \int_{K^*} e^{-\langle x,y\rangle} d\mu(y),$$

where $x \in int(K), K^*$ is the cone dual to K and μ is the Lebesgue measure. The expression we obtained in the case of the cone generated by a Chebyshev system is of the following form:

$$\Phi(x) = \frac{1}{2} \ln \det(\bar{D}(x)),$$

where $\bar{D}(x)$ is a skew-symmetric matrix. The only complication, say, in comparison with the semidefinite programming case is that the entries of $\bar{D}(x)$ are one-dimensional definite integrals. While there are interesting cases, where these integrals can be explicitly calculated,in general, it is important to understand what is the right way to approximate these integrals so that to preserve important properties of modern interior-point algorithms (e.g. complexity estimates). On the other hand, the class of optimization problems involving Chebyshev cones is a subclass of semi-infinite programming problems. Many natural procedures for finding approximate solutions to semi-infinite programming problems rely upon discretizations of semi-infinite constraints. Discretization procedures applied to Chebyshev cones lead to Chebyshev systems over finite (more generally, countable) sets. Thus, it is quite natural to try to calculate universal barrier functions for such systems. Intuitively (and this is rigorously confirmed in the present paper), such barriers should converge to universal barriers of Chebyshev cones on intervals with the refinement of a discretization and, consequently, should provide a natural way to approximate those universal barriers.

The cones generated by Chebyshev systems over finite sets are defined by a finite number (equal to the cardinality of the set) of linear inequality constraints. Obviously,

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one can easily construct a self-concordant barrier for such a cone (as the minus sum of logarithms of linear forms defining these inequalities). However, the so-called barrier parameter for this barrier will be equal to the number of inequality constraints (i.e. the cardinality of the set) and will grow rapidly with the refinement of a discretization. On the contrary, due to a deep result of Yu. Nesterov and A. Nemirovsky [5] the barrier parameter of the universal barrier function is of the order of dimension of the cone ($O(n)$) and hence in our case does not depend on the refinement of a discretization! Since the barrier parameter determines complexity estimates for practically all modern interior algorithms (the smaller parameter, the better estimates), the use of a universal barrier function for polyhedral cones is potentially quite beneficial. The problem, however, is that so far (within the class of polyhedral cones) such a universal barrier function has been calculated only for the positive orthant. More precisely, it is not known how to decompose an arbitrary polyhedral cone into cones linearly isomorphic to the positive orthant in an efficient way.

In the present paper we do calculate the universal barrier function for a broad class of polyhedral cones generated by (weakly) Chebyshev systems over finite sets in the form as computable as in semidefinite programming case. More precisely,

$$\Phi(x) = \frac{1}{2} \ln \det \bar{D}_1(x),$$

where $\bar{D}_1(x)$ is again a skew-symmetric matrix. The entries of $\bar{D}_1(x)$ are essentially Riemann sums for entries of $\bar{D}(x)$ in the case where the corresponding finite set appears as a result of a discretization of an interval. Observe that polynomial splines belong to the class of weak Chebyshev systems and thus covered by the results of the present paper.

We heavily rely upon classical results of M. Krein, A. Nudelman and I. Schoenberg. In our opinion, it provides an additional evidence that the modern theory of interior-point algorithms in the form developed by Yu. Nesterov and A. Nemirovsky has a deep mathematical structure which is currently only partially understood.

2. Chebyshev systems over finite sets. Let $\Delta = \{t_1 < t_2 < \ldots < t_m\}$ be an ordered finite set of real numbers. We say that the functions $u_i : \Delta \to \mathbf{R}, i = 0, \ldots, n$ form a Chebyshev system over Δ, if

$$\det(u_i(t_{j_k})) > 0 \tag{2.1}$$

for any $1 \leq j_1 < j_2 \ldots < j_{n+1} \leq m$. Introduce vectors $v_i \in \mathbf{R^{n+1}}, i = 1, \ldots, m$, where

$$v_i = (u_0(t_i), u_1(t_i), \ldots, u_n(t_i))^T.$$

The condition (2.1) can be rewritten in the form:

$$\det[v_{j_1} v_{j_2} \ldots v_{j_{n+1}}] > 0$$

for any $1 \leq j_1 < j_2 < \ldots j_{n+1} \leq m$.

One can naturally associate a cone with a given Chebyshev system:

$$K = \{x \in \mathbf{R^{n+1}} : x = (x_0, \ldots x_n)^T, \sum_{i=0}^{n} x_i u_i(t_j) \geq 0, \forall j \in [1, m]\}.$$

It is obvious that

$$K = \{x \in \mathbf{R^{n+1}} : \langle x, v_j \rangle \geq 0, \forall j \in [1, m]\}.$$

Here $\langle , \rangle$ is the standard scalar product in $\mathbf{R^{n+1}}$. Hence, the dual cone K^* can be described as

$$K^* = \{\sum_{j=1}^{m} \rho_j v_j : \rho_j \geq 0, j = 1, \ldots, m\}.$$

Our next goal is to describe a combinatorial concept of an index for the finite increasing sequence of integers. It will be used to parameterize the dual cone K^* using the so-called principal representations.

Let θ be such an increasing sequence:

$$\theta = \{1 \leq j_1 < j_2 < \ldots < j_p \leq m\}.$$

If for some $1 \leq l \leq p$ $j_{i+l} = j_l + i, i = 1, \ldots, k$ and $j_l > 1, j_l - 1$ does not belong to $\theta; j_{k+l} < m, j_{k+l} + 1$ does not belong to θ, then we call $j_l, j_{l+1}, \ldots j_{l+k}$ an *internal* block of θ. If $j_l = 1$(respectively, $j_{l+k} = m$), then we call $j_l, j_{l+1}, \ldots j_{l+k}$ a block adjacent to 1 (respectively, to m). According to the parity of k , a block is said to be even or odd. The index of an interior block θ' containing k elements is defined as:

$$\epsilon(\theta') = k,$$

if k is even, and

$$\epsilon(\theta') = k + 1,$$

if k is odd. If a block θ' containing k elements adjoins either 1 or m, then $\epsilon(\theta') = k$. Finally, it is obvious that every θ can be partitioned into blocks $\theta_s, s = 1, 2, \ldots t$ for some t. We define the index of θ (notation:$\epsilon(\theta)$) as

$$\epsilon(\theta) = \sum_{i=1}^{t} \epsilon(\theta_s). \tag{2.2}$$

The next Proposition immediately follows from definitions.

PROPOSITION 1. *We have:*

$$card(\theta) \leq \epsilon(\theta).$$

Moreover, $card(\theta) = \epsilon(\theta)$ if and only if all internal blocks are even. Here $card(\theta)$ is the number of elements in θ.

DEFINITION 1. *A sequence $\theta \subset [1, m]$ is called full if $card(\theta) = \epsilon(\theta)$.*

Let $x \in K^*$,

$$x = \sum_{i=1}^{p} \rho_i v_{j_i}, \tag{2.3}$$

$1 \leq j_1 < j_2 \dots j_p \leq m, \rho_i > 0, i = 1, \dots p$. We say that (2.3)is a principal representation of x if

$$\epsilon\{j_1, \dots j_p\} = n + 1.$$

For a proof of the following Theorem see e.g. [4], [6].

THEOREM 1. *Every $x \in int(K^*)$ admits exactly two principal representations:*

- *the block adjacent to m is even (in particular, empty). This is the so-called lower principal representation;*
- *the block adjacent to m is odd. This is the so-called upper principal representation.*

Given $\theta = \{j_1 < \dots < j_p\}$,where $1 \leq j_1, j_p \leq m$, denote by K^*_θ the cone:

$$K^*_\theta = \{\sum_{i=1}^{p} \rho_i v_{j_i} : \rho_i > 0\}.$$

PROPOSITION 2. *Let Θ_u (respectively,Θ_l) be the set of all full subsets of $[1, m]$ of the index $n + 1$ such that the block adjacent to m is odd (respectively, even). Then*

$$S_u = \cup_{\theta \in \Theta_u} K^*_\theta \subset int(K^*),$$

$$S_l = \cup_{\theta \in \Theta_l} K^*_\theta \subset int(K^*).$$

Moreover, the Lebesgue measure of $int(K^*) \setminus S_u$ (respectively, $int(K^*) \setminus S_l$) is equal to zero. Besides,

$$K^*_\theta \cap K^*_{\theta'} = \emptyset$$

for $\theta, \theta' \in \Theta_u$ (respectively, Θ_l).

Proof Let $\theta \subset [1, m], \theta \neq [1, m]$. Then $\epsilon(\theta) \geq card(\theta)$ and equality takes place if and only if θ is full. Let $\epsilon(\theta) = n + 1$. If $card(\theta) < n + 1$, then $\dim K^*_\theta < n + 1$ and, hence, $\mu(K^*_\theta) = 0$, where μ is the Lebesgue measure on $\mathbf{R^{n+1}}$. On the other hand, if $card(\theta) = n + 1$, then θ is full, $\dim K^*_\theta = n + 1$ (since the vectors $v_j, j \in \theta$ form a basis in $\mathbf{R^{n+1}}$). But then $K^*_\theta \subset int(K^*)$. Using the uniqueness of upper and lower principal representations , we immediately conclude that $K^*_\theta \cap K^*_{\theta'} = \emptyset, \theta, \theta' \in \Theta_u$ (respectively, Θ_l), $\theta \neq \theta'$. The result follows.

EXAMPLE 1. *Let $n = 2, m = 5$. We have*

$$\Theta_u = \{\theta_1, \theta_2, \theta_3\},$$

$$\theta_1 = \{1, 2, 5\}, \theta_2 = \{2, 3, 5\}, \theta_3 = \{3, 4, 5\};$$

$$\Theta_l = \{\theta_4, \theta_5, \theta_6\},$$

$$\theta_4 = \{1, 2, 3\}, \theta_5 = \{1, 4, 5\}, \theta_6 = \{1, 3, 4\}.$$

We are now in position to calculate the characteristic function of K.

THEOREM 2. *Let $x \in int(K)$. Then*

$$I(x) = \int_{K^*} e^{-\langle x,y \rangle} d\mu(y) = \sum_{\theta=\{j_1,\dots,j_{n+1}\} \in \Theta} \frac{\det[v_{j_1}, \dots, v_{j_{n+1}}]}{\prod_{l=1}^{n+1} \langle v_{j_l}, x \rangle},$$

where $\Theta = \Theta_u$ *or* $\Theta = \Theta_l$.

REMARK 1. *Since* $x \in int(K)$, *we have* $\langle x, v_j \rangle > 0, \forall j \in [1, m]$.

Proof Consider the case $\Theta = \Theta_u$ (the case $\Theta = \Theta_l$ is absolutely similar). By Theorem 1

$$I(x) = \sum_{\theta \in \Theta_u} \int_{K^*_\theta} e^{-\langle x,y \rangle} d\mu(y).$$

If $y \in K^*_\theta, \theta = \{j_1, \dots j_{n+1}\}$, we have:

$$y = \sum_{k=1}^{n+1} \rho_k v_{j_k}, \rho_k > 0.$$

Hence,

$$\langle x, y \rangle = \sum_{k=1}^{n+1} \rho_k \langle x, v_{j_k} \rangle,$$

$$d\mu(y) = \det[v_{j_1}, \dots, v_{j_{n+1}}] d\rho_1 d\rho_2 \dots d\rho_{n+1}.$$

We used the fact that $\det[v_{j_1}, \dots v_{j_{n+1}}] > 0$ while changing variables. Hence,

$$\int_{K^*_\theta} e^{-\langle x,y \rangle} d\mu(y) = \int_0^{+\infty} \dots \int_0^{+\infty} \prod_{k=1}^{n+1} e^{-\rho_k \langle x, v_{j_k} \rangle} \det[v_{j_1}, \dots v_{j_{n+1}}] d\rho_1 \dots d\rho_{n+1} = \frac{\det[v_{j_1}, \dots v_{j_{n+1}}]}{\prod_{k=1}^{n+1} \langle x, v_{j_k} \rangle}.$$

The result follows.

3. "Pfaffian form of universal barrier functions". In principle, Theorem 2, provides an explicit description for the universal barrier function for the cone K. However, it is difficult to compare this description and the results of [1]. Thus, we need to find a more "computable" form of our universal barrier function which we would be able to compare with results of [1].

Given $x \in int(K)$, introduce vectors

$$a_i = \frac{v_i}{\langle x, v_i \rangle} - \frac{v_{i+1}}{\langle x, v_{i+1} \rangle}, i = 1, 2, \dots m,$$

where by definition $v_{m+1} = 0$.

THEOREM 3. *We have:*

$$\sum_{\{j_1 < j_2 < \dots j_{n+1}\} \in \Theta_l} \frac{\det[v_{j_1}, \dots v_{j_{n+1}}]}{\prod_{k=1}^{n+1} \langle x, v_{j_k} \rangle} = \sum_{1 \le l_1 < l_2 \dots < l_{n+1} \le m} \det[a_{l_1}, a_{l_2}, \dots a_{l_{n+1}}].$$

The proof of this theorem will be given in the Appendix. Here we illustrate this result by an example.

EXAMPLE 2. *Let* $m = 6, n = 3$. *We have:*

$$\Theta_l = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\},$$

$$\theta_1 = \{1,2,3,4\}, \theta_2 = \{1,2,4,5\}, \theta_3 = \{1,2,5,6\},$$

$$\theta_4 = \{2,3,4,5\}, \theta_5 = \{2,3,5,6\}, \theta_6 = \{3,4,5,6\}.$$

With notations introduced above we have:

$$\sum_{\{j_1<j_2<j_3<j_4\}\in\Theta_l} \frac{\det[v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}]}{\prod_{k=1}^{4}\langle x, v_{j_k}\rangle} =$$

$$\det[a_1, a_2, a_3, a_4 + a_5 + a_6] + \det[a_1, a_2 + a_3, a_4, a_5 + a_6] + \det[a_1, a_2 + a_3 + a_4, a_5, a_6] +$$

$$\det[a_2, a_3, a_4, a_5 + a_6] + \det[a_2, a_3 + a_4, a_5, a_6] + \det[a_3, a_4, a_5, a_6] =$$

$$\sum_{1\leq l_1<l_2<l_3<l_4\leq 6} \det[a_{l_1}, a_{l_2}, a_{l_3}, a_{l_4}].$$

Theorem 4. *We have:*

$$\sum_{\{j_1<j_2<\ldots j_{n+1}\}\in\Theta_u} \frac{\det[v_{j_1}, \ldots v_{j_{n+1}}]}{\prod_{k=1}^{n+1}\langle x, v_{j_k}\rangle} = \sum_{1\leq l_1<l_2\ldots<l_n<m} \det[a_{l_1}, a_{l_2}, \ldots a_{l_n}, v_m]/\langle x, v_m\rangle.$$

Example 3. *Let $m = 6, n = 2$. We have:*

$$\Theta_u = \{\theta_1, \theta_2, \theta_3, \theta_4\},$$

$$\theta_1 = \{1,2,6\}, \theta_2 = \{2,3,6\}, \theta_3 = \{3,4,6\},$$

$$\theta_4 = \{4,5,6\}.$$

With notations introduced above:

$$\sum_{\{j_1<j_2<j_3\}\in\Theta_u} \frac{\det[v_{j_1}, v_{j_2}, v_{j_3}]}{\prod_{k=1}^{3}\langle x, v_{j_k}\rangle} =$$

$$\frac{1}{\langle x, v_6\rangle}(\det[a_1, a_2 + a_3 + a_4 + a_5, v_6] + \det[a_2, a_3 + a_4 + a_5, v_6] +$$

$$\det[a_3, a_4 + a_5, v_6] + \det[a_4, a_5, v_6]) =$$

$$\sum_{1\leq j_1<j_2<j_3\leq 5} \frac{\det[a_{j_1}, a_{j_2}, a_{j_3}, v_6]}{\langle x, v_6\rangle}.$$

Let $b_1, b_2, \dots b_m$ be vectors in an even-dimensional vector space $\mathbf{R^{2r}}, m \geq 2r$. Let, further,

$$b_i = (b_i(1), b_i(2), \dots, b_i(2r))^T,$$

$$d(\alpha, \beta) = \sum_{1 \leq i < j \leq m} \det \begin{bmatrix} b_i(\alpha) & b_j(\alpha) \\ b_i(\beta) & b_j(\beta) \end{bmatrix},$$

$\alpha, \beta = 1, 2, \dots, 2r$. Let D be a skew-symmetric $2r \times 2r$ matrix such that

$$D(\alpha, \beta) = d(\alpha, \beta), \alpha, \beta = 1, 2, \dots 2r.$$

The next proposition is due to S. Okada [7]

PROPOSITION 3. *We have:*

$$Pf(D) = \sum_{1 \leq j_1 < j_2 < \dots < j_{2r} \leq m} \det[b_{j_1}, b_{j_2}, \dots b_{j_{2r}}].$$

Here $Pf(D)$ stands for the Pfaffian of an even-dimensional skew-symmetric matrix D.

REMARK 2. *One substantial for us property of Pfaffians is that:*

$$Pf(D)^2 = \det(D).$$

Hence, $\ln Pf(D) = \frac{1}{2} \ln \det(D)$. For a good introductory discussion of major properties of Pfaffians we recommend [2].

LEMMA 1. *Let $\|z(i,j)\|, i, j = 1, 2, \dots N+1$ be a skew-symmetric matrix such that $z(i, N+1) = 0, i = 1, \dots N$. Then:*

$$S = \sum_{1 \leq i < j \leq N} (z(i,j) + z(i+1, j+1) - z(i, j+1) - z(i+1, j)) = \sum_{i=1}^{N-1} z(i, i+1).$$

Proof. We have:

$$S = \sum_{1 \leq i < j \leq N} z(i,j) + \sum_{2 \leq i < j \leq N+1} z(i,j) - \sum_{2 \leq i \leq j \leq N} z(i,j) - \sum_{1 \leq i < j-1 \leq N} z(i,j).$$

Combining the first and the third (respectively, the second and the fourth) terms and taking into account that $z(i,i) = 0$, we obtain:

$$S = \sum_{j=2}^{N} z(1,j) + \sum_{i=2}^{N} z(i, i+1) - \sum_{j=3}^{N+1} z(1,j) =$$

$$\sum_{i=1}^{N} z(i, i+1) - z(1, N+1).$$

The result follows.

We are now in position to describe the characteristic function of a cone generated by a Chebyshev system over a finite set in the form similar to [1]. Recall that such a system is determined by a finite set of vectors $v_i, i = 1, \ldots m$ in $\mathbf{R^{n+1}}$ satisfying (2.1),where

$$v_i = (v_i(0), v_i(1), \ldots v_i(n))^T,$$

$$v_i(k) = u_k(t_i), k = 0, 1, \ldots, n.$$

THEOREM 5. *Let $x \in int(K)$ and n is odd. Then*

$$I(x) = \int_{K^*} e^{-\langle x,y \rangle} d\mu(y) = Pf(D(x)),$$

where $D(x) = \|d(\alpha, \beta)\|, \alpha, \beta = 0, 1, \ldots, n$

$$d(\alpha, \beta) = \sum_{i=1}^{m-1} \frac{\det \begin{bmatrix} v_i(\alpha) & v_{i+1}(\alpha) \\ v_i(\beta) & v_{i+1}(\beta) \end{bmatrix}}{\langle x, v_i \rangle \langle x, v_{i+1} \rangle} =$$

$$\sum_{i=1}^{m-1} \frac{\det \begin{bmatrix} u_\alpha(t_i) & u_\alpha(t_{i+1}) - u_\alpha(t_i) \\ u_\beta(t_i) & u_\beta(t_{i+1}) - u_\beta(t_i) \end{bmatrix}}{\langle x, v_i \rangle \langle x, v_{i+1} \rangle}.$$

Proof By Theorems 2,3 and Proposition 3, we have:

$$I(x) = Pf(D), D = \|d(\alpha, \beta)\|,$$

$$d(\alpha, \beta) = \sum_{1 \le i < j \le m} \det \begin{bmatrix} a_i(\alpha) & a_j(\alpha) \\ a_i(\beta) & a_j(\beta) \end{bmatrix},$$

where

$$a_i = \frac{v_i}{\langle x, v_i \rangle} - \frac{v_{i+1}}{\langle x, v_{i+1} \rangle}, i = 1, 2, \ldots m.$$

Let

$$z(i, j) = \det \begin{bmatrix} \tilde{v}_i(\alpha) & \tilde{v}_j(\alpha) \\ \tilde{v}_i(\beta) & \tilde{v}_j(\beta) \end{bmatrix},$$

where

$$\tilde{v}_i = \frac{v_i}{\langle x, v_i \rangle}.$$

Then

$$d(\alpha,\beta)=\sum_{1\le i<j\le m}(z(i,j)+z(i+1,j+1)-z(i,j+1)-z(i+1,j)).$$

The result follows from Lemma 1.

Let $v_1, v_2, \ldots, v_m$ be a Chebyshev system in $\mathbf{R^{n+1}}$ and L be a linear isomorphism of $\mathbf{R^{n+1}}$ such that $\det L > 0$. Then $Lv_1, Lv_2, \ldots, Lv_m$ is also a Chebyshev system in $\mathbf{R^{n+1}}$, since

$$\det[Lv_{j_1},\ldots Lv_{j_{n+1}}]=\det L\det[v_{j_1},\ldots v_{j_{n+1}}].$$

Moreover, if we denote by K_L the corresponding cone generated by $Lv_1, \ldots, Lv_m$, i.e.

$$K_L=\{x\in\mathbf{R^{n+1}}:\langle x,Lv_i\rangle\ge 0,\forall i=1,\ldots,m\}$$

, then

$$K_L=L^{-T}K, K_L^*=LK^*.$$

In other words, the cones K and K_L are linearly isomorphic. Hence, there characteristic functions coincide up to a multiplicative constant. One can always find such an L so that

$$Lv_m=(0,\ldots,0,1)^T.$$

THEOREM 6. *Let $x \in int(K), n$ is even and $v_m = (0, \ldots, 0, 1)^T$. Then:*

$$I(x)=\int_{K^*}e^{-\langle x,y\rangle}d\mu(y)=Pf(D_1(x))/\langle x,v_m\rangle,$$

where

$$D_1(x)=\|d_1(\alpha,\beta)\|,\alpha,\beta=0,1,\ldots,n-1,$$

$$d_1(\alpha,\beta)=\sum_{i=1}^{m-2}\frac{\det\begin{bmatrix}v_i(\alpha)&v_{i+1}(\alpha)\\v_i(\beta)&v_{i+1}(\beta)\end{bmatrix}}{\langle x,v_i\rangle\langle x,v_{i+1}\rangle}=$$

$$\sum_{i=1}^{m-2}\frac{\det\begin{bmatrix}u_\alpha(t_i)&u_\alpha(t_{i+1})-u_\alpha(t_i)\\u_\beta(t_i)&u_\beta(t_{i+1})-u_\beta(t_i)\end{bmatrix}}{\langle x,v_i\rangle\langle x,v_{i+1}\rangle}.$$

Proof By Theorems 2,4, we have:

$$I(x)=\sum_{1\le l_1<l_2<\ldots l_n\le m-1}\det[a_{l_1},a_{l_2},\ldots,a_{l_n},v_m]/\langle x,v_m\rangle.$$

Taking into account $v_m = (0, \ldots 0, 1)^T$ and expanding determinants over the last column, we obtain:

$$I(x) = \sum_{1 \leq l_1 < l_2 < \ldots l_n \leq m-1} \det[\tilde{a}_{l_1}, \ldots \tilde{a}_{l_n}]/\langle x, v_m\rangle.$$

Here $\tilde{a}_i \in \mathbf{R^n}$,

$$\tilde{a}_i(j) = a_i(j) = \frac{v_i(j)}{\langle x, v_i\rangle} - \frac{v_{i+1}(j)}{\langle x, v_i\rangle},$$

$j = 0, 1, \ldots n-1$. We can now finish the proof exactly as in Theorem 5.

DEFINITION 2. *We say that the vectors* $v_1, \ldots, v_m \in \mathbf{R^{n+1}}$ *form a periodic Chebyshev system if* $v_1 = v_m$ *and*

$$\det[v_{j_1}, \ldots, v_{j_{n+1}}] > 0, \forall 1 \leq j_1 < j_2 < \ldots j_{n+1} \leq m$$

with the exception $j_1 = 1$ *and* $j_{n+1} = m$.

LEMMA 2. *If* $v_1, \ldots, v_m \in \mathbf{R^{n+1}}$ *is a periodic Chebyshev system, then* n *is even.*

Proof Take $j_{n+1} = m, j_n = m-1, \ldots, j_1 = m-n$.Then

$$\det[v_{j_1}, \ldots v_{j_{n+1}}] = (-1)^n \det[v_{j_{n+1}}, v_{j_1}, v_{j_2}, \ldots v_{j_n}] =$$

$$(-1)^n \det[v_1, v_{j_1}, \ldots, v_{j_n}].$$

Both the first and the last determinant should be positive. Hence, n is even.

If $v_1, \ldots v_m$ form a periodic Chebyshev system, then $v_2, v_3, \ldots, v_m$ form a usual Chebyshev system and cones generated by these systems are obviously coincide. Hence, we can apply Theorem 6 to calculate the universal barrier function for the cone generated by a periodic Chebyshev system.

EXAMPLE 4. *Let* $u_0, \ldots, u_n$ *be a Chebyshev system of continuously differentiable functions on the interval* $[a, b]$. *Let*

$$C = \{(x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i u_i \geq 0, \forall t \in [a, b]\}.$$

In [1] we calculated the universal barrier function for C. *Suppose that* n *is odd. Then, given* $x \in int(C)$,

$$I_C(x) = Pf(D_2(x)), D_2(x) = \|d_2(\alpha, \beta)\|,$$

$\alpha, \beta = 0, 1, \ldots n$, *where*

$$d_2(\alpha, \beta) = \int_a^b \frac{\det \begin{bmatrix} u_\alpha(t) & \dot{u}_\alpha(t) \\ u_\beta(t) & \dot{u}_\beta(t)) \end{bmatrix}}{x(t)^2} dt,$$

$$x(t) = \sum_{i=0}^{n} x_i u_i(t).$$

Choose $a \leq t_1 < t_2 < \ldots t_m \leq b$ *for some* $m \geq n+1$ *It is clear that* $u_0, \ldots, u_n$ *form a Chebyshev system over the finite set* $\Delta = \{t_1, \ldots, t_m\}$. *It is quite obvious that* $C \subset K = K_\Delta$. *Hence, we can compare* $I_C(x)$ *and* $I_K(x)$. *By Theorem 5 we see that* $d(\alpha, \beta)$ *is essentially a Riemann sum for* $d_2(\alpha, \beta)$ *(observe that* $\langle x, v_i \rangle = x(t_i), \forall i$ *). The difference between the corresponding formulas for the case n is even is explained by the fact that in Theorem 6 we used the upper principal representation, whereas in Theorem 5 of [1] we used the lower principal representation. More precisely,let, say, we divide the interval* $[a, b]$ *in* 2^k *equal parts by the points:*

$$t_\nu = a + \frac{(b-a)(\nu - 1)}{2^k}, \nu = 1, \ldots, m = 2^k + 1.$$

Let, further, $\Delta_k = \{t_0 < t_1 < \ldots t_{2^k+1}\}$. *Denote by* K_{Δ_k} *the cone generated by the corresponding Chebyshev system. It is clear that* $K_{\Delta_{K_1}} \supset K_{\Delta_2} \supset \ldots \supset C$ *and hence ,* $K^*_{\Delta_1} \subset K^*_{\Delta_2} \subset \ldots C^*$.

THEOREM 7. *Let* $x \in int(C)$. *Then the sequence* $I_{K_{\Delta_k}}(x), k = 1, 2, \ldots,$*is monotonically increasing and*

$$I_{K_{\Delta_k}} \to I_C(x), k \to \infty.$$

Moreover, the convergence is uniform on any compact subset in $int(C)$. *The result easily follows from the fact that entries of the corresponding skew-symmetric matrices as described in Theorems 5,6 converge to corresponding integrals* $d_2(\alpha, \beta)$.

4. Extensions. Suppose that instead of (2.1) the following weaker condition is satisfied:

$$\det(u_i(t_{j_k})) \geq 0, \forall 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m.$$

If the corresponding $n+1$ by m matrix

$$[v_1, \ldots v_m]$$

has the maximal rank $n+1$ (which is equivalent to say that $\det[v_{j_1}, \ldots v_{j_{n+1}}] > 0$ for at least one ordered set $1 \leq j_1 < j_2 < \ldots l_{j_{n+1}} \leq m$), then $v_1, \ldots, v_m$ is called *weak Chebyshev system*(using a terminology from [3]). We will assume that the cone K

$$K = \{x \in \mathbf{R^{n+1}} : \langle x, v_i \rangle \geq 0, \forall i\}$$

is pointed, i.e. $int(K) \neq \emptyset$ and K does not contain straight lines (observe that if $v_1, \ldots, v_m$ is a Chebyshev system than this is always the case [4]). Our last assumption is that $v_i \neq 0, \forall i$. One can easily see that under these assumptions

$$int(K) = \{x \in \mathbf{R^{n+1}} : \langle x, v_i \rangle > 0, \forall i\}.$$

As it is well known , the dual cone K^* is also pointed. Following a classical argument of Schoenberg [9],consider for a given $0 < q < 1$ an $m \times m$ matrix

$$X(q) = \|x_{ij}(q)\|, x_{ij}(q) = q^{(i-j)^2} i, j = 1, 2, \ldots m.$$

The major property of $X(q)$ which we are going to use is that all minors of $X(q)$ are positive. This is easily follows from the identity:

$$X(q) = diag(q, q^{2^2}, \ldots, q^{i^2}, \ldots, q^{m^2}) W(q) diag(q, q^{2^2}, \ldots, q^{m^2}),$$

where $W(q) = \|w_{ij}(q)\|, w_{ij}(q) = (q^{-2i})^j, ij, = 1, 2, \ldots m$ is a Vandermonde matrix. Consider $(n+1) \times m$ matrix

$$[v_1(q), \ldots v_m(q)] = [v_1, \ldots, v_m] X(q).$$

Observe that

$$v_i(q) = \sum_{j=1}^{m} x_{ij}(q) v_j. \tag{4.1}$$

Denote by $\delta(i_1, \ldots i_{n+1}; j_1, \ldots j_{n+1})(q)$ the minor of $X(q)$ corresponding to rows $i_1 < i_2 < \ldots < i_{n+1}$ and columns $j_1 < j_2 < \ldots j_{n+1}$. By Binet-Cauchy formula:

$$\det[v_{i_1}(q), \ldots, v_{i_{n+1}}(q)] = \sum_{1 \le j_1 < j_2 < \ldots < j_{n+1} \le m} \delta(j_1, \ldots, j_{n+1}; i_1, \ldots, i_{n+1})(q)) \det[v_{j_1}, \ldots, v_{j_{n+1}}],$$

$1 \le i_1 < i_2 < \ldots < i_{n+1} \le m$.

Hence, if at least one $\det[v_{j_1}, \ldots v_{j_{n+1}}] > 0$, (which is the case if K is pointed), then $v_1(q), \ldots v_m(q)$ form a Chebyshev system for any $0 < q < 1$. Since $X(q) \to I$, when $q \to 0$ (I is the identity matrix), we can use this construction to calculate the characteristic function for the cone K. Denote by $K(q)$ the cone generated by the Chebyshev system $v_1(q), \ldots, v_m(q)$. It is obvious from (4.1) , that $K \subset K(q), int(K) \subset int(K(q)), 0 < q < 1$. Thus, $K(q)^* \subset K^*, int(K(q)^*) \subset int(K^*)$. Given $A \subset \mathbf{R^{n+1}}$, denote by $\chi_A : \mathbf{R^{n+1}} \to \mathbf{R}$ the function such that $\chi_A(y) = 1$ if $y \in A, \chi_A(y) = 0$ otherwise.

We will need the following geometrically evident Lemma.

Lemma 3. *Let $M = cone(w_1, w_2, \ldots w_l)$ be a finitely generated cone with a nonempty interior in a finite-dimensional vector space V. Then $x \in int(M)$ if and only if it admits a representation of the form:*

$$x = \sum_{i=1}^{l} \lambda_i w_i,$$

where all λ_i are strictly positive.

Proof Let $e_1, \ldots, e_l$ be a canonical basis in $\mathbf{R^l}$. Consider a linear map $B : \mathbf{R^l} \to V, Be_i = w_i, i = 1, \ldots, l$. Then $M = B(\mathbf{R^l_+})$ and hence $int(M) = B(int(\mathbf{R^l_+}))$ (see e.g. [8]). The result follows.

Lemma 4. *For any sequence $q_1 > q_2 > \ldots$,converging to zero, we have:*

$$\lim \chi_{int(K(q_i)^*)}(y) = \chi_{int(K^*)}(y), i \to \infty, \forall y \in \mathbf{R^{n+1}}. \tag{4.2}$$

Proof Let $y \in int(K^*)$. Then by Lemma 2 y admits a representation of the form:

$$y = \sum_{i=1}^{m} \lambda_i v_i, \tag{4.3}$$

where all λ_i are strictly positive. Let us show that $y \in int(K(q)^*)$ for all sufficiently small q. By Lemma 2 it suffices to indicate a representation

$$y = \sum_{i=1}^{m} \lambda_i(q) v_i(q) \tag{4.4}$$

with $\lambda_i(q) > 0, \forall i$. Comparing (4.2) with (4.3), we see that one can take $\lambda_i(q)$ to be solutions of the system of linear equations:

$$X(q)\mu = \lambda, \tag{4.5}$$

where $\mu = (\mu_1, \dots, \mu_m)^T, \lambda = (\lambda_1, \dots, \lambda_m)^T$. But $X(q)$ tends to the identity matrix when q tends to zero. Hence, $X(q)^{-1}$ tends to the identity matrix, when q tends to zero. But then all components of the solution to (4.4) will be positive for sufficiently small q (since all λ_i are positive). This provides a representation (4.4) for all sufficiently small q. Thus, $y \in int(K(q)^*)$ for all sufficiently small q. The result follows.

THEOREM 8. *Let $x \in int(K)$. Then (in the notation of previous Lemma):*

$$I_{K_{q_i}}(x) \to I_K(x), i \to \infty$$

. **Proof** We have:

$$I_{K_{q_i}}(x) = \int_{K^*_{q_i}} e^{-\langle x,y \rangle} d\mu(y) = \int_{int(K^*_{q_i})} e^{-\langle x,y \rangle} d\mu(y) =$$

$$\int_{\mathbf{R^{n+1}}} \chi_{int(K^*_{q_i})}(y) e^{-\langle x,y \rangle} d\mu(y) \to \int_{\mathbf{R^{n+1}}} \chi_{int(K^*)}(y) e^{-\langle x,y \rangle} d\mu(y) =$$

$$\int_{K^*} e^{-\langle x,y \rangle} d\mu(y).$$

The convergence follows by Lemma 4 , Lebesgue dominated convergence theorem and by (obvious) inequalities:

$$0 \leq \chi_{int(K^*_{q_i})} \leq \chi_{int(K^*)}.$$

COROLLARY 1. *Theorems 2-6 hold true for a weak Chebyshev system $v_1, \dots, v_m$ satisfying two additional conditions:*

- *The corresponding cone K is proper;*
- *All vectors v_i are nonzero.*

Proof Since Theorem 2 is true for each cone K_q,it is also true for K:it suffices to take limit $q \to 0$ and apply previous Theorem . The remaining Theorems are derived from Theorem 2 exactly as in the case of a Chebyshev system.

Example 5. *Consider the following system of functions:*

$$t^l, t^{l-1}, \ldots, t, 1, (t-x_1)_+^l, (t-x_2)_+^l, \ldots, (t-x_r)_+^l$$

on the interval $[-1,1]$*. Here* $-1 < x_1 < x_2 < \ldots < x_r < 1$ *and* $x_+ = \max\{x, 0\}$*. The linear combinations of these functions are called spline polynomials of degree* l *with knots* $x_1, \ldots, x_r$*. These functions form a weak Chebyshev system (see e.g. [3]). Thus, we can apply our results to its discretizations.*

5. Appendix. Proof of Theorem 3. (Compare with [6]) For $i = 1, \ldots, m$, denote $\frac{v_i}{\langle x, v_i \rangle}$ by v_i' and observe that, for $i < j$, $v_i' - v_j' = a_i + a_{i+1} + \cdots + a_{j-1}$. Therefore,

$$\frac{\det[v_{j_1}, \ldots v_{j_{n+1}}]}{\prod_{k=1}^{n+1} \langle x, v_{j_k} \rangle} = \det[v'_{j_1}, \ldots v'_{j_{n+1}}] = \det[v'_{j_1} - v'_{j_2}, \ldots, v'_{j_{n+1}} - v'_{m+1}]$$

$$= \sum_{j_1 \le l_1 < j_2 \le l_2 < j_3 \le \cdots \le j_{n+1} \le l_{n+1} \le m} \det[a_{l_1}, a_{l_2}, \ldots, a_{l_{n+1}}].$$

To complete the proof, one needs to show that, for every collection of indices $I := \{1 \le l_1 < l_2 \ldots < l_{n+1} \le m\}$, there exists exactly one $(n+1)$-tuple $\theta(I) = \{j_1 < j_2 < \ldots j_{n+1}\} \in \Theta_l$ such that $j_1 \le l_1 < j_2 \le l_2 < j_3 \le \ldots \le j_{n+1} \le l_{n+1} \le m$. This can be done by induction on m and n. Indeed, let k be the smallest integer such that $l_{k+1} > k+1$. Then $l_i = i$ for $i = 1, ..., k$ and, therefore, $\theta(I)$ must also satisfy $j_i = i$ for $i = 1, ..., k$. If $\theta(I) \in \Theta_l$, then the parity of the size of the block that contains 1 is equal to the parity of $n+1$. Thus, if $k \equiv n+1 (\mathrm{mod} 2)$, we must have $k+1 < j_{k+1} \le l_{k+1}$, otherwise $j_{k+1} = k+1$ and $l_{k+1} < j_{k+2} \le l_{k+2}$.

Let k' be equal to $k+1$ if $k \equiv n+1 (\mathrm{mod} 2)$ and to $k+2$ otherwise. We see that $j_{k'}$ is the smallest index of the second block in $\theta(I)$. Since this block must be of even size and since $j_{k'} \le l_{k'} < j_{k'+1}$, we conclude that $j_{k'} = l_{k'}$ and $j_{k'+1} = l_{k'}+1 \le l_{k'+1}$. Now if we define m', n' and l_i', j_i' $(i = 1, \ldots, n-k')$ by setting $m' = m - l_{k'+1}$, $n' = n - k' - 1$, $l_{k'+i+1} = l_{k'+1} + l_i'$ and $j_{k'+i+1} = l_{k'+1} + j_i'$, then $1 \le l_1' < \ldots < l'n' + 1 \le m'$ and $\theta(I)$ satisfies needed properties if and only iff the collection $j_1' < \ldots < j_{n+1}'$ belongs to Θ_l and $j_1' \le l_1' < j_2' \le l_2' < j_3' \le \ldots \le j'_{n'+1} \le l'_{n'+1} \le m'$. The statement now follows from the induction assumption.

Remark 3. *The proof of Theorem 4 is completely analogous to the proof of Theorem 3.*

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