Sensitivity analysis of parameterized variational inequalities

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Abstract

We discuss in this paper continuity and differentiability properties of solutions of parameterized variational inequalities (generalized equations). To this end we use an approach of formulating variational inequalities in a form of optimization problems and applying a general theory of perturbation analysis of parameterized optimization problems.

Key words: variational inequalities, gap functions, sensitivity analysis, second order regularity, second order growth condition, locally upper Lipschitz and Hölder continuity, directional differentiability, prox-regularity

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1 Introduction

In this paper we discuss continuity and differentiability properties of solutions of the parameterized variational inequalities

$$F(x, u) \in N_K(x). \tag{1.1}$$

Here K is a nonempty convex closed subset of $\mathbb{R}^n$, $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a mapping, U is a normed space, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$, and $N_K(x)$ denotes the normal cone to K at $\bar{x}$,

$$N_K(\bar{x}) := \{y : \langle y, x - \bar{x} \rangle \le 0, \ \forall x \in K\}, \quad \text{if } \bar{x} \in K,$$

and $N_K(\bar{x}) := \emptyset$ if $\bar{x} \notin K$. We denote variational inequality (1.1) by $VI(K, F_u)$ and by $\mathrm{Sol}(K, F_u)$ its set of solutions, i.e., $\bar{x} \in \mathrm{Sol}(K, F_u)$ iff $\bar{x} \in K$ and $\langle F(\bar{x}, u), x - \bar{x} \rangle \le 0$ for all $x \in K$. For a reference value $u_0 \in U$ of the parameter vector, we often drop the corresponding subscript in the above notation. In particular, $F(\cdot) := F(\cdot, u_0)$ and $VI(K, F)$ corresponds to the reference variational inequality

$$F(x) \in N_K(x). \tag{1.2}$$

It is well known that for optimization problems, $VI(K, F)$ represents first order optimality conditions. That is, consider the optimization problem

$$\operatorname*{Min}_{x \in K} f(x), \tag{1.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. We have that if x_0 is a locally optimal solution of (1.3), then $x_0 \in \mathrm{Sol}(K, F)$ with $F(\cdot) := -\nabla f(\cdot)$.

There is a large literature on all aspects of the theory of variational inequalities. It will be beyond the scope of this paper to give a survey of all relevant results. In that respect we may refer to the recent comprehensive monograph by Facchinei and Pang [5] and references therein. We also do not investigate existence of solutions of $VI(K, F_u)$ and concentrate on continuity and differentiability properties of such solutions if they do exist. For a discussion of various conditions ensuring nonemptiness of the solution set $\mathrm{Sol}(K, F_u)$ we refer to [5, section 2.2]. For example, a simple sufficient condition for existence of a solution of $VI(K, F_u)$ is that $F(\cdot, u)$ is continuous and the set K is bounded, and hence compact, [10]. In case the set K is *polyhedral*, perturbation theory of $VI(K, F_u)$ is thoroughly developed notably by Robinson [15],[16],[17]. Much less is known about continuity and differentiability properties of the solution multifunction $u \mapsto \mathrm{Sol}(K, F_u)$ for a general, not necessarily polyhedral, set K. Some results of that type were obtained in [21] by a reduction approach. It is clear from a general perturbation theory of parameterized optimization problems (see [2] and references therein) and results presented in [21] that an additional term representing curvature of the set K should appear in the corresponding formulas.

In this paper we use an approach of formulating $VI(K, F_u)$ in a form of an optimization problem by employing a (regularized) gap function. Consequently, we investigate properties of the solution mapping by applying some recent results from sensitivity analysis of optimization problems. For perturbation theory of optimization problems we use, as a reference, Bonnans and Shapiro [2]. We also discuss the case where the set K is given in the form

$$K := \{x : G(x) \in Q\} = G^{-1}(Q), \tag{1.4}$$

where $G : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable mapping and $Q \subset \mathbb{R}^m$ is a closed convex set. Of course, so defined the set K may be nonconvex. We then define $N_K(\bar{x})$ as the negative polar cone of the tangent cone[1] $T_K(\bar{x})$ to K at x_0. Sensitivity analysis of such variational inequalities (generalized equations) was studied in Robinson [18] for the case where Q is polyhedral and a nondegeneracy condition holds, and in Shapiro [21] also under a nondegeneracy assumption and cone reducibility of the set Q. Klatte [7] and Klatte and Kummer [8],[9] studied the case where $Q := \mathbb{R}^n_+$ and the mapping $G(x, u)$ may also depend on the parameter vector. Sensitivity analysis of variational inequalities in terms of various coderivative mappings was studied in Levy [11],[12] and Levy and Rockafellar [13].

Unless stated otherwise it will be assumed throughout the paper that $F(\cdot, \cdot)$ is *locally Lipschitz continuous*. We use the following notation and terminology. The space $\mathbb{R}^n$ is equipped with the Euclidean norm $\|x\| := \langle x, x\rangle^{1/2}$. It is said that $F(\cdot, \cdot)$ is directionally differentiable, at a point $(x_0, u_0) \in \mathbb{R}^n \times U$, if the limit

$$F'((x_0, u_0), (h, p)) := \lim_{t \downarrow 0} \frac{F(x_0 + th, u_0 + tp) - F(x_0, u_0)}{t}$$

exists for all $(h, p) \in \mathbb{R}^n \times U$. If $F(\cdot, \cdot)$ is differentiable, we denote by $DF(x, u)$ its differential at a point (x, u), i.e., $DF(x, u)(h, p) = \nabla_x F(x, u)h + \nabla_u F(x, u)p$. By $D^2G(x)(h, h)$ we denote the quadratic form associated with the second order derivative of a mapping $G : \mathbb{R}^n \to \mathbb{R}^m$ at x. For a point $x \in \mathbb{R}^n$ we denote by $B(x, r) := \{y : \|y - x\| \le r\}$ the ball of radius r centered at x, by $\text{dist}(x, K)$ the distance from x to the set K, and by $P_K(x)$ the metric projection of x onto K, i.e., $P_K(x)$ is the closest point of K to x. Of course, $\text{dist}(x, K) = \|x - P_K(x)\|$. For a set $S \subset \mathbb{R}^n$ we denote by $\text{cl}(S)$ its topological closure, by

$$I_S(x) := \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{if } x \notin S, \end{cases}$$

[1]For a nonconvex set K there are several concepts of tangent cone $T_K(\bar{x})$. Unless stated otherwise we assume that $T_K(\bar{x})$ is the contingent (Bouligand) cone.

its indicator function, and by

$$\sigma(x, S) := \sup_{y \in S} \langle x, y \rangle$$

its support function. An extended real valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and its domain $\operatorname{dom} f := \{x : f(x) < +\infty\}$ is nonempty. If f is convex, we denote by $\partial f(x)$ its subdifferential at $x \in \operatorname{dom} f$. For a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ we denote by $A^* : \mathbb{R}^m \to \mathbb{R}^n$ its adjoint mapping, i.e., $\langle y, Ax \rangle = \langle A^* y, x \rangle$ for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. By $\mathcal{S}^n$ we denote the space of $n \times n$ symmetric matrices.

2 Preliminary results

With the variational inequality $VI(K, F)$ is associated the *(regularized) gap function*

$$\gamma(x) := \inf_{y \in K} \left\{ \langle F(x), x - y \rangle + \tfrac{1}{2} \|x - y\|^2 \right\}. \tag{2.1}$$

Clearly for $y = x$ we have that value of the function in the right hand side of (2.1) is zero, and hence $\gamma(x) \le 0$ for any $x \in K$. It is possible to show ([5, Theorem 10.2.3]) that $x_0 \in K$ is a solution of $VI(K, F)$ iff $\gamma(x_0) = 0$. That is, $x_0 \in \operatorname{Sol}(K, F)$ iff $\gamma(x_0) = 0$ and x_0 is an optimal solution of the optimization problem

$$\operatorname*{Max}_{x \in K} \gamma(x). \tag{2.2}$$

Let us discuss some properties of the function $\gamma(x)$. It will be convenient to write this gap function in the form $\gamma(x) = \vartheta(x, F(x))$, where

$$\vartheta(x, a) := \inf_{y \in K} \left\{ g(x, a, y) := \langle a, x - y \rangle + \tfrac{1}{2} \|x - y\|^2 \right\}. \tag{2.3}$$

We can write the function $g(x, a, y)$ in the form

$$g(x, a, y) = \tfrac{1}{2} \|x + a - y\|^2 - \tfrac{1}{2} \|a\|^2.$$

It follows that the minimization problem in the right hand side of (2.3) has unique optimal solution

$$\bar{y}(x, a) = P_K(x + a). \tag{2.4}$$

Therefore, $\vartheta(x, a)$, and hence $\gamma(x)$, is real valued for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$. Let us also observe that if $x \in K$ and $a \in N_K(x)$, then $P_K(x + a) = x$, and hence $\bar{y}(x, a) = x$. In particular, if $x_0 \in \operatorname{Sol}(K, F)$, then $\bar{y}(x_0, F(x_0)) = x_0$.

Since the set K is convex, the mapping $P_K(\cdot)$ is Lipschitz continuous (with Lipschitz constant one), and hence $\bar{y}(\cdot,\cdot)$ is Lipschitz continuous. It follows then by Danskin theorem, [4], that $\vartheta(\cdot,\cdot)$ is differentiable with $D\vartheta(x,a) = Dg(x,a,\bar{y})$, where $\bar{y} = \bar{y}(x,a)$. By straightforward calculations we obtain that

$$\nabla_x \vartheta(x,a) = a + x - \bar{y}(x,a) \text{ and } \nabla_a \vartheta(x,a) = x - \bar{y}(x,a), \tag{2.5}$$

and hence $D\vartheta(\cdot,\cdot)$ is Lipschitz continuous. Consider a point $x_0 \in \mathrm{Sol}(K,F)$, and let $x = x_0 + h$. We have that $\bar{y}(x_0, F(x_0)) = x_0$, and since $F(\cdot)$ is assumed to be locally Lipschitz continuous, $F(x) = F(x_0) + r(h)$ with $r(h) = O(\|h\|)$. Consequently, for $a_0 := F(x_0)$,

$$\begin{array}{rcl} \gamma(x) - \gamma(x_0) & = & \vartheta(x_0 + h, a_0 + r(h)) - \vartheta(x_0, a_0) \\ & = & D\vartheta(x_0,a_0)(h, r(h)) + O(\|h\|^2 + \|r(h)\|^2) \\ & = & \langle F(x_0), h\rangle + O(\|h\|^2). \end{array} \tag{2.6}$$

It follows that $\gamma(\cdot)$ is differentiable at x_0 and $\nabla\gamma(x_0) = F(x_0)$.

Let us discuss now second order differentiability properties of $\vartheta(\cdot,\cdot)$ and $\gamma(\cdot)$. In general the metric projection $P_K(\cdot)$ is not differentiable everywhere, and hence $\vartheta(\cdot,\cdot)$ is not twice differentiable. Therefore we study second order directional derivatives of $\vartheta(\cdot,\cdot)$ and $\gamma(\cdot)$. It is said that a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *second order Hadamard directionally differentiable*, at a point x, if for any $h \in \mathbb{R}^n$ the limit

$$\lim_{\substack{t \downarrow 0 \\ h' \to h}} \frac{f(x + th') - f(x) - tDf(x)h'}{\frac{1}{2}t^2}$$

exists. In case it exists, we denote this limit by $f''(x,h)$. Note that if $f''(x,h)$ exists, it is continuous in h.

Consider the second order tangent set to K at a point $x \in K$ in a direction h:

$$T_K^2(x,h) := \left\{ w \in \mathbb{R}^n : \mathrm{dist}(x + th + \tfrac{1}{2}t^2 w, K) = o(t^2) \right\}.$$

Note that the set $T_K^2(x,h)$ can be nonempty only if $h \in T_K(x)$, and for any $t > 0$,

$$T_K^2(x, th) = t^2 T_K^2(x,h). \tag{2.7}$$

Definition 2.1 *It is said that the set K is* second order regular*, at a pont $\bar{x} \in K$, if for any $h \in T_K(\bar{x})$ and any sequence $x_k \in K$ of the form $x_k := \bar{x} + t_k h + \frac{1}{2}t_k^2 w_k$, where $t_k \downarrow 0$ and $t_k w_k \to 0$, the following condition holds*

$$\lim_{k\to\infty} \mathrm{dist}\left(w_k, T_K^2(\bar{x}, h)\right) = 0.$$

The above concept of second order regularity (for general, not necessarily convex, sets) was developed in [1],[2]. The class of second order regular sets is quite large, it contains polyhedral sets, cones of positive semidefinite matrices, etc. Note that second order regularity of K at $\bar{x}$ implies that $T^2_K(\bar{x},h)$ is nonempty for any $h \in T_K(\bar{x})$.

Consider the optimization (minimization) problem in the right hand side of (2.3). At a point $(x_0, a_0, \bar{y})$, where $x_0 \in K$, $a_0 \in N_K(x_0)$ and $\bar{y} = x_0$, the function $g(\cdot,\cdot,\cdot)$ has the following second order Taylor expansion:

$$g(x_0 + d, a_0 + r, \bar{y} + h) = \langle a_0, d - h\rangle + \langle r, d - h\rangle + \tfrac{1}{2}\|d - h\|^2.$$

(Note that the above expansion is exact since the function $g(\cdot,\cdot,\cdot)$ is quadratic.) Therefore, the corresponding so-called *critical cone*, associated with the problem of minimization of $g(x_0, a_0, \cdot)$ over the set K, is defined as

$$C(x_0) := \{h : \langle F(x_0), h\rangle = 0,\ h \in T_K(x_0)\}. \tag{2.8}$$

We have the following result, [2, Theorem 4.133].

Proposition 2.1 *Let $x_0 \in K$, $a_0 \in N_K(x_0)$, and suppose that the set K is second order regular at x_0. Then $\vartheta(\cdot,\cdot)$ is second order Hadamard directionally differentiable at (x_0, a_0) and $\vartheta''((x_0, a_0), (d, r))$ is equal to the optimal value of the problem:*

$$\operatorname*{Min}_{h \in C(x_0)} \left\{ 2\langle r, d - h\rangle + \|d - h\|^2 - \sigma\left(a_0, T^2_K(x_0, h)\right)\right\}. \tag{2.9}$$

From now on we assume that that the mapping $F(\cdot)$ is *directionally differentiable* at x_0. Then, since it is assumed that $F(\cdot)$ is locally Lipschitz continuous, $F(\cdot)$ is directionally differentiable at x_0 in the Hadamard sense, i.e.,

$$\lim_{\substack{t \downarrow 0 \\ d' \to d}} \frac{F(x_0 + td') - F(x_0)}{t} = F'(x_0, d), \tag{2.10}$$

and $F'(x_0, \cdot)$ is Lipschitz continuous.

Proposition 2.2 *Let $x_0 \in \mathrm{Sol}(K, F)$ and suppose that the set K is second order regular at x_0. Then $\gamma(\cdot)$ is second order Hadamard directionally differentiable at x_0 and $\gamma''(x_0, d) = \nu(d)$, where $\nu(d)$ denotes the optimal value of the problem:*

$$\operatorname*{Min}_{h \in C(x_0)} \left\{ 2\langle F'(x_0, d), d - h\rangle + \|d - h\|^2 - \sigma\left(F(x_0), T^2_K(x_0, h)\right)\right\}. \tag{2.11}$$

Proof. Because of (2.10) and since $D_a\vartheta(x_0, F(x_0)) = 0$ and $D\vartheta(x, a)$ is Lipschitz continuous, we have that for any sequences $t_k \downarrow 0$ and $d_k \to d$,

$$\vartheta(x_0 + t_k d_k, F(x_0 + t_k d_k)) = \vartheta\left(x_0 + t_k d_k, F(x_0) + t_k F'(x_0, d_k)\right) + o(t_k^2),$$

and $F'(x_0, d_k) \to F'(x_0, d)$. Consequently, the limit

$$\lim_{k\to\infty} \frac{\vartheta(x_0 + t_k d_k, F(x_0 + t_k d_k)) - \vartheta(x_0, F(x_0)) - t_k \langle F(x_0), d_k\rangle}{\frac{1}{2} t_k^2}$$

is equal to $\vartheta''\left((x_0, F(x_0)), (d, F'(x_0, d))\right)$. Together with Proposition 2.1, this completes the proof. ∎

Consider a point $x_0 \in \mathrm{Sol}(K, F)$ and let us discuss properties of the function

$$\phi(\cdot) := -\sigma\left(F(x_0), T_K^2(x_0, \cdot)\right). \tag{2.12}$$

Convexity of the set K implies that the (extended real valued) function $\phi(\cdot)$ is convex (see [2, Proposition 3.48]). We also have that (cf., [3])

$$T_K^2(x_0, h) + T_{T_K(x_0)}(h) \subset T_K^2(x_0, h) \subset T_{T_K(x_0)}(h), \tag{2.13}$$

$$T_{T_K(x_0)}(h) = \mathrm{cl}\{T_K(x_0) + \mathrm{sp}(h)\}, \tag{2.14}$$

where $\mathrm{sp}(h)$ denotes the linear space generated by vector h. Since for any $h \in C(x_0)$, $\langle F(x_0), h\rangle = 0$ and $\langle F(x_0), y\rangle \le 0$ for any $y \in T_K(x_0)$, it follows that

$$\sigma\left(F(x_0), T_K^2(x_0, h)\right) \le 0, \quad \forall\, h \in C(x_0). \tag{2.15}$$

Therefore $\phi(h) \ge 0$ for any $h \in C(x_0)$, and hence $\phi(h)$ is finite valued for any $h \in C(x_0)$ provided that the second order tangent set $T_K^2(x_0, h)$ is nonempty. Note that, in particular, $T_K^2(x_0, h)$ is nonempty for all $h \in T_K(x_0)$ if K is second order regular at x_0. For $h \notin T_K(x_0)$ we have that $T_K^2(x_0, h) = \emptyset$, and hence $\phi(h) = +\infty$. By the left hand side inclusion of (2.13) and by (2.14) we have that $\phi(h) = -\infty$ if $h \in T_K(x_0) \setminus C(x_0)$ and $T_K^2(x_0, h)$ is nonempty. We also have that $T_K^2(x_0, 0) = T_K(x_0)$ and hence $\phi(0) = 0$. Since we are interested in values $\phi(h)$ only for $h \in C(x_0)$ let us consider

$$\hat{\phi}(h) := \begin{cases} -\sigma\left(F(x_0), T_K^2(x_0, h)\right), & \text{if } h \in C(x_0), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.16}$$

The function $\hat{\phi}(\cdot)$ is proper, coincides with $\phi(\cdot)$ on the set $C(x_0)$, and hence is convex. Also we have by (2.7) that

$$\hat{\phi}(th) = t^2 \hat{\phi}(h) \quad \text{for any } \; h \in \mathbb{R}^n \; \text{ and } \; t > 0. \tag{2.17}$$

The first-order necessary condition for x_0 to be an optimal solution of (2.2) is that $\nabla\gamma(x_0) \in N_K(x_0)$. Since $\nabla\gamma(x_0) = F(x_0)$, this follows, of course, from the assumption $x_0 \in \mathrm{Sol}(K, F)$. Also the critical cone associated with problem (2.2) at the point x_0 is the same as the one defined in (2.8).

Definition 2.2 *We say that the* second order growth *condition holds, for the problem* (2.2) *at* x_0*, if there exists a constant* $c > 0$ *and a neighborhood* N *of* x_0 *such that*

$$-\gamma(x) \geq c\|x - x_0\|^2, \quad \forall\, x \in K \cap N. \tag{2.18}$$

Of course, since $\gamma(x_0) = 0$, condition (2.18) implies that x_0 is a *locally unique* optimal solution of the problem (2.2).

Suppose that the set K is second order regular at x_0. It follows from Proposition 2.2 that the function $-\gamma(\cdot)$ is second order regular and (see [2, Remark 4.134])

$$\gamma''(x_0; d, w) = D\gamma(x_0)w + \nu(d), \tag{2.19}$$

where

$$\gamma''(x_0; d, w) := \lim_{t \downarrow 0} \frac{\gamma(x_0 + td + \frac{1}{2}t^2 w) - \gamma(x_0) - tD\gamma(x_0)d}{\frac{1}{2}t^2} \tag{2.20}$$

denotes the parabolic second order directional derivative. We have then the following necessary and sufficient condition for the second order growth property (2.18):

$$\inf_{w \in T_K^2(x_0, d)} (-\gamma)''(x_0; d, w) > 0, \quad \forall\, d \in C(x_0) \setminus \{0\}, \tag{2.21}$$

(see [2, Proposition 3.105]). Since $D\gamma(x_0)w = \langle F(x_0), w\rangle$ and (2.19), the above condition (2.21) can be written in the following equivalent form

$$\nu(d) + \sigma\left(F(x_0), T_K^2(x_0, d)\right) < 0, \quad \forall\, d \in C(x_0) \setminus \{0\}. \tag{2.22}$$

Consider the following conditions:

(C1) To every $d \in C(x_0) \setminus \{0\}$ corresponds $h \in C(x_0)$ such that

$$2\langle F'(x_0, d), d - h\rangle + \|d - h\|^2 + \hat{\phi}(h) < \hat{\phi}(d). \tag{2.23}$$

(C2) The system

$$0 \in -2F'(x_0, d) + \partial\hat{\phi}(d) \tag{2.24}$$

has only one solution $d = 0$.

Note that since $F'(x_0, 0) = 0$ and $d = 0$ is a minimizer of $\hat{\phi}(\cdot)$, $d = 0$ is always a solution of (2.24).

Theorem 2.1 *Let* $x_0 \in \mathrm{Sol}(K, F)$ *and consider* $\hat{\phi}(\cdot)$ *defined in* (2.16)*. Suppose that the set* K *is second order regular at* x_0*. Then conditions* (C1) *and* (C2) *are equivalent to each other and are necessary and sufficient for the second order growth condition* (2.18) *to hold.*

Proof. Because of (2.11), condition (C1) is equivalent to condition (2.22) and hence, by the above discussion, is necessary and sufficient for the second order growth condition (2.18). Now for a given $d \in C(x_0)$ consider the function

$$\psi(h) := 2\langle F'(x_0, d), d - h\rangle + \|d - h\|^2 + \hat{\phi}(h). \tag{2.25}$$

Clearly for $h = d$ we have that $\psi(d) = \hat{\phi}(d)$. Since $\psi(h) = +\infty$ for any $h \in \mathbb{R}^n \setminus C(x_0)$, we obtain that condition (C1) means that $\inf_{h\in\mathbb{R}^n} \psi(h) < \hat{\phi}(d)$. Equivalently this can be formulated as that d is not a minimizer of $\psi(\cdot)$ over $\mathbb{R}^n$. Since $\psi(\cdot)$ is convex, we have that d is a minimizer of $\psi(\cdot)$ iff $0 \in \partial\psi(d)$. Moreover,

$$\partial\psi(d) = -2F'(x_0, d) + \partial\hat{\phi}(d),$$

and hence (2.24) is a necessary and sufficient condition for d to be a minimizer of $\psi(\cdot)$. This shows equivalence of conditions (C1) and (C2), and hence completes the proof. ∎

For any $d \notin C(x_0)$ we have that $\hat{\phi}(d) = +\infty$ and hence $\partial\hat{\phi}(d) = \emptyset$. Therefore, the inclusion (2.24) should be verified only for $d \in C(x_0)$. Consider the following condition:

(C3) For all $d \in C(x_0) \setminus \{0\}$ it holds that $\varphi(d) \neq 0$, where

$$\varphi(d) := -\langle d, F'(x_0, d)\rangle + \hat{\phi}(d). \tag{2.26}$$

Proposition 2.3 *Condition* (C3) *implies conditions* (C1) *and* (C2)*. Suppose, further, that $F(\cdot)$ is differentiable at x_0 and the Jacobian matrix $\nabla F(x_0)$ is symmetric, and $\varphi(d) \geq 0$ for all $d \in C(x_0)$. Then conditions* (C1),(C2) *and* (C3) *are equivalent.*

Proof. Suppose that condition (C3) holds. Consider $d \in C(x_0) \setminus \{0\}$. In order to show the implication (C3) $\Rightarrow$ (C1) it will suffice to verify that d is not a minimizer of the function $\psi(\cdot)$ defined in (2.25). By (2.17) we have that for $t > -1$,

$$\psi((1 + t)d) = -2t\langle d, F'(x_0, d)\rangle + t^2\|d\|^2 + (1 + t)^2\hat{\phi}(d) = \psi(d) + q(t, d),$$

where

$$q(t, d) := 2\varphi(d)t + (\|d\|^2 + \hat{\phi}(d))t^2.$$

It follows that $q(t, d) < 0$ for all $t > 0$ sufficiently close to zero if $\varphi(d) < 0$, and $q(t, d) < 0$ for all $t < 0$ sufficiently close to zero if $\varphi(d) > 0$. This implies that d is not a minimizer of $\psi(\cdot)$.

Suppose now that $\varphi(d) \geq 0$ for all $d \in C(x_0)$, $F(\cdot)$ is differentiable at x_0 and $\nabla F(x_0) \in \mathcal{S}^n$. We have that if d is a minimizer of $\varphi(d)$ over $d \in C(x_0)$, then (2.24) holds by the first order necessary conditions. Therefore, condition (C2) implies that

$d = 0$ is the unique minimizer of $\varphi(d)$ over $d \in C(x_0)$, and hence that $\varphi(d) > 0$ for all $d \in C(x_0) \setminus \{0\}$. That is, (C2) implies (C3). Since (C3) $\Rightarrow$ (C2), it follows that (C2) and (C3) are equivalent. ∎

The following is a consequence of Theorem 2.1 and Proposition 2.3.

Corollary 2.1 *Suppose that the set K is second order regular at $x_0 \in \mathrm{Sol}(K, F)$, and either condition* (C2) *or* (C3) *holds. Then x_0 is a locally unique solution of $VI(K, F)$.*

If the set K is polyhedral, then for any $h \in T_K(x_0)$ we have that $x_0 + th \in K$ for all $t > 0$ small enough, and hence $0 \in T_K^2(x_0, h)$. It follows then by (2.13) that $T_K^2(x_0, h) = T_{T_K(x_0)}(h)$, and hence by (2.14) that $\sigma\left(F(x_0), T_K^2(x_0, h)\right) = 0$ for any $h \in C(x_0)$. In that case the function $\hat{\phi}(\cdot)$ becomes the indicator function of the set $C(x_0)$, and hence the system (2.24) takes the form:

$$0 \in -F'(x_0, d) + N_{C(x_0)}(d), \tag{2.27}$$

and $\varphi(d) = -\langle d, F'(x_0, d)\rangle$ for all $d \in C(x_0)$. Therefore, for polyhedral set K condition $\varphi(d) > 0$ for all $d \in C(x_0) \setminus \{0\}$ is equivalent to the condition:

$$\langle d, F'(x_0, d)\rangle < 0, \quad \forall\, d \in C(x_0) \setminus \{0\}. \tag{2.28}$$

It is shown in [5, Proposition 3.3.4] that condition (2.28) implies local uniqueness of solution x_0. Also if K is polyhedral and $F(\cdot)$ is affine, then the condition: "the system (2.27) has only one solution $d = 0$", is necessary and sufficient for local uniqueness of x_0 ([5, Proposition 3.3.7]). Since $\hat{\phi}(d) \geq 0$ for all $d \in C(x_0)$, in general condition (2.28) is stronger than condition (C3).

Remark 2.1 Consider the optimization problem (1.3) and the associated variational inequality representing first order optimality conditions for (1.3). Let x_0 be a locally optimal solution of (1.3), and hence $x_0 \in \mathrm{Sol}(K, F)$ with $F(\cdot) := -\nabla f(\cdot)$. Suppose that the function $f(\cdot)$ is twice continuously differentiable. Then for $d \in C(x_0)$ the function $\varphi(d)$ takes the form

$$\varphi(d) = D^2 f(x_0)(d, d) - \sigma\left(F(x_0), T_K^2(x_0, d)\right). \tag{2.29}$$

By second order necessary conditions we have that $\varphi(d) \geq 0$ for all $d \in C(x_0)$. Suppose, further, that the set K is second order regular at x_0. Then the second order growth condition for the optimization problem (1.3) holds at x_0 iff $\varphi(d) > 0$ for all $d \in C(x_0) \setminus \{0\}$. It follows that the second order growth condition (2.18) holds iff the corresponding second order growth condition for the optimization problem (1.3) is satisfied.

3 Continuity and differentiability properties of solutions

Consider the parameterized variational inequality (1.1). In this section we discuss continuity and differentiability properties of the multifunction $u \mapsto \mathrm{Sol}(K, F_u)$ at a point $x_0 \in \mathrm{Sol}(K, F_{u_0})$. We assume from now on that the mapping $F(\cdot,\cdot)$ is directionally differentiable at the point (x_0, u_0). The gap function associated with (1.1) is $\gamma(x,u) := \vartheta(x, F(x,u))$. By derivations similar to (2.6) one can show that $\gamma(\cdot,\cdot)$ is differentiable at (x_0, u_0) and

$$\nabla_x \gamma(x_0, u_0) = F(x_0) \;\text{ and }\; \nabla_u \gamma(x_0, u_0) = 0. \tag{3.1}$$

Suppose that the set K is second order regular at x_0. Then, similarly to Proposition 2.2, it is possible to show that $\gamma(x,u)$ is second order Hadamard directinally differentiable at (x_0, u_0) and $\gamma''((x_0,u_0),(d,p)) = \nu(d,p)$, where $\nu(d,p)$ denotes the optimal value of the problem

$$\operatorname*{Min}_{h \in C(x_0)} \left\{ 2\langle F'((x_0,u_0),(d,p)), d - h\rangle + \|d-h\|^2 - \sigma\left(F(x_0), T_K^2(x_0,h)\right)\right\}. \tag{3.2}$$

The main result of this section is given in the following theorem.

Theorem 3.1 *Suppose that the set K is second order regular at x_0. For a vector $p \in U$ consider a path $u(t) := u_0 + tp + o(t)$, $t \geq 0$. Let $t_k \downarrow 0$, $u_k := u(t_k)$, $\bar{x}_k \in \mathrm{Sol}(K, F_{u_k})$ and $\bar{d}$ be an accumulation point of the sequence $(\bar{x}_k - x_0)/t_k$. Then $\bar{d}$ is a solution of the system*

$$0 \in -2F'((x_0,u_0),(d,p)) + \partial\hat{\phi}(d). \tag{3.3}$$

Proof. Consider

$$\theta_k := \sup_{x \in K} \gamma(x, u_k). \tag{3.4}$$

For any $d \in C(x_0)$ and $w \in T_K^2(x_0,d)$ there is a sequence $\tilde{x}_k \in K$ such that $\tilde{x}_k = x_0 + t_k d + \frac{1}{2}t_k^2 w + o(t_k^2)$. We have then

$$\theta_k \geq \gamma(\tilde{x}_k, u_k) = \tfrac{1}{2}t_k^2\left[\langle F(x_0), w\rangle + \nu(d,p)\right] + o(t_k^2).$$

It follow that for any $d \in C(x_0)$,

$$\liminf_{k\to\infty} (\tfrac{1}{2}t_k^2)^{-1}\theta_k \geq \langle F(x_0), w\rangle + \nu(d,p), \quad \forall\, w \in T_K^2(x_0,d).$$

By taking supremum of the right hand side of the above inequality with respect to $w \in T_K^2(x_0, d)$, we obtain

$$\liminf_{k\to\infty} (\tfrac{1}{2}t_k^2)^{-1}\theta_k \geq \nu(d,p) + \sigma\left(F(x_0), T_K^2(x_0,d)\right), \quad \forall\, d \in C(x_0). \tag{3.5}$$

On the other hand suppose that the sequence t_k is such that $(\bar{x}_k - x_0)/t_k \to \bar{d}$. It follows that $\bar{d} \in T_K(x_0)$. Also we have that $F(\bar{x}_k, u_k) \in N_K(\bar{x}_k)$, and hence

$$\langle F(\bar{x}_k, u_k), (\bar{x}_k - x_0)/t_k \rangle \geq 0.$$

By passing to the limit we obtain $\langle F(x_0), \bar{d}\rangle \geq 0$. Moreover, $F(x_0) \in N_K(x_0)$, and hence by similar arguments $\langle F(x_0), \bar{d}\rangle \leq 0$. It follows that $\langle F(x_0), \bar{d}\rangle = 0$, that is $\bar{d} \in C(x_0)$.

We can write $\bar{x}_k = x_0 + t_k\bar{d} + \frac{1}{2}t_k^2 w_k$, where $w_k := (\frac{1}{2}t_k^2)^{-1}(\bar{x}_k - x_0 - t_k\bar{d})$. Note that $t_k w_k \to 0$, and hence by second order regularity of K at x_0, $\operatorname{dist}\left(w_k, T_K^2(x_0,\bar{d})\right) \to 0$. Then

$$\theta_k = \gamma(\bar{x}_k, u_k) = \tfrac{1}{2}t_k^2\left[\langle F(x_0), w_k\rangle + \nu(\bar{d}, p)\right] + o(t_k^2),$$

and hence

$$\limsup_{k\to\infty} (\tfrac{1}{2}t_k^2)^{-1}\theta_k \leq \nu(\bar{d},p) + \sigma\left(F(x_0), T_K^2(x_0,\bar{d})\right). \tag{3.6}$$

We obtain that

$$\lim_{k\to\infty} (\tfrac{1}{2}t_k^2)^{-1}\theta_k = \nu(\bar{d},p) + \sigma\left(F(x_0), T_K^2(x_0,\bar{d})\right), \tag{3.7}$$

and hence $\bar{d}$ is an optimal solution of the optimization problem

$$\operatorname*{Max}_{d\in C(x_0)} \nu(d,p) + \sigma\left(F(x_0), T_K^2(x_0,d)\right). \tag{3.8}$$

Now, since it is assumed that the set $\mathrm{Sol}(K, F_{u_k})$ is nonempty, we have that $\theta_k = 0$, $k \in \mathbb{N}$. It follows then that the optimal value of problem (3.8) is zero. Problem (3.8) can be written in the following max-min form

$$\operatorname*{Max}_{d\in C(x_0)} \operatorname*{Min}_{h\in C(x_0)} \xi_p(d,h), \tag{3.9}$$

where

$$\begin{aligned}\xi_p(d,h) := \;& 2\langle F'((x_0,u_0)(d,p)), d-h\rangle + \|d-h\|^2 \\ & +\sigma\left(F(x_0), T_K^2(x_0,d)\right) - \sigma\left(F(x_0), T_K^2(x_0,h)\right).\end{aligned}$$

Clearly for $h = d$ we have that $\xi_p(d, h) = 0$. Since the function $\xi_p(\bar{d}, \cdot)$ is strictly convex, it follows that this function attains its minimum (equal to zero), over $h \in C(x_0)$, at the unique point $\bar{h} = \bar{d}$. Since $F'((x_0, u_0), (\cdot, p))$ is Lipschitz continuous, we obtain in a way similar to (2.6) that

$$\nabla_d \nu(\bar{d}, p) = 2F'((x_0, u_0), (\bar{d}, p)). \tag{3.10}$$

Then (3.3) represents first order optimality (necessary) condition for $\bar{d}$ to be an optimal solution of problem (3.8). This completes the proof. ∎

The following result about directional differentiability of a solution mapping $\bar{x}(u) \in \mathrm{Sol}(K, F_u)$ is a consequence of the above theorem.

Corollary 3.1 *For a vector $p \in U$ and a path $u(t) := u_0 + tp + o(t)$, let $\bar{x}(t) \in \mathrm{Sol}(K, F_{u(t)})$ be such that $\|\bar{x}(t) - x_0\| = O(t)$ for all $t > 0$ sufficiently small. Suppose that the set K is second order regular at x_0 and the system* (3.3) *has unique solution $\bar{d} = \bar{d}(p)$. Then $(\bar{x}(t) - x_0)/t$ converges to $\bar{d}$ as $t \downarrow 0$.*

For polyhedral set K the system (3.3) takes the form (compare with (2.27)):

$$0 \in -F'((x_0, u_0), (d, p)) + N_{C(x_0)}(d). \tag{3.11}$$

For polyhedral set K and differentiable $F(\cdot, \cdot)$, the linearized system (3.11) was considered and directional differentiability of the solution mapping was established, under an assumption of strong regularity, in Robinson [16]. Also for polyhedral set K and locally Lipschitz $F(\cdot, \cdot)$ the result of Theorem 3.1 follows from results presented in Klatte [7] and Klatte and Kummer [9].

Definition 3.1 *It is said that the set K is* cone reducible *at $x_0 \in K$, if there exist a neighborhood V of x_0, a convex closed pointed cone $Q \subset \mathbb{R}^m$ and a twice continuously differentiable mapping $\Xi : V \to \mathbb{R}^m$ such that:* (i) $\Xi(x_0) = 0 \in \mathbb{R}^m$, (ii) *the derivative mapping $D\Xi(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is onto, and* (iii) $K \cap V = \{x \in V : \Xi(x) \in Q\}$.

The above concept of cone reducibility is discussed in detail in [2, sections 3.4.4 and 4.6.1]. If the set K is cone reducible at x_0, then the function $\hat{\phi}(d)$ can be represented as the restriction of a quadratic function $q(d) = \langle d, Ad \rangle$, $A \in \mathcal{S}^n$, to the set $C(x_0)$ (cf., [2, pp. 242]). In that case the system (3.3) takes the form:

$$0 \in -F'((x_0, u_0), (d, p)) + Ad + N_{C(x_0)}(d). \tag{3.12}$$

In some cases (e.g., for the cones of positive semidefinite matrices, see example 4.2) this quadratic function, and hence its gradient $\nabla q(d) = 2Ad$, can be calculated in a closed form. For cone reducible set K and differentiable $F(\cdot, \cdot)$, the system (3.12) and the result of Theorem 3.2 were derived in [21] by a different method.

Let us discuss now locally upper Lipschitz continuity of the solution multifunction $\mathbb{M}(u) := \mathrm{Sol}(K, F_u)$ at the point $x_0 \in \mathbb{M}(u_0)$.

Definition 3.2 *It is said that* $\mathbb{M}(\cdot)$ *is* locally upper Lipschitz *at* (x_0, u_0) *if there exist positive number* ρ *and neighborhoods* V *and* W *of* x_0 *and* u_0*, respectively, such that*

$$\mathbb{M}(u) \cap V \subset B\left(x_0, \rho\|u - u_0\|\right), \quad \forall\, u \in W. \tag{3.13}$$

The result of Theorem 3.1 has the following interpretation. Denote by $\mathbb{S}(p)$ the set of all solutions of the system (3.3). A multifunction $D\mathbb{M}(u_0|x_0) : U \rightrightarrows \mathbb{R}^n$ is called the *contingent derivative* of $\mathbb{M}$, at (u_0, x_0), if it is defined as follows: $h \in D\mathbb{M}(u_0|x_0)(w)$ iff there are sequences $w_k \to w$, $h_k \to h$ and $t_k \downarrow 0$ such that $x_0 + t_k h_k \in \mathbb{M}(u_0 + t_k w_k)$. That is, the graph of $D\mathbb{M}(u_0|x_0)(\cdot)$ is the contingent (Bouligand) cone to the graph of $\mathbb{M}(\cdot)$ at (u_0, x_0). The result of Theorem 3.1 means that, under the assumption of second order regularity, the following inclusion holds

$$D\mathbb{M}(u_0|x_0)(p) \subset \mathbb{S}(p). \tag{3.14}$$

The multifunction $\mathbb{M}(\cdot)$ is locally upper Lipschitz at (x_0, u_0) iff $D\mathbb{M}(u_0|x_0)(0) = \{0\}$ (King and Rockafellar [6]). Because of the inclusion (3.14) we have that if $\mathbb{S}(0) = \{0\}$, then $D\mathbb{M}(u_0|x_0)(0) = \{0\}$. Clearly condition $\mathbb{S}(0) = \{0\}$ is exactly the condition (C2). Therefore, we obtain the following result.

Theorem 3.2 *Let* $x_0 \in \mathrm{Sol}(K, F)$*. Suppose that the set* K *is second order regular at* x_0*, and either condition* (C2) *or* (C3) *holds. Then the solution multifunction* $u \mapsto \mathrm{Sol}(K, F_u)$ *is locally upper Lipschitz at* (x_0, u_0)*.*

For polyhedral set K the system (2.24), used in condition (C2), can be written in the form (2.27). For polyhedral set K and differentiable $F(\cdot)$, the linearized system (2.27) was discussed and the corresponding locally upper Lipschitz behavior of the solution multifunction was derived in Robinson [15, Theorem 4.1]. Also for polyhedral set $K = \mathbb{R}^n_+$ and Lipschitz continuous $F(\cdot)$, an extension of the system (2.27) and upper Lipschitz continuity of $\mathrm{Sol}(K, F_u)$ was derived in Klatte [7, Theorem 4] and Klatte and Kummer [9, Theorem 8.30] in a framework of generalized Kojima functions. As it was mentioned earlier, for a nonpolyhedral set K condition (2.28) is stronger than condition (C3). Condition (2.28) was used in [5, Proposition 5.1.6 and Corollary 5.1.8]. For cone reducible set K and differentiable $F(\cdot)$, the result of Theorem 3.2 was derived in [21] by a different method.

It was shown in remark 2.1 that, under the assumption of second order regularity, for the variational inequality associated with the optimization problem (1.3), condition (C2) is equivalent to the corresponding second order growth condition for the problem (1.3). It is said that a parameterization $f(x, u)$ of the objective function of problem (1.3) includes the tilt perturbation if $f(x, u) = f_1(x, u_1) + \langle u_2, x\rangle$, $u = (u_1, u_2) \in U_1 \times \mathbb{R}^n$. It is possible to show that for a parameterization of problem

(1.3) which includes the tilt perturbation, the corresponding second order growth condition is necessary for the locally upper Lipschitz continuity of $\mathrm{Sol}(K, F_u)$ (see the proof of Theorem 5.36 in [2]). Therefore, at least for variational inequalities associated with optimization problems, the locally upper Lipschitz continuity of $\mathrm{Sol}(K, F_u)$ implies condition (C2) for a sufficiently rich parameterization.

4 Generalized equations

In this section we assume that the set K is given in the form (1.4). Unless stated otherwise we assume that the mapping $G : \mathbb{R}^n \to \mathbb{R}^m$ is twice continuously differentiable, the set $Q \subset \mathbb{R}^m$ is nonempty convex and closed, and Robinson's constraint qualification holds at the point $x_0 \in K$. In the considered finite dimensional case, Robinson's constraint qualification can be formulated in the form

$$DG(x_0)\mathbb{R}^n + T_Q(z_0) = \mathbb{R}^m, \tag{4.1}$$

where $z_0 := G(x_0)$. Then the tangent cone to K at x_0 can be written as follows

$$T_K(x_0) = \{h : DG(x_0)h \in T_Q(z_0)\}.$$

Therefore, $F(x_0) \in N_K(x_0)$ iff the optimal value of the problem

$$\mathrm{Max}\langle F(x_0), h\rangle \quad \text{subject to} \;\; DG(x_0)h \in T_Q(z_0),$$

is zero. By calculating the dual of the above problem (cf., [2, p.151]), we obtain that $F(x_0) \in N_K(x_0)$ iff there exists $\lambda \in \mathbb{R}^m$ satisfying the following conditions

$$F(x_0) - DG(x_0)^*\lambda = 0 \;\; \text{and} \;\; \lambda \in N_Q(G(x_0)). \tag{4.2}$$

We refer to the system (4.2) as *generalized equations.*

Denote by $\Lambda(x_0)$ the set of all λ satisfying (4.2). By the above discussion we have that $x_0 \in \mathrm{Sol}(K, F)$ iff the set $\Lambda(x_0)$ is nonempty. Moreover, because of Robinson's constraint qualification, the set $\Lambda(x_0)$ is bounded and hence compact. Now by a chain rule ([3],[2, Proposition 3.33]), we have

$$T_K^2(x_0, h) = DG(x_0)^{-1}\left[T_Q^2(z_0, DG(x_0)h) - D^2G(x_0)(h,h)\right]. \tag{4.3}$$

It follows that the term $-\sigma\left(F(x_0), T_K^2(x_0,h)\right)$ is equal to the optimal value of the following problem

$$\begin{array}{ll} \mathrm{Min}_{w\in\mathbb{R}^n} & \langle -F(x_0), w\rangle \\ \text{subject to} & DG(x_0)w + D^2G(x_0)(h,h) \in T_Q^2(z_0, DG(x_0)h). \end{array} \tag{4.4}$$

The dual of this problem is

$$\operatorname*{Max}_{\lambda\in\Lambda(x_0)} \left\{ \langle \lambda, D^2G(x_0)(h,h)\rangle - \sigma\left(\lambda, T_Q^2(z_0, DG(x_0)h)\right)\right\}, \tag{4.5}$$

and (under Robinson's constraint qualification) the optimal values of (4.4) and (4.5) are equal to each other (cf., [2, p. 175]). We obtain that for any $d \in C(x_0)$,

$$\hat{\phi}(d) = \sup_{\lambda\in\Lambda(x_0)} \left\{ \langle \lambda, D^2G(x_0)(d,d)\rangle - \sigma\left(\lambda, T_Q^2(z_0, DG(x_0)d)\right)\right\}. \tag{4.6}$$

We also have that if the set Q is second order regular at z_0 and Robinson's constraint qualification (4.1) holds, then K is second order regular at x_0.

Suppose, further, that the following property holds at the point z_0:

$$\operatorname{dist}(z_0 + tw, Q) = o(t^2) \quad \text{for } t > 0 \text{ and every } w \in T_Q(z_0). \tag{4.7}$$

This holds, for example, if the set Q is polyhedral. Condition (4.7) means that $0 \in T_Q^2(z_0, w)$ for any $w \in T_Q(z_0)$. It follows then by (2.13) that $T_Q^2(z_0, w) = T_{T_Q(z_0)}(w)$, and hence because of (2.14) that

$$\sigma(\lambda, T_Q^2(z_0, w)) = 0 \text{ for all } h \in C(x_0) \text{ and } \lambda \in \Lambda(x_0),$$

(cf., [2, p. 177]). In that case, for any $d \in \mathbb{R}^n$,

$$\hat{\phi}(d) = \sup_{\lambda\in\Lambda(x_0)} \langle \lambda, D^2G(x_0)(d,d)\rangle + I_{C(x_0)}(d). \tag{4.8}$$

Condition (4.7) also implies that the set Q is second order regular at z_0 (see [2, p.203]).

In the present situation results of the previous sections can be applied in a straightforward way if the set $K = G^{-1}(Q)$ is convex. However, convexity of the set K is not ensured by the assumed conditions. Therefore, in order to push the corresponding proofs through we need some modifications.

Definition 4.1 *It is said that the set K is* prox-regular *at x_0 if there exists a neighborhood V of x_0 and a positive constant α such that*

$$\operatorname{dist}(y - x, T_K(x)) \le \alpha \|y - x\|^2 \quad \text{for all } x, y \in K \cap V. \tag{4.9}$$

Property (4.9) was introduced in [20] under the name "$O(2)$-convexity". The term "prox-regularity" was suggested in [14] and [19] where this concept was defined in a somewhat different, although equivalent, form. It easily follows from the Robinson-Ursescu stability theorem that if Robinson's constraint qualification holds and $DG(\cdot)$ is Lipschitz continuous in a neighborhood of x_0, then K is prox-regular at x_0 (see [20, pp. 134-135]). It is possible to show that prox-regular sets have the following important property ([20, Theorem 2.2]).

Proposition 4.1 *Suppose that K is prox-regular at $x_0 \in K$. Then $P_K(x)$ exists and is unique and locally Lipschitz continuous for all x in a neighborhood of x_0.*

For a constant $\kappa > 0$ and a neighborhood V of x_0 consider the function

$$\gamma_{\kappa,V}(x,u) := \inf_{y \in K \cap V} \left\{ g_\kappa(x,y,u) := \langle F(x,u), x-y \rangle + \tfrac{1}{2}\kappa \|x-y\|^2 \right\}. \tag{4.10}$$

Clearly $\gamma_{\kappa,V}(x,u) \le 0$ for all $x \in K \cap V$. We also have here that the optimal solution of the right hand side problem in (4.10) is given by

$$\bar{y}_\kappa(x,u) = P_K(x + \kappa^{-1} F(x,u)).$$

It follows by Proposition 4.1 that for κ large enough, $\bar{y}_\kappa(x,u)$ is unique and Lipschitz continuous for all (x,u) in a neighborhood of (x_0,u_0).

Lemma 4.1 *Suppose that the set K is prox-regular at the point x_0. Then there exist constant $\kappa > 0$ and neighborhoods V and W of x_0 and u_0, respectively, such that for any $u \in W$ and $\bar{x} \in V$, we have that $\bar{x} \in \mathrm{Sol}(K, F_u)$ iff $\bar{x} \in K$ and $\gamma_{\kappa,V}(\bar{x},u) = 0$.*

Proof. Consider a point $\bar{x} \in \mathrm{Sol}(K, F_u)$. By the definition we have that $\bar{x} \in K$ and $\langle F(\bar{x},u), h \rangle \le 0$ for any $h \in T_K(\bar{x})$. Together with (4.9) this implies that

$$\langle F(\bar{x},u), \bar{x} - y \rangle \ge -\alpha \|F(\bar{x},u)\| \, \|y - \bar{x}\|^2$$

for all $y \in K \cap V$. We can choose the neighborhoods V and W such that $\|F(x,u)\| \le b$ for some $b > 0$ and all $(x,u) \in V \times W$. Then for $\kappa > 2\alpha b$ we obtain that $g_{\kappa,V}(\bar{x},\cdot,u)$ attains its minimum over $K \cap V$ at $y = \bar{x}$. It follows that $\gamma_{\kappa,V}(\bar{x},u) = 0$.

Conversely, suppose that $\bar{x} \in K$ and $\gamma_{\kappa,V}(\bar{x},u) = 0$. Then $\bar{x}$ is a maximizer of $\gamma_{\kappa,V}(\cdot,u)$ over $K \cap V$ and $\bar{y}_\kappa(\bar{x},u) = \bar{x}$ is an optimal solution of the right hand side of (4.10). Moreover, for κ large enough the minimizer $\bar{y}_\kappa(\bar{x},u)$ is unique, and hence by Danskin theorem

$$\nabla_x \gamma_{\kappa,V}(\bar{x},u) = F(\bar{x},u).$$

It follows then by first order optimality conditions that $F(\bar{x},u) \in N_K(\bar{x})$, and hence $\bar{x} \in \mathrm{Sol}(K, F_u)$. ∎

The above lemma shows that, under Robinson's constraint qualification, we can reformulate (locally) $VI(K, F_u)$ as an optimization problem similar to (2.2) for sufficiently large constant κ. Since for κ large enough, $\bar{y}_\kappa(x,u)$ is unique and locally Lipschitz continuous, the expansions (2.11) and (3.2) still hold, with $\|d-h\|^2$ replaced by $\kappa\|d-h\|^2$, provided that the set K is second order regular at x_0.

Finally, in the previous sections we used convexity of the function $\hat{\phi}(\cdot)$. The function $\hat{\phi}(\cdot)$, given in (4.6), is defined as maximum of the sum of quadratic and convex functions. Since the set $\Lambda(x_0)$ is compact, the corresponding quadratic functions can be represented in the form

$$\langle \lambda, D^2G(x_0)(d,d)\rangle = f_\lambda(d) - \beta\|d\|^2,$$

such that $f_\lambda(\cdot)$ is convex for all $\lambda \in \Lambda(x_0)$ and β large enough. It follows that $\hat{\phi}(\cdot)$ can be represented as difference of an extended real valued convex function and the quadratic function $\beta\|\cdot\|^2$. The subdifferential $\partial\hat{\phi}(d)$ is then defined in a natural way as the subdifferential of the corresponding convex function minus $\nabla(\beta\|d\|^2) = 2\beta d$. In particular, for $\hat{\phi}(\cdot)$ given in (4.8) and $d \in C(x_0)$ we have

$$\partial\hat{\phi}(d) = \text{conv}\left\{\cup_{\lambda\in\Lambda^*(x_0,d)}[2D^2G(x_0)d]^*\lambda\right\} + N_{C(x_0)}(d), \tag{4.11}$$

where $\Lambda^*(x_0,d) := \arg\max_{\lambda\in\Lambda(x_0)}\langle \lambda, D^2G(x_0)(d,d)\rangle$.

Similar to Theorem 3.1 we have the following result.

Theorem 4.1 *Suppose that the set K is given in the form* (1.4)*, Robinson's constraint qualification holds at the point x_0, and the set Q is second order regular at $G(x_0)$. For a vector $p \in U$ consider a path $u(t) := u_0 + tp + o(t)$, $t \geq 0$. Let $t_k \downarrow 0$, $u_k := u(t_k)$, $\bar{x}_k \in \mathrm{Sol}(K, F_{u_k})$ and $\bar{d}$ be an accumulation point of the sequence $(\bar{x}_k - x_0)/t_k$. Then $\bar{d}$ is a solution of the system* (3.3) *with $\hat{\phi}(\cdot)$ defined in* (4.6).

Again the above theorem has an interpretation that, under the specified assumptions, the inclusion (3.14) holds, and hence the multifunction $u \mapsto \mathrm{Sol}(K, F_u)$ is locally upper Lipschitz at (x_0, u_0) if condition (C2) is satisfied.

Let us also note that the above analysis can be extended to situations where the set K is given in the form $K := S \cap G^{-1}(Q)$, with S being a convex closed subset of $\mathbb{R}^n$. That is, consider the mapping $\mathcal{G}(x,u) := (x, G(x,u))$ and write $K = \mathcal{G}^{-1}(\mathcal{Q})$, where $\mathcal{Q} := S \times Q$.

Example 4.1 Suppose that the set K is defined by a finite number of constraints

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, ..., m\}, \tag{4.12}$$

where $g_i(\cdot)$ are twice continuously differentiable real valued functions. We can formulate this in the form (1.4) by defining $G(\cdot) := (g_1(\cdot), ..., g_m(\cdot))$ and $Q := \mathbb{R}^m_-$. The set Q here is polyhedral, and hence K is second order regular and (see (4.8))

$$\hat{\phi}(d) = \sup_{\lambda\in\Lambda(x_0)}\left\{\sum_{i=1}^m \lambda_i D^2 g_i(x_0)(d,d)\right\} + I_{C(x_0)}(d). \tag{4.13}$$

For the set $Q := \mathbb{R}^m_-$, Theorem 4.1 (and its consequence about locally upper Lipschitz continuity of the solution mapping) corresponds to results in [8],[9].

Example 4.2 Suppose that $G : \mathbb{R}^n \to \mathcal{S}^p$ and $Q := \mathcal{S}^p_-$, where $\mathcal{S}^p_-$ denotes the cone of $p \times p$ symmetric negative semidefinite matrices. The set $\mathcal{S}^p_-$ is second order regular and for $Z \in \mathcal{S}^p_-$ and $\Omega \in \mathcal{S}^p_+$,

$$\sigma\left(\Omega, T^2_Q(Z,H)\right) = \begin{cases} 2\,\mathrm{tr}(\Omega H Z^\dagger H), & \text{if } \mathrm{tr}(\Omega Z) = 0,\ \mathrm{tr}(\Omega H) = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.14}$$

where $Z^\dagger$ denotes the Moore-Penrose pseudoinverse of Z, [2, p.487].

Let us finally consider situations where the set $K = K(u)$ also depends on the parameter vector u. That is, let

$$K(u) := \{x \in \mathbb{R}^n : G(x,u) \in Q\}, \tag{4.15}$$

where $G : \mathbb{R}^n \times U \to \mathbb{R}^m$ is a twice continuously differentiable mapping. Similar to (4.2) we consider the following system of generalized equations

$$F(x,u) - D_x G(x,u)^* \lambda = 0 \ \text{ and } \ \lambda \in N_Q(G(x,u)). \tag{4.16}$$

We denote by $\mathbb{M}(u)$ the set of solutions of (4.16), that is, $x \in \mathbb{M}(u)$ iff there exists $\lambda \in \mathbb{R}^m$ satisfying (4.16). Consider a point $x_0 \in \mathbb{M}(u_0)$ and function $\gamma_{\kappa,V}(x,u)$ defined in the same way as in (4.10) with K replaced by $K(u)$. We assume that Robinson's constraint qualification, with respect to the mapping $G(\cdot, u_0)$, holds at the point x_0. Since $DG(x,u)$ is locally Lipschitz continuous, it follows by Robinson-Ursescu stability theorem that property (4.9), in the definition of prox-regularity, holds for all u in a neighborhood of u_0 and $K = K(u)$, with constant α and neighborhood V independent of u. Therefore, we have by Lemma 4.1 that there exist constant $\kappa > 0$ and neighborhoods V and W of x_0 and u_0, respectively, such that for any $u \in W$ and $\bar{x} \in V$, it holds that $\bar{x} \in \mathbb{M}(u)$ iff $\bar{x} \in K$ and $\gamma_{\kappa,V}(\bar{x}, u) = 0$.

The second order growth condition, for $x = x_0$ and $u = u_0$, is defined here in the same way as in (2.18) for the function $\gamma(\cdot) = \gamma_{\kappa,V}(\cdot, u_0)$ and the set $K = K(u_0)$. Similar to Theorem 2.1 we have here that if the set Q is second order regular at $G(x_0, u_0)$, then for κ large enough the second order growth condition (2.18) holds iff condition (C2) is satisfied.

We say that the multifunction $\mathbb{M}(\cdot)$ is *locally upper Hölder* at (x_0, u_0) (of degree 1/2) if there exist positive number ρ and neighborhoods V and W of x_0 and u_0, respectively, such that

$$\mathbb{M}(u) \cap V \subset B\left(x_0, \rho \|u - u_0\|^{1/2}\right), \quad \forall\, u \in W. \tag{4.17}$$

Proposition 4.2 *Suppose that $F(\cdot,\cdot)$ is continuously differentiable, Robinson's constraint qualification, with respect to the mapping $G(\cdot, u_0)$, holds at the point x_0, the set Q is second order regular at $G(x_0, u_0)$, and for sufficiently large κ the second order growth condition* (2.18) *is satisfied. Then the multifunction $\mathbb{M}(\cdot)$ is locally upper Hölder at (x_0, u_0).*

Proof. Suppose that the second order growth condition (2.18) holds at x_0, with the corresponding constant c and neighborhood N. Consider a solution $\hat{x}(u) \in \mathbb{M}(u) \cap N$. We can choose the constant κ large enough and the neighborhoods V and W, such that $\hat{x}(u)$ is a maximizer of $\gamma_{\kappa,V}(\cdot, u)$ over $K(u) \cap V$ for all $u \in W$. By [2, Proposition 4.37] we have then that the following estimate holds:

$$\|\hat{x}(u) - x_0\| \leq c^{-1}\ell + 2\delta + c^{1/2}(\eta_1\delta + \eta_2\delta)^{1/2}, \tag{4.18}$$

Here $\ell = \ell(u)$ is a Lipschitz constant of the function $\chi(\cdot, u) := \gamma_{\kappa,V}(\cdot, u) - \gamma_{\kappa,V}(\cdot, u_0)$ on a subset of N containing x_0 and $\hat{x}(u)$, η_1 and $\eta_2 = \eta_2(u)$ are Lipschitz constants of $\gamma_{\kappa,V}(\cdot, u_0)$ and $\gamma_{\kappa,V}(\cdot, u)$, respectively, on N, and $\delta = \delta(u)$ is the Hausdorff distance between $K(u_0) \cap N$ and $K(u) \cap N$. By Robinson-Ursescu stability theorem we have that for the neighborhood N sufficiently small $\delta(u) = O(\|u - u_0\|)$. The Lipschitz constant $\eta_2(u)$ is bounded for all u in a neighborhood of u_0. Let us estimate $\ell(u)$. By (2.5) we have

$$\begin{aligned} D\ell(x, u) = (D_xF(x, u)^* - D_xF(x, u_0)^*)(x - \bar{y}(x, u_0)) + \\ D_xF(x, u)^*(\bar{y}(x, u_0) - \bar{y}(x, u)) + [F(x, u) - F(x)] + [\bar{y}(x, u_0) - \bar{y}(x, u)]. \end{aligned} \tag{4.19}$$

We have that $\bar{y}(x_0, u_0) = x_0$ and (see [2, pp.434-435]) that the mapping $(x, u) \to P_{K(u)}(x)$, and hence the mapping $(x, u) \to P_{K(u)}(x + \kappa^{-1}F(x, u))$ are locally upper Hölder at (x_0, u_0). Therefore, by choosing the corresponding subset of N consisting of two points x_0 and $\hat{x}(u)$, we can bound the norm of the first term in the right hand side of (4.19), on that subset, by $\beta\|\hat{x}(u) - x_0\|^{1/2}$ for some $\beta > 0$. Moreover, by continuity of $DF(x, u)$, the constant β can be arbitrary small for all u in a sufficiently small neighborhood of u_0. In particular, we can choose a neighborhood of u_0 such that $\beta \leq c/2$. The other three terms in the right hand side of (4.19) are of order $O(\|u - u_0\|^{1/2})$ uniformly in $x \in N$, for a sufficiently small neighborhood N. The proof can be completed then by applying the estimate (4.18). ∎

Note that without additional assumptions, the power constant $\frac{1}{2}$ in the above locally upper Hölder continuity of $\mathbb{M}(\cdot)$ cannot be improved. For optimization problems this is discussed in [2, section 4.5.1].

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References

[1] J.F. Bonnans, R. Cominetti and A. Shapiro, Second order optimality

conditions based on parabolic second order tangent sets, *SIAM J. Optimization*, 9 (1999), 466-492.

[2] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[3] R. Cominetti, Metric regularity, tangent sets and second order optimality conditions, *Applied Mathematics and Optimization*, 21 (1990), 265-287.

[4] J.M. Danskin, *The Theory of Max-Min and its Applications to Weapon Allocation Problems*, Econometrics and Operations Research **5**, Berlin and New York, Springer-Verlag, 1967.

[5] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer, New York, 2003.

[6] A. King and R.T. Rockafellar, Sensitivity analysis for nonsmooth generalized equations, *Mathematical Programming*, 55 (1992), 193-212.

[7] D. Klatte, Upper Lipschitz behavior of solutions to perturbed $C^{1,1}$ programs, *Mathematical Programming*, 88 (2000), 285-311.

[8] D. Klatte and B. Kummer, Generalized Kojima-functions and Lipschitz stability of critical points, *Computational Optimization and Applications*, 13 (1999), 61-85.

[9] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization. Regularity Calculus, Methods and Applications*, Kluwer Academic Publishers, Dordrecht, 2002.

[10] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[11] A.B. Levy, Implicit function theorems for sensitivity analysis of variational conditions, *Mathematical Programming*, 74 (1996), 333-350. [Errata: *Mathematical Programming*, 86 (1999), 439-441]

[12] A.B. Levy, Solution sensitivity from general principles, *SIAM Journal Control and Optimization*, 40 (2001), 1-38.

[13] A.B. Levy and R.T. Rockafellar, Sensitivity analysis for solutions to generalized equations, *Transactions of the American Mathematical Society*, 345 (1994), 661-671.

[14] R.A. Poliquin and R.T. Rockafellar, Prox-regular functions in variational analysis, *Trans. Amer. Math. Soc.*, 348 (1996), 1805-1838.

[15] S.M. Robinson, Generalized equations and their solutions, Part II: applications to nonlinear programming, *Mathematical Programming Study*, 19 (1982), 200-221.

[16] S.M. Robinson, Implicit B-differentiability in generalized equations. Technical Report #2854, Mathematics Research Center, University of Wisconsin, 1985.

[17] S.M. Robinson, Normal maps induced by linear transformations, *Mathematics of Operations Research*, 17 (1992), 691-714.

[18] S.M. Robinson, Constraint nondegeneracy in variational analysis, *Mathematics of Operations Research*, to appear.

[19] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Springer-Verlag, New York, 1998.

[20] A. Shapiro, Existence and differentiability of metric projections in Hilbert spaces, *SIAM J. Optimization*, 4 (1994), 130-141.

[21] A. Shapiro, Sensitivity analysis of generalized equations, *Journal of Mathematical Sciences*, 20 (2002), 116-127.