Error bounds and limiting behavior of weighted paths associated with the SDP map $X^{1/2}SX^{1/2}$

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Abstract

This paper studies the limiting behavior of weighted infeasible central paths for semidefinite programming obtained from centrality equations of the form $X^{1/2}SX^{1/2} = \nu W$, where W is a fixed positive definite matrix and $\nu > 0$ is a parameter, under the assumption that the problem has a strictly complementary primal-dual optimal solution. It is shown that a weighted central path as a function of $\sqrt{\nu}$ can be extended analytically beyond 0 and hence that the path converges as $\nu \downarrow 0$. Characterization of the limit points of the path and its normalized first-order derivatives are also provided. As a consequence, it is shown that a weighted central path can have two types of behavior, namely: either it converges as $\Theta(\nu)$ or as $\Theta(\sqrt{\nu})$ depending on whether the matrix W on a certain scaled space is block diagonal or not, respectively. We also derive an error bound on the distance between a point lying in a certain neighborhood of the central path and the set of primal-dual optimal solutions. Finally, in the light of the results of this paper, we give a characterization of a sufficient condition proposed by Potra and Sheng which guarantees the superlinear convergence of a class of primal-dual interior point SDP algorithms.

Key words: Limiting behavior, weighted central path, error bound, superlinear convergence, semidefinite programming.

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1 Introduction

Let $\mathcal{S}^n$ denote the space of $n \times n$ real symmetric matrices. We consider the semidefinite programming (SDP) problem

$$(P) \quad \begin{array}{ll} \text{minimize} & C \bullet X \\ \text{subject to} & \mathcal{A}X = b, \\ & X \succeq 0, \end{array} \tag{1}$$

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and its associated dual SDP problem

$$(D) \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \mathcal{A}^* y + S = C, \\ & S \succeq 0, \end{array} \tag{2}$$

where the data consists of $C \in \mathcal{S}^n$, $b \in \Re^m$ and a linear operator $\mathcal{A} : \mathcal{S}^n \to \Re^m$, the primal variable is $X \in \mathcal{S}^n$, and the dual variable consists of $(S, y) \in \mathcal{S}^n \times \Re^m$. For a matrix $V \in \mathcal{S}^n$, the notation $V \succeq 0$ means that V is positive semidefinite. Given a fixed positive definite matrix $W \in \mathcal{S}^n$, $\Delta b \in \Re^m$ and $\Delta C \in \mathcal{S}^n$, our interest in this paper is to study the set of solutions of the following system of nonlinear equations parametrized by the parameter $\nu > 0$:

$$\mathcal{A}X = b + \nu \Delta b, \quad X \succ 0, \tag{3}$$

$$\mathcal{A}^* y + S = C + \nu \Delta C, \quad S \succ 0, \tag{4}$$

$$X^{1/2} S X^{1/2} = \nu W. \tag{5}$$

Under suitable conditions on $(W, \Delta C, \Delta b)$, it has been shown in Monteiro and Zanjácomo [31] that the above system has a unique solution, denoted by $p(\nu) \equiv (X(\nu), S(\nu), y(\nu))$, for every $\nu \in (0, 1]$. We refer to the path $\nu \in (0, 1] \to p(\nu)$ as the $(W, \Delta C, \Delta b)$-weighted central path associated with (P) and (D). The main objective of this paper is to analyze the limiting behavior of this path as $\nu \downarrow 0$.

When $(W, \Delta C, \Delta b) = (I, 0, 0)$, the path $\nu \in (0, 1] \to p(\nu)$ is a part of the central path associated with (P) and (D). Properties of the central path have been extensively studied in several papers due to the important role it plays in the development of interior-points algorithms for cone programming, nonlinear programming and complementarity problems. Early works dealing with the well-definedness, differentiability and limiting behavior of weighted central paths in the context of the linear programming and monotone complementarity problems include [1, 2, 3, 8, 9, 10, 11, 16, 22, 23, 24, 26, 27, 29, 32, 36, 37, 38, 39, 40].

Using the fact that every real algebraic variety has a triangulation, Kojima et al. [15] showed that the central path associated with a monotone linear complementarity problem converges to a solution. In [19], Kojima et al. claims that similar arguments as the ones used in [15] can also be used to show that the central path of a monotone linear semidefinite complementarity problem (which is equivalent to SDP) converges to a solution of the problem. More generally, Drummond and Peterzil [8] established convergence of the central path for analytic convex nonlinear SDP problems. An alternative proof based on a deep result from algebraic geometry (see for example Lemma 3.1 of Milnor [25]) of the convergence of the central path for an SDP problem was given by Halická et al. [14]. Characterization of the limit point of the central path has been obtained by De Klerk et al. [6] and Luo et al. [21] for SDP problems possessing strictly complementary primal-dual optimal solutions. Using an approach based on the implicit function theorem described in Stoer and Wechs [37, 38], Halická [12] showed that the central path of an SDP problem possessing a strictly complementary primal-dual optimal solution can be extended analytically as a function of $\nu > 0$ to $\nu = 0$. For more general SDP problems, the above issues regarding the central path still remain open but some advances have been made on a few papers. These include De Klerk et al. [5] and Goldfarb and Scheinberg [7] who proved that any cluster point of the central path must be a maximally complementary optimal solution. Also, Halická et al. [13] and Sporre and Forsgren [36] provided partial characterizations of the limit point of the central path as being the analytic center of some convex subset of the optimal solution set and the unique solution of a perturbed log barrier problem over the optimal solution set, respectively. Finally, the recent paper by Cruz Neto et al. [4], which appeared after the release of the first version of the present work, establishes the convergence of the central path for a special class of SDPs which do not satisfy the strict complementarity condition.

Generalization of the notion of weighted central paths from linear programming to SDP problems is a delicate issue. While for a linear programming a weighted central path can be characterized as optimal solutions of certain weighted logarithmic barrier problems, this characterization does not seem to be a good source to obtain a suitable notion of weighted central paths for SDP. Instead, Monteiro and Zanjácomo [31] (see also Monteiro and Pang [28]) work directly with a system consisting of (3), (4) and an equation of the form $\Phi(X,S)=\nu W$, for some suitable map $\Phi : D \subseteq \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n$, and show that this system has a unique solution for every $\nu \in (0,1]$. Special instances of the map Φ for which the above result applies include the map $(X,S) \to (XS+SX)/2$ and $(X,S) \to X^{1/2}SX^{1/2}$.

Independently to the present work, Preiß and Stoer [35] have proved that the weighted central paths associated with the map $(X,S) \to (XS+SX)/2$ is analytically extendible as functions of $\nu \in (0,1]$ to $\nu = 0$ (see also Lu and Monteiro [20] for another proof of this result). In this paper, we will be interested only in the second map and its corresponding weighted central paths, i.e., the path of solutions of systems of the form (3)-(5). More specifically, we will investigate the asymptotic properties of the weighted central paths $\nu \in (0,1] \to p(\nu)$ and their derivatives for the special class of SDPs possessing strictly complementary primal-dual optimal solutions. Using a suitable change of variables together with the technique described in [37, 38] based on the implicit function theorem, we prove in Section 3 that the path $t \in (0,1] \to p(t^2)$ can be extended analytically to $t = 0$ and we also characterize the limit point of $p(\nu)$ as $\nu \downarrow 0$. In Section 4, we characterize the limit of the normalized derivative $p'(\nu)/\|p(\nu)\|$ as $\nu \downarrow 0$. As a consequence, we show that a weighted central path can have two types of behavior, namely: it converges either as $\Theta(\nu)$ or as $\Theta(\sqrt{\nu})$, depending on whether the matrix W on a certain scaled space is block diagonal or not, respectively. Using these results, we derive in Section 5 an error bound on the distance between a point lying in a certain neighborhood of the central path and the set of primal-dual optimal solutions. Finally, we consider in Section 6 a sufficient condition proposed by Potra and Sheng [33], which guarantees the superlinear convergence of a large class of primal-dual interior point SDP algorithms, and obtain a characterization of it in terms of the results obtained in this paper.

The organization of this paper is as follows. Section 2 introduces the assumptions made throughout the paper and discusses some preliminary known results about weighted central paths. Sections 3-6 establish the results mentioned in the previous paragraph. Finally, we end the paper by providing some concluding remarks in Section 7.

1.1 Notation

The space of symmetric $n \times n$ matrices will be denoted by S^n. Given matrices X and Y in $\Re^{p\times q}$, the standard inner product is defined by $X \bullet Y \equiv \operatorname{tr}(X^TY)$, where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. The Euclidean norm and its associated operator norm, i.e., the spectral norm, are both denoted by $\|\cdot\|$. The Frobenius norm of a $p \times q$-matrix X is defined as $\|X\|_F \equiv \sqrt{X \bullet X}$. Given a point f and a set F in a finite dimensional normed vector space, the distance from f to F is defined as $\operatorname{dist}(f,\ F) \equiv \inf_{\tilde f \in F} \|f - \tilde f\|$. If $X \in \mathcal{S}^n$ is positive semidefinite (resp., definite), we write $X \succeq 0$ (resp., $X \succ 0$). The cone of positive semidefinite (resp., definite) matrices is denoted by $\mathcal{S}^n_+$ (resp., $\mathcal{S}^n_{++}$). Either the identity matrix or operator will be denoted by I. The image (or range) space of a linear operator $\mathcal{A}$ will be denoted by $\text{Im}(\mathcal{A})$; the dimension of the subspace $\text{Im}(\mathcal{A})$, referred to as the rank of $\mathcal{A}$, will be denoted by $\text{rank}(\mathcal{A})$. Given a linear operator $\mathcal{F} : E \to F$ between two finite dimensional inner product spaces $(E, \langle \cdot, \cdot \rangle_E)$ and $(F, \langle \cdot, \cdot \rangle_F)$, its *adjoint* is the unique operator $\mathcal{F}^* : F \to E$ satisfying $\langle \mathcal{F}(u), v \rangle_F = \langle u, \mathcal{F}^*(v) \rangle_E$ for all $u \in E$ and $v \in F$.

Given functions $f : \Omega \to E$ and $g : \Omega \to \Re_{++}$, where Ω is an arbitrary set and E is a normed vector space, and a subset $\tilde{\Omega} \subset \Omega$, we write $f(w) = \mathcal{O}(g(w))$ for all $w \in \tilde{\Omega}$ to mean that there exists $M \geq 0$ such that $\|f(w)\| \leq M g(w)$ for all $w \in \tilde{\Omega}$; moreover, for a function $U : \Omega \to \mathcal{S}^n_{++}$, we write $U(w) = \Theta(g(w))$ for all $w \in \tilde{\Omega}$ if $U(w) = \mathcal{O}(g(w))$ and $U(w)^{-1} = \mathcal{O}(1/g(w))$ for all $w \in \tilde{\Omega}$. The latter condition is equivalent to the existence of a constant $M > 0$ such that

$$\frac{1}{M} I \preceq \frac{1}{g(w)} U(w) \preceq M I, \quad \forall w \in \Omega.$$

2 Preliminaries

In this section, we describe the assumptions that will be used in our presentation. We also describe the weighted central path that will be the subject of our study in this paper. Some preliminary results about this path are also stated including conditions for its well-definedness.

Throughout this paper we will be dealing with the pair of dual SDPs (P) and (D) (see (1) and (2), respectively). Denote the feasible sets of (P) and (D) by $\mathcal{F}_P$ and $\mathcal{F}_D$, respectively. Throughout our presentation we make the following assumptions on the pair of problems (P) and (D).

A.1 $\mathcal{A} : \mathcal{S}^n \to \Re^m$ is an onto linear operator;

A.2 There exists a pair of strictly complementary primal-dual optimal solution for (P) and (D), that is a triple $(X^*, S^*, y^*) \in \mathcal{F}_P \times \mathcal{F}_D$ satisfying $X^* S^* = 0$ and $X^* + S^* \succ 0$.

We will assume that Assumptions **A.1** and **A.2** are in force throughout our presentation. Hence, we will state our results without explicitly mentioning them.

Assumption **A.1** is not really crucial for our analysis but it is convenient to ensure that the variables S and y are in one-to-one correspondence. We will see that the dual weighted central path can always be defined in the S-space. The goal of Assumption **A.1** is just to ensure that this path can also be extended to the y-space.

Assumption **A.2** is the one that is commonly used in the analysis of superlinear convergence of interior-point algorithms and it plays an important role in our analysis. In fact, it is a very challenging problem to generalize the analysis of this paper to the case where Assumption **A.2** is dropped or simply relaxed.

By assumption **A.2**, since $X^* S^* = S^* X^* = 0$, we can diagonalize X^* and S^* simultaneously, i.e. find an orthonormal $P \in \Re^{n \times n}$ such that $P^T X^* P$ and $P^T S^* P$ are both diagonal. Performing the change of variables $\hat{X} = P^T X P$ and $(\hat{S}, \hat{y}) = (P^T S P, y)$ on problems (P) and (D) yield another pair of primal and dual SDPs which has a primal-dual optimal solution $(\hat{X}^*, \hat{S}^*, \hat{y}^*)$ such that $\hat{X}^*$ and $\hat{S}^*$ are both diagonal. To simplify our notation, we will assume without loss of generality that the original (P) and (D) already have a primal-dual optimal solution (X^*, S^*, y^*) such that

$$X^* = \begin{bmatrix} \Lambda_B & 0 \\ 0 & 0 \end{bmatrix}, \quad S^* = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_N \end{bmatrix}, \tag{6}$$

where $\Lambda_B \equiv \text{diag}(\lambda_1, \cdots, \lambda_K)$, $\Lambda_N \equiv \text{diag}(\lambda_{K+1}, \cdots, \lambda_n)$ for some integer $0 \le K \le n$ and some scalars $\lambda_i > 0$, $i = 1, 2, \cdots, n$. Here the subscripts B and N signify the "basic" and "nonbasic" subspaces (following the terminology of linear programming). Throughout this paper, the decomposition of any $n \times n$ matrix V is always made with respect to the above partition B and N, namely:

$$V = \begin{bmatrix} V_B & V_{BN} \\ V_{NB} & V_N \end{bmatrix},$$

so that V_{BN} and V_{NB} denote the off-diagonal block of V. If $V_{BN} = 0$ and $V_{NB} = 0$, V is called block diagonal, otherwise it is called non-block diagonal.

Notice that $X \in \mathcal{F}_P$ is an optimal solution of (P) if and only if $XS^* = 0$. Hence, by assumption **A.2**, the primal optimal solution set $\mathcal{F}_P^*$ is given by

$$\mathcal{F}_P^* \equiv \{X \in \mathcal{F}_P : \, X_{BN} = 0, \; X_{NB} = 0 \; \text{ and } \; X_N = 0\}.$$

Analogously, the dual optimal solution set $\mathcal{F}_D^*$ is given by

$$\mathcal{F}_D^* \equiv \{(S, y) \in \mathcal{F}_D : \, S_{BN} = 0, \; S_{NB} = 0 \; \text{ and } \; S_B = 0\}.$$

Define the linear map $\mathcal{G} : \mathcal{S}^n \times \mathcal{S}^n \times \Re^m \to \mathcal{S}^n \times \Re^m$ by

$$\mathcal{G}(X, S, y) \equiv (\mathcal{A}^* y + S - C, \; \mathcal{A} X - b) \tag{7}$$

and the sets $\mathcal{G}_{++}$ and $\mathcal{W}$ by

$$\mathcal{G}_{++} \equiv \mathcal{G}(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m), \tag{8}$$

$$\mathcal{W} \equiv \left\{ W \in \mathcal{S}^n_{++} : \; \|W - \nu I\| < \nu/\sqrt{2} \text{ for some } \nu > 0 \right\}. \tag{9}$$

Given $(W, \Delta C, \Delta b) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$, in this paper we are interested in the solutions of the system of nonlinear equations (3)-(5) parametrized by the parameter $\nu > 0$. The following result gives condition on $(W, \Delta C, \Delta b)$ for system (3)-(5) to have a unique solution for each $\nu \in (0, 1]$.

Proposition 2.1 *Assume that $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$. Then, for any $\nu \in (0, 1]$, system (3)-(5) has a unique solution, denoted by $(X(\nu), S(\nu), y(\nu))$. Moreover, the path $\nu \in (0, 1] \to (X(\nu), S(\nu), y(\nu))$ is analytic.*

Proof. By **A.2** and the assumption that $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$, we easily see that $\nu(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$ for all $\nu \in (0, 1]$. The first conclusion of the proposition now follows from Theorem 1(b) of Monteiro and Zanjácomo [31] by letting F, Φ and $\mathcal{V}$ in that theorem be defined as $F = \mathcal{G}$, $\Phi(X, S) = X^{1/2} S X^{1/2}$ for all $(X, S) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+$ and $\mathcal{V} = \mathcal{W}$. The second conclusion follows by applying the analytic version of the implicit function theorem to system (3)-(5) viewed as a function of (X, S, y, ν) and using the fact that the assumption $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$ implies that the Jacobian of this system with respect to (X, S, y) is nonsingular at $(X(\nu), S(\nu), y(\nu), \nu)$ for every $\nu \in (0, 1]$. (See Theorem 2.4 of [30] and the paragraph following it.) ∎

For a given $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$, the path $\nu \in (0, 1] \to (X(\nu), S(\nu), y(\nu))$ will be referred to as the $(W, \Delta C, \Delta b)$-weighted central path. In view of the above proposition, we will assume throughout Sections 2-4 that the following condition is true, without explicitly mentioning it in the statements of the results.

A.3 $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$.

The next result gives some estimates on the size of the blocks of $X(\nu)$ and $S(\nu)$.

Lemma 2.2 *For all $\nu > 0$ sufficiently small, we have:*

$$X_B(\nu) = \mathcal{O}(1), \qquad S_N(\nu) = \mathcal{O}(1), \tag{10}$$
$$X_N(\nu) = \mathcal{O}(\nu), \qquad S_B(\nu) = \mathcal{O}(\nu), \tag{11}$$
$$X_{BN}(\nu) = \mathcal{O}(\sqrt{\nu}), \qquad S_{BN}(\nu) = \mathcal{O}(\sqrt{\nu}). \tag{12}$$

Proof. Assume that $\nu > 0$ is sufficiently small and, for notational convenience, let $X \equiv X(\nu)$ and $S \equiv S(\nu)$. Also, let (X^*, S^*, y^*) be as in condition A.2. Since $(\Delta C, \Delta b) \in \mathcal{G}_{++}$, we have $(\Delta C, \Delta b) = \mathcal{G}(X^0, S^0, y^0)$ for some $(X^0, S^0, y^0) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$. It is easy to see that $A(X - \nu X^0 - (1-\nu)X^*) = 0$ and $S - \nu S^0 - (1-\nu)S^* \in \mathrm{Im}(\mathcal{A}^*)$, and hence that

$$(X - \nu X^0 - (1-\nu)X^*) \bullet (S - \nu S^0 - (1-\nu)S^*) = 0. \tag{13}$$

This equality together with (6) and the fact that $X^* \bullet S^* = 0$, $X \bullet S = \nu\,\mathrm{tr}(W)$ and all the quantities X, X^0, X^*, S, S^0, S^* are in $\mathcal{S}^n_+$ imply that

$$X \bullet S^0 + X^0 \bullet S \le \mathrm{tr}(W) + \xi(\nu) \tag{14}$$

and

$$X_N \bullet S_N^* + X_B^* \bullet S_B = X \bullet S^* + X^* \bullet S \le \frac{\nu(\mathrm{tr}(W) + \xi(\nu))}{1-\nu}, \tag{15}$$

where $\xi(\nu) \equiv \nu(X^0 \bullet S^0) + (1-\nu)(X^0 \bullet S^* + X^* \bullet S^0)$. The above two inequalities together with the fact that the matrices $X^0, S^0, X_B^*, S_N^*, X_N, S_B$ are all positive definite clearly imply that (10) and (11) hold. Using the fact that $X(\nu) \succ 0$ and $S(\nu) \succ 0$, we obtain that $X_{ij}^2(\nu) \le X_{ii}(\nu)X_{jj}(\nu)$ and $S_{ij}^2(\nu) \le S_{ii}(\nu)S_{jj}(\nu)$ for all i, j. These inequalities together with (10) and (11) imply (12). ∎

The next result gives estimates on the size of the blocks of the matrix $X^{1/2}(\nu) \equiv [X(\nu)]^{1/2}$.

Lemma 2.3 *Let $U(\nu) \equiv X^{1/2}(\nu)$ for all $\nu \in (0,1]$. Then, for all $\nu > 0$ sufficiently small, we have:*

$$U(\nu) = \begin{pmatrix} U_B(\nu) & U_{BN}(\nu) \\ U_{NB}(\nu) & U_N(\nu) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\sqrt{\nu}) \\ \mathcal{O}(\sqrt{\nu}) & \mathcal{O}(\sqrt{\nu}) \end{pmatrix}.$$

Proof. For notational convenience, let $U = U(\nu)$. Since $X = UU$, we have $X_B = U_B U_B + U_{BN} U_{BN}^T$ and $X_N = U_N U_N + U_{NB} U_{NB}^T$. Hence,

$$n\|X_B\| \ge \mathrm{tr} X_B = \mathrm{tr}\left(U_B U_B + U_{BN} U_{BN}^T\right) = \|U_B\|_F^2 + \|U_{BN}\|_F^2 \ge \max\{\|U_B\|^2, \|U_{BN}\|^2\},$$
$$n\|X_N\| \ge \mathrm{tr} X_N = \mathrm{tr}\left(U_N U_N + U_{NB} U_{NB}^T\right) = \|U_N\|_F^2 + \|U_{NB}\|_F^2 \ge \max\{\|U_N\|^2, \|U_{NB}\|^2\},$$

from which the result follows in view of (10) and (11). ∎

We end this section by stating a convergence result of the $(W, \Delta C, \Delta b)$-weighted central path to a primal-dual optimal solution of (1) and (2). We do not provide a proof for it since it is similar to the one given in the Appendix of Halická et al. [14].

Proposition 2.4 *There exists some $\epsilon > 0$ and an analytic function $\nu : [0,\epsilon) \to (0,1)$ such that $\nu(0) = 0$ and the path $t \in (0,\epsilon) \to (X(\nu(t)), S(\nu(t)), y(\nu(t)))$ is analytic at $t = 0$. In particular, $(X(\nu(t)), S(\nu(t)), y(\nu(t)))$ converges to some primal-dual optimal solution (X^*, S^*, y^*) as $t \downarrow 0$.*

We observe that Proposition 2.4 holds even without requiring Assumption **A.2**. As a consequence, its main advantage is that it holds for any SDP problem. Its main drawbacks are that it neither gives a characterization of the limit point (X^*, S^*, y^*) nor describes how fast $\nu(t)$ converges to 0. These issues and others will be addressed in the remaining sections of this paper in the context of SDPs satisfying Assumption **A.2**.

3 Analyticity of the weighted central path

In the parametrization introduced in the previous section, the weighted central path in general cannot be extended analytically to an interval of the form $(-\epsilon, \infty)$, for some $\epsilon > 0$ (see Corollary 4.3). However, in this section we will show that the re-parametrized weighted central path $t \to p(t^2)$ can be extended analytically to an interval as above.

For the sake of brevity, it is convenient to introduce the following definition.

Definition 1 *Let $w : (0,\delta) \to E$ be a given function where $\delta > 0$ and E is a finite dimensional normed vector space. The function w is said to be analytic at 0 if there exist $\epsilon > 0$ and an analytic function $\psi : (-\epsilon, \epsilon) \to E$ such that $w(t) = \psi(t)$ for all $t \in (0, \epsilon)$.*

The basic result that we use to establish that a function $w : (0,\delta) \to E$ is analytic at 0 is the following corollary of the analytic version of the implicit function theorem.

Proposition 3.1 *Let $w : (0,\delta) \to E$ be a given function where $\delta > 0$ and E is a finite dimensional normed vector space. Assume that there exists an analytic function $H : \mathcal{O} \times (-\epsilon, \epsilon) \to E$, where $\epsilon > 0$ and $\mathcal{O}$ is an open subset of E, such that $w = w(t)$ is the unique solution of $H(w,t) = 0$ in $\mathcal{O}$, for every $t \in (0,\epsilon)$. Assume also there exists $\bar{w} \in \mathcal{O}$ such that $H(\bar{w}, 0) = 0$ and $H'_w(\bar{w}, 0)$ is nonsingular. Then,*

i) $w = \bar{w}$ is the unique solution of the system $H(w, 0) = 0$;

ii) w is analytic at 0 and, as a consequence, $\lim_{t\downarrow 0} w(t) = \bar{w}$ and the limits of all the derivatives of $w(t)$ as $t \downarrow 0$ exist.

The following theorem is one of the main results of this section. Its proof will be given at the end of this section.

Theorem 3.2 *The re-parametrized $(W, \Delta C, \Delta b)$-weighted central path $t \in (0,1] \to (X(t^2), S(t^2), y(t^2))$ is analytic and also analytic at $t = 0$. As a consequence, the $(W, \Delta C, \Delta b)$-weighted central path $\nu \in (0,1] \to (X(\nu), S(\nu), y(\nu))$ converges.*

A key step towards showing the above result is a reformulation of the weighted central path system (3)-(5) as we now discuss. First, observe that (3), (4) and the equations

$$USU = t^2 W, \tag{16}$$
$$UU = X. \tag{17}$$

have $(U,X,S,y) = (U(t^2), X(t^2), S(t^2), y(t^2))$ as its unique solution in $\mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$, where $U(t^2) \equiv [X(t^2)]^{1/2}$. Letting

$$D_B(t) \equiv \begin{bmatrix} I/t & 0 \\ 0 & I \end{bmatrix}, \qquad D_N(t) \equiv \begin{bmatrix} I & 0 \\ 0 & I/t \end{bmatrix}, \tag{18}$$

and noting that $D_B(t)D_N(t) = I/t$ for every $t \in (0,1]$, we easily see that $U, X, S \in \mathcal{S}^n_{++}$ satisfies (16) and (17) if and only if U, $\tilde{X} \equiv D_N(t) X D_N(t)$ and $\tilde{S} \equiv D_B(t) S D_B(t)$ satisfies

$$[U D_N(t)]\, \tilde{S}\, [D_N(t)\, U] = W, \tag{19}$$

$$[D_N(t)\, U]\, [U D_N(t)] = \tilde{X}. \tag{20}$$

Now, let

$$\begin{aligned} \mathcal{U}^n &= \left\{ U \in \Re^{n\times n} : U_B \in \mathcal{S}^{|B|},\ U_N \in \mathcal{S}^{|N|},\ U_{NB} = 0 \right\}, \\ \mathcal{U}^n_{++} &= \{ U \in \mathcal{U}^n : U_B \succ 0,\ U_N \succ 0 \}. \end{aligned}$$

and define $\mathcal{L} : \mathcal{U}^n \to \Re^{n\times n}$ as

$$\mathcal{L}(U) = \begin{bmatrix} 0 & 0 \\ U_{BN}^T & 0 \end{bmatrix}, \quad \forall\, U \in \mathcal{U}^n.$$

It then follows that $(U, \tilde{X}, \tilde{S})$ satisfies (19) and (20) if and only if $(\tilde{U}, \tilde{X}, \tilde{S})$ with

$$\tilde{U} \equiv \begin{bmatrix} U_B & U_{BN}/t \\ 0 & U_N/t \end{bmatrix},$$

satisfies the equations

$$\left[\tilde{U} + t\,\mathcal{L}(\tilde{U})\right] \tilde{S} \left[\tilde{U} + t\,\mathcal{L}(\tilde{U})\right]^T = W, \tag{21}$$

$$\left[\tilde{U} + t\,\mathcal{L}(\tilde{U})\right]^T \left[\tilde{U} + t\,\mathcal{L}(\tilde{U})\right] = \tilde{X}. \tag{22}$$

Indeed, the above claim follows from the identity

$$U D_N(t) = \begin{bmatrix} U_B & U_{BN}/t \\ U_{NB} & U_N/t \end{bmatrix} = \tilde{U} + t\,\mathcal{L}(\tilde{U}).$$

The above arguments establish the following key result.

Proposition 3.3 *Let $(X^*, S^*, y^*) \in \mathcal{F}^*_P \times \mathcal{F}^*_D$ be given. Then, for every $t \in (0,1]$, the system defined by (21), (22) and the linear equations*

$$\mathcal{A}\left(D_N(t)^{-1} \tilde{X} D_N(t)^{-1} - X^* \right) = t^2 \Delta b, \tag{23}$$

$$D_B(t)^{-1} \tilde{S}\, D_B(t)^{-1} - S^* \in t^2 \Delta C + \mathrm{Im}(\mathcal{A}^*). \tag{24}$$

has a unique solution, denoted by $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$, in $\mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$. Moreover, the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ is analytic and, for every $t \in (0,1]$,

$$\tilde{X}(t) = D_N(t) X(t^2) D_N(t), \qquad \tilde{S}(t) = D_B(t) S(t^2) D_B(t), \tag{25}$$

$$\tilde{U}(t) = \begin{bmatrix} U_B(t^2) & U_{BN}(t^2)/t \\ 0 & U_N(t^2)/t \end{bmatrix}. \tag{26}$$

The next result states some basic properties about the accumulation points of the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ as t approaches 0.

Proposition 3.4 *The path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ remains bounded as t approaches 0 and any accumulation point $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ of this path as t approaches 0 is in $\mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ and satisfies the equations*

$$\tilde{U}\tilde{S}\tilde{U}^T = W, \tag{27}$$
$$\tilde{U}^T\tilde{U} = \tilde{X}. \tag{28}$$

Proof. By (18) and (25), we have

$$\tilde{X}(t) = \begin{bmatrix} X_B(t^2) & X_{BN}(t^2)/t \\ X_{NB}(t^2)/t & X_N(t^2)/t^2 \end{bmatrix}, \quad \tilde{S}(t) = \begin{bmatrix} S_B(t^2)/t^2 & S_{BN}(t^2)/t \\ S_{NB}(t^2)/t & S_N(t^2) \end{bmatrix}, \tag{29}$$

which, together with Lemma 2.2, imply that $(\tilde{X}(t), \tilde{S}(t))$ remains bounded as t approaches 0. Relation (26) and Lemma 2.3 imply that $\tilde{U}(t)$ also remains bounded as t approaches 0.

Consider now an accumulation point $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ of the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ as t approaches 0. By (25) and (26), we see that $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t)) \in \mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ for all $t \in (0,1]$, and hence we must have $\tilde{U}^* \in \mathcal{U}^n$, $\tilde{X}^* \succeq 0$, $\tilde{S}^* \succeq 0$, $\tilde{U}^*_B \succeq 0$ and $\tilde{U}^*_N \succeq 0$. Thus, to conclude that $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*) \in \mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$, it suffices to show that $\tilde{U}^*$, $\tilde{X}^*$ and $\tilde{S}^*$ are all invertible. Indeed, since $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ satisfies (21) and (22), we conclude upon letting $t \downarrow 0$ that $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ satisfies (27) and (28). This conclusion together with the fact that $W \succ 0$ implies that $\tilde{U}^*$, $\tilde{X}^*$ and $\tilde{S}^*$ are all invertible. ■

Our next goal is to show that the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t))$ is analytic at $t = 0$. The basic tool we use to establish this fact is the implicit function theorem applied to a specific system of equations parametrized by the parameter $t \in \Re$. A first natural candidate for such a system seems to be the one given by (21), (22), (23) and (24). However, the main drawback of this system is that its Jacobian with respect to $(\tilde{U}, \tilde{X}, \tilde{S})$ is generally singular for $t = 0$ (even though for $t \in (0,1)$ it is always nonsingular). The main cause for this phenomenon is that the "rank" of the linear equations (23) and (24) changes when t becomes 0.

We will now show how the linear equations (23) and (24) can be reformulated into equivalent linear equations for every $t \in (0,1]$. Moreover, the new linear equations have the property that their rank remains constant for every $t \in \Re$. First note that the linear operator $\mathcal{A} : \mathcal{S}^n \to \Re^m$ can be expressed as

$$\mathcal{A}(X) = \mathcal{A}_B(X_B) + \mathcal{A}_{BN}(X_{BN}) + \mathcal{A}_N(X_N) \equiv (\mathcal{A}_B \ \mathcal{A}_{BN} \ \mathcal{A}_N) \begin{pmatrix} X_B \\ X_{BN} \\ X_N \end{pmatrix}, \tag{30}$$

for some linear operators $\mathcal{A}_B : \mathcal{S}^{|B|} \to \Re^m$, $\mathcal{A}_{BN} : \Re^{|B| \times |N|} \to \Re^m$ and $\mathcal{A}_N : \mathcal{S}^{|N|} \to \Re^m$.

A well-known result from linear algebra says that any matrix can be put into row-echelon form after a sequence of elementary row operations. A similar type of argument allows one to establish the following result.

Lemma 3.5 *Let $\mathcal{A} : \mathcal{S}^n \to \Re^m$ be an onto linear operator. Assume that*

$$i_1 = \text{rank}(\mathcal{A}_B), \quad i_2 = \text{rank}(\mathcal{A}_B \ \mathcal{A}_{BN}) - i_1, \quad i_3 = \text{rank}(\mathcal{A}) - (i_1 + i_2) = m - (i_1 + i_2).$$

Then there exists an isomorphism $T : \Re^m \to \Re^m$ such that

$$(T \circ \mathcal{A})(X) = \begin{pmatrix} \mathcal{A}_{11}(X_B) & + \mathcal{A}_{12}(X_{BN}) & + \mathcal{A}_{13}(X_N) \\ & \mathcal{A}_{22}(X_{BN}) & + \mathcal{A}_{23}(X_N) \\ & & \mathcal{A}_{33}(X_N) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} X_B \\ X_{BN} \\ X_N \end{pmatrix},$$

for some linear operators

$$\begin{array}{ll} \mathcal{A}_{11} : \mathcal{S}^{|B|} \to \Re^{i_1}, & \mathcal{A}_{12} : \Re^{|B|\times|N|} \to \Re^{i_1}, \\ \mathcal{A}_{13} : \mathcal{S}^{|N|} \to \Re^{i_1}, & \mathcal{A}_{22} : \Re^{|B|\times|N|} \to \Re^{i_2}, \\ \mathcal{A}_{23} : \mathcal{S}^{|N|} \to \Re^{i_2}, & \mathcal{A}_{33} : \mathcal{S}^{|N|} \to \Re^{i_3} \end{array}$$

such that $\text{rank}(\mathcal{A}_{11}) = i_1, \ \text{rank}(\mathcal{A}_{22}) = i_2, \ \text{rank}(\mathcal{A}_{33}) = i_3$.

We can now reformulate the linear system (23) with the use of Lemma 3.5 as follows. Using the fact that

$$D_N^{-1} \tilde{X} D_N^{-1} - X^* = \begin{bmatrix} \tilde{X}_B - X_B^* & t\tilde{X}_{BN} \\ t\tilde{X}_{NB} & t^2 \tilde{X}_N \end{bmatrix}$$

and Lemma 3.5, we easily see that (23) is equivalent to the linear system

$$\begin{pmatrix} \mathcal{A}_{11} & t\mathcal{A}_{12} & t^2\mathcal{A}_{13} \\ 0 & t\mathcal{A}_{22} & t^2\mathcal{A}_{23} \\ 0 & 0 & t^2\mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} \tilde{X}_B - X_B^* \\ \tilde{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = t^2 \begin{pmatrix} \widetilde{\Delta b_1} \\ \widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix}$$

where $\widetilde{\Delta b} \equiv T(\Delta b)$. Dividing the second and third blocks of rows in the above system by t and t^2 respectively, we obtain the following system

$$\begin{pmatrix} \mathcal{A}_{11} & t\mathcal{A}_{12} & t^2\mathcal{A}_{13} \\ 0 & \mathcal{A}_{22} & t\mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} \tilde{X}_B - X_B^* \\ \tilde{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = \begin{pmatrix} t^2\widetilde{\Delta b_1} \\ t\widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix}. \tag{31}$$

Note that the linear system (31) is equivalent to (23) for every $t \in (0, 1]$. Hence, $\tilde{X}(t)$ satisfies (31) for every $t \in (0, 1]$. A nice feature of (31) is that the operator on its left hand side does not lose full rankness as t becomes 0. We state this fact in the following proposition.

Proposition 3.6 *Let $\mathcal{A}_t : \mathcal{S}^n \to \Re^m$ be the operator such that $\mathcal{A}_t(\tilde{X})$ is defined by the left hand side of (31). Then, $t \in \Re \to \mathcal{A}_t$ is a continuous map such* $\text{rank}(\mathcal{A}_t) = m$ *for every $t \in \Re$.*

The linear system (24) can also be reformulated with the aid of Lemma 3.5 as follows. First note that by Lemma 3.5 we have

$$\text{Im}(\mathcal{A}^*) = \text{Im}\left[(T \circ \mathcal{A})^*\right] = \text{Im}\left[\begin{pmatrix} \mathcal{A}_{11}^* & 0 & 0 \\ \mathcal{A}_{12}^* & \mathcal{A}_{22}^* & 0 \\ \mathcal{A}_{13}^* & \mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix}\right] = \text{Im}\left[\begin{pmatrix} t^2\mathcal{A}_{11}^* & 0 & 0 \\ t^2\mathcal{A}_{12}^* & t\mathcal{A}_{22}^* & 0 \\ t^2\mathcal{A}_{13}^* & t\mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix}\right],$$

for every $t \in (0,1]$. Hence, for every $t \in (0,1]$, (24) is equivalent to

$$\begin{pmatrix} t^2\tilde{S}_B \\ t\tilde{S}_{BN} \\ \tilde{S}_N - S_N^* \end{pmatrix} \in t^2 \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \\ \Delta C_N \end{pmatrix} + \mathrm{Im}\left[\begin{pmatrix} t^2\mathcal{A}_{11}^* & 0 & 0 \\ t^2\mathcal{A}_{12}^* & t\mathcal{A}_{22}^* & 0 \\ t^2\mathcal{A}_{13}^* & t\mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix}\right].$$

Dividing the first and second block of rows in the above system by t^2 and t, respectively, we obtain the system

$$\begin{pmatrix} \tilde{S}_B \\ \tilde{S}_{BN} \\ \tilde{S}_N - S_N^* \end{pmatrix} \in \begin{pmatrix} \Delta C_B \\ t\Delta C_{BN} \\ t^2\Delta C_N \end{pmatrix} + \mathrm{Im}\left[\begin{pmatrix} \mathcal{A}_{11}^* & 0 & 0 \\ t\mathcal{A}_{12}^* & \mathcal{A}_{22}^* & 0 \\ t^2\mathcal{A}_{13}^* & t\mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix}\right], \tag{32}$$

which is equivalent to (24), and hence satisfied by $\tilde{S}(t)$, for all $t \in (0,1]$.

Using the definition of $\mathcal{A}_t$ and the fact that $\tilde{X}(t)$ and $\tilde{S}(t)$ satisfy (31) and (32), respectively, for every $t \in (0,1]$, we conclude that there exists a function $\tilde{y} : (0,1] \to \Re^m$ such that $(\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ satisfies

$$\mathcal{A}_t(\tilde{X} - X^*) = \begin{pmatrix} t^2\widetilde{\Delta b_1} \\ t\widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix}, \quad \mathcal{A}_t^*\, \tilde{y} + (\tilde{S} - S^*) = \begin{pmatrix} \Delta C_B \\ t\Delta C_{BN} \\ t^2\Delta C_N \end{pmatrix}, \tag{33}$$

for every $t \in (0,1]$. Moreover, using Proposition 3.6 and the fact that $\{\tilde{S}(t) : t \in (0,1]\}$ is bounded, we easily see that $\{\tilde{y}(t) : t \in (0,1]\}$ is also bounded. We have thus established the following result.

Proposition 3.7 *There exists a curve $\tilde{y} : \Re_{++} \to \Re^m$ such that $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ is the unique solution of (21), (22) and (33) in $\mathcal{U}_{++}^n \times \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \Re^m$ for every $t \in (0,1]$. Moreover, the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ remains bounded as t approaches 0 and any of its accumulation points are in $\mathcal{U}_{++}^n \times \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \Re^m$.*

The system formed by (21), (22) and (33) is the one which we will use to establish that the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ is analytic at $t = 0$. This will follow by Proposition 3.1 if we can establish that the Jacobian of this system with $t = 0$ with respect to $(\tilde{U}, \tilde{X}, \tilde{S}, \tilde{y})$ is nonsingular as long as $(\tilde{U}, \tilde{X}, \tilde{S})$ is well-centered in the sense that $\|\tilde{U}S\tilde{U} - \nu I\| < \nu/\sqrt{2}$ for some $\nu \in (0,1]$. The nonsingularity of this Jacobian can be easily seen to be equivalent to showing that $(\widetilde{\Delta U}, \widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0,0,0,0) \in \mathcal{U}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ is the only solution of the following linear system:

$$\begin{aligned} \widetilde{\Delta U}\,\tilde{S}\,\tilde{U}^T + \tilde{U}\widetilde{\Delta S}\,\tilde{U}^T + \tilde{U}\tilde{S}\widetilde{\Delta U}^T &= 0, \\ \widetilde{\Delta U}^T\tilde{U} + \tilde{U}^T\widetilde{\Delta U} &= \widetilde{\Delta X}, \\ \mathcal{A}_0\,\widetilde{\Delta X} &= 0, \\ \mathcal{A}_0^*\,\widetilde{\Delta y} + \widetilde{\Delta S} &= 0. \end{aligned} \tag{34}$$

Before establishing the above fact, we state and prove two technical results.

Lemma 3.8 *For any $U \in \mathcal{U}_{++}^n$ and $H \in \mathcal{S}^n$, there exists a unique matrix $V \in \mathcal{U}^n$ such that*

$$H = U^TV + V^TU. \tag{35}$$

Moreover,

$$\|VU^{-1}\|_F \le \frac{\|U^{-T}HU^{-1}\|_F}{\sqrt{2}}. \tag{36}$$

Proof. The first part of the lemma follows from the fact that the linear map $\Psi_U \; : \mathcal{U}^n \to \mathcal{S}^n$ defined by $\Psi_U(V) = U^TV + V^TU$ for all $V \in \mathcal{U}^n$ is an isomorphism. Indeed, since its domain and co-domain have the same dimension, it suffices to show that Ψ_U is one-to-one, or equivalently that $U^TV + V^TU = 0$ implies $V = 0$. In turn, this last implication follows from the fact that any solution V of (35) satisfies (36) (simply set $H = 0$ in (36) to conclude that $V = 0$). To show the last claim, we multiply both sides of (35) on the left by U^{-T} and on the right by U^{-1} to obtain

$$U^{-T}HU^{-1} = VU^{-1} + (VU^{-1})^T. \tag{37}$$

Letting $\tilde{U} \equiv VU^{-1}$ and squaring both sides of the above equation, we obtain

$$\|U^{-T}HU^{-1}\|_F^2 = \|\tilde{U} + \tilde{U}^T\|_F^2 = 2\|\tilde{U}\|_F^2 + 2\mathrm{tr}(\tilde{U}^2). \tag{38}$$

Since

$$\begin{aligned} \tilde{U} = VU^{-1} &= \begin{bmatrix} V_B & V_{BN} \\ 0 & V_N \end{bmatrix} \begin{bmatrix} U_B^{-1} & -U_B^{-1}U_{BN}U_N^{-1} \\ 0 & U_N^{-1} \end{bmatrix} \\ &= \begin{bmatrix} V_BU_B^{-1} & -V_BU_B^{-1}U_{BN}U_N^{-1} + V_{BN}U_N^{-1} \\ 0 & V_NU_N^{-1} \end{bmatrix}, \end{aligned}$$

we have

$$\begin{aligned} \mathrm{tr}(\tilde{U}^2) &= \mathrm{tr}((V_BU_B^{-1})^2) + \mathrm{tr}((V_NU_N^{-1})^2), \\ &= \|U_B^{-1/2}V_BU_B^{-1/2}\|_F^2 + \|U_N^{-1/2}V_NU_N^{-1/2}\|_F^2 \ge 0. \end{aligned} \tag{39}$$

Hence, by (38) and (39), we see that (36) holds. ■

Lemma 3.9 *Suppose that* $\gamma \in [0, 1/\sqrt{2})$ *and that* $(U,S) \in \mathcal{U}_{++}^n \times \mathcal{S}^n$ *is such that* $\|USU^T - \nu I\| \le \gamma\nu$ *for some* $\nu > 0$*. For some* $H \in \mathcal{S}^n$*, if* $(\Delta U, \Delta X, \Delta S)$ *satisfies*

$$\Delta USU^T + U\Delta SU^T + US\Delta U^T = H, \tag{40}$$

$$\Delta U^TU + U^T\Delta U = \Delta X, \tag{41}$$

$$\Delta X \bullet \Delta S = 0, \tag{42}$$

then

$$\max\{\nu\|U^{-T}\Delta XU^{-1}\|_F, \|U\Delta SU^T\|_F\} \le \frac{\|H\|_F}{(1-\sqrt{2}\gamma)}. \tag{43}$$

Proof. Multiplying both sides of (41) on the left by U^{-T} and on the right by U^{-1} to obtain

$$U^{-T}\Delta U^T + \Delta UU^{-1} = U^{-T}\Delta XU^{-1}.$$

By this equality and (40), we have

$$\nu U^{-T}\Delta XU^{-1} + U\Delta SU^T = H - \Delta UU^{-1}(USU^T - \nu I) - (USU^T - \nu I)U^{-T}\Delta U^T. \tag{44}$$

Taking the Frobenius norm on both sides of this equality and using (36) and (42), we obtain

$$\begin{aligned}
\max \quad & \{\nu\|U^{-T}\Delta X U^{-1}\|_F, \|U\Delta S U^T\|_F\} \\
\leq \quad & \left(\nu^2\|U^{-T}\Delta X U^{-1}\|_F^2 + \|U\Delta S U^T\|_F^2\right)^{1/2} \\
= \quad & \left\|H - \Delta U U^{-1}(USU^T - \nu I) - (USU^T - \nu I)U^{-T}\Delta U^T\right\|_F \\
\leq \quad & \|H\|_F + 2\|\Delta U U^{-1}\|_F\|USU^T - \nu I\| \\
\leq \quad & \|H\|_F + \sqrt{2}\gamma\nu\|U^{-T}\Delta X U^{-1}\|_F, \qquad (45)
\end{aligned}$$

which clearly implies that

$$\nu\|U^{-T}\Delta X U^{-1}\|_F \leq \frac{\|H\|_F}{(1-\sqrt{2}\gamma)}. \qquad (46)$$

Using this last inequality to bound the right hand side of (45), we obtain (43). ■

As an immediate consequence of the above lemma, we obtain the following corollary.

Corollary 3.10 *Assume that* $(\tilde{U}, \tilde{S}) \in \mathcal{U}^n_{++} \times \mathcal{S}^n$ *is such that* $\|USU^T - \nu I\| < \nu/\sqrt{2}$ *for some* $\nu > 0$. *Then, system (34) has* $(\widetilde{\Delta U}, \widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0,0,0,0)$ *as its unique solution.*

Proof. The last two equations of system (34) imply that $\widetilde{\Delta X} \bullet \widetilde{\Delta S} = 0$. Using this identity and the first two equations of (34), by Lemma 3.9 we easily obtain that $\widetilde{\Delta X} = 0$ and $\widetilde{\Delta S} = 0$, which together with the second equation of (34) and Lemma 3.8 implies $\widetilde{\Delta U} = 0$. Also, $\widetilde{\Delta y} = 0$ follows from the fact that $\mathcal{A}^*$ is one-to-one and the last equation of (34). ■

We are now ready to establish the analyticity of the path $t \in (0,1] \to (\tilde{U}(t), \tilde{X}(t), \tilde{y}(t), \tilde{S}(t))$.

Theorem 3.11 *Let* $(X^*, S^*, y^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$ *be given. There hold:*

i) the path $t \in (0,1] \to \tilde{p}(t) \equiv (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$, *where* $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ *is the unique solution of (21), (22), (33) in* $\mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$, *is analytic and also analytic at 0; consequently,* $\tilde{p}(t)$ *and all its k-th order derivatives,* $k \geq 1$, *converge as* $t \downarrow 0$;

ii) $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*) \equiv \lim_{t\downarrow 0}(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ *is the unique solution of the system defined by (27), (28) and*

$$\mathcal{A}_0(\tilde{X} - X^*) = \begin{pmatrix} 0 \\ 0 \\ \widetilde{\Delta b_3} \end{pmatrix}, \quad \mathcal{A}_0^*\,\tilde{y} + (\tilde{S} - S^*) = \begin{pmatrix} \Delta C_B \\ 0 \\ 0 \end{pmatrix}; \qquad (47)$$

iii) $(\widetilde{\delta U}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^*) \equiv \lim_{t\downarrow 0}(\dot{\tilde{U}}(t), \dot{\tilde{X}}(t), \dot{\tilde{S}}(t), \dot{\tilde{y}}(t))$ *is the unique solution of the linear system defined by*

$$\widetilde{\delta U}\,\tilde{S}^*(\tilde{U}^*)^T + \tilde{U}^*\tilde{S}^*\,\widetilde{\delta U}^T + \tilde{U}^*\widetilde{\delta S}\,(\tilde{U}^*)^T = -\left[\mathcal{L}(\tilde{U}^*)\tilde{S}^*(\tilde{U}^*)^T + \tilde{U}^*\tilde{S}^*\mathcal{L}(\tilde{U}^*)^T\right], \qquad (48)$$

$$\widetilde{\delta U}^T\tilde{U}^* + (\tilde{U}^*)^T\widetilde{\delta U} - \widetilde{\delta X} = -\left[\mathcal{L}(\tilde{U}^*)^T\tilde{U}^* + (\tilde{U}^*)^T\mathcal{L}(\tilde{U}^*)\right], \qquad (49)$$

$$\mathcal{A}_0\,\widetilde{\delta X} = -\mathcal{B}_0\tilde{X}^* + \begin{pmatrix} 0 \\ \widetilde{\Delta b_2} \\ 0 \end{pmatrix}, \quad \mathcal{A}_0^*\widetilde{\delta y} + \widetilde{\delta S} = -\mathcal{B}_0^*\tilde{y}^* + \begin{pmatrix} 0 \\ \Delta C_{BN} \\ 0 \end{pmatrix}, \qquad (50)$$

where

$$\mathcal{B}_0 \equiv \begin{pmatrix} 0 & \mathcal{A}_{12} & 0 \\ 0 & 0 & \mathcal{A}_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The proof of theorem is based on Proposition 3.1. Indeed, let $E = \mathcal{U}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$, $\mathcal{O} = \mathcal{U}^n_{++} \times \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$, $\delta = \epsilon = 1$, $w : (0,1) \to E$ denote the path $t \in (0,1) \to (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ and $H(w,t) = H(\tilde{U}, \tilde{X}, \tilde{S}, \tilde{y}, t)$ be the map determined by system (21), (22), (33). By Proposition 3.7, the path $\tilde{p}(t) = (\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ has an accumulation point $w^* = (\tilde{U}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ in $\mathcal{O}$ and, by Corollary 3.10, it follows that $H'_w(w^*, 0)$ is nonsingular since (27) with $(\tilde{U}, \tilde{X}, \tilde{S}) = (\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ implies that $\|\tilde{U}^* \tilde{S}^* (\tilde{U}^*)^T - \nu I\| = \|W - \nu I\| < \nu/\sqrt{2}$. Hence, i) and ii) follow directly from Proposition 3.1. Differentiating the identity $H(\tilde{p}(t), t) = 0$ with respect to t and letting $t \downarrow 0$, we conclude that $\delta w = \delta w^* \equiv (\widetilde{\delta U}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^*)$ satisfies

$$H'_w(w^*, 0)\delta w = -H'_t(w^*, 0).$$

Statement iii) now follows from the fact that $H'_w(w^*, 0)$ is nonsingular and the latter system is equivalent to (48)-(50). ■

The proof of Theorem 3.2 is now obvious. Indeed, the analyticity of the map $t \to (X(t^2), S(t^2))$ follows from (29) and the analyticity of $t \to (\tilde{X}(t), \tilde{S}(t))$. The analyticity of $t \to y(t^2)$ follows from the analyticity of $t \to S(t^2)$ and Assumption A.1. The last statement of the theorem is obvious.

In the remainder of this paper, we will let $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ and $(\widetilde{\delta U}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^*)$ denote the limits of $(\tilde{U}(t), \tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ and $(\dot{\tilde{U}}(t), \dot{\tilde{X}}(t), \dot{\tilde{S}}(t), \dot{\tilde{y}}(t))$, respectively, as $t \downarrow 0$ (as in Theorem 3.11 above). Observe that Theorem 3.11 provides a characterization of $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ as being the unique solution of a certain system of equations which arises by first performing some transformations to the original weighted central path system, and then setting $t = 0$ in the resulting system. Hence, it is reasonable to expect that the linear equations (47) can be entirely described in terms of the original data $(W, \mathcal{A}, C, \Delta C, b, \Delta b)$. Indeed, the following result gives this alternative description of (47).

Theorem 3.12 *$(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ is the unique solution of the system given by (27), (28) and the linear equations*

$$\mathcal{A}_B(\tilde{X}_B) = b, \quad \mathcal{A}_{BN}(\tilde{X}_{BN}) \in \mathrm{Im}(\mathcal{A}_B), \quad \mathcal{A}_N(\tilde{X}_N) \in \Delta b + \mathrm{Im}(\mathcal{A}_B \ \mathcal{A}_{BN}), \tag{51}$$

$$\tilde{S}_B \in \Delta C_B + \mathrm{Im}(\mathcal{A}_B^*), \quad \begin{pmatrix} 0 \\ \tilde{S}_{BN} \end{pmatrix} \in \mathrm{Im}\left[\begin{pmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \end{pmatrix}\right], \quad \begin{pmatrix} 0 \\ 0 \\ \tilde{S}_N \end{pmatrix} \in C + \mathrm{Im}\left[\begin{pmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \\ \mathcal{A}_N^* \end{pmatrix}\right]. \tag{52}$$

Proof. From Theorem 3.11(ii), it suffices to show that (47) is equivalent to (51) and (52). Since the first equation of (47) is the same as (31) with $t = 0$, we have that the first equation of (47) holds if and only if

$$\mathcal{A}_{11}(\tilde{X}_B) = \mathcal{A}_{11}(X_B^*), \quad \mathcal{A}_{22}(\tilde{X}_{BN}) = 0, \quad \mathcal{A}_{33}(\tilde{X}_N) = \widetilde{\Delta b}_3. \tag{53}$$

By Lemma 3.5, the first identity in (53) can be written as

$$(T \circ \mathcal{A}) \begin{pmatrix} \tilde{X}_B \\ 0 \\ 0 \end{pmatrix} = (T \circ \mathcal{A}) \begin{pmatrix} X_B^* \\ 0 \\ 0 \end{pmatrix},$$

and hence it is equivalent to $\mathcal{A}_B(\tilde{X}_B) = \mathcal{A}_B(X_B^*) = b$, in view of relation (30) and the fact that T is an isomorphism. By Lemma 3.5 and the fact that $\mathcal{A}_{11}$ is onto, the second identity in (53) holds if and only if

$$(T \circ \mathcal{A}) \begin{pmatrix} \hat{X}_B \\ \tilde{X}_{BN} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for some $\hat{X}_B \in \mathcal{S}^{|B|}$, and hence it is equivalent to $\mathcal{A}_{BN}(\tilde{X}_{BN}) \in \operatorname{Im}(\mathcal{A}_B)$, in view of (30) and the fact that T is an isomorphism. Using Lemma 3.5 again and the fact that $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are onto, we easily see that the last identity in (53) holds if and only if

$$(T \circ \mathcal{A}) \begin{pmatrix} \check{X}_B \\ \check{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta b_1} \\ \widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix} = T(\Delta b)$$

for some $(\check{X}_B, \check{X}_{BN}) \in \mathcal{S}^{|B|} \times \Re^{|B| \times |N|}$, and hence it is equivalent to $\mathcal{A}_N(\tilde{X}_N) \in \Delta b + \operatorname{Im}(\mathcal{A}_B\ \mathcal{A}_{BN})$, in view of (30) and the fact that T is an isomorphism. We have thus shown that the first equation of (47) is equivalent to (51).

The fact that the second equation of (47) holds if and only if (52) holds can be proved in a similar way as above. ■

The following result gives an alternative characterization of $(\widetilde{\delta U}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*)$ involving the original data $(W, \mathcal{A}, C, \Delta C, b, \Delta b)$.

Theorem 3.13 *$(\widetilde{\delta U}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*)$ is the unique solution of the linear system of equations (48), (49) and*

$$\begin{bmatrix} \mathcal{A}_B & \mathcal{A}_{BN} \end{bmatrix} \begin{bmatrix} \widetilde{\delta X_B} \\ \tilde{X}_{BN}^* \end{bmatrix} = 0, \ \begin{bmatrix} \mathcal{A}_{BN} & \mathcal{A}_N \end{bmatrix} \begin{bmatrix} \widetilde{\delta X}_{BN} \\ \tilde{X}_N^* \end{bmatrix} \in \Delta b + \operatorname{Im}(\mathcal{A}_B), \ \mathcal{A}_N(\widetilde{\delta X}_N) \in \operatorname{Im}(\mathcal{A}_B\ \mathcal{A}_{BN}) \tag{54}$$

$$\widetilde{\delta S}_B \in \operatorname{Im}(\mathcal{A}_B^*), \quad \begin{pmatrix} \tilde{S}_B^* \\ \widetilde{\delta S}_{BN} \end{pmatrix} \in \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \end{pmatrix} + \operatorname{Im}\left[\begin{pmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \end{pmatrix}\right], \quad \begin{pmatrix} 0 \\ \tilde{S}_{BN}^* \\ \widetilde{\delta S}_N \end{pmatrix} \in \operatorname{Im}\left[\begin{pmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \\ \mathcal{A}_N^* \end{pmatrix}\right]. \tag{55}$$

Proof. From Theorem 3.11(iii), it suffices to show that (50) is equivalent to (54) and (55). Observe that the first equation of (50) can be written as

$$\begin{array}{rcl} \mathcal{A}_{11}(\widetilde{\delta X}_B) + \mathcal{A}_{12}(\tilde{X}_{BN}^*) & = & 0, \\ \mathcal{A}_{22}(\widetilde{\delta X}_{BN}) + \mathcal{A}_{23}(\tilde{X}_N^*) & = & \widetilde{\Delta b_2}, \\ \mathcal{A}_{33}(\widetilde{\delta X}_N) & = & 0. \end{array} \tag{56}$$

Using Lemma 3.5, the fact that $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are onto and the identities $\mathcal{A}_{22}\tilde{X}_{BN}^* = 0$ and $\mathcal{A}_{33}\tilde{X}_N^* = \widetilde{\Delta b_3}$ which hold in view of Theorem 3.11(ii), we easily see that the above three equations are respectively equivalent to

$$(T \circ \mathcal{A}) \begin{pmatrix} \widetilde{\delta X_B} \\ \tilde{X}_{BN}^* \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ (T \circ \mathcal{A}) \begin{pmatrix} \hat{X}_B \\ \widetilde{\delta X}_{BN} \\ \tilde{X}_N^* \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta b_1} \\ \widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix}, \ (T \circ \mathcal{A}) \begin{pmatrix} \check{X}_B \\ \check{X}_{BN} \\ \widetilde{\delta X}_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for some $\hat{X}_B, \check{X}_B \in \mathcal{S}^{|B|}$ and $\check{X}_{BN} \in \Re^{|B|\times|N|}$. The latter conditions in turn are equivalent to (54) in view of (30) and the facts that $\widetilde{\Delta b} = T(\Delta b)$ and T is an isomorphism.

Using similar arguments as to ones used above, it can be shown that the second equation of (50) holds if and only if (55) holds. ∎

4 Limiting behavior of the derivative of the weighted central path

In this section, we will first characterize the limit of the normalized derivatives of a weighted central path as ν approaches 0. We then show that a weighted central path can have two types of behavior, namely: either it converges as $\Theta(\nu)$ or as $\Theta(\sqrt{\nu})$ depending on whether the matrix W on a certain scaled space is block diagonal or not, respectively.

Theorem 4.1 $\lim_{\nu\downarrow 0} \sqrt{\nu}\ (\dot{X}(\nu), \dot{S}(\nu), \dot{y}(\nu))$ *exists and satisfies*

$$\lim_{\nu\downarrow 0} \sqrt{\nu}\,\dot{X}(\nu) = \begin{bmatrix} \widetilde{\delta X}^*_B/2 & \tilde{X}^*_{BN}/2 \\ \tilde{X}^*_{NB}/2 & 0 \end{bmatrix}, \quad \lim_{\nu\downarrow 0} \sqrt{\nu}\,\dot{S}(\nu) = \begin{bmatrix} 0 & \tilde{S}^*_{BN}/2 \\ \tilde{S}^*_{NB}/2 & \widetilde{\delta S}^*_N/2 \end{bmatrix}. \tag{57}$$

Proof. By (29), we have

$$X(t^2) = \begin{bmatrix} \tilde{X}_B(t) & t\tilde{X}_{BN}(t) \\ t\tilde{X}_{NB}(t) & t^2\tilde{X}_N(t) \end{bmatrix}, \quad S(t^2) = \begin{bmatrix} t^2\tilde{S}_B(t) & t\tilde{S}_{BN}(t) \\ t\tilde{S}_{NB}(t) & \tilde{S}_N(t) \end{bmatrix} \tag{58}$$

Differentiating both sides with respect to t, letting $t \downarrow 0$, and using Theorem 3.11, we obtain (57) upon letting $\nu = t^2$. ∎

We establish one technical lemma as follows, which gives a characterization of block diagonal weighted matrix W. This lemma will play a crucial role in further analyzing the limiting behavior of derivatives of the weighted central path.

Lemma 4.2 *The following statements hold:*

i) $\tilde{X}^*_{BN} \bullet \tilde{S}^*_{BN} = 0$*;*

ii) $\tilde{X}^*_{BN} = \tilde{S}^*_{BN} = 0$ *if and only if* $W_{BN} = 0$.

Proof. Statement i) follows from the fact that $\tilde{X}^*_{BN}$ and $\tilde{S}^*_{BN}$ satisfy the second equations in (51) and (52), respectively, which can be easily seen to determine two orthogonal complementary subspaces in $\Re^{|B|\times|N|}$.

We now show ii). Using the fact that $(\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ satisfies (27) and (28), it is easy to see that

$$W_{BN} = \tilde{U}^*_B \tilde{S}^*_{BN} \tilde{U}^*_N + \tilde{U}^*_{BN} \tilde{S}^*_N \tilde{U}^*_N, \quad \tilde{X}^*_B = (\tilde{U}^*_B)^2, \quad \tilde{X}^*_{BN} = \tilde{U}^*_B \tilde{U}^*_{BN}. \tag{59}$$

By Proposition 3.4, we know that $(\tilde{X}^*, \tilde{S}^*, \tilde{U}^*) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times U^n_{++}$, and hence $\tilde{X}^*_B \succ 0$, $\tilde{S}^*_N \succ 0$, $\tilde{U}^*_B \succ 0$, $\tilde{U}^*_N \succ 0$. Thus, the last relation in (59) implies that

$$\tilde{X}^*_{BN} = 0 \iff \tilde{U}^*_{BN} = 0. \tag{60}$$

Assume first that $\tilde{X}^*_{BN} = \tilde{S}^*_{BN} = 0$. Then, (60) and the first equation in (59) immediately imply that $W_{BN} = 0$. Assume now that $W_{BN} = 0$. Using (59), we obtain

$$\begin{aligned} \tilde{X}^*_B \tilde{S}^*_{BN} + \tilde{X}^*_{BN} \tilde{S}^*_N &= (\tilde{U}^*_B)^2 \tilde{S}^*_{BN} + \tilde{U}^*_B \tilde{U}^*_{BN} \tilde{S}^*_N \\ &= \tilde{U}^*_B \left(\tilde{U}^*_B \tilde{S}^*_{BN} \tilde{U}^*_N + \tilde{U}^*_{BN} \tilde{S}^*_N \tilde{U}^*_N \right) (\tilde{U}^*_N)^{-1} \\ &= \tilde{U}^*_B W_{BN} (\tilde{U}^*_N)^{-1} \; = \; 0. \end{aligned}$$

Multiplying the above equation on the left by $(\tilde{X}^*_B)^{-1/2}$ and on the right by $(\tilde{S}^*_N)^{-1/2}$, squaring both sides of the resulting expression and using i), we conclude that

$$\|(\tilde{X}^*_B)^{1/2} \tilde{S}^*_{BN} (\tilde{S}^*_N)^{-1/2}\|^2_F + \|(\tilde{X}^*_B)^{-1/2} \tilde{X}^*_{BN} (\tilde{S}^*_N)^{1/2}\|^2_F = 0,$$

from which it follows that $\tilde{X}^*_{BN} = \tilde{S}^*_{BN} = 0$. ∎

From Lemma 4.2 and Theorem 4.1, the following corollary follows.

Corollary 4.3 *If $W_{BN} \neq 0$ then at least one of the limits in (57) is nonzero and*

$$\|(X(\nu), S(\nu), y(\nu)) - (X^*, S^*, y^*)\| = \Theta(\sqrt{\nu}).$$

Proof. Assume that $W_{BN} \neq 0$. By Lemma 4.2(ii), we have that either $\tilde{X}^*_{BN} \neq 0$ or $\tilde{S}^*_{BN} \neq 0$, which together with (57) implies the first claim of the corollary. The second claim follows directly from (57) and the equality

$$\lim_{\nu \downarrow 0} \frac{(X(\nu), S(\nu), y(\nu)) - (X^*, S^*, y^*)}{\sqrt{\nu}} = \lim_{\nu \downarrow 0} 2\sqrt{\nu} \left(\dot{X}(\nu), \dot{S}(\nu), \dot{y}(\nu) \right),$$

which holds due to Theorem 3.2. ∎

From Corollary 4.3, we immediately see that the weighted central path as a function of ν in general cannot be extended analytically to an interval of the form $(-\epsilon, \infty)$, for some $\epsilon > 0$. Theorem 4.1 and Corollary 4.3 give a precise characterization of how the primal-dual weighted central path approaches its limit (X^*, S^*, y^*) for the case when W is non-block diagonal, that is $W_{BN} \neq 0$. However, it is still possible for one of the limits in (57) to be equal to zero in this situation. The following result claims that in this case the corresponding primal or dual weighted central path converges towards (X^*, S^*, y^*) at a $\Theta(\nu)$ rate of convergence.

Theorem 4.4 *The following statements hold:*

i) If $\lim_{\nu \downarrow 0} \sqrt{\nu}\, \dot{X}(\nu) = 0$ then $X(\nu) - X^* = \Theta(\nu)$ and

$$\lim_{\nu \downarrow 0} \dot{X}(\nu) = \begin{bmatrix} \widetilde{\delta^{(2)} X}^*_B/2 & \widetilde{\delta X}^*_{BN} \\ (\widetilde{\delta X}^*_{BN})^T & \tilde{X}^*_N \end{bmatrix}, \tag{61}$$

where $\widetilde{\delta^{(2)} X^*_B} \equiv \lim_{t \downarrow 0} \ddot{\tilde{X}}_B(t)$;

ii) If $\lim_{\nu\downarrow 0}\sqrt{\nu}\,\dot{S}(\nu) = 0$ *then* $\|(S(\nu), y(\nu)) - (S^*, y^*)\| = \Theta(\nu)$ *and*

$$\lim_{\nu\downarrow 0}\dot{S}(\nu) = \begin{bmatrix} \widetilde{\delta^{(2)} S}_B^*/2 & \widetilde{\delta S}_{BN}^* \\ (\widetilde{\delta S}_{BN}^*)^T & \tilde{S}_N^* \end{bmatrix}, \tag{62}$$

where $\widetilde{\delta^{(2)} S}_B^* \equiv \lim_{t\downarrow 0}\ddot{\tilde{S}}_B(t)$.

Proof. To prove i), assume that $\lim_{\nu\downarrow 0}\sqrt{\nu}\,\dot{X}(\nu) = 0$. By Theorem 4.1, we must have $\widetilde{\delta X}_B^* = 0$ and $\tilde{X}_{BN}^* = 0$. Differentiating both sides of the first identity in (58) with respect to t and then dividing the resulting identity by $2t$, we obtain that

$$\dot{X}(t^2) = \begin{bmatrix} \dot{\tilde{X}}_B(t)/(2t) & \tilde{X}_{BN}(t)/(2t) + \dot{\tilde{X}}_{BN}(t)/2 \\ \tilde{X}_{BN}(t)^T/(2t) + \dot{\tilde{X}}_{BN}(t)^T/2 & \tilde{X}_N(t) + t\dot{\tilde{X}}_N(t)/2 \end{bmatrix}.$$

Using the fact that $\widetilde{\delta X}_B^* = 0$ and $\tilde{X}_{BN}^* = 0$ and and using Theorem 3.11, we obtain (61) upon letting $\nu = t^2 \downarrow 0$. The conclusion that $X(\nu) - X^* = \Theta(\nu)$ follows immediately from (61) and the fact $\tilde{X}_N^* \succ 0$. Using the same arguments as above and assumption **A.1**, we can similarly show ii). ■

The remainder of this section considers the case when W is block diagonal, that is $W_{BN} = 0$. We will show in this case that two limits in (57) are equal to zero, and hence that $\lim_{\nu\downarrow 0}(\dot{X}(\nu), \dot{S}(\nu), \dot{y}(\nu))$ exists and $\|(X(\nu), S(\nu), y(\nu)) - (X^*, S^*, y^*)\| = \Theta(\nu)$ due to Theorem 4.4.

Note that to establish the above claim, it suffices to show that $\widetilde{\delta X}_B^* = 0$ and $\widetilde{\delta S}_N^* = 0$ in view of Lemma 4.2(ii). Before showing this fact, we state two technical results from Monteiro and Tsuchiya [30].

Lemma 4.5 (Lemma 2.1 of [30]) *For every* $A \in \mathcal{S}^n_{++}$ *and* $H \in \mathcal{S}^n$*, the equation* $AU + UA = H$ *has a unique solution* $U \in \mathcal{S}^n$*. Moreover, this solution satisfies* $\|AU\|_F \le \|H\|_F/\sqrt{2}$.

Lemma 4.6 (Lemma 2.3 of [30]) *Suppose that* $\gamma \in [0, 1/\sqrt{2})$ *and that* $(U, S) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ *is such that* $\|USU - \nu I\| \le \gamma\nu$ *for some* $\nu > 0$*. If* $(\Delta X, \Delta S, \Delta U) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathcal{S}^n$ *is a solution of the following system*

$$\begin{aligned} \Delta U S U + U S \Delta U + U \Delta S U &= H, \\ \Delta U U + U \Delta U &= \Delta X, \\ \Delta X \bullet \Delta S &= 0 \end{aligned}$$

and $H \in \mathcal{S}^n$*, then*

$$\max\left\{\nu\|U^{-1}\Delta X U^{-1}\|_F, \|U\Delta S U\|_F\right\} \le \frac{\|H\|_F}{(1-\sqrt{2}\gamma)}.$$

We are now ready to show that $\widetilde{\delta X}_B^* = 0$ and $\widetilde{\delta S}_N^* = 0$.

Lemma 4.7 *If* $W_{BN} = 0$ *then*

$$\begin{aligned} \widetilde{\delta X}_B^* = \widetilde{\delta S}_B^* = \widetilde{\delta U}_B^* = 0, \\ \widetilde{\delta X}_N^* = \widetilde{\delta S}_N^* = \widetilde{\delta U}_N^* = 0. \end{aligned}$$

Proof. From Lemma 4.2(ii), we know that $\tilde{X}^*_{BN} = \tilde{S}^*_{BN} = 0$. Using this identity and the fact that $(\widetilde{\delta X}^*_B, \widetilde{\delta S}^*_B)$ and $(\widetilde{\delta X}^*_N, \widetilde{\delta S}^*_N)$ satisfy the first and third equations of (54) and (55), respectively, we obtain that

$$\widetilde{\delta X}^*_B \bullet \widetilde{\delta S}^*_B = 0, \qquad \widetilde{\delta X}^*_N \bullet \widetilde{\delta S}^*_N = 0. \tag{63}$$

By (60), $\tilde{X}^*_{BN} = 0$ implies $\tilde{U}^*_{BN} = 0$, and thus $\mathcal{L}(\tilde{U}^*) = 0$. Hence, by (48) and (49), we have

$$\begin{aligned} \widetilde{\delta U}^* \tilde{S}^* (\tilde{U}^*)^T + \tilde{U}^* \tilde{S}^* (\widetilde{\delta U}^*)^T + \tilde{U}^* \widetilde{\delta S}^* (\tilde{U}^*)^T &= 0, \\ (\widetilde{\delta U}^*)^T \tilde{U}^* + (\tilde{U}^*)^T \widetilde{\delta U}^* &= \widetilde{\delta X}^*. \end{aligned}$$

These equations together with the fact that $\tilde{S}^*_{BN} = 0$ and $\tilde{U}^*_{BN} = 0$ can be easily seen to imply that

$$\widetilde{\delta U}^*_B \; \tilde{S}^*_B \; \tilde{U}^*_B + \tilde{U}^*_B \; \tilde{S}^*_B \; \widetilde{\delta U}^*_B + \tilde{U}^*_B \; \widetilde{\delta S}^*_B \; \tilde{U}^*_B = 0, \tag{64}$$

$$\widetilde{\delta U}^*_B \; \tilde{U}^*_B + \tilde{U}^*_B \; \widetilde{\delta U}^*_B = \widetilde{\delta X}^*_B. \tag{65}$$

Moreover, by (27) $(\tilde{U}, \tilde{X}, \tilde{S}) = (\tilde{U}^*, \tilde{X}^*, \tilde{S}^*)$ and the fact $\tilde{S}^*_{BN} = \tilde{U}^*_{BN} = 0$, we have

$$\tilde{U}^*_B \tilde{S}^*_B \tilde{U}^*_B = W_B,$$

which together with the assumption that $\|W - \nu I\| < \nu/\sqrt{2}$ for some $\nu > 0$ and the fact that $\tilde{U}^* \in \mathcal{U}^n_{++}$ implies

$$\|\tilde{U}^*_B \tilde{S}^*_B \tilde{U}^*_B - \nu I\| = \|W_B - \nu I\| < \nu/\sqrt{2}$$

and $(\tilde{U}^*_B, \ \tilde{S}^*_B) \in \mathcal{S}^{|B|}_{++} \times \mathcal{S}^{|B|}_{++}$. Using the conclusions above, relations (64) and (65), the first identity in (63), together with Lemma 4.5 and Lemma 4.6 with $H = 0$, we conclude that $\widetilde{\delta X}^*_B = \widetilde{\delta S}^*_B = \widetilde{\delta U}^*_B = 0$. Using similar arguments, we can also show that $\widetilde{\delta X}^*_N = \widetilde{\delta S}^*_N = \widetilde{\delta U}^*_N = 0$. ∎

As a consequence of the results obtained above, we have the following theorem when $W_{BN} = 0$.

Theorem 4.8 *If $W_{BN} = 0$ then the primal-dual weighted central path $(X(\nu), y(\nu), S(\nu))$ satisfies:*

i) $\lim_{\nu \downarrow 0} \sqrt{\nu} \; (\dot{X}(\nu), \dot{S}(\nu)) = 0$;

ii) $\|(X(\nu), S(\nu), y(\nu)) - (X^*, S^*, y^*)\| = \Theta(\nu)$;

ii) $\lim_{\nu \downarrow 0} (\dot{X}(\nu), \dot{S}(\nu), \dot{y}(\nu))$ *exists and (61) and (62) hold.*

Proof. Using Lemma 4.2 ii), Lemma 4.7 and the condition $W_{BN} = 0$, we obtain that $\tilde{X}^*_{BN} = \tilde{S}^*_{BN} = 0$, $\widetilde{\delta X}^*_B = 0$ and $\widetilde{\delta S}^*_N = 0$. Consequently, by Theorem 4.1, i) immediately follows. Statements ii) and iii) follow directly from i) and Theorem 4.4. ∎

5 Error bound analysis

By strengthening some of the results of the previous sections, in this section we derive a new error bound on the distance of a point lying in a certain neighborhood of the central path to the primal-dual optimal set.

For any given nonempty compact set $\mathcal{K} \subset \mathcal{G}_{++}$ and constants $\gamma, \tau > 0$, define

$$\mathcal{N}(\gamma, \tau, \mathcal{K}) \equiv \left\{ (X, S, y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m : \mathcal{G}(X, S, y) \in \tau\mathcal{K}, \; \|X^{1/2} S X^{1/2} - \tau I\| \le \gamma\tau \right\},$$

where the map $\mathcal{G}$ and the set $\mathcal{G}_{++}$ are defined in (7) and (8), respectively.

Observe that the set $\cup_{\tau>0}\mathcal{N}(\gamma, \tau, \mathcal{K})$ forms a neighborhood of the primal-dual central path. This neighborhood and related ones have been frequently used in the development of primal-dual interior point algorithms for SDP.

The following result gives a new error bound on the distance of a point lying in $\mathcal{N}(\gamma, \tau, \mathcal{K})$ to the primal-dual optimal set $\mathcal{F}^*_P \times \mathcal{F}^*_D$. Its proof will be given at the end of this section after we have derived stronger versions of the results of the previous sections.

Theorem 5.1 *Let $\gamma \in (0, 1/\sqrt{2})$ and any nonempty compact set $\mathcal{K} \subset \mathcal{G}_{++}$ be given. Then, there exists a constant $M = M(\gamma, \mathcal{K}) > 0$ such that*

$$\mathrm{dist}((X, S, y), \; \mathcal{F}^*_P \times \mathcal{F}^*_D) \le M(\tau + \sqrt{\tau}\|W_{BN}\|), \tag{66}$$

for every $\tau \in (0, 1]$ and $(X, S, y) \in \mathcal{N}(\gamma, \tau, \mathcal{K})$, where $W = W(X, S, \tau) \equiv X^{1/2} S X^{1/2}/\tau$.

In view of Proposition 2.1, for each $(\nu, W, \Delta C, \Delta b) \in (0, 1] \times \mathcal{W} \times \mathcal{G}_{++}$, the system of nonlinear equations (3)-(5) has a unique solution, which in this section we denote by $(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b), y(\nu, W, \Delta C, \Delta b))$ in order to emphasize and study its dependence on $(W, \Delta C, \Delta b)$. Moreover, in view of Theorem 3.2, the limit

$$\lim_{\nu \downarrow 0} (X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b), y(\nu, W, \Delta C, \Delta b)),$$

denoted by $(X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b), y(0, W, \Delta C, \Delta b))$, exists for every $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{G}_{++}$. Hence, the functions $X(\cdot,\cdot,\cdot,\cdot)$, $S(\cdot,\cdot,\cdot,\cdot)$ and $y(\cdot,\cdot,\cdot,\cdot)$ are well-defined in the set of points $[0, 1] \times \mathcal{W} \times \mathcal{G}_{++}$. In an obvious way, we can also define the functions $\tilde{X}(t, W, \Delta C, \Delta b)$, $\tilde{S}(t, W, \Delta C, \Delta b)$ and $\tilde{y}(t, W, \Delta C, \Delta b)$ over the set of points $(t, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W} \times \mathcal{G}_{++}$.

It turns out that re-parametrizations of the above functions are analytic according to the following definition. We say that that a function $f : \Omega \subseteq E \to F$, where E, F are two finite dimensional normed vector spaces, is analytic if there exists an open set $\mathcal{O} \subseteq E$ containing Ω and an analytic function $\tilde{f} : \mathcal{O} \to F$ such that $\tilde{f}$ restricted to Ω is equal to f.

Theorem 5.2 *There hold:*

i) the map $(t, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W} \times \mathcal{G}_{++} \to (\tilde{X}(t, W, \Delta C, \Delta b), \tilde{S}(t, W, \Delta C, \Delta b), \tilde{y}(t, W, \Delta C, \Delta b))$ is analytic;

ii) the map $(t, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W} \times \mathcal{G}_{++} \to (X(t^2, W, \Delta C, \Delta b), S(t^2, W, \Delta C, \Delta b), y(t^2, W, \Delta C, \Delta b))$ is analytic.

Proof. The proof of the theorem is identical to the proof of Theorem 3.11 and Theorem 3.2, except that when invoking the implicit function theorem, we should view $(t, W, \Delta C, \Delta b)$ as the parameter vector. ∎

Let

$$\mathcal{W}^b \equiv \{W \in \mathcal{W} \ : \ W_{BN} = 0\},$$

where $\mathcal{W}$ is defined in (9). One important result that we will establish next is that the function $(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}^b \times \mathcal{G}_{++} \to (X'(t^2, W, \Delta C, \Delta b), S'(t^2, W, \Delta C, \Delta b))$ is analytic. We emphasize that this result only holds over the smaller set $[0,1] \times \mathcal{W}^b \times \mathcal{G}_{++}$. Note also that this result does not follow immediately from Theorem 5.2(ii) since the derivative of the function in Theorem 5.2(ii) is not equal to the above function.

We now state a simple but crucial technical result needed to establish the result stated in the previous paragraph.

Proposition 5.3 *Let $f : I \times E \to F$ be a given analytic function, where $I \subset \Re$ is an interval and E, F are two finite dimensional normed vector spaces. Then, for any $t^* \in I$, the function $g : I \times E \to F$ defined as*

$$g(t,u) = \begin{cases} \frac{f(t,u) - f(t^*,u)}{t - t^*}, & \text{if } t \neq t^*; \\ \frac{\partial f}{\partial t}(t^*, u), & \text{if } t = t^*, \end{cases}$$

is analytic.

We are now ready to establish the result alluded to just before Proposition 5.3.

Lemma 5.4 *There hold:*

i) for any $(W, \Delta C, \Delta b) \in \mathcal{W}^b \times \mathcal{G}_{++}$, we have

$$\lim_{t \downarrow 0} \left(t \frac{\partial X}{\partial \nu}(t^2, W, \Delta C, \Delta b) \, , \, t \frac{\partial S}{\partial \nu}(t^2, W, \Delta C, \Delta b) \right) = 0;$$

ii) $(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}^b \times \mathcal{G}_{++} \to (X'(t^2, W, \Delta C, \Delta b), S'(t^2, W, \Delta C, \Delta b))$ is analytic.

Proof. In view of Theorem 4.8(i), we easily see that i) holds. Since partial derivatives of an analytic function are also analytic, it follows from Theorem 5.2(ii) that the functions

$$(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}^b \times \mathcal{G}_{++} \to \left(t \tfrac{\partial X}{\partial \nu}(t^2, W, \Delta C, \Delta b) \, , \, t \tfrac{\partial S}{\partial \nu}(t^2, W, \Delta C, \Delta b) \right),$$
$$(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}^b \times \mathcal{G}_{++} \to \left(\tfrac{\partial X}{\partial W}(t^2, W, \Delta C, \Delta b) \, , \, \tfrac{\partial S}{\partial W}(t^2, W, \Delta C, \Delta b) \right)$$

are both analytic. Using i) and Proposition 5.3, we conclude that the first function above divided by t is also analytic. We have thus shown that ii) holds. ■

For $\gamma > 0$, let

$$\mathcal{W}(\gamma) \equiv \{W \in \mathcal{S}^n_{++} : \|W - I\| \leq \gamma\}, \quad \mathcal{W}^b(\gamma) \equiv \{W \in \mathcal{W}(\gamma) : W_{BN} = 0\}.$$

We can easily see that if $\gamma < 1/\sqrt{2}$ then $\mathcal{W}(\gamma)$ and $\mathcal{W}^b(\gamma)$ are convex compact subsets of $\mathcal{W}$ and $\mathcal{W}^b$, respectively. For the remainder of this section, we let $\mathcal{K} \subset \mathcal{G}_{++}$ be any given nonempty compact set.

The next two results provide estimates on the sizes of the blocks of the matrices $X(\nu, W, \Delta C, \Delta b)$ and $S(\nu, W, \Delta C, \Delta b)$ first when $(W, \Delta C, \Delta b) \in \mathcal{W}^b(\gamma) \times \mathcal{K}$ and then for a general $(W, \Delta C, \Delta b) \in \mathcal{W}(\gamma) \times \mathcal{K}$.

Lemma 5.5 *Let $\gamma \in (0, 1/\sqrt{2})$ be given. Then, for all $(\nu, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}^b(\gamma) \times \mathcal{K}$, there holds*

$$\|(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b)) - (X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b))\| = \mathcal{O}(\nu).$$

Proof. By the mean value theorem, we have

$$\begin{aligned}&\|(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b)) - (X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b))\| \\ &\leq \sup_{\theta \in [0,1]} \|(X'(\theta\nu, W, \Delta C, \Delta b), S'(\theta\nu, W, \Delta C, \Delta b))\|\, \nu\end{aligned}$$

By Theorem 5.4(ii) and the fact that $\mathcal{W}^b(\gamma) \times \mathcal{K}$ is compact, there exists a constant $M = M(\gamma, \mathcal{K}) > 0$ such that $\|(X'(\theta\nu, W, \Delta C, \Delta b), S'(\theta\nu, W, \Delta C, \Delta b))\| \leq M$ for all $(\theta, \nu, W, \Delta C, \Delta b) \in [0,1] \times [0,1] \times \mathcal{W}^b(\gamma) \times \mathcal{K}$. Hence, the lemma follows. ■

Lemma 5.6 *For all $(\nu, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}(\gamma) \times \mathcal{K}$, there hold*

$$X(\nu, W, \Delta C, \Delta b) - X(0, W^b, \Delta C, \Delta b) = \begin{pmatrix} \mathcal{O}(\|W_{BN}\|) & \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) \\ \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) & \mathcal{O}(\nu\|W_{BN}\|) \end{pmatrix} + \mathcal{O}(\nu), \tag{67}$$

$$S(\nu, W, \Delta C, \Delta b) - S(0, W^b, \Delta C, \Delta b) = \begin{pmatrix} \mathcal{O}(\nu\|W_{BN}\|) & \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) \\ \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) & \mathcal{O}(\|W_{BN}\|) \end{pmatrix} + \mathcal{O}(\nu) \tag{68}$$

where

$$W^b \equiv \begin{pmatrix} W_B & 0 \\ 0 & W_N \end{pmatrix}. \tag{69}$$

Proof. By Theorem 5.2(i), we know that $(\tilde{X}(t, W, \Delta C, \Delta b), \tilde{S}(t, W, \Delta C, \Delta b))$ is analytic over $[0,1] \times \mathcal{W} \times \mathcal{K}$. Hence, its derivative function $(\tilde{X}'(t, W, \Delta C, \Delta b), \tilde{S}'(t, W, \Delta C, \Delta b))$ is analytic, and hence continuous, over $[0,1] \times \mathcal{W} \times \mathcal{K}$. Since $[0,1] \times \mathcal{W}(\gamma) \times \mathcal{K}$ is compact, there exists a constant $M = M(\gamma, \mathcal{K}) > 0$ such that $\|(\tilde{X}'(\nu, W, \Delta C, \Delta b), \tilde{S}'(\nu, W, \Delta C, \Delta b))\| \leq M$ for all $(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}(\gamma) \times \mathcal{K}$. This together with (69) and the mean value theorem implies

$$\begin{aligned}\|\tilde{X}(t, W, \Delta C, \Delta b) - \tilde{X}(t, W^b, \Delta C, \Delta b)\| &= \sup_{\theta \in [0,1]} \|\tilde{X}'(t, \theta W + (1-\theta) W^b, \Delta C, \Delta b)\|\ \|W - W^b\| \\ &\leq M\|W_{BN}\| = \mathcal{O}(\|W_{BN}\|),\end{aligned}$$

for all $(t, W, \Delta C, \Delta b) \in [0,1] \times \mathcal{W}(\gamma) \times \mathcal{K}$. This estimate and the fact that $\tilde{X}(t, W, \Delta C, \Delta b) = D_N(t) X(t^2, W, \Delta C, \Delta b) D_N(t)$ for all $t \in (0,1]$ and $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{K}$ imply

$$\begin{aligned}&X(t^2, W, \Delta C, \Delta b) - X(t^2, W^b, \Delta C, \Delta b) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \left(\tilde{X}(t, W, \Delta C, \Delta b) - \tilde{X}(t, W^b, \Delta C, \Delta b)\right) \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(\|W_{BN}\|) & \mathcal{O}(t\|W_{BN}\|) \\ \mathcal{O}(t\|W_{BN}\|) & \mathcal{O}(t^2\|W_{BN}\|) \end{pmatrix}.\end{aligned}$$

Noting that

$$\begin{aligned}&X(t^2, W, \Delta C, \Delta b) - X(0, W^b, \Delta C, \Delta b)\\&= \Big(X(t^2, W, \Delta C, \Delta b) - X(t^2, W^b, \Delta C, \Delta b)\Big) + \Big(X(t^2, W^b, \Delta C, \Delta b) - X(0, W^b, \Delta C, \Delta b)\Big)\end{aligned}$$

and using the above estimate together with Lemma 5.5, we immediately obtain (67) upon letting $\nu = t^2$. The estimate (68) can be proved in a similar way. ■

We are now ready to state and prove the main result of this section.

Theorem 5.7 *For all* $(\nu, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W}(\gamma) \times \mathcal{K}$*, there hold*

$$X(\nu, W, \Delta C, \Delta b) - X(0, W, \Delta C, \Delta b) = \begin{pmatrix} \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) & \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) \\ \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) & \mathcal{O}(\nu\|W_{BN}\|) \end{pmatrix} + \mathcal{O}(\nu), \tag{70}$$

$$S(\nu, W, \Delta C, \Delta b) - S(0, W, \Delta C, \Delta b) = \begin{pmatrix} \mathcal{O}(\nu\|W_{BN}\|) & \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) \\ \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) & \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\|) \end{pmatrix} + \mathcal{O}(\nu) \tag{71}$$

Proof. We will prove (70) only since the proof of (71) is similar. Since both $X(0, W, \Delta C, \Delta b)$ and $X(0, W^b, \Delta C, \Delta b)$ are in $\mathcal{F}_P^*$, we have $X_{BN}(0, W, \Delta C, \Delta b) = X_{BN}(0, W^b, \Delta C, \Delta b) = 0$ and $X_N(0, W, \Delta C, \Delta b) = X_N(0, W^b, \Delta C, \Delta b) = 0$ for any $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{K}$. Hence, in view of Lemma 5.6, it suffices to show that for all $(\nu, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W}(\gamma) \times \mathcal{K}$, we have

$$X_B(\nu, W, \Delta C, \Delta b) - X_B(0, W, \Delta C, \Delta b) = \mathcal{O}(\sqrt{\nu}\,\|W_{BN}\| + \nu). \tag{72}$$

Indeed, using the fact that $\tilde{X}_B(t, W, \Delta C, \Delta b)$ is analytic over the compact set $[0, 1] \times \mathcal{W}(\gamma) \times \mathcal{K}$ due to Theorem 5.2(i), we conclude that

$$\begin{aligned} X_B(t^2, W, \Delta C, \Delta b) - X_B(0, W, \Delta C, \Delta b) &= \tilde{X}_B(t, W, \Delta C, \Delta b) - \tilde{X}_B(0, W, \Delta C, \Delta b) \\ &= \frac{\partial}{\partial t}\tilde{X}_B(0, W, \Delta C, \Delta b)t + \mathcal{O}(t^2), \end{aligned} \tag{73}$$

for every $(t, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W}(\gamma) \times \mathcal{K}$. Moreover, by Lemma 4.7, we have

$$\frac{\partial \tilde{X}_B}{\partial t}(0, W^b, \Delta C, \Delta b) = 0,$$

for any $(W, \Delta C, \Delta b) \in \mathcal{W} \times \mathcal{K}$, where W^b is defined in (69). Hence, for every $(W, \Delta C, \Delta b) \in \mathcal{W}(\gamma) \times \mathcal{K}$, we have

$$\frac{\partial}{\partial t}\tilde{X}_B(0, W, \Delta C, \Delta b) = \frac{\partial}{\partial t}\tilde{X}_B(0, W^b, \Delta C, \Delta b) + \mathcal{O}(\|W - W^b\|) = \mathcal{O}(\|W_{BN}\|). \tag{74}$$

Combining (73) and (74), we obtain (72) upon letting $\nu = t^2$. ■

The proof of Theorem 5.1 now follows from Assumption **A.1** and Theorem 5.7 with $\nu = \tau$, $W = X^{1/2}SX^{1/2}/\tau$, $(X, S) = (X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b))$ and the fact $(X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b), y(0, W, \Delta C, \Delta b)) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$.

6 Superlinear convergence criteria

In this section, we consider a sufficient condition introduced by Potra and Sheng [33], which guarantees the superlinear convergence of a class of primal-dual interior point algorithms for SDP, and show that it is equivalent to a natural condition about the matrix $W(X, S, \tau)$ of Theorem 5.1.

For sake of concreteness, we will focus our attention on the algorithm and results obtained in Potra and Sheng (see Algorithm 3.1 in [34]), but we remark that our discussion also applies to a broader class of algorithms. Potra and Sheng [34] have developed a primal-dual infeasible-interior-point algorithm which, for some $\alpha \in (0, 1/2]$, generates a sequence of iterates $\{(X^k, S^k, y^k)\} \subseteq \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$ satisfying

$$\|W^k - I\|_F \leq \alpha, \quad r_p^k = \frac{\tau_k}{\tau_0} r_p^0, \quad r_d^k = \frac{\tau_k}{\tau_0} r_d^0, \tag{75}$$

for some sequence $\{\tau_k\} \subset \Re_{++}$ converging to 0 at least Q-linearly, where

$$\begin{aligned} r_p^k &\equiv \mathcal{A}X^k - b, \\ r_d^k &\equiv \mathcal{A}^* y^k + S^k - C, \\ W^k &\equiv \frac{(X^k)^{1/2} S^k (X^k)^{1/2}}{\tau_k}, \end{aligned}$$

for all $k \geq 0$. The derived linear rate of convergence of the sequence $\{\tau_k\}$ is sufficient to guarantee polynomial convergence of their method under some suitable conditions on the initial point (X^0, S^0, y^0). Observe that the first condition in (75) implies that $\tau_k = \Theta(X^k \bullet S^k/n)$, and hence asymptotic convergence of $\{X^k \bullet S^k/n\}$ can be derived from the one obtained for $\{\tau_k\}$.

Some sufficient conditions have been developed in the literature which guarantee the Q-superlinear convergence of $\{\tau_k\}$ to zero. One such condition is the tangential condition proposed by Kojima et al. [17], namely

$$\lim_{k \to \infty} W^k = I. \tag{76}$$

Another such condition, and the one which will be the main subject of this section, is the one that has been proposed by Potra and Sheng [33], namely

$$\lim_{k \to \infty} X^k S^k / \sqrt{\tau_k} = 0. \tag{77}$$

We remark that Potra and Sheng [33]) have shown that the tangential condition (76) implies their condition (77). Moreover, they have also established the following superlinear convergence result.

Proposition 6.1 (Theorem 6.1 of [33]) *If (77) holds, then the sequence $\{\tau_k\}$ generated by Algorithm 3.1 of [34] converges to zero Q-superlinearly. Moreover, if $X^k S^k = \mathcal{O}(\tau_k^{0.5+\sigma})$ for some constant $\sigma > 0$, then $\{\tau_k\}$ converges to zero with Q-order at least* $1 + \min\{\sigma, 0.5\}$.

A natural relaxation of the tangential condition (76) is the condition that

$$\lim_{k \to \infty} W_{BN}^k = 0. \tag{78}$$

Surprisingly, the following result shows that it is equivalent to Potra and Sheng's condition (77).

Proposition 6.2 *Let $\theta_k \equiv \|X^k S^k\|/\sqrt{\tau_k}$. Then, $\|W^k_{BN}\| + \sqrt{\tau_k} = \Theta(\theta_k + \sqrt{\tau_k})$.*

Proof. We first show that $\|W^k_{BN}\| + \sqrt{\tau_k} = \mathcal{O}(\theta_k + \sqrt{\tau_k})$. By Lemma 4.2 of [33], we have

$$X^k = \begin{pmatrix} \Theta(1) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\tau_k) \end{pmatrix}, \quad S^k = \begin{pmatrix} \mathcal{O}(\tau_k) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \Theta(1) \end{pmatrix}.$$

and hence

$$\frac{X^k S^k}{\sqrt{\tau_k}} = \frac{1}{\sqrt{\tau_k}} \begin{pmatrix} \Theta(1) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\tau_k) \end{pmatrix} \begin{pmatrix} \mathcal{O}(\tau_k) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \Theta(1) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(1) \\ \mathcal{O}(\tau_k) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix}.$$

According to the definition of θ_k , we then conclude that

$$\frac{X^k S^k}{\sqrt{\tau_k}} = \begin{pmatrix} \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\theta_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix}. \tag{79}$$

By Lemma 4.5 of [33], we have

$$(X^k)^{1/2} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix}, \quad (X^k)^{-1/2} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1/\sqrt{\tau_k}) \end{pmatrix}, \tag{80}$$

which together with (79) imply

$$\begin{aligned} W^k &= \frac{(X^k)^{1/2} S^k (X^k)^{1/2}}{\tau_k} = \frac{1}{\sqrt{\tau_k}} (X^k)^{-1/2} \left(\frac{X^k S^k}{\sqrt{\tau_k}} \right) (X^k)^{1/2} \\ &= \frac{1}{\sqrt{\tau_k}} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1/\sqrt{\tau_k}) \end{pmatrix} \begin{pmatrix} \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\theta_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix}. \end{aligned}$$

Since the (B,N)-block of the matrix in the right hand side of the above identity is $\mathcal{O}(\theta_k + \sqrt{\tau_k})$, we conclude that $\|W^k_{BN}\| + \sqrt{\tau_k} = \mathcal{O}(\theta_k + \sqrt{\tau_k})$.

Next we show that $\theta_k + \sqrt{\tau_k} = \mathcal{O}(\|W^k_{BN}\| + \sqrt{\tau_k})$. By the first condition in (75), we have

$$W^k = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\|W^k_{BN}\|) \\ \mathcal{O}(\|W^k_{BN}\|) & \mathcal{O}(1) \end{pmatrix},$$

which together with (80) implies that

$$\begin{aligned} \frac{X^k S^k}{\sqrt{\tau_k}} &= \sqrt{\tau_k}\, (X^k)^{1/2} \left(\frac{(X^k)^{1/2} S^k (X^k)^{1/2}}{\tau_k} \right) (X^k)^{-1/2} = \sqrt{\tau_k}\, (X^k)^{1/2} W^k (X^k)^{-1/2} \\ &= \sqrt{\tau_k} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\sqrt{\tau_k}) \\ \mathcal{O}(\sqrt{\tau_k}) & \mathcal{O}(\sqrt{\tau_k}) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\|W^k_{BN}\|) \\ \mathcal{O}(\|W^k_{BN}\|) & \mathcal{O}(1) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1/\sqrt{\tau_k}) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}\left(\sqrt{\tau_k} + \sqrt{\tau_k}\, \|W^k_{BN}\|\right) & \mathcal{O}\left(\sqrt{\tau_k} + \|W^k_{BN}\|\right) \\ \mathcal{O}\left(\tau_k + \tau_k \|W^k_{BN}\|\right) & \mathcal{O}\left(\sqrt{\tau_k} + \sqrt{\tau_k}\, \|W^k_{BN}\|\right) \end{pmatrix}. \end{aligned}$$

This together with the definition of θ_k implies that $\theta_k + \sqrt{\tau_k} = \mathcal{O}(W_{BN}^k + \sqrt{\tau_k})$. ∎

In view of the equivalence between (77) and (78), it follows that a sufficient condition for the superlinear convergence of the path-following algorithm outlined in this section is that the sequence of matrices $\{W^k\}$ approaches the set of block diagonal matrices. Clearly, this a much weaker condition than (76), which requires this sequence to approach the identity matrix.

7 Concluding remarks

In this section we provide some final remarks related to the results derived in this paper.

Under the assumptions of this paper, we have shown that the re-parametrized $(W, \Delta C, \Delta b)$-weighted central path $(X(t^2), S(t^2), y(t^2))$ is analytic at $t = 0$ and that the condition $W_{BN} = 0$ implies that $\lim_{\nu \downarrow 0}(\dot{X}(\nu), \dot{S}(\nu), \dot{y}(\nu))$ exists. Based on the latter conclusion, it is natural to wonder whether the path $(X(\nu), S(\nu), y(\nu))$ is analytic at $\nu = 0$ when W is block-diagonal. Note that the answer to this question is affirmative when $(W, \Delta C, \Delta b) = (I, 0, 0)$, i.e., the weighted central path is exactly the central path (see Halická [12]).

In this paper, we have proved that the rate of convergence of the $(W, \Delta C, \Delta b)$-weighted central path $(X(\nu), S(\nu), y(\nu))$ towards the optimal solution set is $\mathcal{O}(\sqrt{\nu})$ (and $\mathcal{O}(\nu)$ when $W_{BN} = 0$). In contrast, Preiß and Stoer [35] have shown that the rate of convergence of the weighted central paths associated with the map $(XS + SX)/2$ is always $\mathcal{O}(\nu)$(see also Lu and Monteiro [20]). An error bound of this type has also been shown by Kojima et al. [18], where it is shown that an interior-point algorithm based on a centering condition associated with the $(XS + SX)/2$-map does not need to approach the central path tangentially in order to converge superlinearly. On the other hand, the iterates of all superlinearly-convergent interior-point algorithms based on centering conditions associated with the map $X^{1/2}SX^{1/2}$ that have been proposed in the literature are required to approach the central path tangentially. The latter requirement is natural in view of the fact that it forces $(X^k)^{1/2}S^k(X^k)^{1/2}$ to approach a block diagonal matrix (the identity matrix), and hence it reduces the bound on the distance of (X^k, S^k, y^k) to the optimal solution set from the usual $\mathcal{O}(\sqrt{\mu_k})$ to $o(\sqrt{\mu_k})$ (see Theorem 5.1).

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