Asymptotic behavior of the central path for a special class of degenerate SDP problems

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Abstract

This paper studies the asymptotic behavior of the central path $(X(\nu), S(\nu), y(\nu))$ as $\nu \downarrow 0$ for a class of degenerate semidefinite programming (SDP) problems, namely those that do not have strictly complementary primal-dual optimal solutions and whose "degenerate diagonal blocks" $X_{\mathcal{T}}(\nu)$ and $S_{\mathcal{T}}(\nu)$ of the central path are assumed to satisfy $\max\{\|X_{\mathcal{T}}(\nu)\|, \|S_{\mathcal{T}}(\nu)\|\} = \mathcal{O}(\sqrt{\nu})$. We establish the convergence of the central path towards a primal-dual optimal solution, which is characterized as being the unique optimal solution of a certain log-barrier problem. A characterization of the class of SDP problems which satisfy our assumptions are also provided. It is shown that the re-parametrization $t > 0 \to (X(t^4), S(t^4), y(t^4))$ of the central path is analytic at $t = 0$. The limiting behavior of the derivative of the central path is also investigated and it is shown that the order of convergence of the central path towards its limit point is $\mathcal{O}(\sqrt{\nu})$. Finally, we apply our results to the convex quadratically constrained convex programming (CQCCP) problem and characterize the class of CQCCP problems which can be formulated as SDPs satisfying the assumptions of this paper. In particular, we show that CQCCP problems with either a strictly convex objective function or at least one strictly convex constraint function lie in this class.

Key words: Limiting behavior, central path, semidefinite programming, convex quadratic programming, convex quadratically constrained programming.

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1 Introduction

In this paper we will study the asymptotic behavior of the central path $(X(\nu), S(\nu), y(\nu)$ for a class of degenerate semidefinite programming (SDP) problems, namely those that do not have strictly complementary primal-dual optimal solutions and whose "degenerate diagonal blocks" $X_{\mathcal{T}}(\nu)$ and $S_{\mathcal{T}}(\nu)$ of the central path are assumed to satisfy $\max\{\|X_{\mathcal{T}}(\nu)\|, \|S_{\mathcal{T}}(\nu)\|\} = \mathcal{O}(\sqrt{\nu})$.

Properties of the central path have been extensively studied on several papers due to the important role it plays in the development of interior-points algorithms for cone programming, nonlinear programming and complementarity problems. Early works dealing with the well-definedness, differentiability and limiting behavior of weighted central paths in the context of the linear programming and the monotone complementarity problem include [1, 2, 3, 5, 6, 7, 12, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 28, 29].

Convergence of the central path for a SDP problem towards a primal-dual optimal solution has been established by Kojima et al. [13] (see also Halická et al. [10]) using a deep algebraic geometry result (see Lemma 3.1 of Milnor [19]). Characterization of the limit point of the central path has been obtained by Luo et al. [14] for SDP problems possessing strictly complementary primal-dual optimal solutions. Also, for this special class of SDP problems, Halická [8] has shown that the central path can be extended analytically "beyond" $\nu = 0$. For more general SDP problems, the above issues regarding the central path still remain open although some progress have been made on a few papers. These include Goldfarb and Scheinberg [4] who proved that the limit point of the central path must be a maximally complementary optimal solution and Halická et al. [9], and Sporre and Forsgren [25] who provided a partial characterization of the limit point of the central path as being the analytic center of some convex subset of the optimal solution set.

The organization of our paper is as follows. In Subsection 1.1, we list some basic notation and terminology used in our presentation. In Section 2, we review the notion of the central path, introduce the assumptions that will be in our presentation and state some basic results about the central path and its underlying structure. In Section 3, we derive some important estimates on the off-diagonal blocks of the central path. In Section 4, we establish the convergence of the central path towards a primal-dual optimal solution, which is characterized as being the unique optimal solution of a certain log-barrier problem. We also characterize the class of SDP problems which satisfy our initial assumption on the degenerate diagonal blocks of the central path. In Section 5, we look at a different scaled version of the central path and, as a by-product, we conclude that the re-parametrized central path $t > 0 \to (X(t^4), S(t^4), y(t^4))$ is analytic at $t = 0$. We also analyze the limiting behavior of the derivative of the central path and conclude that the order of convergence of the central path towards its limit point is $\mathcal{O}(\sqrt{\nu})$. In Section 6, we apply our results to the convex quadratically constrained convex programming (CQCCP) problem and characterize the class of CQCCP problems which can be formulated as SDPs satisfying the assumptions of this paper. In particular, we show that CQCCP problems with either a strictly convex objective function or at least one strictly convex constraint function lie in this class.

1.1 Notation and terminology

The following notation is used throughout our presentation. If J is a subset of Υ, we sometimes denote its complement with respect to Υ by $\bar{J}$. $\Re^p$ denotes the p-dimensional Euclidean space and, for a given subset I of $\{1,\ldots,n\}$, $\Re^I$ denotes the set of all real tuples $(x_i : i \in I)$ indexed by I. The (i,j)-th entry of a matrix $Q \in \Re^{p\times q}$ is denoted by Q_{ij} and the j-th column is denoted by Q_j. The set of all symmetric $p \times p$ matrices is denoted by $\mathcal{S}^p$. The cone of positive semidefinite (resp., definite) $p \times p$ symmetric matrices is denoted by $\mathcal{S}^p_+$ (resp., $\mathcal{S}^p_{++}$). For P, $Q \in \mathcal{S}^p$, $Q \succeq P$ (or $P \preceq Q$) means that $Q - P \in \mathcal{S}^p_+$ and $Q \succ P$ (or $P \prec Q$) means that $Q - P \in \mathcal{S}^p_{++}$. The trace of a matrix $Q \in \Re^{p\times p}$ is denoted by $\operatorname{tr} Q \equiv \sum_{i=1}^p Q_{ii}$. Given P and Q in $\Re^{p\times q}$, the inner product between them is defined as $P \bullet Q \equiv \operatorname{tr} P^T Q = \sum_{i=1,j=1}^p P_{ij} Q_{ij}$. The Frobenius norm of the matrix Q is defined as $\|Q\| \equiv (Q \bullet Q)^{1/2}$. The image (or range) space and the null space of a linear operator $\mathbb{P}$ will be denoted by $\mathrm{Im}(\mathbb{P})$ and $\mathrm{Null}(\mathbb{P})$ respectively; the dimension of the subspace $\mathrm{Im}(\mathbb{P})$, referred to as the rank of $\mathbb{P}$, will be denoted by $\mathrm{rank}(\mathbb{P})$. Given a linear operator $\mathcal{F} : E \to F$ between two finite dimensional inner product spaces $(E, \langle\cdot,\cdot\rangle_E)$ and $(F, \langle\cdot,\cdot\rangle_F)$, its *adjoint* is the unique operator $\mathcal{F}^* : F \to E$ satisfying $\langle \mathcal{F}(u), v\rangle_F = \langle u, \mathcal{F}^*(v)\rangle_E$ for all $u \in E$ and $v \in F$. Given functions $f : \Omega \to \mathfrak{E}$ and $g : \Omega \to (0, +\infty)$, where Ω is an arbitrary set and $\mathfrak{E}$ is a normed vector space, and a subset $\tilde{\Omega} \subset \Omega$, we write $f(w) = \mathcal{O}(g(w))$ for all $w \in \tilde{\Omega}$ to mean that, for some constant $M > 0$, $\|f(w)\| \le M g(w)$ for all $w \in \tilde{\Omega}$.

Given a matrix $X \in \Re^{n\times n}$ and a subset $\mathcal{J}$ of $\{(k,\ell) : \ell, k = 1, \cdots n\}$, we let $X_{\mathcal{J}} \equiv (X_{k\ell} : (k,\ell) \in \mathcal{J})$ and think of it as a "submatrix" of X. When $\mathcal{J} = B \times N$, where B and N are two subsets of $\{1,\ldots,n\}$, we will denote $X_{\mathcal{J}}$ simply by X_{BN}; moreover, if $\mathcal{J} = B \times B$ then $X_{\mathcal{J}}$ is denoted simply by $X_{\mathcal{B}}$ with the understanding that $\mathcal{B} = B \times B$. A subset $\mathcal{J} \subset \{(k,\ell) : \ell, k = 1, \cdots n\}$ is said to be *symmetric* if $(k,\ell) \in \mathcal{J}$ implies that $(\ell, k) \in \mathcal{J}$. For a symmetric set $\mathcal{J}$ of $\{(k,\ell) : \ell, k = 1, \cdots n\}$, we will denote by $\mathcal{S}^{\mathcal{J}}$ the set of all "symmetric matrices" $(X_{j\ell} : \ell, k = 1, \cdots n)$ satisfying $X_{j\ell} = X_{\ell j}$ for all $(j,\ell) \in \mathcal{J}$, and will often denote an element X of $\mathcal{S}^{\mathcal{J}}$ by $X_{\mathcal{J}}$ to emphasize its indexing by $\mathcal{J}$. Clearly, any element of $\mathcal{S}^{\mathcal{J}}$ is a usual matrix in the case where $\mathcal{J}$ is a Cartesian product of two subsets of $\{1, \cdots n\}$. $\mathcal{S}^{\mathcal{J}}$ can be thought as a subset of $\mathcal{S}^n$ by identifying $X \in \mathcal{S}^{\mathcal{J}}$ with the matrix $Y \in \mathcal{S}^n$ such that $Y_{\mathcal{J}} = X$ and $Y_{\bar{\mathcal{J}}} = 0$. Given a map $\mathbb{U} : \mathcal{S}^n \to E$ and a symmetric subset $\mathcal{J}$ of $\{(k,\ell) : \ell, k = 1, \cdots n\}$, we denote by $\mathbb{U}_{\mathcal{J}}$ the restriction of $\mathbb{U}$ to $\mathcal{S}^{\mathcal{J}}$. Also, given a map $\mathbb{A} : \mathcal{S}^n \to \Re^m$, a symmetric subset $\mathcal{J}$ of $\{(k,\ell) : \ell, k = 1, \cdots n\}$ and a subset I of $\{1, \cdots, m\}$, we denote by $\mathbb{A}_{I\mathcal{J}} : \mathcal{S}^{\mathcal{J}} \to \Re^I$ the map defined for every $X \in \mathcal{S}^{\mathcal{J}}$ by $\mathbb{A}_{I\mathcal{J}}(X) = (u_i : i \in I)$, where $u = \mathbb{A}_{\mathcal{J}}(X)$. For given vector spaces $\mathfrak{E}_1, \cdots, \mathfrak{E}_q$ and $\mathfrak{F}_1, \cdots, \mathfrak{F}_p$ and given linear operators $\mathbb{P}_{ij} : \mathfrak{E}_j \to \mathfrak{F}_i$, for $i = 1, \cdots, p$ and $j = 1, \cdots, q$, the *matrix operator* of the $\mathbb{P}_{ij}$'s, denoted by

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_{11} & \cdots & \mathbb{P}_{1q} \\ \vdots & \ddots & \vdots \\ \mathbb{P}_{p1} & \cdots & \mathbb{P}_{pq} \end{pmatrix},$$

or simply by $(\mathbb{P}_{ij})_{1,1}^{p,q}$, is the linear operator $\mathbb{P} : \mathfrak{E}_1 \times \cdots \times \mathfrak{E}_q \to \mathfrak{F}_1 \times \cdots \times \mathfrak{F}_p$ defined as

$$\mathbb{P}\,(x_1, \cdots, x_q) = \begin{pmatrix} \mathbb{P}_{11} & \cdots & \mathbb{P}_{1q} \\ \vdots & \ddots & \vdots \\ \mathbb{P}_{p1} & \cdots & \mathbb{P}_{pq} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{q} \mathbb{P}_{1j} x_j \\ \vdots \\ \sum_{j=1}^{q} \mathbb{P}_{pj} x_j \end{pmatrix}.$$

for every $(x_1, \ldots, x_q) \in \mathfrak{E}_1 \times \cdots \times \mathfrak{E}_q$. It is easy to verify that the adjoint of above operator is the matrix operator $(\mathbb{P}_{ji}^*)_{1,1}^{q,p}$.

2 Preliminaries

In this section, we describe our problem and the assumptions that will be used throughout the paper. We also describe the central path that will be the subject of our study in this paper. Some preliminary results about this path are also stated besides previous results.

We consider the semidefinite programming problem

$$(P) \qquad \min \{C \bullet X : \mathbb{A}X = b,\ X \succeq 0\},$$

and its associated dual SDP

$$(D) \qquad \max \{b^T y : \mathbb{A}^* y + S = C,\ S \succeq 0\},$$

where the data consists of $C \in \mathcal{S}^n$, $b \in \Re^m$ and a linear operator $\mathbb{A} : \mathcal{S}^n \to \Re^m$, the primal variable is $X \in \mathcal{S}^n$, and the dual variable consists of $(S, y) \in \mathcal{S}^n \times \Re^m$. We write $\mathcal{F}(P)$ and $\mathcal{F}(D)$ for the sets of feasible solutions to (P) and (D) respectively, and correspondingly $\mathcal{F}^0(P)$ and $\mathcal{F}^0(D)$ for the sets of strictly feasible solutions to (P) and (D) respectively; here "strictly" means that X or S is required to be positive definite. We also write $\mathcal{F}^*(P)$ and $\mathcal{F}^*(D)$ for the sets of optimal solutions of (P) and (D) respectively.

Throughout this paper, we assume that the following two conditions hold without explicitly mentioning them in the statements of our results.

A1) $\mathbb{A} : \mathcal{S}^n \to \Re^m$ is a surjective linear operator;

A2) $\mathcal{F}^0(P) \neq \varnothing$ and $\mathcal{F}^0(D) \neq \varnothing$.

Assumption **A.1** is not really crucial for our analysis but it is convenient to ensure that the variables S and y are in one-to-one correspondence. We will see that the dual central path can always be defined in the S-space. The goal of Assumption **A.1** is just to ensure that this path is also well-defined in the y-space. Assumption **A.2** ensures that both (P) and (D) have optimal solutions and that the optimal values of (P) and (D) are equal. It is also important to ensure the existence of the central path.

Our interest in this paper is to study the set of solutions of the following system of nonlinear equations parametrized by the parameter $\nu > 0$:

$$XS = \nu I, \quad \nu > 0. \tag{1}$$

$$\mathbb{A}^* y + S = C, \quad S \succ 0, \tag{2}$$

$$\mathbb{A}X = b, \quad X \succ 0, \tag{3}$$

When $\nu = 0$, the set of solutions $(X, S, y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m$ of (1)-(3) is exactly the set $\mathcal{F}^*(P) \times \mathcal{F}^*(D)$. Moreover, for each $\nu > 0$, it is well-known that system (1)-(3) has a unique solution in $\mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m$, which we denote by $(X(\nu), S(\nu), y(\nu))$ (see for example Monteiro and Todd [22]). The *central path* is the path $\nu > 0 \to (X(\nu), S(\nu), y(\nu))$, which is known to be an analytic map.

A point $(X^*, S^*, y^*) \in \mathcal{F}^*(P) \times \mathcal{F}^*(D)$ is said to be a *maximally complementary solution pair* if it maximizes $\text{rank}(X) + \text{rank}(S)$ over $\mathcal{F}^*(P) \times \mathcal{F}^*(D)$. It is known that the set of maximally complementary solution pairs coincides with the relative interior of $\mathcal{F}^*(P) \times \mathcal{F}^*(D)$. Kojima et al. [13] (see also Halická et al. [10]) have shown that the the central path converges to a point in $\mathcal{F}^*(P) \times \mathcal{F}^*(D)$ as $\nu \downarrow 0$ and Goldfarb and Scheinberg [4] has shown that its limit point is a maximally complementary solution pair.

Let (X^*, S^*, y^*) be a maximally complementary solution pair and assume that P is a nonsingular matrix such that $P^T X^* P$ and $P^{-1} S^* P^{-T}$ are both diagonal matrices. Since $X^* S^* = 0$, and hence the matrices X^* and S^* commute, we know that there exists an orthonormal matrix $Q \in \Re^{n \times n}$ such that $Q^T X^* Q$ and $Q^T S^* Q$ are both diagonal. Hence, the existence of a matrix P as above is guaranteed by simply letting $P = Q$. Performing the change of variables $\tilde{X} = P^T X P$ and $(\tilde{S}, \tilde{y}) = (P^{-1} S P^{-T}, y)$ on problems (P) and (D) yield another pair of primal and dual SDPs which has a maximally complementary solution pair $(\tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ such that $\tilde{X}^*$ and $\tilde{S}^*$ are both diagonal. We observe that the central path in the original space corresponds to the central path in the scaled space, i.e. the map $\nu > 0 \to (P^T X(\nu) P, P^{-1} S(\nu) P^{-T}, y(\nu))$ is exactly the central path in the scaled space. Hence, there is no loss of generality if we introduce the above scaling and study the central path in the scaled space. To keep the same notation we have been using so far, we will assume without loss of generality that the original (P) and (D) already have a maximally complementary solution pair (X^*, S^*, y^*) such that

$$X^* = \begin{pmatrix} X^*_{\mathcal{B}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S^*_{\mathcal{N}} \end{pmatrix}, \tag{4}$$

where $X^*_{\mathcal{B}} \in \mathcal{S}^{|B|}_{++}$ and $S^*_{\mathcal{N}} \in \mathcal{S}^{|N|}_{++}$. Clearly, $|B| + |N| \leq n$. Here the subscripts B and N are the subsets of $\{1, \ldots, n\}$ consisting of the row (or column) indices of the rows of X^* and S^* containing the rows of $X^*_{\mathcal{B}}$ and $S^*_{\mathcal{N}}$ respectively. We define $T \equiv \{1, \ldots, n\} \setminus (B \cup N)$. Throughout this paper, we make the following extra assumption.

A3) $T \neq \varnothing$.

In the other words, assumption **A3** means that there exists no strictly complementary primal-dual optimal solution, i.e. a pair $(\bar{X}, \bar{S}, \bar{y}) \in \mathcal{F}^*(P) \times \mathcal{F}^*(D)$ such that $\bar{X} + \bar{S} \succ 0$. We observe that the assumption **A3** implies that $N \neq \varnothing$ and $B \neq \varnothing$. To see that, suppose for contradiction that $B = \varnothing$. Then, by (4) we have $X^* = 0$, and hence $b = 0$. Clearly, this implies that $\mathcal{F}^*(D) = \mathcal{F}(D)$. Since $\mathcal{F}^0(D) \neq \varnothing$ by **A2**, it follows that $\mathcal{F}^*(D)$ contains a positive definite matrix, yielding the contradiction that $T = \varnothing$. Hence, we must have $B \neq \varnothing$. The proof that $N \neq \varnothing$ is similar.

Notice that $(X, S, y) \in \mathcal{F}^*(P) \times \mathcal{F}^*(D)$ if and only if $(X, S, y) \in \mathcal{F}(P) \times \mathcal{F}(D)$, $XS^* = 0$ and $X^*S = 0$. Hence, using (4) and the fact that (X^*, S^*, y^*) is a maximally complementary solution pair, it is easy to see that

$$\mathcal{F}^*(P) = \{X \in \mathcal{F}(P) : X_{\bar{\mathcal{B}}} = 0\}, \quad \mathcal{F}^*(D) = \{(S, y) \in \mathcal{F}(D) : S_{\bar{\mathcal{N}}} = 0\}. \tag{5}$$

Given a $(X, S, y) \in \mathcal{F}(P) \times \mathcal{F}(D)$, we will consider throughout the paper the decompositions of X and S according to the optimal decomposition (4) as follows:

$$X = \begin{pmatrix} X_{\mathcal{B}} & X_{BT} & X_{BN} \\ X_{TB} & X_{\mathcal{T}} & X_{TN} \\ X_{NB} & X_{NT} & X_{\mathcal{N}} \end{pmatrix}, \quad S = \begin{pmatrix} S_{\mathcal{B}} & S_{BT} & S_{BN} \\ S_{TB} & S_{\mathcal{T}} & S_{TN} \\ S_{NB} & S_{NT} & S_{\mathcal{N}} \end{pmatrix}.$$

The next result states some basic properties about the order of convergence of the several blocks of $X(\nu)$ and $S(\nu)$ as $\nu \downarrow 0$.

Lemma 1. Let (B, T, N) be a optimal partition. Then

$$X_{\mathcal{B}}(\nu) = \mathcal{O}(1), \qquad S_{\mathcal{N}}(\nu) = \mathcal{O}(1), \tag{6}$$

$$X_{\mathcal{N}}(\nu) = \mathcal{O}(\nu), \qquad S_{\mathcal{B}}(\nu) = \mathcal{O}(\nu), \tag{7}$$

$$\|X_{BN}(\nu)\| = \mathcal{O}\left(\sqrt{\nu}\right), \qquad \|S_{BN}(\nu)\| = \mathcal{O}\left(\sqrt{\nu}\right), \tag{8}$$

$$\|X_{TB}(\nu)\| = \mathcal{O}\left(\|X_{\mathcal{T}}(\nu)\|^{1/2}\right), \qquad \|S_{TB}(\nu)\| = \mathcal{O}\left(\sqrt{\nu}\|S_{\mathcal{T}}(\nu)\|^{1/2}\right), \tag{9}$$

$$\|X_{TN}(\nu)\| = \mathcal{O}\left(\sqrt{\nu}\|X_{\mathcal{T}}(\nu)\|^{1/2}\right), \qquad \|S_{TN}(\nu)\| = \mathcal{O}\left(\|S_{\mathcal{T}}(\nu)\|^{1/2}\right). \tag{10}$$

Proof. The proof of (6) and (7) is similar to the one of Lemma 3.2 of Luo, Sturm and Zhang (1998). Using the fact that $X(\nu) \succ 0$ and $S(\nu) \succ 0$, we obtain that $X_{ii}(\nu) > 0$, $S_{ii}(\nu) > 0$,

$$\sqrt{X_{ii}(\nu)X_{jj}(\nu)} \geq |X_{ij}(\nu)| \text{ and } \sqrt{S_{ii}(\nu)S_{jj}(\nu)} \geq |S_{ij}(\nu)|, \tag{11}$$

for all $i, j \in \{1, \cdots, n\}$. The estimates in (8), (9) and (10) follow from (6), (7) and (11). □

Note that the estimates on the order of convergence of the off-diagonal blocks (9) and (10) are functions of $\|X_{\mathcal{T}}(\nu)\|$ and $\|S_{\mathcal{T}}(\nu)\|$. For a general SDP problem, it is an open and difficult problem to accurately predict the exact order of these blocks based on the description of the problem. To make the problem more tractable, we will assume throughout most of the paper that the following estimates hold:

A4) $X_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$, and $S_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$.

Note that **A4** is a particular case of the following (apparently) weaker condition.

A4') $\|X_{\mathcal{T}}(\nu)\|\|S_{\mathcal{T}}(\nu)\| = \mathcal{O}(\nu)$.

We will prove in Section 4 that **A4** and **A4'** are actually equivalent. Note that this prevents estimates like $\|X_{\mathcal{T}}(\nu)\| = \mathcal{O}(\nu^{1/4})$ and $\|X_{\mathcal{T}}(\nu)\| = \mathcal{O}(\nu^{3/4})$ to hold.

3 Some technical results

Our main goal in this section is to show that the estimates in (9) and (10) can be improved when either conditions **A4** or **A4'** is in force. The main results obtained in this section are stated in Theorem 1 and Corollary 1. In fact, we recommend the reader on a first reading to skip their highly technical proofs and only read their statements.

Lemma 2. Let $(J, \bar{J})$ be a partition of the index set $\{1, \cdots, n\}$. If $X, S \in \mathcal{S}^n_{++}$ is such that $XS = \nu I$ for some $\nu > 0$, then

a) $\left\| X_{\mathcal{J}}^{-1/2} X_{J\bar{J}} S_{\bar{\mathcal{J}}}^{1/2} \right\|^2 = -S_{J\bar{J}} \bullet X_{\bar{J}J} = \left\| X_{\bar{\mathcal{J}}}^{1/2} S_{\bar{J}J} S_{\mathcal{J}}^{-1/2} \right\|^2$;

b) $S_{\mathcal{J}}/\nu \succeq X_{\mathcal{J}}^{-1}$.

Proof. The equality $XS = \nu I$ implies

$$X_{\mathcal{J}} S_{J\bar{J}} + X_{J\bar{J}} S_{\bar{\mathcal{J}}} = 0, \tag{12}$$
$$X_{\mathcal{J}} S_{\mathcal{J}} + X_{J\bar{J}} S_{\bar{J}J} = \nu I. \tag{13}$$

Multiplying (12) on the left by $X_{J\bar{J}} X_{\mathcal{J}}^{-1}$ and taking the trace of both sides of the resulting expression, we obtain the first equality in a). Multiplying (12) on the right by $S_{\bar{\mathcal{J}}}^{-1} S_{\bar{J}J}$ and taking the trace of both sides of the resulting expression, we obtain the second equality in a). By (12), we have $X_{J\bar{J}} = -X_{\mathcal{J}} S_{J\bar{J}} S_{\bar{\mathcal{J}}}^{-1}$. This expression together with (13) then imply statement b) as follows:

$$\frac{1}{\nu} S_{\mathcal{J}} \;=\; X_{\mathcal{J}}^{-1} - \frac{1}{\nu} X_{\mathcal{J}}^{-1} X_{J\bar{J}} S_{\bar{J}J} \;=\; X_{\mathcal{J}}^{-1} + \frac{1}{\nu} S_{J\bar{J}} S_{\mathcal{J}}^{-1} S_{J\bar{J}}^T \;\succeq\; X_{\mathcal{J}}^{-1}.$$

□

Lemma 3. For every $J \subset \{1, \ldots, n\}$, we have $\liminf_{\nu \to 0} \|X_{\mathcal{J}}(\nu)\| \, \|S_{\mathcal{J}}(\nu)\|/\nu > 0$. As a consequence,

$$\liminf_{\nu \to 0} \frac{\|X_{\mathcal{J}}(\nu)\|}{\nu} > 0, \qquad \liminf_{\nu \to 0} \frac{\|S_{\mathcal{J}}(\nu)\|}{\nu} > 0. \tag{14}$$

Proof. By Lemma 2(b), we have

$$\|X_{\mathcal{J}}(\nu)\| \geq \nu \|S_{\mathcal{J}}(\nu)^{-1}\| \geq \nu \frac{\|I\|}{\|S_{\mathcal{J}}(\nu)\|},$$

which clearly implies that $\|X_{\mathcal{J}}(\nu)\| \, \|S_{\mathcal{J}}(\nu)\|/\nu > \|I\|$ for all $\nu > 0$. Relation (14) follows from the previous relation and the fact that $\max\{\|X_{\mathcal{J}}(\nu)\|, \|S_{\mathcal{J}}(\nu)\|\} = \mathcal{O}(1)$. □

Lemma 4. Let $(J, \bar{J})$ be a partition of the index set $\{1, \cdots, n\}$. If $U, V \in \mathcal{S}^n_+$ is such that $V \succeq U^2$, then

$$U = \begin{pmatrix} U_{\mathcal{J}} & U_{J\bar{J}} \\ U_{\bar{J}J} & U_{\bar{\mathcal{J}}} \end{pmatrix} = \begin{pmatrix} \mathcal{O}\left(\|V_{\mathcal{J}}\|^{1/2}\right) & \mathcal{O}(\phi) \\ \mathcal{O}(\phi) & \mathcal{O}\left(\|V_{\bar{\mathcal{J}}}\|^{1/2}\right) \end{pmatrix}, \tag{15}$$

where $\phi = \phi(V) \equiv \min\{\|V_{\mathcal{J}}\|^{1/2}, \|V_{\bar{\mathcal{J}}}\|^{1/2}\}$.

Proof. The assumption that $V \succeq U^2$ implies that $V_{\mathcal{J}} \succeq (U^2)_{\mathcal{J}} = U_{\mathcal{J}} U_{\mathcal{J}} + U_{J\bar{J}} U^T_{J\bar{J}}$, and hence

$$n\|V_{\mathcal{J}}\| \geq \operatorname{tr} V_{\mathcal{J}} \geq \operatorname{tr}\left(U_{\mathcal{J}} U_{\mathcal{J}} + U_{J\bar{J}} U^T_{J\bar{J}}\right) = \|U_{\mathcal{J}}\|^2 + \|U_{J\bar{J}}\|^2 \geq \max\{\|U_{\mathcal{J}}\|^2, \|U_{J\bar{J}}\|^2\}.$$

Since we can prove the inequality $n\|V_{\bar{\mathcal{J}}}\| \geq \max\{\|U_{\bar{\mathcal{J}}}\|^2, \|U_{J\bar{J}}\|^2\}$ in a similar way, relation (15) follows. □

Lemma 5. There holds

$$\max\left\{ \frac{\|X_{TB}(\nu)\|}{\|X_{\mathcal{T}}(\nu)\|^{1/2}}, \frac{\|X_{TN}(\nu)\|}{\nu^{1/2}\,\|X_{\mathcal{T}}(\nu)\|^{1/2}}, \frac{\|S_{TB}(\nu)\|}{\nu^{1/2}\,\|S_{\mathcal{T}}(\nu)\|^{1/2}}, \frac{\|S_{TN}(\nu)\|}{\|X_{\mathcal{T}}(\nu)\|^{1/2}} \right\} = \mathcal{O}\left(h(\nu)\right). \tag{16}$$

where $h : (0, +\infty) \to [0, +\infty)$ is defined by

$$h(\nu) = \frac{|X_{TN}(\nu) \bullet S_{TN}(\nu) + X_{TB}(\nu) \bullet S_{TB}(\nu)|^{1/2}}{\nu^{1/2}}, \quad \forall \nu > 0. \tag{17}$$

Proof. Lemma 2(a) with $J = T$ implies

$$\left\| X_{\mathcal{T}}^{-1/2}(\nu) X_{T\bar{T}}(\nu) S_{\bar{\mathcal{T}}}^{1/2}(\nu) \right\| = |X_{TN}(\nu) \bullet S_{TN}(\nu) + X_{TB}(\nu) \bullet S_{TB}(\nu)|^{1/2},$$

which together with the definition of h implies

$$\left\| X_{T\bar{T}}(\nu) S_{\bar{\mathcal{T}}}^{1/2}(\nu) \right\| = \mathcal{O}\left(h(\nu)\, \nu^{1/2} \, \|X_{\mathcal{T}}(\nu)\|^{1/2} \right). \tag{18}$$

By Lemma 2(b) with $J = \bar{T}$, we know that $X_{\bar{\mathcal{T}}}(\nu)/\nu \succeq S_{\bar{\mathcal{T}}}(\nu)^{-1}$. Since $X_{\mathcal{B}}(\nu)/\nu = \mathcal{O}(\nu^{-1})$ and $X_{\mathcal{N}}(\nu)/\nu = \mathcal{O}(1)$ due to Lemma 1, it follows from Lemma 4 that

$$S_{\bar{\mathcal{T}}}(\nu)^{-1/2} = \begin{pmatrix} \mathcal{O}\left(\nu^{-1/2}\right) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}. \tag{19}$$

Noting that

$$(X_{TB}(\nu)\ \ X_{TN}(\nu)) = X_{T\bar{T}}(\nu) = \left(X_{T\bar{T}}(\nu) S_{\bar{\mathcal{T}}}^{1/2}(\nu)\right) S_{\bar{\mathcal{T}}}^{-1/2}(\nu)$$

and using the estimates (18) and (19), we easily see that

$$X_{TB}(\nu) = \mathcal{O}\left(h(\nu)\, \|X_{\mathcal{T}}(\nu)\|^{1/2}\right), \quad X_{TN}(\nu) = \mathcal{O}\left(h(\nu)\, \nu^{1/2}\, \|X_{\mathcal{T}}(\nu)\|^{1/2}\right)$$

holds. In a similar way, we can prove that

$$S_{TB}(\nu) = \mathcal{O}\left(h(\nu)\, \nu^{1/2}\, \|X_{\mathcal{T}}(\nu)\|^{1/2}\right), \quad S_{TN}(\nu) = \mathcal{O}\left(h(\nu)\, \|X_{\mathcal{T}}(\nu)\|^{1/2}\right).$$

We have thus shown that (16) holds. □

Theorem 1. If condition **A4'** holds, then the limits

$$\lim_{\nu\to 0} \frac{X_{TB}(\nu)}{\|X_{\mathcal{T}}(\nu)\|^{1/2}}, \quad \lim_{\nu\to 0} \frac{X_{TN}(\nu)}{\nu^{1/2}\|X_{\mathcal{T}}(\nu)\|^{1/2}}, \quad \lim_{\nu\to 0} \frac{S_{TB}(\nu)}{\nu^{1/2}\|S_{\mathcal{T}}(\nu)\|^{1/2}}, \quad \text{and} \quad \lim_{\nu\to 0} \frac{S_{TN}(\nu)}{\|S_{\mathcal{T}}(\nu)\|^{1/2}}$$

are all equal to 0.

Proof. By Lemma 5, it is sufficient to prove that $\lim_{\nu\to 0} h(\nu) = 0$. Let $\mathcal{K} \equiv \mathcal{N} \cup TN \cup NT$. Applying Hoffman lemma to the linear system $\mathbb{A}X = b$, $X_{\mathcal{K}} = 0$, we conclude that there exists a set of points $\{\bar{X}(\nu) : \nu > 0\}$ such that

$$\mathbb{A}\bar{X}(\nu) = b, \quad \bar{X}_{\mathcal{K}}(\nu) = 0, \quad \forall \nu > 0, \tag{20}$$

$$\|X_{\bar{\mathcal{K}}}(\nu) - \bar{X}_{\bar{\mathcal{K}}}(\nu)\| \ = \ \mathcal{O}(\|X_{\mathcal{K}}(\nu)\|) \ = \ \mathcal{O}\left(\nu^{1/2}\|X_{\mathcal{T}}(\nu)\|^{1/2}\right), \tag{21}$$

where the last equality in (21) follows from (7), (10) and (14) with $\mathcal{J} = \mathcal{T}$. Also, applying Hoffman lemma to the linear system $S \in \mathrm{Im}\,(\mathbb{A}^*) + C$, $S_{\bar{\mathcal{N}}} = 0$, we conclude that there exists a set of points $\{\bar{S}(\nu) : \nu > 0\}$ such that

$$\bar{S}(\nu) \in \mathrm{Im}\,(\mathbb{A}^*) + C, \quad \bar{S}_{\bar{\mathcal{N}}}(\nu) = 0, \quad \forall \nu > 0, \tag{22}$$

$$\|S_{\mathcal{N}}(\nu) - \bar{S}_{\mathcal{N}}(\nu)\| \ = \ \mathcal{O}(\|S_{\bar{\mathcal{N}}}(\nu)\|) \ = \ \mathcal{O}\left(\|S_{\mathcal{T}}(\nu)\|^{1/2}\right), \tag{23}$$

where the last equality in (23) follows from (7), (8), (9), (10) and (14) with $\mathcal{J} = \mathcal{T}$. By (20), (22) and the fact that $(X(\nu), S(\nu), y(\nu))$ satisfies (2) and (3), we conclude that $X(\nu) - \bar{X}(\nu) \in \mathrm{Null}(\mathbb{A})$ and $S(\nu) - \bar{S}(\nu) \in \mathrm{Im}(\mathbb{A})$, and hence $(X(\nu) - \bar{X}(\nu)) \bullet (S(\nu) - \bar{S}(\nu)) = 0$. This equality together with (20) and (22) then imply that

$$\left(X_{\bar{\mathcal{K}}}(\nu) - \bar{X}_{\bar{\mathcal{K}}}(\nu)\right) \bullet S_{\bar{\mathcal{K}}}(\nu) + 2X_{TN}(\nu) \bullet S_{TN}(\nu) + X_{\mathcal{N}}(\nu) \bullet \left(S_{\mathcal{N}}(\nu) - \bar{S}_{\mathcal{N}}(\nu)\right) = 0,$$

and hence

$$2\,|X_{TN}(\nu) \bullet S_{TN}(\nu)| \leq \|X_{\bar{\mathcal{K}}}(\nu) - \bar{X}_{\bar{\mathcal{K}}}(\nu)\|\,\|S_{\bar{\mathcal{K}}}(\nu)\| + \|X_{\mathcal{N}}(\nu)\|\|S_{\mathcal{N}}(\nu) - \bar{S}_{\mathcal{N}}(\nu)\|. \tag{24}$$

By (7), (8), (9) and (14) with $\mathcal{J} = \mathcal{T}$, we have

$$\|S_{\bar{\mathcal{K}}}(\nu)\| = \mathcal{O}\left(\max\left\{\nu^{1/2}, \|S_{\mathcal{T}}(\nu)\|\right\}\right), \text{ and } \|X_{\mathcal{N}}(\nu)\| = \mathcal{O}(\nu). \tag{25}$$

Substituting the estimates (21), (23) and (25) into the inequality (24) and using condition **A4'**, we obtain

$$\begin{aligned}|X_{TN}(\nu) \bullet S_{TN}(\nu)| &= \mathcal{O}\left(\nu^{1/2}\|X_{\mathcal{T}}(\nu)\|^{1/2} \max\left\{\nu^{1/2}, \|S_{\mathcal{T}}(\nu)\|\right\} + \nu\|S_{\mathcal{T}}(\nu)\|^{1/2}\right). \\ &= \mathcal{O}\left(\nu \max\left\{\|X_{\mathcal{T}}(\nu)\|^{1/2}, \|S_{\mathcal{T}}(\nu)\|^{1/2}\right\}\right).\end{aligned}$$

Since we can similarly prove that $|X_{TB}(\nu) \bullet S_{TB}(\nu)|$ can be bounded by the same term above, we conclude that

$$|X_{TN}(\nu) \bullet S_{TN}(\nu) + X_{TB}(\nu) \bullet S_{TB}(\nu)| = \mathcal{O}\left(\nu \max\left\{\|X_{\mathcal{T}}(\nu)\|^{1/2}, \|S_{\mathcal{T}}(\nu)\|^{1/2}\right\}\right),$$

which in view of (17) implies that $h(\nu) = \mathcal{O}(\max\{\|X_{\mathcal{T}}(\nu)\|^{1/4}, \|S_{\mathcal{T}}(\nu)\|^{1/4}\})$. This clearly implies that $\lim_{\nu\to 0} h(\nu) = 0$. □

Corollary 1. If condition **A4** holds, then the limits

$$\lim_{\nu\to 0} \frac{X_{TB}(\nu)}{\nu^{1/4}}, \quad \lim_{\nu\to 0} \frac{X_{TN}(\nu)}{\nu^{3/4}}, \quad \lim_{\nu\to 0} \frac{S_{TB}(\nu)}{\nu^{3/4}}, \quad \text{and} \quad \lim_{\nu\to 0} \frac{S_{TN}(\nu)}{\nu^{1/4}},$$

are all equal to 0.

Proof. Since **A4** implies **A4'**, the conclusion of Theorem 1 holds. This fact together with **A4** can be easily seen to imply the theorem. □

4 Convergence of the central path

In this section we will study the limiting behavior of the central path $\nu > 0 \to (X(\nu), S(\nu), y(\nu))$ as ν approaches zero. Towards this end, we will introduce a crucial change of variables that will play an important role in this and the next section. We will also be able to characterize those SDP problems which satisfy condition **A4** and show that the latter is equivalent to condition **A4'**.

For $t > 0$, let P_t and D_t denote the block diagonal matrices given by $P_t \equiv \mathrm{Diag}\,(I_B, t^{-1}I_T, t^{-2}I_N)$ and $D_t \equiv \mathrm{Diag}\,(t^{-2}I_B, t^{-1}I_T, I_N)$. Consider the following re-parametrization of the central path given by

$$(\tilde{X}(t), \tilde{S}(t)) \equiv \left(P_t X(t^4) P_t\,,\, D_t S(t^4) D_t\right), \quad \forall t > 0. \tag{26}$$

Then,

$$\tilde{X}(t) = \begin{pmatrix} X_{\mathcal{B}}(t^4) & t^{-1}X_{BT}(t^4) & t^{-2}X_{BN}(t^4) \\ t^{-1}X_{TB}(t^4) & t^{-2}X_{\mathcal{T}}(t^4) & t^{-3}X_{TN}(t^4) \\ t^{-2}X_{NB}(t^4) & t^{-3}X_{NT}(t^4) & t^{-4}X_{\mathcal{N}}(t^4) \end{pmatrix} \tag{27}$$

and

$$\tilde{S}(t) = \begin{pmatrix} t^{-4}S_{\mathcal{B}}(t^4) & t^{-3}S_{BT}(t^4) & t^{-2}S_{BN}(t^4) \\ t^{-3}S_{TB}(t^4) & t^{-2}S_{\mathcal{T}}(t^4) & t^{-1}S_{TN}(t^4) \\ t^{-2}S_{NB}(t^4) & t^{-1}S_{NT}(t^4) & S_{\mathcal{N}}(t^4) \end{pmatrix}. \tag{28}$$

In view of the way the above blocks are scaled by different powers of t, it is natural to introduce the following groups of "blocks":

$$\mathcal{J}_1 = \mathcal{B}, \quad \mathcal{J}_2 = BT \cup TB, \quad \mathcal{J}_3 = \mathcal{T} \cup BN \cup NB, \quad \mathcal{J}_4 = TN \cup NT \;\text{ and }\; \mathcal{J}_5 = \mathcal{N}. \tag{29}$$

The following result gives another characterization of the path $(\tilde{X}(t), \tilde{S}(t))$ which provides the basis of our analysis.

Lemma 6. Suppose condition **A4** holds and let $(X^*, S^*, y^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$ be given. Then, the following statements hold:

a) for every $t > 0$, $(\tilde{X}(t), \tilde{S}(t))$ is the unique solution in $\mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ of the system given by

$$\tilde{X}\tilde{S} = I, \tag{30}$$
$$D_t^{-1}\tilde{S}\,D_t^{-1} - S^* \in \mathrm{Im}(\mathbb{A}^*); \tag{31}$$
$$P_t^{-1}\tilde{X}P_t^{-1} - X^* \in \mathrm{Null}(\mathbb{A}), \tag{32}$$

b) the path $t > 0 \to (\tilde{X}(t), \tilde{S}(t))$ remains bounded as t approaches 0 and any accumulation point $(\tilde{X}^*, \tilde{S}^*)$ of this path as t approaches 0 satisfies (30) and the following linear equations

$$
\text{(SP)} \quad \begin{cases} \mathbb{A}_{\mathcal{J}_1} \tilde{X}_{\mathcal{J}_1} = b, \\ \tilde{X}_{\mathcal{J}_2} = 0, \\ \mathbb{A}_{\mathcal{J}_3} \tilde{X}_{\mathcal{J}_3} \in \mathrm{Im}\left(\mathbb{A}_{\mathcal{J}_{1:2}}\right), \\ \tilde{X}_{\mathcal{J}_4} = 0, \\ \tilde{X}_{\mathcal{J}_5} \in \mathrm{Im}\left(\mathbb{A}_{\mathcal{J}_{1:4}}\right), \end{cases} \qquad \text{(SD)} \quad \begin{cases} \tilde{S}_{\mathcal{J}_1} \in \mathrm{Im}\left(\mathbb{A}^*_{\mathcal{J}_1}\right), \\ \tilde{S}_{\mathcal{J}_2} = 0, \\ (0,\, \tilde{S}_{\mathcal{J}_3}) \in \mathrm{Im}\left(\mathbb{A}^*_{\mathcal{J}_{1:2}},\, \mathbb{A}^*_{\mathcal{J}_3}\right), \\ \tilde{S}_{\mathcal{J}_4} = 0, \\ (0,\, \tilde{S}_{\mathcal{J}_5}) \in (C_{\mathcal{J}_{1:4}},\, C_{\mathcal{J}_5}) + \mathrm{Im}\left(\mathbb{A}^*_{\mathcal{J}_{1:4}},\, \mathbb{A}^*_{\mathcal{J}_5}\right), \end{cases}
$$

where $\mathcal{J}_{1:2} \equiv \mathcal{J}_1 \cup \mathcal{J}_2$ and $\mathcal{J}_{1:4} \equiv \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \mathcal{J}_4$.

Proof. We first prove a). Fix $t > 0$ and let $(X, S) \equiv (X(t^4), S(t^4))$ and $(\tilde{X}, \tilde{S}) \equiv (\tilde{X}(t), \tilde{S}(t))$. Using the definition of the central path and the fact that $\mathbb{A}X^* = b$ and $S^* \in C + \mathrm{Im}\, \mathbb{A}^*$, it is easy to see that (X, S) is the unique point satisfying

$$
X - X^* \in \mathrm{Null}\, \mathbb{A}, \quad S - S^* \in \mathrm{Im}\, \mathbb{A}^*, \quad XS = t^4 I. \tag{33}
$$

Note that by (26), we have $\tilde{X} = P_t X P_t$ and $\tilde{S} = D_t S D_t$. Using these relations and the identity $P_t D_t = I/t^2$, it is now easy to see that (X, S) satisfies (33) if and only if $(\tilde{X}, \tilde{S})$ satisfies (30)-(32), from which a) follows.

We next prove b). Using Lemma 1 and Corollary 1, it is easy to see that the path $t > 0 \to (\tilde{S}(t), \tilde{S}(t))$ remains bounded as $t \downarrow 0$ and that any accumulation point $(\tilde{X}^*, \tilde{S}^*)$ of this path as $t \downarrow 0$ satisfies the the second and fourth relations of systems (SP) and (SD). The fact that $(\tilde{X}^*, \tilde{S}^*)$ satisfies (30) follows immediately from a). It remains to prove that $(\tilde{X}^*, \tilde{S}^*)$ satisfies the first, third and fifth relations of (SP) and (SD). We will prove this fact only for $\tilde{S}^*$ since the proof for $\tilde{X}^*$ is similar. Let $\{t_k\} \subset \Re_{++}$ be a sequence converging to 0 such that $\tilde{S}^* = \lim_{k \to +\infty} \tilde{S}(t_k)$. Since $S(t_k^4) - S^* \in \mathrm{Im}\, \mathbb{A}^*$ for all k and $S^*_{\mathcal{J}_1} = 0$, it follows that $\tilde{S}_{\mathcal{J}_1}(t_k) = S_{\mathcal{J}_1}(t_k^4)/t_k^4 \in \mathrm{Im}\left(\mathbb{A}^*_{\mathcal{J}_1}\right)$ for all k, and hence that the first relation of (SD) holds. A similar argument shows that

$$
\left(\frac{S_{\mathcal{J}_{1:2}}(t_k^4)}{t_k^2},\, \tilde{S}_{\mathcal{J}_3}(t_k) \right) \;=\; \left(\frac{S_{\mathcal{J}_{1:2}}(t_k^4)}{t_k^2},\, \frac{S_{\mathcal{J}_3}(t_k^4)}{t_k^2} \right) \in \mathrm{Im}(\mathbb{A}^*_{\mathcal{J}_{1:2}},\, \mathbb{A}^*_{\mathcal{J}_3}),
$$

from which the third relation of (SD) follows upon noting that $\lim_{t \to 0} S_{\mathcal{J}_{1:2}}(t^4)/t^2 = 0$, due to Lemma 1 and Corollary 1. Finally, the fifth relation of (SD) follows from the relation

$$
(S_{\mathcal{J}_{1:4}}(t_k^4), \tilde{S}_{\mathcal{J}_5}(t_k)) = (S_{\mathcal{J}_{1:4}}(t_k^4), S_{\mathcal{J}_5}(t_k^4)) \in C + \mathrm{Im}(\mathbb{A}^*_{\mathcal{J}_{1:4}},\, \mathbb{A}^*_{\mathcal{J}_5})
$$

and the fact that $\lim_{t \to 0} S_{\mathcal{J}_{1:4}}(t^4) = 0$. $\square$

Let

$$T(P) \equiv \left\{ \tilde{X} \in S^n_{++} : \text{satisfying (SP)} \right\}, \quad T(D) \equiv \left\{ \tilde{S} \in S^n_{++} : \text{ satisfying (SD)} \right\}.$$

The following result shows that the path $(\tilde{X}(t), \tilde{S}(t))$ converges as t approaches zero and provides a characterization of its limit point as being an optimal solution of a certain log-barrier problem over the set $T(P) \times T(D)$.

Theorem 2. Suppose condition **A4** holds and let $\left(\bar{X}, \bar{S}, \bar{y}\right) \in \mathcal{F}^*(P) \times \mathcal{F}^*(D)$ be given. Then, the path $(\tilde{X}(t), \tilde{S}(t))$ converges to $(\tilde{X}^*, \tilde{S}^*)$, where

$$\tilde{X}^* \equiv \operatorname{argmax} \left\{ \log\det(\tilde{X}) - \bar{S} \bullet \tilde{X} : \ \tilde{X} \in T(P) \right\}, \tag{34}$$

$$\tilde{S}^* \equiv \operatorname{argmax} \left\{ \log\det(\tilde{S}) - \bar{X} \bullet \tilde{S} : \ \tilde{S} \in T(D) \right\}. \tag{35}$$

In particular, the central path converges.

Proof. We will prove only (34) since the proof of (35) is similar. Since (34) has a unique solution, it is sufficient to show that any accumulation point $\tilde{X}^*$ of the path $\tilde{X}(t)$ as $t \downarrow 0$ satisfies the optimality condition for (34), that is $[(\tilde{X}^*)^{-1} - \bar{S}] \bullet \Delta X = 0$ for every $\Delta X \in \mathcal{S}^n$ satisfying the homogeneous system corresponding to (SP). Indeed, using the fact that $\tilde{S}^* = (\tilde{X}^*)^{-1}$, $\Delta X_{\mathcal{J}_2} = 0$, $\Delta X_{\mathcal{J}_4} = 0$ and $\bar{S}_{J_i} = 0$ for $i = 1, 2, 3, 4$, we obtain

$$[(\tilde{X}^*)^{-1} - \bar{S}] \bullet \Delta X = \tilde{S}^*_{\mathcal{J}_1} \bullet \Delta X_{\mathcal{J}_1} + \tilde{S}^*_{\mathcal{J}_3} \bullet \Delta X_{\mathcal{J}_3} + (\tilde{S}^*_{\mathcal{J}_5} - \bar{S}_{\mathcal{J}_5}) \bullet \Delta X_{\mathcal{J}_5}.$$

Now, using the fact that ΔX is a solution of (SP) with $b = 0$ and $\tilde{S}^* - \bar{S}$ is a solution of (SD) with $C = 0$, we can easily see that each one of the inner products in the right hand side of the above expression is equal to zero, from which the lemma follows. □

In the rest of this section, we will characterize the class of SDPs satisfying condition **A4** and also show that this condition is equivalent to **A4'**. We first state the following very intuitive result.

Lemma 7. Let a convex set $\emptyset \neq \mathcal{C} \subset \mathcal{S}^n_{++}$ be given. If the problem

$$\max \left\{ \log\det(X) : X \in \mathcal{C} \right\}, \tag{36}$$

has an optimal solution then $\mathcal{C}$ is bounded.

Proof. Let $\tilde{X}$ be an optimal solution of the above problem. This is equivalent to the condition that

$$\tilde{X}^{-1} \bullet (X - \tilde{X}) \leq 0, \quad \forall X \in \text{cl}\, \mathcal{C}. \tag{37}$$

Let H be a direction of recession of $\mathrm{cl}\,\mathcal{C}$ so that $\tilde{X} + \tau H \in \mathrm{cl}\, C \subset \mathcal{S}^n_+$ for every $\tau > 0$. In view of (37), it follows that $\tilde{X}^{-1} \bullet H \le 0$. Letting $\tilde{\lambda} > 0$ denote the minimum eigenvalue of $\tilde{X}^{-1}$ and noting that $\tilde{X}^{-1} - \tilde{\lambda} I \succeq 0$, we obtain for every $\tau > 0$ that

$$\begin{aligned} n &\ge \tilde{X}^{-1} \bullet \left(\tilde{X} + \tau H\right) = \left(\tilde{X}^{-1} - \tilde{\lambda} I\right) \bullet \left(\tilde{X} + \tau H\right) + \tilde{\lambda} I \bullet \left(\tilde{X} + \tau H\right) \\ &\ge \tilde{\lambda} I \bullet \left(\tilde{X} + \tau H\right) \ge \tilde{\lambda} \|\tilde{X} + \tau H\| \ge \tilde{\lambda}\left(\tau \|H\| - \|\tilde{X}\|\right). \end{aligned}$$

The last inequality holds for all $\tau > 0$ only if $\|H\| = 0$, or equivalently $H = 0$. The boundedness of $\mathrm{cl}\,\mathcal{C}$ now follows from Proposition 2.2.3 of Chapter III of Hiriart-Urruty and Lemaréchal [11]. □

We will now derive a necessary condition for condition **A4** to hold. Later, we will establish that this condition is also sufficient.

Theorem 3. If condition **A4** holds, then the system

$$\mathbb{A}_{\mathcal{T}}(\Delta X_{\mathcal{T}}) \in \mathrm{Im}\,(\mathbb{A}_{\mathcal{J}_{1:2}}), \quad \Delta X_{\mathcal{T}} \in S^{|T|}_+, \tag{38}$$

$$(0,\, \Delta S_{\mathcal{T}}) \in \mathrm{Im}\,(\mathbb{A}^*_{\mathcal{K}},\, \mathbb{A}^*_{\mathcal{T}}), \quad \Delta S_{\mathcal{T}} \in S^{|T|}_+, \tag{39}$$

where $\mathcal{K} \equiv \mathcal{J}_{1:2} \cup BN \cup NB$, has $(\Delta X_{\mathcal{T}}, \Delta S_{\mathcal{T}}) = (0, 0)$ as its unique solution.

Proof. We will only prove that $\Delta X_{\mathcal{T}} = 0$ is the unique solution of (38). The proof that $\Delta S_{\mathcal{T}} = 0$ is the unique solution of (39) is similar. Let

$$\mathcal{C}(P) \equiv \left\{X \in T(P) :\, X_{\mathcal{J}_1} = \tilde{X}^*_{\mathcal{J}_1},\, X_{\mathcal{J}_5} = \tilde{X}^*_{\mathcal{J}_5}\right\},$$

and observe that, in view of Theorem 2, $\tilde{X}^*$ is an optimal solution of (36) with $\mathcal{C} = \mathcal{C}(P)$. Hence, it follows from Lemma 7 that $\mathcal{C}(P)$, and its closure, is bounded. By Proposition 2.2.3 of Chapter III of Hiriart-Urruty and Lemaréchal [11], we conclude that $\Delta X = 0$ is the only direction of recession of $\mathrm{cl}\,\mathcal{C}(P)$. Since it is easy to see that the directions of recession of $\mathrm{cl}\,\mathcal{C}(P)$ consist of those ΔX such that $\Delta X_{\bar{\mathcal{T}}} = 0$ and $\Delta X_{\mathcal{T}}$ satisfies (38), it follows that $\Delta X_{\mathcal{T}} = 0$ is the unique solution of (38). □

At a first sight, there seems to be a lack of symmetry between (38) and (39). However, (38) and (39) are equivalent to the existence of matrices $U_{\mathcal{B}} \in \mathcal{S}^{|B|}$, $U_{TB} \in \Re^{|T|\times|B|}$, $W_{TN} \in \Re^{|T|\times|N|}$ and $W_{\mathcal{N}} \in \mathcal{S}^{|N|}$ such that

$$\begin{pmatrix} U_{\mathcal{B}} & U_{TB}^T & 0 \\ U_{TB} & \Delta X_{\mathcal{T}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Null}\,(\mathbb{A}) \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta S_{\mathcal{T}} & W_{TN} \\ 0 & W_{TN}^T & W_{\mathcal{N}} \end{pmatrix} \in \mathrm{Im}\,(\mathbb{A}^*).$$

Note that the latter conditions illustrates well the symmetry between (38) and (39).

The following result gives some useful properties about the systems (38) and (39).

Lemma 8. The following statements hold:

i) system (38) has no strictly feasible solution; if, in addition, $\mathbb{A}_{\mathcal{J}_2} = 0$ then $\Delta X_{\mathcal{T}} = 0$ is the only solution of system (38);

ii) system (39) has no strictly feasible solution; if, in addition, $\mathbb{A}_{\mathcal{J}_4} = 0$ then $\Delta S_{\mathcal{T}} = 0$ is the only solution of system (38);

Proof. We will only prove i) since the proof of ii) is similar. Suppose for contradiction that system (38) has a feasible solution $\Delta X_{\mathcal{T}} \in \mathcal{S}^{|T|}_{++}$. This means that there exist $\Delta X_{\mathcal{B}} \in \mathcal{S}^{|B|}$ and $\Delta X_{BT} \in \Re^{|B|\times|T|}$ such that

$$\Delta X = \begin{pmatrix} \Delta X_{\mathcal{B}} & \Delta X_{BT} & 0 \\ \Delta X_{BT}^T & \Delta X_{\mathcal{T}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \operatorname{Null} \mathbb{A}. \tag{40}$$

Hence $\mathbb{A}(X^* + \tau \Delta X) = b$ for every $\tau \in \Re$. Using the fact that $X^*_{\mathcal{B}} \succ 0$ and $\Delta X_{\mathcal{T}} \succ 0$, we easily see that $X^* + \tau\Delta X \succeq 0$ for every $\tau > 0$ sufficiently small. Since $S^*_{\mathcal{J}_{1:4}} = 0$ and $(X^* + \tau\Delta X)_{\mathcal{J}_5} = 0$, it follows that $(X^* + \tau\Delta X) \bullet S^* = 0$ for all $\tau \in \Re$. Therefore, we conclude that $X^* + \tau\Delta X \in \mathcal{F}^*(P)$ for every $\tau > 0$ sufficiently small. However, this contradicts the description of $\mathcal{F}^*(P)$ given by (5) since $(X^* + \tau\Delta X)_{\mathcal{T}} = \Delta X_{\mathcal{T}} \neq 0$.

Assume now that $\mathbb{A}_{\mathcal{J}_2} = 0$. If system (38) had a feasible solution $0 \neq \Delta X_{\mathcal{T}} \in \mathcal{S}^{|T|}_{+}$, this would enable us to take $\Delta X_{BT} = 0$ in (40) to obtain the desired contradiction by using similar arguments to ones used in the previous paragraph. □

The next lemma essentially establishes that the converse of Theorem 3 holds.

Lemma 9. The following statements hold:

i) if $X_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$ does not hold, then there exists an accumulation point $\widehat{\Delta X}_{\mathcal{T}}$ of

$$\left\{ \Delta X_{\mathcal{T}}(\nu) \equiv \frac{X_{\mathcal{T}}(\nu)}{\|X_{\mathcal{T}}(\nu)\|} : \nu \in (0,1] \right\}, \tag{41}$$

which is a solution of (38); hence, $\widehat{\Delta X}_{\mathcal{T}} \neq 0$ and $\widehat{\Delta X}_{\mathcal{T}} \notin \mathcal{S}^{|T|}_{++}$;

ii) if $S_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$ does not hold, then there exists an accumulation point $\widehat{\Delta S}_{\mathcal{T}}$ of

$$\left\{ \Delta S_{\mathcal{T}}(\nu) \equiv \frac{S_{\mathcal{T}}(\nu)}{\|S_{\mathcal{T}}(\nu)\|} : \nu \in (0,1] \right\}, \tag{42}$$

which is a solution of (38); hence, $\widehat{\Delta S}_{\mathcal{T}} \neq 0$ and $\widehat{\Delta S}_{\mathcal{T}} \notin \mathcal{S}^{|T|}_{++}$.

Proof. We will prove only i) since the proof of ii) is similar. First note that the assumption means that $\lim_{k\to+\infty} {\nu_k}^{1/2}/\|X_{\mathcal{T}}(\nu_k)\| = 0$ for some sequence of positive numbers $\{\nu_k\}$ converging to zero. By passing to a subsequence if necessary, we may assume that $\Delta X_{\mathcal{T}}(\nu_k)$ converges, say to $\widehat{\Delta X}_{\mathcal{T}}$. Clearly, $0 \neq \widehat{\Delta X}_{\mathcal{T}} \in \mathcal{S}_+^{|T|}$. Now, let $\mathcal{L} = BN \cup NB \cup \mathcal{J}_4 \cup \mathcal{J}_5$. Since, by Lemma 1, we have $X_{\mathcal{L}}(\nu) = \mathcal{O}(\nu^{1/2})$, we conclude that

$$\lim_{k\to+\infty} \frac{X_{\mathcal{L}}(\nu_k)}{\|X_{\mathcal{T}}(\nu_k)\|} = \lim_{k\to+\infty} \frac{X_{\mathcal{L}}(\nu_k)}{{\nu_k}^{1/2}} \frac{{\nu_k}^{1/2}}{\|X_{\mathcal{T}}(\nu_k)\|} = 0. \tag{43}$$

Using the fact that $b \in \mathrm{Im}(\mathbb{A}_B)$ and $\mathbb{A}X(\nu) = b$, we obtain that $\mathbb{A}_{\mathcal{T}} X_{\mathcal{T}}(\nu_k) + \mathbb{A}_{\mathcal{L}} X_{\mathcal{L}}(\nu_k) \in \mathrm{Im}(\mathbb{A}_{\mathcal{J}_{1:2}})$. Dividing this expression by $\|X_{\mathcal{T}}(\nu_k)\|$, letting $k \to \infty$ and using (43), we conclude that $\mathbb{A}_{\mathcal{T}} \widehat{\Delta X}_{\mathcal{T}} \in \mathrm{Im}(\mathbb{A}_{\mathcal{J}_{1:2}})$. We have thus shown that $\widehat{\Delta X}_{\mathcal{T}}$ is a nontrivial solution of (38). By Lemma 8(i), we know that $\widehat{\Delta X}_{\mathcal{T}} \notin \mathcal{S}_{++}^{|T|}$. □

The following result follows as an immediate consequence of Lemma 9.

Corollary 2. The following statements hold:

i) if $\Delta X_{\mathcal{T}} = 0$ is the only solution of system (38), then $X_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$;

ii) if $\Delta S_{\mathcal{T}} = 0$ is the only solution of system (39), then $S_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$.

The following result gives some sufficient conditions for condition **A4**, or part of it, to hold.

Corollary 3. The following statements hold:

i) if $|T| = 1$, then the condition **A4** holds;

ii) if $\mathbb{A}_{\mathcal{J}_2} = 0$, then the condition $X_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$ holds;

iii) if $\mathbb{A}_{\mathcal{J}_4} = 0$, then the condition $S_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$ holds.

Proof. The statement i) it follows from Lemma 9, by noting that $\Delta X_{\mathcal{T}}(\nu) = 1$ and $\Delta S_{\mathcal{T}}(\nu) = 1$, for all $\nu > 0$. The other one it follows from Lemma 8 and Corollary 2. □

Lemma 10. Assume that the condition **A4'** holds. Then, any accumulation points $\widehat{\Delta X}_{\mathcal{T}}$ and $\widehat{\Delta S}_{\mathcal{T}}$ of (41) and (42), respectively, are in $\mathcal{S}_{++}^{|T|}$.

Proof. The equality $X(\nu)S(\nu) = \nu I$ implies

$$X_{TB}(\nu)S_{BT}(\nu) + X_{\mathcal{T}}(\nu)S_{\mathcal{T}}(\nu) + X_{TN}(\nu)S_{NT}(\nu) = \nu I.$$

Dividing the above identity by ν, we obtain

$$\alpha(\nu)\left(\frac{X_{TB}(\nu)}{\|X_{\mathcal{T}}(\nu)\|^{1/2}}\frac{S_{BT}(\nu)}{\|S_{\mathcal{T}}(\nu)\|^{1/2}\nu^{1/2}}+\alpha(\nu)\Delta X_{\mathcal{T}}(\nu)\Delta S_{\mathcal{T}}(\nu)+\frac{X_{TN}(\nu)}{\|X_{\mathcal{T}}(\nu)\|^{1/2}\nu^{1/2}}\frac{S_{NT}(\nu)}{\|S_{\mathcal{T}}(\nu)\|^{1/2}}\right)=I,$$

where $\alpha(\nu)\equiv(\|X_{\mathcal{T}}(\nu)\|\,\|S_{\mathcal{T}}(\nu)\|/\nu)^{1/2}$. By condition **A4'** and Lemma 3 with $J=T$, we conclude that $\alpha(\nu)$ is bounded above and away from zero as $\nu\downarrow 0$. Hence, letting $\nu\downarrow 0$ in the above expression and using Theorem 1, we conclude that

$$\lim_{\nu\to 0}\alpha(\nu)^2\Delta X_{\mathcal{T}}(\nu)\Delta S_{\mathcal{T}}(\nu)=I.$$

The lemma now is easily seen to follow from the last relation and the fact that $\alpha(\nu)$ is bounded above as $\nu\downarrow 0$. □

We have already shown in Theorem 3 and Corollary 2 that condition **A4** is equivalent to $(\Delta X_{\mathcal{T}},\Delta S_{\mathcal{T}})=(0,0)$ being the only solution of (38) and (39). We record this fact in the next result, which also establishes that two conditions are in turn equivalent to condition **A4'**.

Theorem 4. The following statements are equivalent:

i) condition **A4** holds;

ii) condition **A4'** holds;

iii) $(\Delta X_{\mathcal{T}},\Delta S_{\mathcal{T}})=(0,0)$ is the unique solution of system (38)-(39).

Proof. In view of the comments made on the paragraph preceding the theorem and the fact that **A4** clearly implies **A4'**, it suffices to prove that ii) implies i). Assume for contradiction that **A4'** holds but **A4** does not. Without loss of generality, we may assume that $X_{\mathcal{T}}(\nu)=\mathcal{O}(\nu^{1/2})$ does not hold. Then, Lemma 9(i) implies the existence of an accumulation point $\widehat{\Delta X}_{\mathcal{T}}\notin\mathcal{S}^{|T|}_{++}$ of $\{\Delta X_{\mathcal{T}}(\nu):\nu>0\}$ as $\nu\downarrow 0$. However, in view of condition **A4'**, Lemma 10 implies that $\widehat{\Delta X}_{\mathcal{T}}$ must be in $\mathcal{S}^{|T|}_{++}$, yielding the desired contradiction. □

5 Convergence of the derivative of the central path

It is well known that the path $(X(\nu),S(\nu))$ is analytic in the interval $(0,+\infty)$ (see for example [22]). However, the central path in this parametrization can not be extended analytically to an interval of the form $(-\epsilon,+\infty)$, for some $\epsilon>0$. In this section we will show that the re-parametrized central path $t\to(X(t^4),S(t^4))$ can be extended analytically to an interval of this form. Using this analyticity result, we also derive results about the order of convergence of the central path towards the set $\mathcal{F}^*(P)\times\mathcal{F}^*(D)$ and the limiting behavior of the normalized derivative of this path.

Throughout this section, we assume that condition **A4** is in force. Hence, we will not explicitly mention it in the statements of the results of this section.

For the sake of shortness, it is convenient to introduce the following definition.

Definition 1. Let $w : (0, +\infty) \to E$ be a given function where E is a finite dimensional normed vector space. The function w is said to be *analytic at* 0 if there exist $\epsilon > 0$ and an analytic function $\psi : (-\epsilon, \epsilon) \to E$ such that $w(t) = \psi(t)$ for all $t \in (0, \epsilon)$.

The basic result that we use to establish that a function $w : (0, +\infty) \to E$ is analytic at 0 is the following corollary of the implicit function theorem.

Proposition 1. Let $w : (0, +\infty) \to E$ be a given function where E is a finite dimensional normed vector space. Assume that there exists an analytic function $H : \Lambda \times (-\delta, \delta) \to E$, where $\delta > 0$ and Λ is an open subset of E, such that $w = w(t)$ is the unique solution of $H(w, t) = 0$ in Λ, for every $t \in (0, \delta)$. Assume also there exists $\bar{w} \in \Lambda$ such that $H(\bar{w}, 0) = 0$ and $H'_w(\bar{w}, 0)$ is nonsingular. Then, w is analytic at 0 and, as a consequence, $\lim_{t \downarrow 0} w(t) = \bar{w}$ and the limits of all the derivatives of $w(t)$ as $t \downarrow 0$ exist.

Our first goal will be to show that the path $t > 0 \to (\tilde{X}(t), \tilde{S}(t))$ defined by (26) is analytic at $t = 0$. Our point of departure will be the fact that $(\tilde{X}(t), \tilde{S}(t))$ is the unique solution of system (30)-(32), for every $t > 0$. Our approach will be to apply Proposition 1 to a specific system of equations characterizing the path $t > 0 \to (\tilde{X}(t), \tilde{S}(t))$. The utilization of system (30)-(32) towards this end is not appropriate since its Jacobian with respect to $(\tilde{X}, \tilde{S})$ is generally singular at $t = 0$ (even though for $t > 0$ it is always nonsingular). The main cause of this phenomenon is that the "rank" of the linear equations (31) and (32) may change as t becomes 0.

We will now show how the linear equations (31) and (32) can be reformulated into equivalent linear system whose rank is constant for every $t \in \Re$. We start by recalling a standard result from linear algebra but stated in terms of operators.

Lemma 11. Let $\mathbb{A} : \mathcal{S}^n \to \Re^m$ be an onto linear operator. Let $(\mathcal{J}_1, \cdots, \mathcal{J}_p)$ be given a partition of the set $\{(k, \ell) : k, \ell = 1, \cdots, n\}$. Then, there exist a partition $(I_1, \cdots, I_p)$ of the set $\{1, \cdots, m\}$ (possibly, with some $I_i = \varnothing$), an isomorphism $\mathbb{U} : \Re^m \to \Re^{I_1} \times \cdots \times \Re^{I_p}$, and a collection of linear operators $\tilde{\mathbb{A}}_{I_i \mathcal{J}_j} : \mathcal{S}^{\mathcal{J}_j} \to \Re^{I_i}$, $i \le j \in \{1, \ldots, p\}$, whose diagonal ones $\tilde{\mathbb{A}}_{I_i \mathcal{J}_i}$, $i = 1, \ldots, p$, are all onto, satisfying

$$(\mathbb{U} \circ \mathbb{A})X = \left(\sum_{j=1}^{p} \tilde{\mathbb{A}}_{I_1 \mathcal{J}_j} X_{\mathcal{J}_j}, \cdots, \sum_{j=i}^{p} \tilde{\mathbb{A}}_{I_i \mathcal{J}_j} X_{\mathcal{J}_j}, \cdots, \sum_{j=p}^{p} \tilde{\mathbb{A}}_{I_p \mathcal{J}_j} X_{\mathcal{J}_j} \right), \quad \forall X \in \mathcal{S}^n,$$

or equivalently, after we identify $\mathcal{S}^p$ with $\mathcal{S}^{\mathcal{J}_1} \times \cdots \times \mathcal{S}^{\mathcal{J}_p}$,

$$\mathbb{U} \circ \mathbb{A} = \begin{pmatrix} \tilde{\mathbb{A}}_{I_1\mathcal{J}_1} & \tilde{\mathbb{A}}_{I_1\mathcal{J}_2} & \cdots & \tilde{\mathbb{A}}_{I_1\mathcal{J}_p} \\ 0 & \tilde{\mathbb{A}}_{I_2\mathcal{J}_2} & & \vdots \\ \vdots & & \ddots & \tilde{\mathbb{A}}_{I_{p-1}\mathcal{J}_p} \\ 0 & \cdots & 0 & \tilde{\mathbb{A}}_{I_p\mathcal{J}_p} \end{pmatrix}. \tag{44}$$

The next result describes a suitable system of equations which characterizes the re-parametrized central path and whose rank does not change as t becomes zero.

Lemma 12. Let $(X^*, S^*, y^*) \in \mathcal{F}^*(P) \times \mathcal{F}^*(P)$ be given. Consider the partition $(\mathcal{J}_1, \ldots, \mathcal{J}_5)$ defined in (29) and the corresponding partition $(I_1, \ldots, I_5)$ and collection of operators $\tilde{\mathbb{A}}_{I_i\mathcal{J}_j} : \mathcal{S}^{\mathcal{J}_j} \to \Re^{I_i}$, $i \leq j \in \{1, \ldots, 5\}$, as in the Lemma 11. Then, there exists an analytic curve $\tilde{y} : \Re \to \Re^m$ such that, for every $t > 0$, $(\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ is the unique solution in $\mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$ of the system

$$-\tilde{X}^{-1} + \tilde{S} = 0, \tag{45}$$

$$\tilde{\mathbb{A}}_t^* \tilde{y} + \tilde{S} - S^* = 0, \tag{46}$$

$$\tilde{\mathbb{A}}_t \left(\tilde{X} - X^*\right) = 0, \tag{47}$$

where

$$\tilde{\mathbb{A}}_t \equiv \begin{pmatrix} \tilde{\mathbb{A}}_{I_1\mathcal{J}_1} & t\tilde{\mathbb{A}}_{I_1\mathcal{J}_2} & t^2\tilde{\mathbb{A}}_{I_1\mathcal{J}_3} & t^3\tilde{\mathbb{A}}_{I_1\mathcal{J}_4} & t^4\tilde{\mathbb{A}}_{I_1\mathcal{J}_5} \\ 0 & \tilde{\mathbb{A}}_{I_2\mathcal{J}_2} & t\tilde{\mathbb{A}}_{I_2\mathcal{J}_3} & t^2\tilde{\mathbb{A}}_{I_2\mathcal{J}_4} & t^3\tilde{\mathbb{A}}_{I_2\mathcal{J}_5} \\ 0 & 0 & \tilde{\mathbb{A}}_{I_3\mathcal{J}_3} & t\tilde{\mathbb{A}}_{I_3\mathcal{J}_4} & t^2\tilde{\mathbb{A}}_{I_3\mathcal{J}_5} \\ 0 & 0 & 0 & \tilde{\mathbb{A}}_{I_4\mathcal{J}_4} & t\tilde{\mathbb{A}}_{I_4\mathcal{J}_5} \\ 0 & 0 & 0 & 0 & \tilde{\mathbb{A}}_{I_5\mathcal{J}_5} \end{pmatrix}. \tag{48}$$

Proof. Fix some $t > 0$. We claim that $(\tilde{X}, \tilde{S}) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$ satisfies (30)-(32) if and only if it satisfies (45)-(47) for some $\tilde{y} \in \Re^m$. Using this claim and Lemma 6(a), it follows that the unique solution of (45)-(47) is $(\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$, where $\tilde{y}(t) \equiv (\tilde{\mathbb{A}}_t\tilde{\mathbb{A}}_t^*)^{-1}\tilde{\mathbb{A}}_t(\tilde{S}(t) - S^*)$. Since this curve $\tilde{y}$ is clearly analytic, the lemma follows.

We will now show the above claim. First, note that (45) is obviously equivalent to (30). We will next show that (47) is equivalent to (32) by using Lemma 11. By identifying $\mathcal{S}^n$ with $\mathcal{S}^{\mathcal{J}_1}\times, \cdots, \times\mathcal{S}^{\mathcal{J}_5}$, we have

$$P_t^{-1}\tilde{X}P_t^{-1} - X^* = \left(\tilde{X}_{\mathcal{J}_1} - X^*_{\mathcal{J}_1},\ t\tilde{X}_{\mathcal{J}_2},\ t^2\tilde{X}_{\mathcal{J}_3},\ t^3\tilde{X}_{\mathcal{J}_4},\ t^4\tilde{X}_{\mathcal{J}_5}\right)$$

and hence, in view of (44) with $p = 5$, (31) is equivalent to

$$\begin{pmatrix} \tilde{\mathbb{A}}_{I_1\mathcal{J}_1} & t\tilde{\mathbb{A}}_{I_1\mathcal{J}_2} & t^2\tilde{\mathbb{A}}_{I_1\mathcal{J}_3} & t^3\tilde{\mathbb{A}}_{I_1\mathcal{J}_4} & t^4\tilde{\mathbb{A}}_{I_1\mathcal{J}_5} \\ 0 & t\tilde{\mathbb{A}}_{I_2\mathcal{J}_2} & t^2\tilde{\mathbb{A}}_{I_2\mathcal{J}_3} & t^3\tilde{\mathbb{A}}_{I_2\mathcal{J}_4} & t^4\tilde{\mathbb{A}}_{I_2\mathcal{J}_5} \\ 0 & 0 & t^2\tilde{\mathbb{A}}_{I_3\mathcal{J}_3} & t^3\tilde{\mathbb{A}}_{I_3\mathcal{J}_4} & t^4\tilde{\mathbb{A}}_{I_3\mathcal{J}_5} \\ 0 & 0 & 0 & t^3\tilde{\mathbb{A}}_{I_4\mathcal{J}_4} & t^4\tilde{\mathbb{A}}_{I_4\mathcal{J}_5} \\ 0 & 0 & 0 & 0 & t^4\tilde{\mathbb{A}}_{I_5\mathcal{J}_5} \end{pmatrix} \begin{pmatrix} \tilde{X}_{\mathcal{J}_1} - X^*_{\mathcal{J}_1} \\ \tilde{X}_{\mathcal{J}_2} \\ \tilde{X}_{\mathcal{J}_3} \\ \tilde{X}_{\mathcal{J}_4} \\ \tilde{X}_{\mathcal{J}_5} \end{pmatrix} = 0.$$

Dividing the second, third, fourth and fifth blocks of rows in the above system by t, t^2, t^3 and t^4, respectively, we obtain (47).

Finally, we will show that the condition $\tilde{S} - S^* \in \operatorname{Im} \tilde{\mathbb{A}}^*_t$ is equivalent to (31). First note that Lemma 11 implies that

$$\begin{aligned} \operatorname{Im}(\mathbb{A}^*) = \operatorname{Im}\left[(\mathbb{U} \circ \mathbb{A})^*\right] &= \operatorname{Im}\left[\begin{pmatrix} \tilde{\mathbb{A}}^*_{I_1\mathcal{J}_1} & 0 & 0 & 0 & 0 \\ \tilde{\mathbb{A}}^*_{I_1\mathcal{J}_2} & \tilde{\mathbb{A}}^*_{I_2\mathcal{J}_2} & 0 & 0 & 0 \\ \tilde{\mathbb{A}}^*_{I_1\mathcal{J}_3} & \tilde{\mathbb{A}}^*_{I_2\mathcal{J}_3} & \tilde{\mathbb{A}}^*_{I_3\mathcal{J}_3} & 0 & 0 \\ \tilde{\mathbb{A}}^*_{I_1\mathcal{J}_4} & \tilde{\mathbb{A}}^*_{I_2\mathcal{J}_4} & \tilde{\mathbb{A}}^*_{I_3\mathcal{J}_4} & \tilde{\mathbb{A}}^*_{I_4\mathcal{J}_4} & 0 \\ \tilde{\mathbb{A}}^*_{I_1\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_2\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_3\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_4\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_5\mathcal{J}_5} \end{pmatrix}\right] \\ &= \operatorname{Im}\left[\begin{pmatrix} t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_1} & 0 & 0 & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_2} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_2} & 0 & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_3} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_3} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_3} & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_4} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_4} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_4} & t\tilde{\mathbb{A}}^*_{I_4\mathcal{J}_4} & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_5} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_5} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_5} & t\tilde{\mathbb{A}}^*_{I_4\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_5\mathcal{J}_5} \end{pmatrix}\right]. \end{aligned}$$

Hence, (31) is equivalent to

$$\bar{P}_t^{-1}\tilde{S}\bar{P}_t^{-1} - S^* \;=\; \begin{pmatrix} t^4\tilde{S}_{\mathcal{J}_1} \\ t^3\tilde{S}_{\mathcal{J}_2} \\ t^2\tilde{S}_{\mathcal{J}_3} \\ t\tilde{S}_{\mathcal{J}_4} \\ \tilde{S}_{\mathcal{J}_5} - S^*_{\mathcal{J}_5} \end{pmatrix} \;\in\; \operatorname{Im}\left[\begin{pmatrix} t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_1} & 0 & 0 & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_2} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_2} & 0 & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_3} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_3} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_3} & 0 & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_4} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_4} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_4} & t\tilde{\mathbb{A}}^*_{I_4\mathcal{J}_4} & 0 \\ t^4\tilde{\mathbb{A}}^*_{I_1\mathcal{J}_5} & t^3\tilde{\mathbb{A}}^*_{I_2\mathcal{J}_5} & t^2\tilde{\mathbb{A}}^*_{I_3\mathcal{J}_5} & t\tilde{\mathbb{A}}^*_{I_4\mathcal{J}_5} & \tilde{\mathbb{A}}^*_{I_5\mathcal{J}_5} \end{pmatrix}\right].$$

Dividing the first, second, third and fourth blocks of rows in above system by t^4, t^3, t^2 and t, respectively, we conclude that $\tilde{S} - S^* \in \operatorname{Im} \tilde{\mathbb{A}}^*_t$, or equivalently that (46) holds for some $\tilde{y} \in \Re^m$. □

Theorem 5. The following statements hold:

i) the path $t > 0 \to (\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ is analytic at $t = 0$, and hence, all its higher-order derivatives converge as $t \downarrow 0$;

ii) the path $t > 0 \to (X(t^4), S(t^4), y(t^4))$ is analytic at $t = 0$.

Proof. The proof is based on Proposition 1. Indeed, let $E = \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$, $\Lambda = \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$, $\delta = +\infty$, $w : (0, +\infty) \to E$ denote the path $w(t) = (\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ and $H(w,t) = H(\tilde{X}, \tilde{S}, \tilde{y}, t)$ be the map determined by system (45)-(47). By Theorem 2, we know that the path $(\tilde{X}(t), \tilde{S}(t))$ converges to $(\tilde{X}^*, \tilde{S}^*)$, hence $w(t) = (\tilde{X}(t), \tilde{S}(t), \tilde{y}(t))$ converges to the point $w^* = (\tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ in Λ, where $\tilde{y}^* = (\mathbb{A}_0 \mathbb{A}_0^*)^{-1} \mathbb{A}_0 (\tilde{S}^* - S^*)$, since $\lim_{t\to 0} \tilde{\mathbb{A}}_t = \tilde{\mathbb{A}}_0$ and $\mathbb{A}_0$ has full-rank. We claim that the Jacobian $H'_w(w^*, 0)$ is non-singular, or equivalently, that the only solution of the homogeneous system

$$\tilde{X}^{*-1} \widetilde{\Delta X} \tilde{X}^{*-1} + \widetilde{\Delta S} = 0, \tag{49}$$

$$\tilde{\mathbb{A}}_0^* \widetilde{\Delta y} + \widetilde{\Delta S} = 0, \tag{50}$$

$$\tilde{\mathbb{A}}_0 \widetilde{\Delta X} = 0, \tag{51}$$

is the trivial one. In fact, it follows from (50) and (51) that $\widetilde{\Delta X} \bullet \widetilde{\Delta S} = 0$. Taking the dot-product of the first equation with $\widetilde{\Delta X}$ and using the last relation, we easily see that $\|(\tilde{X}^*)^{-1/2} \widetilde{\Delta X} (\tilde{X}^*)^{-1/2}\| = 0$, an hence that $\widetilde{\Delta X} = 0$. This together with (49), (50) and the fact that $\tilde{\mathbb{A}}_0^*$ has full-rank imply that $\widetilde{\Delta S} = 0$ and $\widetilde{\Delta y} = 0$. We have thus shown that $H'_w(w^*, 0)$ is non-singular. Statement i) now follows from Proposition 1. Statement ii) follows from i), relations (27) and (28), and the fact that $y(t^4)$ can be expressed as an analytic function of $S(t^4)$. ◻

Define

$$\left(\dot{\tilde{X}}(0), \dot{\tilde{S}}(0)\right) \equiv \lim_{t\to 0} \left(\dot{\tilde{X}}(t), \dot{\tilde{S}}(t)\right) \quad \text{and} \quad \left(\ddot{\tilde{X}}(0), \ddot{\tilde{S}}(0)\right) \equiv \lim_{t\to 0} \left(\ddot{\tilde{X}}(t), \ddot{\tilde{S}}(t)\right). \tag{52}$$

We will now investigate the implications of the above theorem regarding the limiting behavior of $(\dot{X}(\nu), \dot{S}(\nu))$ as $\nu \downarrow 0$. We will see that $\lim_{\nu\to 0} \sqrt{\nu}\,(\dot{X}(\nu), \dot{S}(\nu))$ exists, is nonzero and be characterized in terms of $(\tilde{X}^*, \tilde{S}^*)$ and the first and second derivatives in (52).

Theorem 6. There hold

$$\lim_{\nu\to 0} \frac{X(\nu) - X^*}{\sqrt{\nu}} = \lim_{\nu\to 0} 2\sqrt{\nu}\, \dot{X}(\nu) = \begin{pmatrix} 1/2\ddot{\tilde{X}}_{\mathcal{B}}(0) & \dot{\tilde{X}}_{BT}(0) & \tilde{X}_{BN}(0) \\ \dot{\tilde{X}}_{TB}(0) & \tilde{X}_{\mathcal{T}}(0) & 0 \\ \tilde{X}_{NB}(0) & 0 & 0 \end{pmatrix} \neq 0, \tag{53}$$

$$\lim_{\nu\to 0} \frac{S(\nu) - S^*}{\sqrt{\nu}} = \lim_{\nu\to 0} 2\sqrt{\nu}\, \dot{S}(\nu) = \begin{pmatrix} 0 & 0 & \tilde{S}_{BN}(0) \\ 0 & \tilde{S}_{\mathcal{T}}(0) & \dot{\tilde{S}}_{TN}(0) \\ \tilde{S}_{NB}(0) & \dot{\tilde{S}}_{NT}(0) & 1/2\ddot{\tilde{S}}_{\mathcal{N}}(0) \end{pmatrix} \neq 0. \tag{54}$$

Proof. We will prove only (53) since the proof of (54) is similar. It suffices to show that (53) holds with $\nu = t^4$. First, observe that L'Hospital rule implies that

$$\lim_{t\to 0} \frac{X(t^4) - X^*}{t^2} = \lim_{t\to 0} 2t^2 \dot{X}(t^4),$$

as long as the second limit in (53) exists. That the T-block in (53) is nonzero follows from the fact that Lemma 6(a) implies that $\tilde{X}_{\mathcal{T}}\tilde{S}_{\mathcal{T}} = I$, and hence that $\tilde{X}_{\mathcal{T}} \neq 0$ and $\tilde{S}_{\mathcal{T}} \neq 0$. We will now show that the second limit in (53) does indeed exist. By (27), we have

$$X_{\mathcal{J}_j}(t^4) = t^{j-1}\tilde{X}_{\mathcal{J}_j}(t), \quad j = 1, \dots, 5.$$

Derivating the above relation and dividing the resulting expression by $2t$, we obtain

$$2t^2 \dot{X}_{\mathcal{J}_j}(t^4) = \frac{t^{j-2}}{2}\dot{\tilde{X}}_{\mathcal{J}_j}(t) + \frac{j-1}{2}t^{j-3}\tilde{X}_{\mathcal{J}_j}(t), \quad j = 1, \dots, 5. \tag{55}$$

Now, using Theorem 5(i) and the above expression, we easily see that $\lim_{\nu\to 0} 2\sqrt{\nu}\dot{X}_{\mathcal{J}_{3:5}}(\nu) = \lim_{t\to 0} 2t^2\dot{X}_{\mathcal{J}_{3:5}}(t^4) = (\tilde{X}_{\mathcal{J}_3}(0),\ 0,\ 0) \neq 0$, where $\mathcal{J}_{3:5} \equiv \mathcal{T} \cup BN \cup NB \cup TN \cup NT \cup \mathcal{N}$. Since $\lim_{t\to 0}\tilde{X}_{\mathcal{J}_2}(t) = \tilde{X}^*_{\mathcal{J}_2} = 0$, it follows that

$$\lim_{t\to 0} \frac{\tilde{X}_{\mathcal{J}_2}(t)}{t} = \dot{\tilde{X}}_{\mathcal{J}_2}(0). \tag{56}$$

Hence, $\lim_{t\to 0} 2t^2\dot{X}_{\mathcal{J}_2}(t^4) = \dot{\tilde{X}}_{\mathcal{J}_2}(0)$ in view of Theorem 5(i), (55) and (56). An argument similar to one used for the case $j = 2$ can be used to prove that $\lim_{t\to 0} 2t^2\dot{X}_{\mathcal{B}}(t^4) = \ddot{\tilde{X}}_{\mathcal{B}}(0)/2$ as long as we can show that $\dot{\tilde{X}}_{\mathcal{B}}(0) = 0$. The latter condition follows as a consequence of the lemma stated below. ∎

Lemma 13. $\dot{\tilde{X}}_{\mathcal{J}_j}(0) = 0$ and $\dot{\tilde{S}}_{\mathcal{J}_j}(0) = 0$ for $j = 1, 3, 5$.

Proof. Derivating (47) with respect to t and setting $t = 0$ in the resulting expression, we easily see that

$$\begin{aligned}
\tilde{\mathbb{A}}_{I_1\mathcal{J}_1}\dot{\tilde{X}}_{\mathcal{J}_1}(0) + \tilde{\mathbb{A}}_{I_1\mathcal{J}_2}\tilde{X}^*_{\mathcal{J}_2} &= 0,\\
\tilde{\mathbb{A}}_{I_3\mathcal{J}_3}\dot{\tilde{X}}_{\mathcal{J}_3}(0) + \tilde{\mathbb{A}}_{I_3\mathcal{J}_4}\tilde{X}^*_{\mathcal{J}_4} &= 0,\\
\tilde{\mathbb{A}}_{I_5\mathcal{J}_5}\dot{\tilde{X}}_{\mathcal{J}_5}(0) \qquad\qquad\quad &= 0.
\end{aligned}$$

Since by Lemma 6(b), $\tilde{X}^*_{\mathcal{J}_2} = 0$ and $\tilde{X}^*_{\mathcal{J}_4} = 0$, it follows from the above equations and relation (48) with $t = 0$ that

$$\tilde{\mathbb{A}}_0\widetilde{\Delta X}_0 = 0, \quad \text{where} \quad \widetilde{\Delta X}_0 \equiv \left(\dot{\tilde{X}}_{\mathcal{J}_1}(0),\ 0,\ \dot{\tilde{X}}_{\mathcal{J}_3}(0),\ 0,\ \dot{\tilde{X}}_{\mathcal{J}_5}(0)\right). \tag{57}$$

A similar argument in the S-space reveals that

$$\widetilde{\Delta S}_0 \in \operatorname{Im} \tilde{\mathbb{A}}_0^*, \text{ where } \widetilde{\Delta S}_0 \equiv \left(\dot{\tilde{S}}_{\mathcal{J}_1}(0),\ 0,\ \dot{\tilde{S}}_{\mathcal{J}_3}(0),\ 0,\ \dot{\tilde{S}}_{\mathcal{J}_5}(0)\right). \tag{58}$$

Therefore, it follows from (57) and (58) that

$$\widetilde{\Delta X}_0 \bullet \widetilde{\Delta S}_0 = 0. \tag{59}$$

By (45) we have that $\tilde{X}(t)\tilde{S}(t) = I$ for all t. Derivating this expression with respect to t and setting $t = 0$, we obtain

$$\dot{\tilde{X}}(0)\,\tilde{S}^* + \tilde{X}^*\dot{\tilde{S}}(0) = 0. \tag{60}$$

By identifying S^n with $S^{\mathcal{J}_1} \times \cdots \times S^{\mathcal{J}_5}$, we can define matrices $\widetilde{\Delta X}_1 \in \mathcal{S}^n$ and $\widetilde{\Delta S}_1 \in \mathcal{S}^n$ as

$$\widetilde{\Delta X}_1 \equiv \left(0,\ \dot{\tilde{X}}_{\mathcal{J}_2}(0),\ 0,\ \dot{\tilde{X}}_{\mathcal{J}_4}(0),\ 0\right), \text{ and } \widetilde{\Delta S}_1 \equiv \left(0,\ \dot{\tilde{S}}_{\mathcal{J}_2}(0),\ 0,\ \dot{\tilde{S}}_{\mathcal{J}_4}(0),\ 0\right). \tag{61}$$

In view of (57) and (61), we have that (60) is equivalent to the equation

$$\widetilde{\Delta X}_0\,\tilde{S}^* + \tilde{X}^*\widetilde{\Delta S}_0 = -\widetilde{\Delta X}_1\,\tilde{S}^* - \tilde{X}^*\widetilde{\Delta S}_1. \tag{62}$$

Now, it easy to see that the matrices in the left and right hand side of the above equation have block structures given by

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix},$$

respectively. Therefore, both sides of (62) must be zero, and, in particular, $\widetilde{\Delta X}_0\,\tilde{S}^* + \tilde{X}^*\widetilde{\Delta S}_0 = 0$. Now, using this relation together with (59), we easily see that $\widetilde{\Delta X}_0 = 0$ and $\widetilde{\Delta S}_0 = 0$. □

6 Convex quadratically constrained convex programming

In this section we consider the problem of minimizing a convex quadratic function subject to convex quadratic constraints. It is well-know that this problem can be reformulated as an SDP problem. Our goal in this section is to derive sufficient conditions for the resulting SDP reformulation of this problem to satisfy our assumptions **A1**-**A4**, so that all the results developed in the previous sections apply to it. The basic tool we use to verify **A4** is the equivalence between statements i) and iii) of Theorem 4. It turns out that iii) can be guaranteed to hold for an important subclass of convex quadratically constrained convex programming (CQCCP) problems, namely the ones for which either the objective function or one of the active constraints at every optimal solution is strictly convex.

Consider the following CQCCP problem

$$\min_{y\in\Re^m} \{f_0(y) : f_k(y) \le 0, k = 1, \cdots, \ell\}, \tag{63}$$

where, for some $Q_k \in \mathcal{S}^m_+$, $b_k \in \Re^m$ and $\alpha_k \in \Re$, $f_k(y) \equiv y^T Q_k y - b_k^T y - \alpha_k$ for every $y \in \Re^m$ and $k = 0, \cdots, \ell$. Let $\mathcal{C}$ and $\mathcal{C}^*$ denote its set of feasible solutions and optimal solutions, respectively. Throughout this section, we assume that:

B1) there exists $y_0 \in \Re^m$ such that $f_k(y_0) < 0$ for all $k = 1, \cdots, \ell$;

B2) $\mathcal{C}^* \neq \emptyset$ and $\{(-f_1(\bar{y}), \cdots, -f_\ell(\bar{y})) : \bar{y} \in \mathcal{C}^*\}$ is bounded.

We remark that **B1** and **B2** imply that condition **A2** holds. (see for example Proposition 4.2 of Monteiro and Zhou [24]).

Clearly, (63) is equivalent to

$$\max\{-\eta \ : \ f_0(y) \le \eta, \ f_k(y) \le 0, \ k = 1, \cdots, \ell\}. \tag{64}$$

Noting that the conditions $f_0(y) \le \eta$ and $f_k(y) \le 0$ are equivalent to the following semidefinite inequalities

$$\tilde{S}_0(y,\eta) = \begin{pmatrix} I & Q_0^{1/2} y \\ y^T Q_0^{1/2} & b_0^T y + \alpha_0 + \eta \end{pmatrix} \succeq 0, \quad \tilde{S}_k(y) = \begin{pmatrix} I & Q_k^{1/2} y \\ y^T Q_k^{1/2} & b_k^T y + \alpha_k \end{pmatrix} \succeq 0,$$

for $k = 1, \ldots, \ell$, it follows that problem (64), and hence (63), is equivalent to the following special case of the dual SDP problem (D):

$$\max\left\{-\eta \ : \ \tilde{S}(y,\eta) \equiv \text{Diag}\left(\tilde{S}_0(y,\eta), \tilde{S}_1(y), \cdots, \tilde{S}_\ell(y)\right) \succeq 0\right\}.$$

We will now introduce a change of variables which enforce (4) in the new scaled space. Fix some $y^* \in \text{ri}(\mathcal{C}^*)$, and define $P \equiv \text{Diag}(P_0, \cdots, P_\ell)$, where

$$P_k \equiv \begin{pmatrix} I & 0 \\ -y^{*T} Q_k^{1/2} & 1 \end{pmatrix}, \quad k = 0, \cdots, \ell.$$

The scaled dual slack $S(y,\eta) \equiv P\tilde{S}(y,\eta)P^T$ then becomes $S(y,\eta) = \text{Diag}\,(S_0(y,\eta), S_1(y), \cdots, S_\ell(y))$, where

$$S_0(y,\eta) \equiv \begin{pmatrix} I & Q_0^{1/2}(y-y^*) \\ (y-y^*)^T Q_0^{1/2} & h_0(y,y^*) + \eta \end{pmatrix}, \quad S_k(y) \equiv \begin{pmatrix} I & Q_k^{1/2}(y-y^*) \\ (y-y^*)^T Q_k^{1/2} & h_k(y,y^*) \end{pmatrix}, \tag{65}$$

and

$$h_k(y,y^*) \equiv (y^*)^T Q_k y^* - 2(y^*)^T Q_k y + b_k^T y + \alpha_k = -\left[f_k(y^*) + \nabla f_k(y^*)^T(y-y^*)\right].$$

for $k = 0, \cdots, \ell$. The scaled dual SDP problem is then

$$\max\{-\eta \ : \ S(y,\eta) \equiv \text{Diag}\ (S_0(y,\eta), S_1(y), \cdots, S_\ell(y)) \succeq 0\}. \tag{66}$$

Now define $\mathcal{I}^* \equiv \{k : f_k(y^*) = 0\}$ and note that since $y^* \in \text{ri}(\mathcal{C}^*)$, we also have

$$\mathcal{I}^* = \{k : f_k(\bar{y}) = 0, \ \forall\, \bar{y} \in \mathcal{C}^*\}. \tag{67}$$

Hence, from (65) and (67) it follows that

$$S_k(y^*) \quad = \quad \begin{pmatrix} I & 0 \\ 0 & -f_k(y^*) \end{pmatrix} \succ 0, \quad k \notin \mathcal{I}^* \cup \{0\}, \tag{68}$$

$$S_k(\bar{y}) \quad = \quad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall\, \bar{y} \in \mathcal{C}^*, \ \ \forall\, k \in \mathcal{I}^* \cup \{0\}, \tag{69}$$

where in the second relation we used the fact that $\nabla f_k(y^*)^T(\bar{y} - y^*) = 0$ for all $\bar{y} \in \mathcal{C}^*$ and that, every $k \in \mathcal{I}^* \cup \{0\}$, $Q_k^{1/2}\bar{y}$ remains constant as $\bar{y}$ varies in $\mathcal{C}^*$ (see Theorem 1 of Mangasarian [15]).

By (68) and (69), we conclude that: i) for each $k \notin \mathcal{I}^* \cup \{0\}$, the block S_k is part of the block $S_{\mathcal{N}}$, and; ii) for each $k \in \mathcal{I}^* \cup \{0\}$, the leading principal $m \times m$ block of S_k is part of the block $S_{\mathcal{N}}$ and the $(m+1)$-th diagonal element of S_k can be either in $S_{\mathcal{B}}$ or $S_{\mathcal{T}}$. In the following, we will say that $k \in \hat{J}$ for $J = B, T$, if the $(m+1)$-th diagonal element of S_k is in $S_{\mathcal{J}}$. Clearly, $\mathcal{I}^* \cup \{0\} = \hat{B} \cup \hat{T}$. A characterization of these sets requires us to examine the nature of the optimal set of the scaled primal problem.

We will now describe the associated scaled primal problem. Because of the block-diagonal structure of the scaled dual problem (66), we may assume that the primal feasible solutions X have the same block-diagonal structure $X = \text{Diag}\,(X_0, X_1, \ldots, X_\ell)$, where each

$$X_k \equiv \begin{pmatrix} U_k & u_k \\ u_k^T & \lambda_k \end{pmatrix} \in \mathcal{S}_+^{m+1}, \quad k = 0, 1, \ldots, \ell. \tag{70}$$

Moreover, it is easy to see that the set of primal feasible solutions consists of those $X \succeq 0$ as above satisfying

$$\lambda_0 = 1, \quad \sum_{k=0}^{\ell} \lambda_k (2Q_k y^* - b_k) - 2\sum_{k=0}^{\ell} Q_k^{1/2} u_k = 0. \tag{71}$$

The scaled primal problem is then given by

$$\min \left\{ \sum_{k=0}^{\ell} \left[I \bullet U_k - 2(y^*)^T Q_k^{1/2} u_k + \left((y^*)^T Q_k y^* + \alpha_k\right) \lambda_k \right] : \text{(70) and (71) hold} \right\}. \tag{72}$$

By (68), (69) and the complementarity slackness condition, it is easy to see that a primal optimal solution $\bar{X} = \text{Diag}(\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_\ell)$ of the scaled pair of dual problems have the following structure:

$$\bar{X}_k = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\lambda}_k \end{pmatrix}, \quad k = 0, 1 \ldots, \ell, \tag{73}$$

where $\bar{\lambda}_0 = 1$ and $\bar{\lambda} \equiv (\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell) \in \Re^\ell$ satisfies

$$\bar{\lambda}_k f_k(y^*) = 0, \;\; \bar{\lambda}_k \geq 0, \;\; k = 1, \ldots, \ell, \tag{74}$$

$$\nabla f_0(y^*) + \sum_{k=1}^{\ell} \bar{\lambda}_k \nabla f_k(y^*) = 0. \tag{75}$$

We remark that the set $\mathcal{M}(y^*) \equiv \{\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_\ell) :$ (74) and (75) holds$\}$ is exactly the set of the Lagrange multipliers of problem (63). Since this set does not depend on the particular $y^* \in \mathcal{C}^*$ chosen (see for example Proposition 3.1.1 of Chapter VII of [11]), we will henceforth denote it simply by $\mathcal{M}$. From the above discussion, it is now easy to see that the following result holds.

Proposition 2. Assume that $\bar{X} = \text{Diag}(\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_\ell)$ is a feasible solution for the scaled primal SDP problem, or equivalently, that (70) and (71) holds. Then, $\bar{X}$ is optimal if and only if (73) holds and $\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_\ell) \in \mathcal{M}$.

The following result whose proof is now straightforward gives a characterization of the index set $\hat{B}$ (and hence of $\hat{T}$).

Lemma 14. $\hat{B} = \{0\} \cup \{k : \bar{\lambda}_k > 0 \text{ for some } \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_\ell) \in \mathcal{M}\}$.

Based on the above discussion, it is now easy to see that our pair of scaled dual problems (66) and (72) satisfy the requirement (4). We are now ready to state the main result of this section, which provides a characterization for when condition **A4** holds for the pair of dual SDPs (66) and (72).

Theorem 7. Let $(X(\nu), S(\nu))$ denote the central path for the scaled pair of dual problems (66) and (72). Then, the following statements hold:

i) $X_{\mathcal{T}}(\nu) = \mathcal{O}(\sqrt{\nu})$;

ii) condition **A4** holds for the pair of dual SDPs (66) and (72) if and only if

$$b_k^T \Delta y = 0, \quad k \in \hat{B} \setminus \{0\}, \tag{76}$$

$$Q_k \Delta y = 0, \quad k \in \hat{B}, \tag{77}$$

$$(b_k - 2Q_k y^*)^T \Delta y \geq 0, \quad k \in \hat{T}.$$

implies that $(b_k - 2Q_k y^*)^T \Delta y = 0$ for all $k \in \hat{T}$.

Proof. Statement i) and the fact that $\Delta X_{\mathcal{T}} = 0$ is the unique solution of (38) follow from Corollary 3(ii) and Lemma 8(i), respectively, by noting that the structure of the dual problem (66) implies that $\mathbb{A}_{\mathcal{J}_2} = 0$. The special structure of the dual problem implies that $\Delta S_{\mathcal{T}} = \text{Diag}\,(\Delta s_k : k \in \hat{T})$, for some scalars Δs_k, $k \in \hat{T}$. Moreover, it is easy to see that $\Delta S_{\mathcal{T}}$ satisfies (39) if and only if, for some $\Delta y \in \Re^m$, $0 \le \Delta s_k = (b_k - 2Q_k y^*)^T \Delta y$ for all $k \in \hat{T}$ and relations (76) and (77) hold. Statement ii) now follows from the above observations and Theorem 4. □

The following result, which is an immediate consequence of Theorem 7(ii), gives some sufficient conditions for **A4** to hold for the pair of dual SDPs (66) and (72).

Theorem 8. The following statements hold:

i) if $\bigcap \{\text{Null}(Q_k) : k \in \hat{B}\} = \{0\}$, then condition **A4** holds for (66) and (72); in particular, if $Q_k \succ 0$ for some $k = 0, 1, \cdots, \ell$, then condition **A4** holds for (66) and (72);

ii) if $Q_k = 0$ for all $k \in \hat{T}$, then condition **A4** holds for (66) and (72); in particular, if the pair of SDPs (66) and (72) corresponds to a convex quadratic programming, namely problem (63) with $Q_k = 0$ for all $k = 1, \ldots, \ell$, then condition **A4** holds for (66) and (72).

Proof. Since the condition $\bigcap \{\text{Null}(Q_k) : k \in \hat{B}\} = \{0\}$ is equivalent to $\Delta y = 0$ being the unique solution of (77), we conclude that statement i) follows immediately from Theorem 7(ii).

To prove ii), assume that $Q_k = 0$ for all $k \in \hat{T}$. Due to the special structure of the dual problem (66) (see relation (65)), this implies that $\mathbb{A}_{\mathcal{J}_4} = 0$. Therefore, by Corollary 3(iii) and Theorem 7(i), we conclude **A4** holds. □

The following example shows the pair of dual SDPs (66) and (72) corresponding to a general CQCCP problem may not satisfy condition **A4**.

Example 1. Consider the CQCCP problem (63), where

$$f_0(y) = y_1^2 + y_3^2, \;\; f_1(y) = y_1^2 + 5y_2^2 + 4y_1y_2 - y_3, \;\; f_2(y) = y_2^2 + 2y_3^2 + 2y_2y_3 - y_2 - y_3,$$

for every $y = (y_1, y_2, y_3) \in \Re^3$. Note that $y^0 = (0, 0, 1)$ satisfies condition **B1**. Moreover, it is easy to see that $\mathcal{C}^* = \{(0, 0, 0)\}$ and $\mathcal{M} = \{(0, 0)\}$ so that condition **B2** is also satisfied and $(\hat{B}, \hat{N}, \hat{T}) = (\emptyset, \emptyset, \{1, 2\})$ due to Lemma 14. Moreover, it is easy to see that $\Delta y \equiv (0, 1, 0)$ does not satisfy the equivalent condition to **A4** of Theorem 7(ii). We have thus shown that the pair of dual SDPs (66) and (72) corresponding to this CQCCP problem satisfies conditions **A2** and **A3** but not **A4**.

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