Lift-and-Project Ranks and Antiblocker Duality

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Abstract

Recently, Aguilera et al. exposed a beautiful relationship between antiblocker duality and the lift-and-project operator proposed by Balas et al. We present a very short proof of their result that the BCC-rank of the clique polytope is invariant under complementation. The proof of Aguilera et al. relies on their main technical result, which describes a stronger duality property of all intermediate relaxations. We provide a short proof of this result, too, using simpler and more general arguments. As a result, our theorems are slightly more general. We conclude by proving that such properties do not extend to the N_0 and N procedures of Lovász and Schrijver, or to the N_+ procedure unless $\mathcal{P} = \mathcal{NP}$.

Keywords: stable set problem, antiblocker duality, lift-and-project, semidefinite lifting, integer programming, perfect graphs
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1 Introduction

Recently, Aguilera, Escalante, and Nasini [1] exposed a beautiful relationship between the lift-and-project operator proposed by Balas, Ceria, and Cornuéjols [3] and the antiblocker duality [8, 9] (also see [2], for a study of the blocker duality in the same context). Their results are aimed at the relaxation of the stable set polytope defined by the clique inequalities (we refer to this relaxation as the *clique polytope*).

Aguilera et al. [1] proved

Theorem 1 *The minimum number of iterations of the Balas–Ceria–Cornuéjols (BCC) procedure required to obtain the stable set polytope of a graph G from the clique polytope of G is invariant under the complementation of G.*

Here we provide a very short proof of this theorem from the basic principles underlying the BCC procedure and Lovász's perfect graph theorem (a graph is perfect if and only if its complement is) [14]; see our Theorem 6. The proof of Aguilera et al. [1] relies on their main technical result, which describes a stronger property of all intermediate relaxations:

Theorem 2 *Every application of the BCC procedure to the clique polytope of any graph followed by taking the antiblocker of the resulting polytope is an involution.*

We also provide a short proof of this result. The proof of Aguilera et al. utilizes results of Ceria [6] (also see [5]), whereas our proof uses simpler and more general arguments; see Theorems 11 and 12. As a result, our theorems are slightly more general. As we were preparing this note for submission, we learned from [5] that Gerards, Maróti, and Schrijver [10] also obtained short proofs of Theorem 1, 2. Their proofs are very concise and have some similarities to our proofs. These results were independently obtained.

We provide examples, proving that these elegant properties of the BCC procedure do not generalize to the procedures of Lovász–Schrijver [15]. We conclude with a discussion of related computational complexity issues and relaxations involving positive semidefiniteness constraints.

2 Definitions and Fundamentals

Let $P \subseteq [0,1]^d$ be given. We say that P is *integral* if $P_I := \mathrm{conv}\left(P \cap \{0,1\}^d\right)$ is equal to P, i.e. if P is a polytope with only integral extreme points. Define

$$H_i(0) := \left\{x \in \mathbb{R}^d : x_i = 0\right\}, \quad H_i(1) := \left\{x \in \mathbb{R}^d : x_i = 1\right\}.$$

The following operator is usually defined via the corresponding lift-and-project procedure; however, for the purposes of this paper, we can simply define the BCC operator as follows:

$$N_{(i)}(P) := \mathrm{conv}\left\{[P \cap H_i(0)] \cup [P \cap H_i(1)]\right\}.$$

Then clearly

$$P_I \subseteq N_{(i)}(P) \subseteq P.$$

Let $J := \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, d\}$. We can apply $N_{(i)}$ for $i \in J$ successively. As shown in [3], the final polytope is independent of the order in which we apply these individual operators. So we can define without ambiguity:

$$N_{(J)}(P) := N_{(i_k)}\left(N_{(i_{k-1})}\left(\cdots N_{(i_1)}(P)\cdots\right)\right).$$

In particular, Balas et al. [3] proved the following nice geometric property of the operator $N_{(J)}$:

Lemma 3 *For every set* $J \subseteq \{1, 2, \ldots d\}$ *we have*

$$N_{(J)}(P) = \text{conv}\left(P \cap \left\{x \in \mathbb{R}^d : x_j \in \{0,1\} \text{ for all } j \in J\right\}\right). \quad (1)$$

We can define the *rank of* P_I *relative to* P as the smallest $|J|$ such that $N_{(J)}(P) = P_I$, we denote this rank by $\tilde{r}(P)$. Similarly, the rank of an inequality $a^T x \leq \alpha$ valid for P_I is the smallest $|J|$ such that $a^T x \leq \alpha$ is valid for $N_{(J)}(P)$.

From Lemma 3 we can easily derive

Lemma 4 *Let* $J \subseteq \{1, 2, \ldots, d\}$, *and suppose* $P \subseteq [0,1]^d$ *is given. Then* $N_{(J)}(P) = P_I$ *if and only if* $P(J; z) := \{x \in P : x_j = z_j \text{ for every } j \in J\}$ *is integral for every* $z \in \{0,1\}^J$.

Proof: First assume that $P(J; z)$ is not integral for some $z \in \{0,1\}^J$. Then there is a vector $x \in P(J; z) \setminus P(J; z)_I$. If x is in P_I, then it can be written as a convex combination of integral vectors in P, and since $x_j = 0$ or 1 for each $j \in J$, the j-component of each of those vectors must be the same for each $j \in J$, so they are also in $P(J; z)$. But then x is a convex combination of integral vectors in $P(J; z)$, a contradiction. Hence x is not in P_I. Since x is clearly in $N_{(J)}(P)$, this shows that $N_{(J)}(P) \neq P_I$.

Next assume that $P(J; z)$ is integral for every $z \in \{0,1\}^J$. Since the inclusion $N_{(J)}(P) \supseteq P_I$ is clear, it suffices to prove $P_I \supseteq N_{(J)}(P)$. Let $x \in N_{(J)}(P)$. By Lemma 3, x is a convex combination of vectors in $P \cap \left\{x \in \mathbb{R}^d : x_j \in \{0,1\} \text{ for all } j \in J\right\}$. Each of these vectors lies in $P(J; z)$ for some $z \in \{0,1\}^J$, by definition. Since $P(J; z)$ is integral for every $z \in \{0,1\}^J$, these vectors can be written as a convex combination of integral vectors in P. Therefore, x can be written as a convex combination of integral vectors in P; i.e., $x \in P_I$. Thus $P_I \supseteq N_{(J)}(P)$, and we conclude $P_I = N_{(J)}(P)$. ∎

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For any $U \subset V$, let $G - U$ denote the graph obtained by deleting all vertices of U from G. The *neighborhood* of U, i.e. the set of vertices of G that are adjacent to a vertex in U, will be denoted by $\Gamma(U)$. A *clique* is a set of vertices so that every pair of them are joined by an edge. The *clique polytope* of G is defined by

$$\text{CLQ}(G) := \left\{x \in \mathbb{R}_+^V : x(C) \leq 1 \text{ for every clique } C \text{ in } G\right\},$$

where we used the notation $x(C) := \sum_{j \in C} x_j$. Now, for each $v \in V$ the application of the BCC operator gives $N_{(v)}(\text{CLQ}(G))$, and similarly it gives $N_{(U)}(\text{CLQ}(G))$ for each $U \subseteq V$.

It is well-known that $\text{CLQ}(G) = \text{STAB}(G)$ if and only if the graph G is perfect (this follows from the works of Fulkerson [8] and Chvátal [7]). Rephrasing Lemma 4 gives

Corollary 5 *Let* $U \subseteq V$. *Then*

$$N_{(U)}(\text{CLQ}(G)) = \text{STAB}(G)$$

if and only if $G - U$ *is perfect.*

Proof: By Lemma 4 we have $N_{(U)}(\mathrm{CLQ}(G)) = \mathrm{STAB}(G)$ if and only if

$$\mathrm{CLQ}(G, U; z) := \{x \in \mathrm{CLQ}(G) : x_u = z_u \text{ for every } u \in U\}$$

is integral for every $z \in \{0,1\}^U$. So, it suffices to show that this latter condition holds if and only if $G - U$ is perfect.

If $G - U$ is not perfect, then there is a vector $x \in \mathrm{CLQ}(G - U) \setminus \mathrm{STAB}(G - U)$, and then extending x with zeros for the components corresponding to U gives a vector in $\mathrm{CLQ}(G, U; \mathbf{0})$, showing it is not integral.

It is easy to see that for any $z \in \{0,1\}^U$ if we define $U_z := \{v \in U : z_u = 1\}$, then the set $\mathrm{CLQ}(G, U; z)$ is defined by the clique inequalities on $G - U - \Gamma(U_z)$ and $x_u = 0$ for all $u \in \Gamma(U_z)$. If $G - U$ is perfect, then $G - U - \Gamma(U_z)$ is also perfect, so its clique polytope is integral, showing that $\mathrm{CLQ}(G, U; z)$ is integral, finishing the proof. ∎

Let $\tilde{r}(G)$ denote the rank of $\mathrm{STAB}(G)$ relative to $\mathrm{CLQ}(G)$. Corollary 5 helps us give a short proof for the following theorem of Aguilera et al. [1]:

Theorem 6 *For every graph G, $\tilde{r}(G) = \tilde{r}(\overline{G})$.*

Proof: If $\tilde{r}(G) = k$, then there are k vertices $v_1, \ldots, v_k$ such that with $U = \{v_1, \ldots, v_k\}$ we have $N_{(U)}(G) = \mathrm{STAB}(G)$. So, by Corollary 5 the graph $G - U$ is perfect. But then $\overline{G - U} = \overline{G} - U$ is also perfect, so again by the lemma we get that $N_{(U)}(\overline{G}) = \mathrm{STAB}(\overline{G})$, so $\tilde{r}(\overline{G}) \le k = \tilde{r}(G)$. Exchanging the roles of G and $\overline{G}$ we get $\tilde{r}(G) \le \tilde{r}(\overline{G})$, proving the theorem. ∎

Note that the above theorem generalizes Lovász's perfect-graph theorem (the special case $\tilde{r}(G) = \tilde{r}(\overline{G}) = 0$).

3 Polytopes with Integral Antiblocker

For a set $P \subseteq [0,1]^d$, let P^* denote its *antiblocker*:

$$P^* := \left\{ s \in \mathbb{R}^d_+ : x^T s \le 1 \text{ for all } x \in P \right\}.$$

Suppose $P \subseteq [0,1]^d$ is given. Then $P^* \subseteq [0,1]^d$ if and only if $e_j \in P$ for every $j \in \{1, 2, \ldots, d\}$ (e_j denotes the jth unit vector, and e denotes the vector of all ones in $\mathbb{R}^d$).

We will need the following elementary, well-known facts:

Proposition 7 *If $P \subseteq [0,1]^d$, then*

(i) *$P \subseteq \tilde{P}$ implies $\tilde{P}^* \subseteq P^*$;*

(ii) *Suppose $e_j \in P$ for every $j \in \{1, 2, \ldots, d\}$. Then $P^* \subseteq [0,1]^d$ is convex and lower comprehensive (if $0 \le y \le x \in P^*$, then $y \in P^*$).*

(iii) *Suppose P is compact, convex, and $0 \in P$. Then $P^{**} = P$.*

Definition 8 *An operator $N_\sharp$ that maps subsets of $[0,1]^d$ to convex subsets of $[0,1]^d$ such that*

(i) *$N_\sharp(P) \supseteq P_I$ for every $P \subseteq [0,1]^d$, and*

(ii) $N_\sharp(P) \subseteq P$ *for every convex* $P \subseteq [0,1]^d$,

is called a 0-1 monotone operator.

It is clear that if $N_\sharp$ is a 0-1 monotone operator, then iterating it finitely many times yields another (usually stronger) 0-1 monotone operator.

Not all 0-1 monotone operators are interesting. However, note that $N_{(J)}$ is a 0-1 monotone operator for every $J \subseteq \{1, 2, \ldots, d\}$. Also, the operators N_0, N, and N_+ (see [15] for a definition) are 0-1 monotone operators. The related works of Sherali–Adams [16], Lasserre [12], and Bienstock–Zuckerberg [4] all yield various 0-1 monotone operators. Finally, Gomory–Chvátal closures can also be thought of as 0-1 monotone operators.

Proposition 9 *Let* $N_\sharp$ *be a* 0-1 *monotone operator, and let* $P \subseteq [0,1]^d$ *be such that* $P^* = \mathrm{conv}(P^* \cap \{0,1\}^d)$ *(i.e.,* P^* *is an integral polytope in the unit hypercube). Then for any* $k \geq 1$

$$\left\{ N_\sharp^k \left(\left[N_\sharp^k \left(P \right) \right]^* \right) \right\}^* \subseteq P.$$

Proof: By (ii) of Definition 8 we have $N_\sharp^k(P) \subseteq P$, hence $\left[N_\sharp^k(P)\right]^* \supseteq P^*$. Since P^* is an integral polytope in the unit hypercube, using (i) of Definition 8, we arrive at $N_\sharp^k \left(\left[N_\sharp^k \left(P\right)\right]^* \right) \supseteq P^*$. Taking the antiblocker of both sides we obtain the desired result. ∎

Definition 10 *Let* $Q \subseteq [0,1]^d$ *be an integral polytope. A convex set* $P \subseteq [0,1]^d$ *is called a* formulation of Q *if* $\mathrm{conv}\left(P \cap \{0,1\}^d\right) = Q$.

Note that CLQ(G) and STAB(G) are both lower comprehensive and that the operator $N_{(J)}$ preserves the lower comprehensiveness of its argument.

Theorem 11 *Let* $P \in [0,1]^d$ *be a lower comprehensive polytope such that* P^* *is integral. Suppose further that* $\left[N_{(i)}(P)\right]^*$ *is a formulation of* P^*. *Then*

$$\left\{ N_{(i)} \left(\left[N_{(i)}(P) \right]^* \right) \right\}^* = P.$$

That is, the following diagram

$$\begin{array}{ccc} P & \stackrel{*}{\longleftrightarrow} & P^* \\ N_{(i)} \big\downarrow & & \big\uparrow N_{(i)} \\ N_{(i)}(P) & \stackrel{*}{\longleftrightarrow} & \left[N_{(i)}(P)\right]^* \end{array}$$

commutes.

Proof: Since $N_{(i)}$ is a 0-1 monotone operator, by Proposition 9 it suffices to establish the inclusion $N_{(i)} \left(\left[N_{(i)}(P) \right]^* \right) \subseteq P^*$. Without loss of generality let $i = d$. We have

$$N_{(d)} \left(\left[N_{(d)}(P) \right]^* \right) \;=\; \mathrm{conv} \left\{ \left(\left[N_{(d)}(P) \right]^* \cap H_d(0) \right) \cup \left(\left[N_{(d)}(P) \right]^* \cap H_d(1) \right) \right\}. \qquad (2)$$

We first prove

$$\left[N_{(d)}(P)\right]^* \cap H_d(0) \quad = \quad P^* \cap H_d(0). \tag{3}$$

Since $N_{(d)}(P) \subseteq P$ (thus $\left[N_{(d)}(P)\right] \supseteq P^*$), the "$\supseteq$" direction is clear. Let $\bar{s} \in \left[N_{(d)}(P)\right]^* \cap H_d(0)$. Then $\bar{s} \geq 0$, $\bar{s}_d = 0$, and $x^T\bar{s} \leq 1$ for all $x \in (P \cap H_d(0))$. Since P is lower comprehensive, and $\bar{s}_d = 0$, we have $x^T\bar{s} \leq 1$ for every $x \in P$. Therefore $\bar{s} \in P^*$, and we established (3).

Next we claim that $\left[N_{(d)}(P)\right]^* \cap H_d(1) \subseteq P^*$. To see this, let $I \subseteq \{1, 2, \ldots, (d-1)\}$ denote the set of all indices j for which there exists $x \in P$ such that $x_d = 1$ and $x_j > 0$. Then

$$\begin{aligned}
\left[N_{(d)}(P)\right]^* \cap H_d(1) &= \left\{ \begin{pmatrix} \tilde{s} \\ 1 \end{pmatrix} : \tilde{s} \geq 0, \tilde{x}^T\tilde{s} \leq 1 - x_d,\ \forall \begin{pmatrix} \tilde{x} \\ x_d \end{pmatrix} \in \left[(P \cap H_d(0)) \cup (P \cap H_d(1))\right] \right\} \\
&= \left\{ \begin{pmatrix} \tilde{s} \\ 1 \end{pmatrix} : \tilde{s} \geq 0, \tilde{s}_j = 0, \forall j \in I, \tilde{x}^T\tilde{s} \leq 1,\ \forall \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \in (P \cap H_d(0)) \right\} \\
&= \left\{ \begin{pmatrix} \tilde{s} \\ 1 \end{pmatrix} : \begin{pmatrix} \tilde{s} \\ 0 \end{pmatrix} \in P^*, \tilde{s}_j = 0, \forall j \in I \right\},
\end{aligned}$$

where to obtain the last equality we used the fact that P is lower comprehensive. Note that the last set is equal to the face $P^* \cap \left\{s \in \mathbb{R}^d : s_j = 0, \forall j \in I \cup \{d\}\right\}$ of P^* translated by the unit vector e_d. Since P^* is integral, every face of it is integral; therefore, the set in question is integral. Since $\left[N_{(i)}(P)\right]^*$ is a formulation of P^*, the set in question must be contained in P^*. So we conclude that the set inside the convex hull operator in (2) is a subset of P^*, finishing the proof. ∎

A direct generalization of Theorem 11 is

Theorem 12 *Let $P \subseteq [0,1]^d$ be a lower comprehensive polytope such that P^* is integral. Suppose further that $\left[N_{(J)}(P)\right]^*$ is a formulation of P^* for some $J \subseteq \{1, 2, \ldots, d\}$. Then*

$$\left\{N_{(J)}\left(\left[N_{(J)}(P)\right]^*\right)\right\}^* = P.$$

That is, the following diagram

$$\begin{array}{ccc}
P & \overset{*}{\longleftrightarrow} & P^* \\
N_{(J)}\Big\downarrow & & \Big\uparrow N_{(J)} \\
N_{(J)}(P) & \overset{*}{\longleftrightarrow} & \left[N_{(J)}(P)\right]^*
\end{array} \tag{4}$$

commutes.

Proof: We proceed by induction on $|J|$. The case $|J| = 1$ was established by the previous theorem. Consider $J' \subseteq \{1, 2, \ldots, d\}$ such that $|J'| > 1$. Without loss of generality we can assume $d \in J'$, and let $J := J' \setminus \{d\}$. As in the proof of Theorem 11, we have

$$\left[N_{(J')}(P)\right]^* \cap H_d(0) = \left[N_{(J)}(P)\right]^* \cap H_d(0),$$

where we used the fact that $N_{(J)}(P)$ is lower comprehensive (since P is, too). Also, as in the proof of Theorem 11 we obtain

$$\left[N_{(J')}(P)\right]^* \cap H_d(1) \quad = \quad \left\{ \begin{pmatrix} \tilde{s} \\ 1 \end{pmatrix} : \begin{pmatrix} \tilde{s} \\ 0 \end{pmatrix} \in \left[N_{(J)}(P)\right]^*, \tilde{s}_j = 0, \forall j \in I \right\}, \tag{5}$$

where $I \subseteq \{1, 2, \ldots, (d-1)\}$ denotes the set of all indices j for which there exists $x \in N_{(J)}(P)$ such that $x_d = 1$ and $x_j > 0$ (again we used the lower comprehensiveness of $N_{(J)}(P)$). By (5) it is clear that $\left[N_{(J')}(P)\right]^* \cap H_d(1)$ can be expressed as $\left[N_{(J)}(P)\right]^*$ intersected with a face of $[0,1]^d$ and then translated by e_d. By the induction hypothesis

$$N_{(J)}\left(\left[N_{(J)}(P)\right]^*\right) = P^*,$$

therefore $N_{(J)}$ applied to the right-hand side of (5) yields the corresponding (integral) face of P^*. Since $\left[N_{(J')}(P)\right]^*$ is a formulation of P^*, we have the desired equality. ■

The above theorem does not extend to even a single application of the stronger operators N_0 and N (the operator N_0 is defined by $N_0(P) := \cap_{i=1}^d N_{(i)}(P)$, and for the definition and additional properties of N see [15, 13]). To see this, consider the following example:

Example 13 *Let the graph G be as shown on Figure* 1, *and let*

$$P := \mathrm{CLQ}(G) = \mathrm{FRAC}(G) := \left\{x \in [0,1]^V : x_i + x_j \leq 1,\ \forall \{i,j\} \in E\right\}.$$

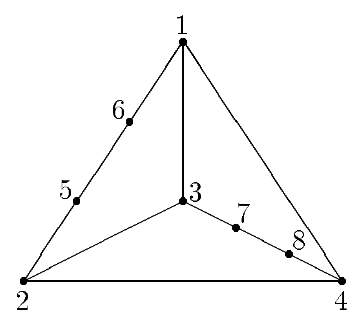

Figure 1: A graph G for which neither N nor N_0 satisfies (4)

It can be easily seen that $N_0(P)$ (and $N(P)$ as well) will be defined by the edge and the 5-cycle inequalities, so its vertices will be the characteristic vectors of stable sets of G and four additional vertices, in which each coordinate is $\frac{1}{3}$ except two (one from vertices 5 and 6 and one from vertices 7 and 8), which are $\frac{2}{3}$. Hence $[N_0(P)]^*$ is defined by the triangle inequalities and four additional inequalities corresponding to these four vertices. Then it is easy (though tedious) to check that $N_0([N_0(P)]^*)$ will still contain the point $\frac{1}{5}(2,2,2,1,1,1,1,1)^T$ (this can be shown by the matrix

$$\frac{1}{5}\begin{pmatrix} 5 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

where it needs to be checked that each column and the difference of the first and any other column is in $M_0([N_0(P)]^*)$; see [13] for the definition of M_0, and note that this matrix is symmetric, so the result applies to N as well). Since this point violates the inequality $\sum_{i=1}^{8} x_i \leq 2$ valid for P^* (it corresponds to the point $\frac{1}{2}e \in \mathrm{FRAC}(G)$), $N_0([N_0(P)]^*)$ is not equal to P^*, so diagram (4) does not commute for N_0 or N.

It is well-known that when P is the clique polytope of a graph, then the antiblocker of its convex hull is a formulation for P^* (since the stable sets correspond to cliques in the complement of the graph). So the conditions of Theorem 12 are automatically satisfied for any J, thus diagram (4) commutes for any J. This consequence was also proved by Aguilera et al. [1]. Next we show that these are the only examples among polytopes whose antiblocker is integral:

Theorem 14 *Let $P \subseteq [0,1]^d$ be a polytope containing 0 such that P^* is integral. If $(P_I)^*$ is a formulation of P^*, then P is the clique polytope of a graph with vertices $\{1,\dots,d\}$.*

Proof: Let $S_* := P^* \cap \{0,1\}^d$, and define the graph $G := (V, E)$ as follows:

$$V := \{1,2,\dots,d\}\,, \qquad E := \{\{i,j\} : e_i + e_j \leq s \text{ for some } s \in S_*\}\,.$$

Since P^* is integral, $S_* \subseteq \{0,1\}^d$, and $0 \in P$, using (iii) of Proposition 7 we get

$$P = \left\{x \in [0,1]^d : s^T x \leq 1 \text{ for all } s \in \overline{S}_*\right\},$$

where $\overline{S}_*$ denotes the set of maximal elements of S_*. We now prove by contradiction that P is the clique polytope of G. Assume $\overline{S}_*$ does not correspond to the maximal cliques in G. By construction, if $s \in \overline{S}_*$, then it describes a maximal clique in G. Therefore there must exist a clique C in G such that the characteristic vector of C, χ_C, does not lie in $\overline{S}_*$. Note that $\mathrm{STAB}(G) \subseteq P \subseteq \mathrm{FRAC}(G)$, and $\mathrm{FRAC}(G)$ is a formulation of $\mathrm{STAB}(G)$. Therefore $\left[N_{(V)}(P)\right]^* = (P_I)^*$ must include χ_C. But $\chi_C \notin \overline{S}_*$, so $(P_I)^*$ is not a formulation of P^*, a contradiction. ∎

Thus we have shown that the diagram in Theorem 12 commutes for every J exactly when P is a clique polytope of a graph. Note, however, that it may commute for some J even for nonclique polytopes. So Theorem 12 is slightly more general than the corresponding theorem in [1]. Indeed, for an application of our theorem with a fixed J, all we need is that $\left[N_{(J)}(P)\right]^*$ be a formulation of P^*.

4 Complexity Issues and SDP Based Relaxations

As we discussed in the previous sections, if we begin with $\mathrm{CLQ}(G)$ as the initial relaxation of $\mathrm{STAB}(G)$ and then apply the BCC procedure to it, a beautiful relationship between the intermediate relaxations and their antiblockers exists. However, given G, optimization (or separation) over $\mathrm{CLQ}(G)$ (or its antiblocker) is $\mathcal{NP}$-hard. So, in this sense, $\mathrm{CLQ}(G)$ is an intractable relaxation of $\mathrm{STAB}(G)$.

If G is triangle-free, then $\mathrm{CLQ}(G) = \mathrm{FRAC}(G)$, which is tractable. Then the above results apply to $\mathrm{FRAC}(G)$ as the initial relaxation. Moreover, we can obtain from any graph G a triangle-free graph G' by subdividing an edge from each triangle of G. Then $\mathrm{FRAC}(G') =$

$\mathrm{CLQ}(G')$. However, the ranks of G and G' may be very different (for a detailed study of rank changes under such graph operations see [13]).

Lovász and Schrijver [15] proved that their N_+ operator (which utilizes semidefiniteness constraints, see [15] for a definition and further details) has the property that

$$N_+(\mathrm{FRAC}(G)) \subseteq \mathrm{CLQ}(G), \text{ for every graph } G.$$

Moreover, we can optimize over $N_+^k(\mathrm{FRAC}(G))$ in polynomial time for every $k = O(1)$. We utilize these facts below:

Theorem 15 *If for every graph G*

$$\left\{ N_+^k \left(\left[N_+^k(\mathrm{CLQ}(G)) \right]^* \right) \right\}^* = \mathrm{CLQ}(G)$$

for some $k \geq 1$ and $k = O(1)$, that is, if the following diagram

$$\begin{array}{ccc} \mathrm{CLQ}(G) & \stackrel{*}{\longleftrightarrow} & \mathrm{STAB}(\overline{G}) \\ N_+^k \big\downarrow & & \big\uparrow N_+^k \\ N_+^k(\mathrm{CLQ}(G)) & \stackrel{*}{\longleftrightarrow} & \left[N_+^k(\mathrm{CLQ}(G)) \right]^* \end{array}$$

commutes for some $k \geq 1$ and $k = O(1)$, then $\mathcal{P} = \mathcal{NP}$.

Proof: We have $N_+(\mathrm{FRAC}(G)) \subseteq \mathrm{CLQ}(G)$ for every graph G. Also, for every G we have $N_+^{|V|/3}(\mathrm{FRAC}(G)) = \mathrm{STAB}(G)$ (see [13]). Since $[\mathrm{STAB}(G)]^* = \mathrm{CLQ}(\overline{G})$, $\left[N_+^{|V|/3}(\mathrm{FRAC}(G)) \right]^*$ is a formulation of $\mathrm{STAB}(\overline{G})$. Since $N_+(\mathrm{FRAC}(G)) \subseteq N_+^{|V|/3}(\mathrm{FRAC}(G))$, using (i) of Proposition 7 we see that $[N_+(\mathrm{FRAC}(G))]^*$ is also a formulation of $\mathrm{STAB}(\overline{G})$. Now suppose that the above diagram commutes for some $k = O(1)$ for all G. Then

$$\begin{aligned} & \mathrm{CLQ}(G) \supseteq N_+(\mathrm{FRAC}(G)) \supseteq N_+^k(\mathrm{CLQ}(G)) \\ \Rightarrow \quad & \mathrm{STAB}(\overline{G}) \subseteq [N_+(\mathrm{FRAC}(G))]^* \subseteq \left[N_+^k(\mathrm{CLQ}(G)) \right]^* \\ \Rightarrow \quad & N_+^k\left([N_+(\mathrm{FRAC}(G))]^* \right) = \mathrm{STAB}(\overline{G}), \end{aligned}$$

where the last implication used the assumption that the diagram commutes. Since $k = O(1)$, by Theorem 1.6 of [15] and the equivalence of separation and optimization (see [11]) for this class of convex sets, we can optimize over $N_+^k\left([N_+(\mathrm{FRAC}(G))]^* \right)$ in polynomial time. Thus we can optimize over $\mathrm{STAB}(G)$ for every graph G in polynomial time. Therefore $\mathcal{P} = \mathcal{NP}$. ∎

References

[1] N. E. Aguilera, M. S. Escalante and G. L. Nasini, A generalization of the perfect graph theorem under the disjunctive index, *Math. Oper. Res.* **27** (2002) 460–469.

[2] N. E. Aguilera, M. S. Escalante and G. L. Nasini, The disjunctive procedure and blocker duality, *manuscript*, 2001.

[3] E. Balas, S. Ceria and G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed 0-1 programs, *Math. Prog. A* **58** (1993) 295–323.

[4] D. Bienstock and M. Zuckerberg, Subset algebra lift operators for 0-1 integer programming, *CORC Report 2002–01*, Columbia University, NY, NY, USA.

[5] S. Ceria, Lift-and-project cuts and perfect graphs, *Math. Prog.*, to appear.

[6] S. Ceria, *Lift-and-Project Methods for Mixed 0-1 Programs*, Ph.D. Dissertation, Carnegie Mellon University, 1993.

[7] V. Chvátal, On certain polytopes associated with graphs, *J. of Combin. Theory Ser. B* **18** (1975) 138–154.

[8] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Prog.* **1** (1971) 168–194.

[9] D. R. Fulkerson, Anti-blocking polyhedra, *J. Combinatorial Theory B* **18** (1972) 50–71.

[10] A. M. H. Gerards, G. Maróti and A. Schrijver, Note on: N. E. Aguilera, M. S. Escalante, G. L. Nasini, "A generalization of the perfect graph theorem under the disjunctive index," *manuscript*, 2002.

[11] M. Grötschel, L. Lovász and A. Schrijver, *Geometric algorithms and combinatorial optimization*, Springer, New York, 1988.

[12] J. B. Lasserre, An explicit exact SDP relaxation for nonlinear 0-1 programs, in *Lecture Notes in Computer Science* (K. Aardal and A. M. H. Gerards, eds.) 2001, pp. 293–303.

[13] L. Lipták and L. Tunçel, The stable set problem and the lift-and-project ranks of graphs, *Math. Prog. B* to appear.

[14] L. Lovász, Normal hypergraphs and the weak perfect graph conjecture, *Discrete Math.* **2** (1972) 253–267.

[15] L. Lovász and A. Schrijver, Cones of matrices and set-functions and 0-1 optimization, *SIAM J. Optim.* **1** (1991) 166–190.

[16] H. D. Sherali and W. P. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM J. Discrete Mathematics* **3** (1990) 411–430.