Capacitated Facility Location Model with Risk Pooling

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Abstract

The Facility Location Model with Risk Pooling (LMRP) extends the uncapacitated fixed charge model to incorporate inventory decisions at the distribution centers (DCs). In this paper, we introduce a capacitated version of the LMRP that handles inventory management at the DCs such that the capacity limitations at the DCs are not exceeded. We consider a logistics system in which a single plant ships one type of product to a set of retailers, each with uncertain demand. Each DC serves as the direct intermediary between the plant and the retailers for the shipment of the product. Safety stock is retained at the DCs to provide appropriate service levels. The Capacitated Facility Location Model with Risk Pooling (CLMRP) simultaneously determines DC locations, shipment sizes and frequencies from the plant to the DCs, the working inventory and safety stock levels at the DCs and the assignment of retailers to the DCs, to minimize the fixed facility location costs, transportation costs and inventory carrying costs. The capacity constraints are defined based on how the inventory is managed. Thus, the relationship between the capacity of a DC and the inventory levels are embedded in the model. CLMRP is capable of evaluating the tradeoff between having more DCs to have sufficient system capacity versus ordering more frequently through the definition of capacity.

The model is formulated as a non-linear integer-programming problem. A Lagrangian relaxation solution algorithm is proposed. The Lagrangian subproblem is also a non-linear integer program. An efficient algorithm is obtained for the linear relaxation of this subproblem.[1]

1 Introduction

Logistics is becoming the most important aspect of business success. Logistics is part of the supply chain process that plans, implements, and controls the efficient, and cost effective movement and storage of goods from the point of production to the point of consumption to meet customer demands. Traditionally, companies have managed the distribution of goods and storage in a disparate way within different functional departments. However, today managers are well aware that optimizing the logistic system as a whole could mean a huge potential for cost savings and a large impact on customer satisfaction. Estimates are that the costs of the three key elements of a supply chain – supply chain-inventory, the distribution center, and freight – can represent 10 to 15 percent of sales in most industries. And the costs of these three key elements of a supply

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chain are highly related. For example, managing inventory costs requires the effective location of distribution centers and then the determination of the optimal amount of storage at these centers. An effective transportation scheme also depends on the location of centers. Being able to integrate the management of these elements of a supply chain is a major challenge and can provide a tremendous advantage to a company in today's increasingly competitive markets.

The logistics literature has tended to treat strategic facility location decisions and tactical inventory management decisions separately. Strategic decisions are considered as long-term and operational decisions are considered as short-term. Hence, the relationship between the strategic and operational elements of a supply chain has been overlooked by the supply chain optimization models. Only recently, Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002) developed a location model with risk pooling (LMRP) that considers working inventory and safety stock costs while making facility location decisions. The motivation behind this model was a study that two of the authors of Daskin, Coullard, and Shen (2002) conducted for a Chicago-based blood bank. The blood bank produced and distributed platelets, a highly perishable and expensive blood product, to a large number of hospitals in the greater Chicago area. Many of the hospitals ordered twice the number of platelets they used each year, resulting in significant inventory costs, as the unused platelets had to be destroyed. Some of the hospitals chose to keep less inventory and to order on an emergency basis which required uneconomical and costly shipment methods. There was clearly a need for a better inventory management system. The idea was to select some hospitals as distributers so that less safety stock would need to be carried in the system. In addition, emergency shipments would be less costly as the platelets would be supplied by a nearby hospital (DC) rather than the blood bank (supplier). An integrated location and inventory optimization model was needed to improve the system efficiency.

Recent developments in information technology and specifically e-commerce also led to an increased need for integrated logistics optimization models. Although B2B exchanges allowed companies certain purchasing and transaction-processing benefits, it is not clear that reductions in inventory, improved service levels, and faster time to market can be easily realized by the companies. Actually, e-commerce end customers have increased expectations on the service level and the speed of delivery, which pushes companies to stock excessive amounts of inventory and to use costly shipment methods. One approach that is undertaken by companies to solve this problem is to ship smaller amounts more frequently. But the number of DCs and their closeness to the demand points affects the order frequency and shipment size. The location model with risk pooling (LMRP; Shen (2000)) succeeds in determining the optimal location of the DCs and the order frequency from the DCs to the customers simultaneously. However, the LMRP assumes infinite capacity at the DCs, which is usually not the case in practice. Having limited capacity may affect the location of the DCs, the inventory that can be stored at the DCs and as a result the order frequency as well as the assignment of customers to DCs.

In this paper, we introduce a capacitated version of the LMRP that handles inventory management at the DCs such that the capacity limitations at the DCs are not exceeded. We consider a system in which a single plant ships one type of product to a set of retailers, each with uncertain demand. Each DC serves as the direct intermediary between the plant and the retailers for the shipment of the product. Safety stock is retained at the DCs to provide appropriate service levels. The objective is to select DC locations, assign the retailers to the DCs, determine the order frequency and safety stock levels at each DC to minimize the total cost of transportation, inventory and location. The capacity constraints are defined based on how the inventory is managed. Thus, the relationship between the capacity of a DC and the inventory levels are embedded in the model.

The capacitated facility location model we present is an extension of the LMRP but its relationship to the classic capacitated facility location problem (CFLP) is worth noting. In the logistics literature, the CFLP and its variants are well studied. The classical capacitated location models don't take inventory management into consideration. When inventory management is ignored, capacity is usually determined exogenously and the relationship between inventory and capacity is not modeled. As a consequence, to have sufficient system capacity more DCs than necessary may be opened. By ordering more frequently, however, we could decrease the average inventory levels and avoid incurring high fixed location costs. Our model is capable of evaluating the tradeoff between having more DCs versus ordering more frequently through the definition of capacity. In this sense, the definition of capacity in our model is more flexible than the definition in the traditional capacitated location models. Also both the traditional capacitated location models and our model does reassignment of retailers to DCs to have sufficient system capacity. But again,in our model reassignment of a set of customers to another DC might not be necessary since ordering more frequently is an option in our model.

The paper is organized as follows. In Section 2 we review some of the literature related to location theory and inventory models. In Section 3 we review the LMRP and describe two solution methods that have been proposed. The model we propose, the capacitated version of the LMRP is formulated in Section 4. The formulation we obtain is a mixed integer non-linear programming model. We outline the solution algorithm for the problem which exploits the structure of the model. Section 5 presents the computational results of our solution algorithms. Section 6 outlines our future research plans.

2 Literature Review

Most research in the logistics literature has studied inventory theory and location theory separately. Inventory theory focuses on the evaluation of inventory replenishment strategies at the distribution centers and the retailers. These studies assume that the strategic location decisions have been made; the number and location of DCs are assumed to be known. Usually, the objective in inventory models is to minimize the inventory costs while providing appropriate service levels. For a detailed study of inventory models, see the texts by Graves, Rinnoy Kan, and Zipkin (1993), Nahmias (1997), and Zipkin (1997). On the other hand, the location theory literature focuses on finding the optimal number and location of DCs and the DC-retailer assignments. Usually, the objective in location models is to minimize fixed facility location and transportation costs, ignoring the inventory related costs are ignored. For a summary of location models, see the texts by Daskin (1995), Drezner (1995), Hurter and Martinich (1989), also the more recent Drezner and Hamacher edited text (2002).

There is a vast literature that considers facility location and customer allocation problems. Typically, the mathematical models for these problems involve two sets of decision variables. The first set is the location variables which determine whether a facility should be located at a candidate facility site. The second set is the assignment (allocation) variables which determine the assignment of customers to the open facilities. The location variables are necessarily integer whereas the assignment variables could be integer or continuous. The assignment variables are defined to be continuous if the location/allocation model allows fractions of customer demand to be served by a facility or if the model naturally results in integer-valued customer assignments, as is the case in the uncapacitated fixed charge location problem or the P-median model. The assignment variables are defined to be integer if the model involves the *single sourcing constraints* which assure that each customer is supplied by only one facility. Earlier research focuses on models with continuous assignment variables (see Akinc and Khumawala (1977), Geoffrion and McBride (1978), Guinard and Spielberg (1979) and Nauss (1978)) which have only a few integer (location) variables. More recent research focuses on models with single sourcing constraints (see Klincewicz and Luss (1986), Pirkul (1987) and Daskin and Jones (1993)) resulting in a pure 0-1 integer programming problem. It should be noted that most location/allocation problems of interest are NP-complete. See Krarup and Pruzan (1983) for a proof of NP-hardness of uncapacitated facility location problem and Garey and Johnson (1979) for a general discussion of complexity theory and NP-complete problems.

Roughly speaking, location/allocation problems can be classified into: (1) uncapacitated facility location problems, and (2) capacitated facility location problems. Various extensions of these problems have been studied in the literature. Models that involve different distribution levels are referred to as *multi-echelon* facility location models. For example a two-echelon system involves the shipment of products from plants to the DCs and then from DCs to the retailers. In this case, the problem involves the location and number of plants and DCs, plant-DC and DC-retailer assignments (see Kaufman, Eede, and Hansen (1977)). A natural extension that researchers have considered is the *multi-commodity* location/allocation model (see Elson 1972, Pirkul 1998). Some researchers also introduced a dynamic aspect into the problem by considering a time horizon and defining location variables for each time period. These models allow the facilities to be established in a time-staged manner. Recently, the focus of research in this area has been on developing a general framework for location/allocation models. Pirkul and Vaidyanathan (1998) developed a capacitated location model that considers a two-echelon system with multiple plants and multiple commodities. Hinojosa, Puerto, and Fenandez (2000) developed a capacitated dynamic location model for a two-echelon system with multiple commodities. Much of the location theory research in the last decade has focused on the capacitated facility location problem and its extensions.

Recently, several models that combine different elements of a logistics system have been introduced. Federgruen and Zipkin (1984), Viswanathan and Mathur (1997), Chan, Federgruen and Simchi-Levi (1995) studied models that combine inventory management and routing decisions. Laporte and Dejax (1989), Berman, Jaillet and Simchi-Levi (1995) and Berger, Coullard, and Daskin (1998) studied models that combine location and routing decisions. A number of integrated location-inventory models have also appeared recently. Barahona and Jensen (1998) solved a location model with a fixed inventory cost through Dantzig-Wolfe decomposition. Erlebacher and Meller (2000) formulated a location-inventory model with highly non-linear objective function but limited computational success was achieved. Teo, Ou, and Goh (2001) developed a $\sqrt{2}$-approximation algorithm for a location model that considers inventory costs but ignores transportation costs.

Shen (2000), Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002) developed the location model with risk pooling (LMRP) that we present in the next section. The authors aim to capture the "risk pooling effects" in this model. Eppen (1979) studied the effects of grouping retailers on inventory costs. Eppen showed that when each of the retailers faces independent Normal demands with mean μ_i and standard deviation σ_i^2, the expected total inventory costs are significantly less in the centralized mode than in the decentralized mode. Specifically, when the demands are independent, in the decentralized mode the expected inventory cost for N retailers is proportional to $\sum_{i=1}^{N} \sigma_i$ whereas in the centralized mode it is proportional to $\sqrt{\sum_{i=1}^{N} \sigma_i^2}$. The LMRP makes use of this result by keeping safety stock for the retailers at their corresponding designated DCs.

3 The Uncapacitated Facility Location Model with Risk Pooling

3.1 Model Formulation and Properties

In this Section, we introduce the location model with risk pooling (LMRP) proposed by Shen (2000), Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002). The model deals with the storage and movement of a single product from a single plant to a set I of retailers through a set J of candidate distribution centers (DCs). Hence, the model represents a two-echelon logistics system. The retailers (customers) face independent, Normal random demands. The DCs hold two types of inventory: the working inventory which depends on the inventory ordering policy adopted, and the safety stock which is maintained to protect the system against possible stockouts during the replenishment lead time. The models assume each DC orders inventory from the plant using an economic order quantity (EOQ) model, which is an approximation to the (Q,r) model with Type I service (Hopp and Spearman (1996), Nahmias (1997)). Note that in this case, the optimal order quantity depends on the mean demand served by the DC, which is a function of DC-retailer assignments. A fixed location cost, f_j, is incurred to establish a DC j. Transportation costs are incurred for the shipment of the product from the plant to the DCs and then from the DCs to the retailers. In short, the LMRP extends the uncapacitated fixed charge problem to incorporate inventory decisions at the DCs.

The LMRP simultaneously determines DC locations, shipment sizes and frequencies from the plant to the DCs, the working inventory and safety stock levels at the DCs and the assignment of retailers to the DCs, to minimize the fixed facility location costs, transportation costs and inventory carrying costs. To model this problem, they define the following notation:

Inputs and Parameters:

Demand Side

μ_i: mean of daily demand at retailer i, for each $i \in I$

σ_i^2: variance of daily demand at retailer i, for each $i \in I$

Costs

f_j: fixed cost of locating a DC at retailer j, for each $j \in J$

F_j: fixed cost of placing an order at DC j, for each $j \in J$

g_j: fixed cost per shipment from the plant to DC j, for each $j \in J$

a_j: per-unit shipment cost from the plant to DC j, for each $j \in J$

h: inventory holding cost per unit of product per year

d_{ij}: per-unit cost to ship from DC j to retailer i, for each $i \in I$ and $j \in J$

Weights

β: weight factor associated with the transportation cost

θ: weight factor associated with the inventory cost

Other Parameters

z_α: standard Normal deviate such that $P(z \leq z_\alpha) = \alpha$

L: lead time from supplier to DCs, in days

χ: number of days in a year

Decision Variables:

X_j :=1, if we locate at candidate DC j,and 0, otherwise, for each $j \in J$

Y_{ij} :=1, if retailer i is served by DC j, and 0, otherwise, for each $i \in I$ and $j \in J$

Using this notation, Shen (2000) formulated the LMRP as follows:

$$\text{Minimize} \quad \sum_{j \in J} f_j X_j + \beta\chi \sum_{j \in J}\sum_{i \in I}(d_{ij} + a_j)\mu_i Y_{ij}$$
$$\sum_{j \in J} \sqrt{2\theta h(F_j + \beta g_j)\chi \sum_{i \in I} \mu_i Y_{ij}} + \theta h z_\alpha \sum_{j \in J} \sqrt{L \sum_{i \in I} \sigma_i^2 Y_{ij}} \qquad (1)$$

$$\text{subject to} \quad \sum_{j \in J} Y_{ij} = 1, \text{for each } i \in I \qquad (2)$$

$$Y_{ij} - X_j \leq 0, \text{for each } i \in I, j \in J \qquad (3)$$

$$Y_{ij} \in \{0,1\}, \text{for each } i \in I, j \in J \qquad (4)$$

$$X_j \in \{0,1\}, \text{for each } j \in J \qquad (5)$$

The objective function (1) minimizes the weighted sum of four costs: the fixed cost of locating DCs, the shipping cost from the DCs to the retailers, the expected working inventory cost and the safety stock costs. Constraints (2) require that each retailer is assigned to exactly one distribution center. Constraints (3) state that retailers can only be assigned to the DCs that are located. Constraints (4)and (5) are standard integrality constraints.

To simplify the notation in the objective function, they define the following notation:

$$\hat{d_{ij}} = \beta \chi \mu_i (d_{ij} + a_j)$$

$$K_j = \sqrt{2\theta h \chi (F_j + \beta g_j)}$$

$$q = \theta h z_\alpha \sqrt{L}$$

Then the objective function (1) becomes:

$$\sum_{j \in J} \left[f_j X_j + \sum_{i \in I} \hat{d_{ij}} Y_{ij} + K_j \sqrt{\sum_{i \in I} \mu_i Y_{ij}} + q \sqrt{\sum_{i \in I} \sigma_i^2 Y_{ij}} \right] \tag{6}$$

Several aspects of this model are worth commenting on. First, the optimal order quantity does not appear explicity as a variable in the model. However, once the optimal location and assignment variables are found, the optimal order quantity can be easily computed as shown in Shen (2000). This optimal inventory cost and the inventory policy given the DC locations is embedded in the objective function. In other words, the LMRP is capable of optimizing inventory management endogenously. Another aspect of this model is that it is structurally identical to the classical Uncapacitated Facility Location Model (UFLP;Balinski, 1965), when $\theta = 0$ in (1). However, when $\theta > 0$, due to the two square terms, the standard approaches for solving UFLP can not be applied to this model. Further, it is not always optimal to assign a retailer to its closest open distribution center. Finally, it is possible for the demand of an open DC i to be served by some other open DC j in the optimal solution. See Shen (2000) for examples of these occurences.

Shen (2000), Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002) solved a special case of this model. They analyzed the case when the variance of demand is proportional to its mean. That is they made the following assumption:

For all $i \in I$, $\sigma_i^2/\mu_i = \gamma$, *for some constant* $\gamma \geq 0$. This assumption is exact when the demands are Poisson. In that case, $\gamma = 1$. Approximating a Poisson demand process by Normally distributed demands can be shown to be good for sufficiently large demand values (Montgomery, Runger, and Hubele 1998). With the assumption that demands are Poisson, the objective function (6) can be further simplified to:

$$
\begin{aligned}
&\sum_{j \in J} \left[f_j X_j + \sum_{i \in I} \hat{d_{ij}} Y_{ij} + K_j \sqrt{\sum_{i \in I} \mu_i Y_{ij}} + q \sqrt{\sum_{i \in I} \sigma_i^2 Y_{ij}} \right] \\
&= \sum_{j \in J} \left[f_j X_j + \sum_{i \in I} \hat{d_{ij}} Y_{ij} + K_j \sqrt{\sum_{i \in I} \mu_i Y_{ij}} + q \sqrt{\sum_{i \in I} \mu_i Y_{ij}} \right] \\
&= \sum_{j \in J} \left[f_j X_j + \sum_{i \in I} \hat{d_{ij}} Y_{ij} + \hat{K_j} \sqrt{\sum_{i \in I} \mu_i Y_{ij}} \right] \qquad (7)
\end{aligned}
$$

where $\hat{K_j} = K_j + q$

Hence, the number of square root terms in the objective function of LMRP reduces from two to one. Shen (2000), Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002) were able to solve this problem by modifying the standard approaches to the UFLP. Later, Jia, Teo, and Shen (2001) solved the model with the two square root terms efficiently, as well. Also Daskin, Coullard, and Shen (2002) proved that when the demands are Poisson, it is not possible to have a retailer located at an open DC served by another DC in the optimal solution. This is a desirable property as in practice the demands in the immediate vicinity of a DC will not be assigned to another DC, even if doing so is optimal. We will also assume that the variance to mean ratio is identical for all retailers for the remainder of the paper.

3.2 Solution Approach

Shen (2000) and Shen, Coullard, and Daskin (2003) presented a set-covering algorithm for the LMRP. Daskin, Coullard, and Shen (2002) presented a Lagrangian Relaxation based solution approach. The pricing problem for the column generation and the sub-problem for the Lagrangian are both equivalent to the following problem:

$$
(\mathbf{SP_j}) = \begin{cases} \text{Minimize} & \tilde{V_j} = \sum_{i \in I} b_i Z_i + \sqrt{\sum_{i \in I} c_i Z_i} \\ \text{Subject to} & Z_i \in \{0,1\},\ \forall i \in I \end{cases}
$$

$$\text{where } c_i \geq 0 \ \forall i, \text{ but some } b_i < 0.$$

This problem is solved for each facility $j \in J$ at each iteration of the algorithms. The coefficients b_i and c_i depend on the LP or Lagrangian dual variables. $\tilde{V}_j$ denotes the *benefit* of facility j and represents the contribution of having facility j open. Shen (2000) and Shen, Coullard, and Daskin (2003) showed that this non-linear integer program can be solved in O(| I |log | I |) time.

4 The Capacitated Facility Location Model with Risk Pooling

4.1 Model Formulation

The model presented in the previous section assumes that the distribution centers have infinite capacity. In practice the distribution centers have limited capacity and it is important to remove the infinite capacity assumption. Also the traditional capacitated facility location problems ignore inventory management, resulting in a more restrictive definition of capacity and thus requiring more DCs to be opened than necessary. Therefore, our purpose is to extend the LMRP to capture capacity limitations in a more complete way.

The goal of CLMRP is to determine the number and location of distribution centers, the assignment of retailers to the distribution centers, the order quantity at the distribution centers and the safety stock level, to minimize the facility location, shipment and inventory costs without exceeding the capacity limits at the DCs. The key issues in this model are that the working inventory, safety stock at the distribution center and shipment quantity from the plant to the DCs depend on the demand seen by the distribution center which is a function of the endogenously-determined DC-retailer assignments.

We assume that the inventory is managed by a (Q,r) model with type I service at each of the DCs. Hence, the DCs will order a fixed order quantity from the supplier to satisfy the demands of its customer set without exceeding the DC capacity limitations. We define the following notation:

C_j: capacity of DC j, for each $j \in J$

r_j: reorder point at DC j, for each $j \in J$

Q_j: reorder quantity at DC j, for each $j \in J$

Clearly, at any point in time the maximum inventory accumulation at a DC should not exceed the DC's capacity. Under a (Q,r) policy, DC j orders Q_j units when the inventory level goes down to r_j. Safety stock is maintained to buffer against possible stockout during the replenishment lead times. The inventory at DC j reaches its maximum when there is no demand during the lead time. This can be also seen from figure 1. Hence, we can conclude that:

1. *Max Accumulation at DC j* $= Q_j + r_j$
2. r_j = *Safety Stock + E[demand during lead time]*

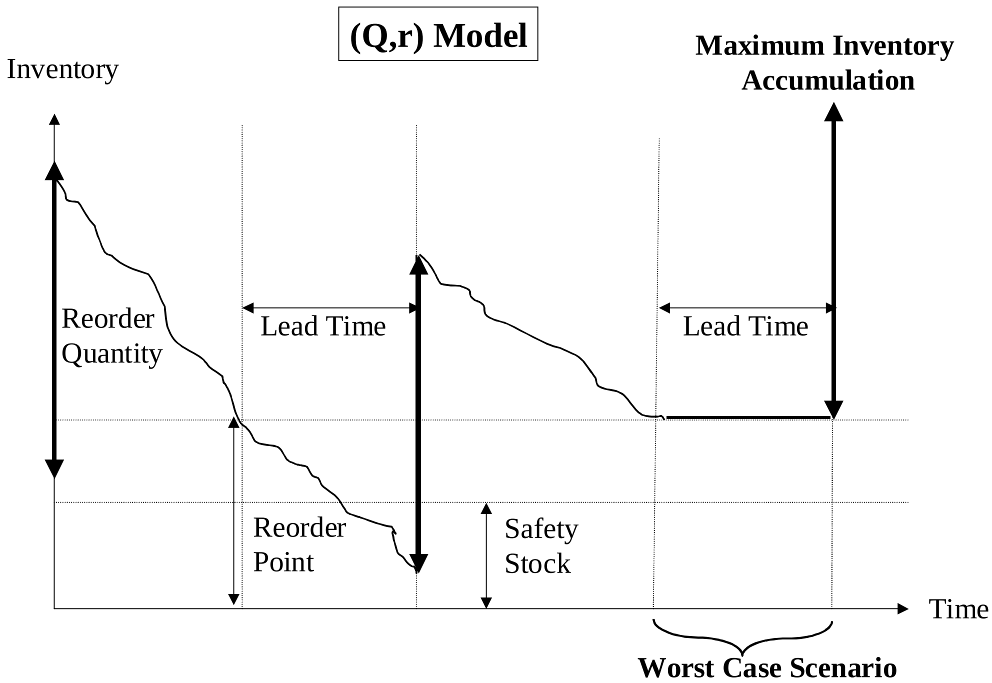

Figure 1: Evolution of Inventory over time in the (Q,r) model

However, the safety stock maintained and the expected demand during a lead time at a DC depend on the customer set assigned to the DC. To begin modeling the problem, let us assume for the moment that the DC-retailer assignments are known. Let S_j be the set of retailers assigned to DC j. And let D_j be the expected annual demand of retailers in S_j, that is: $D_j = \chi \sum_{i \in S_j} \mu_i$. Following the notation in the LMRP and assuming that the demands are Poisson, the safety stock amount for a DC is $z_\alpha \sqrt{L} \sqrt{D_j / \chi}$. Then, the capacity constraint for DC j is:

$$\begin{aligned} Q_j + r_j &\leq C_j \\ &\Leftrightarrow \\ Q_j + z_\alpha \sqrt{L} \sqrt{D_j/\chi} + L \frac{D_j}{\chi} &\leq C_j \end{aligned}$$

To minimize the working inventory costs, we need to solve the following capacitated EOQ problem for DC j, where the reorder quantity, Q_j, is the decision variable:

$$W_j^*(D_j) = \begin{cases} \text{Minimize} & F_j \frac{D_j}{Q_j} + \beta(g_j \frac{D_j}{Q_j} + a_j D_j) + \theta \frac{hQ_j}{2} \\ \text{subject to} & Q_j + z_\alpha \sqrt{L} \sqrt{D_j/\chi} + L \frac{D_j}{\chi} \leq C_j \\ & Q_j > 0 \end{cases}$$

The objective function of this EOQ model is the expected annual working inventory cost. The first term is the total fixed cost of ordering Q_j units. The second term represents the transportation cost from the plant to the DC j. The last term is the cost of holding an average of $Q_j/2$ units of inventory. Let $W_j^*(D_j)$ be the optimal total working inventory cost for DC j, that is the optimal objective value to the nonlinear program above. But, the derivation of the optimal working inventory cost for a DC j assumes that we know the set S_j of customers assigned to DC j. So, we need to solve for S_j endogeneously as well. Following the same notation for the location and assignment variables as in the LMRP, we can see that:

$$D_j = \chi \sum_{i \in S_j} \mu_i = \chi \sum_{i \in I} \mu_i Y_{ij}$$

Now, we can formulate the CLMRP as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j \in J} \left[f_j X_j + \beta\chi \sum_{i \in I} d_{ij} \mu_i Y_{ij} + z_\alpha \sqrt{L} \sqrt{\sum_{i \in I} \mu_i Y_{ij}} \right] + \\ & \sum_{j \in J} \left[(F_j + \beta g_j) \frac{\chi \sum_{i \in I} \mu_i Y_{ij}}{Q_j} + \beta\chi \sum_{i \in I} a_j \mu_i Y_{ij} + \theta \frac{hQ_j}{2} \right] \end{aligned} \tag{8}$$

$$\text{subject to} \quad \sum_{j \in J} Y_{ij} = 1, \ \forall i \in I \tag{9}$$

$$Y_{ij} - X_j \leq 0, \ \forall i \in I, j \in J \tag{10}$$

$$Q_j + \left(z_\alpha \sqrt{L} \sqrt{\sum_{i \in I} \mu_i Y_{ij}} + L \sum_{i \in I} \mu_i Y_{ij} \right) \leq C_j, \ \forall j \in J \tag{11}$$

$$Q_j > 0, \ \forall j \in J \tag{12}$$

$$Y_{ij} \in \{0, 1\}, \ \forall i \in I, j \in J \tag{13}$$

$$X_j \in \{0, 1\}, \ \forall j \in J \tag{14}$$

The objective function (8) sums the fixed cost of locating distribution centers, the DC-retailer transportation cost, the safety stock cost and the working inventory cost. However, this model has nonlinear terms in both the objective function and the constraints unlike LMRP. In addition, to obtain the optimal solution, we need to solve for the location, assignment and order quantity variables simultaneously. This situation makes the CLMRP a much harder problem to solve than the LMRP. To simplify matters, we include the $W_j^*(D_j)$ in the formulation above instead of representing the working inventory cost and the reorder quantity explicitly. Also to simplify the notation further, let $\bar{W}_j^* \left(\chi \sum_{i \in I} \mu_i Y_{ij}\right)$ denote the optimal working inventory and safety stock cost for DC j, as a function of the assigned daily demand. That is

$$\bar{W}_j^* \left(\chi \sum_{i \in I} \mu_i Y_{ij} \right) = W_j^* \left(\chi \sum_{i \in I} \mu_i Y_{ij} \right) + z_\alpha \sqrt{L} \sqrt{\sum_{i \in I} \mu_i Y_{ij}}$$

In other words, we choose to work with the following equivalent CLMRP formulation for the rest of the paper:

$$\textbf{(CLMRP)} \quad \text{Minimize} \quad \sum_{j \in J} \left[f_j X_j + \beta \chi \sum_{i \in I} d_{ij} \mu_i Y_{ij} + \bar{W}_j^* \left(\chi \sum_{i \in I} \mu_i Y_{ij} \right) \right] \tag{15}$$

$$\text{subject to} \quad \sum_{j \in J} Y_{ij} = 1, \ \forall i \in I \tag{16}$$

$$Y_{ij} - X_j \leq 0, \ \forall i \in I, j \in J \tag{17}$$

$$Y_{ij} \in \{0, 1\}, \ \forall i \in I, j \in J \tag{18}$$

$$X_j \in \{0,1\}, \ \forall j \in J \tag{19}$$

We prefer this formulation for two reasons. First, we can solve for the optimal order quantity endogeneously. Second, this formulation allows the optimal working inventory cost function, W_j^*, embedded in the objective function to be due to any inventory policy not only the one described in this section. In this sense, this formulation is more general. Also, note that if it weren't for the third and fourth terms in the objective function, the CLMRP formulation would be identical to the classical UFLP formulation.

4.2 Solution Approach

4.2.1 Obtaining a Lower Bound

As we discussed earlier, Daskin, Coullard, and Shen (2002) solved the LMRP using the Lagrangian relaxation embedded in branch and bound. The algorithm they developed is an extension of the standard Lagrangian relaxation approach for the UFLP. Fisher (1981,1985) provides an excellent discussion of Lagrangian relaxation, and for its application to the UFLP see Daskin (1995). Since CLMRP is a variant of UFLP, we choose to relax the assignment constraints and use Lagrangian relaxation embedded in branch and bound. By relaxing the assignment constraints, we obtain the following Lagrangian problem:

$$\begin{aligned} \text{Max}_\pi \ \text{Min}_{X,Y} \quad & \sum_{j\in J}\left[f_jX_j + \beta\chi\sum_{i\in I} d_{ij}\mu_iY_{ij} + \bar{W}_j^*\left(\chi\sum_{i\in I}\mu_iY_{ij}\right)\right] + \sum_{i\in I}\pi_i(1-\sum_{j\in J}Y_{ij}) \\ & = \sum_{j\in J}\left[f_jX_j + \sum_{i\in I}(\beta\chi d_{ij}\mu_i - \pi_i)Y_{ij} + \bar{W}_j^*\left(\chi\sum_{i\in I}\mu_iY_{ij}\right)\right] + \sum_{i\in I}\pi_i \end{aligned} \tag{20}$$

$$Y_{ij} - X_j \leq 0, \text{for each } i \in I, j \in J \tag{21}$$

$$Y_{ij} \in \{0,1\}, \text{for each } i \in I, j \in J \tag{22}$$

$$X_j \in \{0,1\}, \text{for each } i \in I, j \in J \tag{23}$$

For fixed values of the Lagrangian multipliers, π_i, we want to minimize (20) over the location variables, X_j, and the assignment variables, Y_{ij}. For a given π value, the problem decouples to the following subproblem for each distribution center j:

$$(\textbf{C-SP}_\textbf{j}) = \begin{cases} \text{Minimize} & V_j = f_j + \sum_{i\in I}(\beta\chi d_{ij}\mu_i - \pi_i)Y_{ij} + \bar{W}_j^*\left(\chi\sum_{i\in I}\mu_i Y_{ij}\right) \\ \text{Subject to} & Y_{ij} \in \{0,1\},\ \forall i \in I \end{cases}$$

V_j denotes the *benefit* of facility j and represents the contribution of having facility j open to the objective function (20). To obtain a lower bound for CLMRP, we set $X_j = 1$ if $V_j \leq 0$, otherwise we set $X_j = 0$. If $V_j \geq 0$ for all $j \in J$, we set $X_j = 1$ for the facility j that has the minimum V_j value. Then we set $Y_{ij} = 1$ in (20) –(23), if a) $X_j = 1$ and b) $Y_{ij} = 1$ in ($\textbf{C-SP}_\textbf{j}$) ,otherwise we set $Y_{ij} = 0$ in (20)–(23).

For Lagrangian relaxation to be effective, the subproblem needs to be solved easily. For the subproblem ($\textbf{C-SP}_\textbf{j}$), if the objective function were linear (e.g. no working inventory and safety stock cost as in UFLP), the subproblem could have been simply solved by computing, $V_j = f_j + \sum_{i\in I} min\,(0, (\beta\mu_i d_{ij} - \pi_i))$. Then to obtain a lower bound for the problem, set $X_j = 1$ for those candidate sites for which $V_j \leq 0$. The assignment variables Y_{ij} are set to 1 if $\beta\mu_i d_{ij} - \pi_i \leq 0$ and $X_j = 1$ set to 0 otherwise. However, the optimal working inventory plus the safety stock cost function, $\bar{W}_j^*(.)$, is nonlinear in terms of the assignment variables. Hence, solving the subproblem is not straightforward.

We begin the analysis of problem ($\textbf{C-SP}_\textbf{j}$) with an observation based on its LP relaxation. The LP relaxation of the subproblem, ($\textbf{C-SP}_\textbf{j}$) is:

$$(\textbf{C-SP}_\textbf{j}^\textbf{LP}) = \begin{cases} \text{Minimize} & V_j^{LP} = f_j + \sum_{i\in I} A_i Y_{ij} + \bar{W}_j^*\left(\chi\sum_{i\in I}\mu_i Y_{ij}\right) \\ \text{Subject to} & 0 \leq Y_{ij} \leq 1, \text{for each } i \in I \end{cases}$$

where

$$A_i = \beta\chi\mu_i d_{ij} - \pi_i$$

Lemma 1: Assume that Y_j^* is an optimal solution to ($\textbf{C-SP}_\textbf{j}^\textbf{LP}$). If $Y_{ij}^* > 0$, then $A_i \leq 0$.

Proof: Assume that there exists a $k \in I$ such that $Y_{kj}^* > 0$ and $A_k > 0$ for subproblem ($\textbf{C-SP}_\textbf{j}^\textbf{LP}$). Since $\bar{W}_j^*\left(\chi\sum_{i\in I}\mu_i Y_{ij}\right)$ is monotone non-decreasing (see Appendix), the objective function of the solution $Y_j^{'}$ obtained by setting $Y_{ij}^{'} = Y_{ij}^*$ for each $i \in I\backslash\{k\}$ and $Y_{kj}^{'} = 0$ is strictly less than the objective function value of Y_j^*. Hence Y_j^* is not optimal, contradiction. □

Due to Lemma 1, to find the set of retailers that will be assigned to DC j in the optimal solution to the subproblem ($\mathbf{C\text{-}SP_j^{LP}}$), we can restrict our attention to the retailers $I^- = \{i \in I : b_i \le 0\}$. Now assume that the subset of retailers I^- has been sorted such that:

$$\frac{A_1}{\mu_1} \le \frac{A_2}{\mu_2} \le \cdots \le \frac{A_n}{\mu_n}$$

where $n = |I^-|$.

Theorem 1: There is an optimal solution, Y_j^* to ($\mathbf{C\text{-}SP_j^{LP}}$) with the following three properties:

1. $Y_{ij}^* = 0$ for all $i \in I \setminus I^-$;
2. At most one of the assignment variables Y_{ij}, take on a fractional value;
3. If $Y_{kj}^* > 0$, for some $k \in 1, \ldots, n$, then $Y_{lj}^* = 1$, for all $l \in \{1, \ldots, k-1\}$.

Proof: Property 1 follows from Lemma 1.

We prove Property 2 using an interchange argument. Let Y_j^* be an optimal solution vector to the subproblem, ($\mathbf{C\text{-}SP_j^{LP}}$) that satisfies Property 1 and contains two or more fractional values. Let $Y_{i_1 j}^*$ and $Y_{i_2 j}^*$ be the two assignment variables that take on fractional values in the optimal solution. Then the objective function value of the solution Y_j^* can be computed as follows:

$$Z_j^* = f_j + \sum_{i \in I^- \setminus \{i_1, i_2\}} A_i Y_{ij}^* + A_{i_1} Y_{i_1 j}^* + A_{i_2} Y_{i_2 j}^* + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{i_1, i_2\}} \mu_i Y_{ij}^* + \mu_{i_1} Y_{i_1 j}^* + \mu_{i_2} Y_{i_2 j}^* \right] \right)$$

Now, let us define a new solution, $Y_j^{'}$ to the subproblem, ($\mathbf{C\text{-}SP_j^{LP}}$), as follows:

$$Y_j^{'} = \begin{cases} Y_{ij}^* & \forall i \in I \setminus \{i_1, i_2\} \\ Y_{i_1 j}^{'} & \text{if } i = i_1 \\ Y_{i_2 j}^{'} & \text{if } i = i_2 \end{cases}$$

where

$Y_{i_1 j}^{'} = Y_{i_1 j}^* + \epsilon$ and $Y_{i_2 j}^{'} = Y_{i_2 j}^* - \frac{\mu_{i_1}}{\mu_{i_2}} \epsilon$ for some $\epsilon > 0$, and i_1 is ranked before i_2.

For $Y_j^{'}$ to be a feasible solution to the subproblem, ($\mathbf{C\text{-}SP_j^{LP}}$), we need to have:

1. $Y_{i_1j}^{'} = Y_{i_1j}^{*} + \epsilon \leq 1 \rightarrow \epsilon \leq 1 - Y_{i_1j}^{*}$

2. $Y_{i_2j}^{'} = Y_{i_2j}^{*} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon \geq 0 \rightarrow \epsilon \leq \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})$

Let us set $\epsilon = \min\{1 - Y_{i_1j}^{*}, \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})\}$. Then $Y_j^{'}$ is a feasible solution to the subproblem, ($\mathbf{C\text{-}SP_j^{LP}}$). And the objective function value of the solution $Y_j^{'}$ is:

$$Z_j^{'} = f_j + \sum_{i \in I \setminus \{i_1, i_2\}} A_i Y_{ij}^{*} + A_{i_1}(Y_{i_1j}^{*} + \epsilon) + A_{i_2}(Y_{i_2j}^{*} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon)$$
$$+ \bar{W}_j^{*} \left(\chi \left[\sum_{i \in I \setminus \{i_1, i_2\}} \mu_i Y_{ij}^{*} + \mu_{i_1}(Y_{i_1j}^{*} + \epsilon) + \mu_{i_2}(Y_{i_2j}^{*} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon) \right] \right)$$

$$= f_j + \sum_{i \in I \setminus \{i_1, i_2\}} A_i Y_{ij}^{*} + A_{i_1} Y_{i_1j}^{*} + A_{i_2} Y_{i_2j}^{*} + \epsilon(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2})$$
$$+ \bar{W}_j^{*} \left(\chi \left[\sum_{i \in I \setminus \{i_1, i_2\}} \mu_i Y_{ij}^{*} + \mu_{i_1} Y_{i_1j}^{*} + \epsilon\mu_{i_1} + \mu_{i_2} Y_{i_2j}^{*} - \epsilon\mu_{i_1} \right] \right)$$

Since i_1 is ranked before i_2, we have $\frac{A_{i_1}}{\mu_{i_1}} \leq \frac{A_{i_2}}{\mu_{i_2}}$. Hence, $\epsilon(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2}) \leq 0$ and we have:

$$Z_j^{'} = f_j + \sum_{i \in I \setminus \{i_1, i_2\}} A_i Y_{ij}^{*} + A_{i_1} Y_{i_1j}^{*} + A_{i_2} Y_{i_2j}^{*} + \epsilon(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2})$$
$$+ \bar{W}_j^{*} \left(\chi \left[\sum_{i \in I \setminus \{i_1, i_2\}} \mu_i Y_{ij}^{*} + \mu_{i_1} Y_{i_1j}^{*} + \mu_{i_2} Y_{i_2j}^{*} \right] \right) \leq Z_j^{*}$$

Hence the solution $Y_j^{'}$ is optimal. Now consider the following three possible values that ϵ can take on based on its definition:

- If $(1 - Y_{i_1j}^{*}) < \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})$, then $\epsilon = 1 - Y_{i_1j}^{*}$ and $Y_{i_1j}^{'} = 1$ and $0 < Y_{i_2j}^{'} < 1$
- If $(1 - Y_{i_1j}^{*}) > \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})$, then $\epsilon = \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})$ and $0 < Y_{i_1j}^{'} < 1$ and $Y_{i_2j}^{'} = 0$
- If $(1 - Y_{i_1j}^{*}) = \frac{\mu_{i_2}}{\mu_{i_1}}(Y_{i_2j}^{*})$, then $Y_{i_1j}^{'} = 1$ and $Y_{i_2j}^{'} = 0$

In each case, we have reduced the number of fractional variables by at least one without degrading the objective function value. Note that we may actually improve the objective function value, hereby disproving the assumed optimality of Y_j^{*}. This argument can be repeated until at most one fractional value is included in the solution.

Property 3 follows from a similar interchange argument. □

Note that to prove Theorem 1, we used the fact that optimal working inventory and safety stock cost function, $\bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}\right)$ is monotone non-decreasing. However, it should be noted that properties 2 and 3 hold for any optimal working inventory cost function and safety stocj cost function. To develop an algorithm to solve the subproblem, first we need to be able to determine if there is an assignment variable with fractional optimal value. And if there is one, we need to be able to compute its optimal value. To this end, we prove the following theorem:

Theorem 2: Assume that Y_j^* is an optimal solution to ($\mathbf{C\text{-}SP_j^{LP}}$). If there is an assignment variable Y_{kj}^* that takes on a fractional value, then for that retailer k the following condition holds:

$$A_k = -\frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial Y_{kj}^*}$$

Proof: The Lagrangian function for the problem ($\mathbf{C\text{-}SP_j^{LP}}$) is:

$$L_j(Y,u,v) = f_j + \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}\right) + \sum_{i\in I} A_i Y_{ij} - \sum_{i\in I} u_i Y_{ij} + \sum_{i\in I} v_i (Y_{ij} - 1) \qquad (24)$$

Then the KKT conditions for the problem ($\mathbf{C\text{-}SP_j^{LP}}$) is:

- The Gradient Conditions:

$$\frac{\partial L_j(Y,u,v)}{\partial Y_{ij}} = \frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}\right)}{\partial Y_{ij}} + A_i - u_i + v_i = 0, \text{for each } i \in I \qquad (25)$$

- The Orthogonality Conditions:

$$u_i Y_{ij} = 0, \text{for each } i \in I \qquad (26)$$

$$v_i (Y_{ij} - 1) = 0, \text{for each } i \in I \qquad (27)$$

- The Feasibility Conditions:

$$0 \le Y_{ij} \le 1, \text{for each } i \in I \qquad (28)$$

- The Nonnegativity Conditions:

$$u_i, v_i \geq 0, \text{for each } i \in I \tag{29}$$

If $0 < Y^*_{kj} < 1$, then due to equations (26) and (27) we have $u_k = v_k = 0$. Then from equations (25), we have

$$A_k = -\frac{\partial \bar{W}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right)}{\partial Y^*_{kj}}$$

as desired. □

Theorem 2 basically states that the marginal inventory cost of including retailer i to the customer set of DC j should equal to the shipment cost from DCj to retailer i. Now to simplify the notation, we can rewrite this condition as follows:

By chain rule: $$\frac{\partial \bar{W}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right)}{\partial Y^*_{kj}} = \frac{\partial \bar{W}^*_j (\chi \sum_{i \in I} \mu_i Y^*_{ij})}{\partial (\chi \sum_{i \in I} \mu_i Y^*_{ij})} * \frac{\partial (\chi \sum_{i \in I} \mu_i Y^*_{ij})}{\partial Y^*_{kj}}$$

By letting $$\bar{E}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right) = \frac{\partial \bar{W}^*_j (\chi \sum_{i \in I} \mu_i Y^*_{ij})}{\partial (\chi \sum_{i \in I} \mu_i Y^*_{ij})}$$

Then we have $$\frac{\partial \bar{W}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right)}{\partial Y^*_{kj}} = \bar{E}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right) \chi \mu_k$$

Hence, if there is an assignment, Y_{ij}, that takes on a fractional value in the optimal solution, the following condition holds for that retailer i:

$$A_k + \chi \mu_k \bar{E}^*_j \left(\chi \sum_{i \in I} \mu_i Y^*_{ij}\right) = 0 \tag{30}$$

Note that theorem 2 assumes that the expected optimal working inventory and safety stock cost is differentiable which we show in the appendix. Based on these two theorems, we propose the following algorithm to solve the problem ($\mathbf{C\text{-}SP^{LP}_j}$):

Algorithm 1.

Step 1: Partition I into two sets:

$$\begin{aligned} I^+ &= \{i : A_i \geq 0\} \\ I^- &= \{i : A_i < 0\} \end{aligned}$$

Step 2: Sort the elements of I^- such that

$$\frac{A_1}{\mu_1} \leq \frac{A_2}{\mu_2} \leq \ldots \leq \frac{A_n}{\mu_n}$$

where $n = |I^-|$.

Use this ordering of the elements of $|I^-|$ in the following steps.

Step 3: Let $D_k = \chi \sum_{i=1}^{k} \mu_i$. For $k = 0, \ldots, n$:

Define set $\Delta_k = \left\{ \hat{D}_k \in \Re \mid A_k + \chi\mu_k \bar{E}^*(\hat{D}_k) = 0 \text{ and } D_{k-1} < \hat{D}_k < D_k \right\}$

That is we need to determine whether Y_{kj} can take on a fractional value.

case 1. If set $\Delta_k = \emptyset$ compute the partial sum as follows:

$$S_k = \sum_{i=1}^{k} A_i + \bar{W}^*(D_k)$$

So, if Y_{kj} can not take on a fractional value, then we include retailer k's total demand.

case 2. If set $\Delta_k \neq \emptyset$ then :

$$S_k = \text{Min} \left\{ \hat{S}_k, \sum_{i=1}^{k} A_i + \bar{W}^*(D_k) \right\}$$

where the first term $\hat{S}_k = \text{Min}_{\hat{D}_k \in \Delta_k} \left\{ \sum_{i=1}^{k-1} A_i + A_k(\hat{D}_k - D_{k-1})/\chi\mu_k + \bar{W}^*(\hat{D}_k) \right\}$.

Record the D_k^* that gives the S_k value.

So if Y_{kj} can take a fractional value, then we can choose to serve a fraction of retailer k's demand that would increase the total cost the least. The second term is due to integral solution which results from serving the total demand of retailer k rather than a fractional amount. We take the minimum of these two terms to minimize the cost.

Step 4: Let k^* be the value of k that gives the minimum S_k value. Due to theorem 1:

$$Y_{ij} = \begin{cases} 0, & \text{if } i \in I^+ \\ 1, & \text{if } i \in I^- \text{and } i < k^* \end{cases}$$

And due to theorem 2:

For $i \in I^-$ and $i = k^*$, $Y_{ij} = (D_{k^*}^* - D_{k^*-1})/\chi\mu_{k^*}$ (Note that $Y_{ij} = 1$ if $S_{k^*} \neq \hat{S}_{k^*}$)

Note that Algorithm 1 would work not only for $\bar{W}^*(D)$ due to the capacitated EOQ model that we discussed in the previous section, but for any *differentiable, monotone non-decreasing,* expected total working inventory cost function. It should be noted that with minor modification to Algorithm 1, we could solve the subproblem for any differentiable expected total working inventory cost function. This algorithm would take $O(\mid I \mid log(\mid I \mid)$ if step 3 did not involve finding the "zeros" of a function. Therefore, the efficiency of the algorithm depends on the behavior of the $\bar{W}^*(D)$ function. This algorithm is similar to the one developed in Shen (2000). However, the algorithm in Shen (2000) requires the optimal working inventory cost function to be concave whereas $\bar{W}^*(D)$ is not.

Using Algorithm 1, we can solve the subproblem ($\mathbf{C}$-$\mathbf{SP_j^{LP}}$) and compute the benefit of a facility, $V_j^{LP} = f_j + S_{k^*}$. The solution of ($\mathbf{C}$-$\mathbf{SP_j^{LP}}$) provides a lower bound solution for ($\mathbf{C}$-$\mathbf{SP_j}$) and hence can be used to find a lower bound solution for the original problem CLMRP. We set $X_j = 1$ if $V_j^{LP} \leq 0$, otherwise we set $X_j = 0$. If $V_j^{LP} \geq 0$ for all $j \in J$, we set $X_j = 1$ for the facility j that has the minimum V_j^{LP} value. Then if $X_j = 1$, we set Y_{ij} to the appropriate values in (20)–(23) as outlined in step 4 of algorithm ,otherwise we set $Y_{ij} = 0$ in (20)–(23)

However, the resulting lower bound solution due to ($\mathbf{C}$-$\mathbf{SP_j^{LP}}$) is not guaranteed to provide a tight bound for the CLMRP. Hence, we apply branch and bound on the assignment variables. We have shown in theorem 1 that at most one assignment variable can take on a fractional value in the optimal solution of ($\mathbf{C}$-$\mathbf{SP_j^{LP}}$). So at each node of the branch bound tree, we branch on the retailer with the fractional assignment value, if there is one. At every stage of this procedure, a retailer is first forced to be in the customer set served by the DC and then forced out of the customer set. Branching is done in a depth-first manner. We are able to decrease the number of branch and bound nodes due to the following observation.

Theorem 3: Assume that the subset of retailers $I^- = \{i \in I : A_i \leq 0\}$ has been sorted such that:

$$\frac{A_1}{\mu_1} \leq \frac{A_2}{\mu_2} \leq \cdots \leq \frac{A_m}{\mu_m}$$

If retailer k is forced out at node t of the branch and bound tree, i.e. $Y_{kj} = 0$ and for some $l > k$ if $A_l > A_k$, then retailer l can be also forced out at node t of the branch and bound tree.

Proof: Assume that we are processing node t of the branch and bound tree and that the constraint $Y_{kj} = 0$ is included in the current subproblem. Let $l > k$ having $A_l > A_k$. We will show it is valid to add the constraint $Y_{lj} = 0$ to node t and the nodes below node t. It suffices to show that no optimal solution at or below node t violates this constraint.

Let Y_j^* be an optimal solution vector to the IP (**C-SP$_\mathbf{j}$**). If $Y_{kj}^* = 1$, then we are finished, since this optimal solution is neither at or below node t. Assume $Y_{kj}^* = 0$. We will show $Y_{lj}^* = 0$ by contradiction. Suppose $Y_{lj}^* = 1$. Then the objective function value of the solution Y_j^* can be computed as follows:

$$Z_j^* = f_j + \sum_{i \in I^- \setminus \{k,l\}} A_i Y_{ij}^* + A_k Y_{kj}^* + A_l Y_{lj}^* + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_k Y_{kj}^* + \mu_l Y_{lj}^* \right] \right)$$

$$= f_j + \sum_{i \in I \setminus \{k,l\}} A_i Y_{ij}^* + A_l + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_l \right] \right)$$

Now, let us define a new solution, $Y_j^{'}$ to the subproblem, (**C-SP$_\mathbf{j}$**), as follows:

$$Y_j^{'} = \begin{cases} Y_{ij}^* & \forall i \in I \setminus \{k,l\} \\ Y_{kj}^{'} = 1 & \\ Y_{lj}^{'} = 0 & \end{cases}$$

And the objective function value of the solution $Y_j^{'}$ is:

$$Z_j^{'} = f_j + \sum_{i \in I \setminus \{k,l\}} A_i Y_{ij}^{'} + A_k Y_{kj}^{'} + A_l Y_{lj}^{'} + \bar{W}_j^{'} \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^{'} + \mu_k Y_{kj}^{'} + \mu_l Y_{lj}^{'} \right] \right)$$

$$= f_j + \sum_{i \in I \setminus \{k,l\}} A_i Y_{ij}^* + A_k + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_k \right] \right)$$

$$Z_j^{'} - Z_j^* = A_k - A_l + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_k \right] \right) - \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_l \right] \right)$$

Note that since $0 \geq A_l > A_k$ and $\frac{A_k}{\mu_k} \leq \frac{A_l}{\mu_l}$, we have $\mu_k < \mu_l$ and since $\bar{W}_j^*()$ is a monotone non-decreasing function (see Appendix), then $Z_j^{'} - Z_j^* < 0$. Hence, Y_j^* is not optimal, contradiction.

Also note that when we force out Y_{kj} at some node t of the branch and bound tree and if this causes $Y_{lj} = 0$ at node t but Y_{lj} was forced in a parent node of t, we can prune node t.

4.2.2 Obtaining an Upper Bound

At each iteration of the Lagrangian procedure, we use the current lower bound solution (X', Y') to obtain a feasible solution for CLMRP. First we open a DC j only if $X_j' = 1$ in the current Lagrangian solution. Then we sort the retailers in decreasing order of mean demand, μ_i, and loop through retailers based on this order. We first process the retailers with $\sum_{j \in J} Y_{ij}' > 0$ and assign retailer i to DC j with $Y_{ij}' > 0$ that increases the total cost the least, based on the assignments made so far. Next, we process the retailers with $\sum_{j \in J} Y_{ij} = 0$ and assign retailer i to any open DC that increases the total cost the least. If at the end of this process, there are DCs with no retailers assigned to them then we close those DCs. It is also possible to have some retailers not assigned to any DC if there are not enough open DCs with enough capacity in the Lagrangian solution. However, this should be very rare since we just order more often. One such occurrence is possible when the total safety stock exceeds the total capacity. In that case, we set the upper bound to infinity at that iteration. If all the retailers are assigned to some DC, then the resulting solution is feasible for CLMRP and provides an upper bound on the objective function (15).

If the value of the upper bound that we compute using the procedure outlined above is less than the best known upper bound, we apply a retailer reassignment heuristic to it. This heuristic is essentially the same to that described in Daskin, Coullard, and Shen (2002). The heuristic reassigns a retailer from its currently assigned facility to another open DC if this movement leads to a decrease in the total objective function value. We also implemented a retailer swap heuristic. This heuristic swaps two retailers served by different DCs if this movement leads to a decrease in the total objective function value.

However, mostly neither of these two heuristics improved the objective function value significantly and occassionaly it deteriorated the objective function value. Therefore, we turned off these two improvement heuristics when obtaining the results that are reported in the next section.

At the end of the Lagrangian procedure, if the lower bound is strictly less than the upper bound obtained, we apply a DC-exchange heuristic which is also described in Daskin, Coullard, and Shen (2002). The heuristic swaps a DC currently open in the solution with another DC that is not currently in the solution, if doing so improves the solution. This procedure is a variant of Teitz and Bart's (1968) procedure for the P-median problem. We observed that from time to time this heuristic improves the objective function value significantly.

4.2.3 Variable Fixing

After performing the DC exchange heuristic, if the lower bound is not equal to the upper bound obtained within some pre-specified tolerance, then we apply a variable fixing technique at the root node of the branch and bound tree (for a general description of this technique and its application, see Wolsey, 1998). For the current set of Lagrangian multipliers, π, let LB denote the lower bound and let V_j be the DC benefit of DC j. Also let UB denote the current upper bound. Then the two following rules can be used to fix the location variables:

1. If $X_j = 0$ in the current solution to the Lagrangian problem using the current multipliers π and if $LB + V_j > UB$ then DC j can not be part of the optimal solution. Hence, we can fix $X_j = 0$.

2. If $X_j = 1$ in the current solution to the Lagrangian problem using the current multipliers π and if $LB - V_j > UB$ then DC j must be part of the optimal solution. Hence, we can fix $X_j = 1$.

These rules can be applied only if we have computed V_j for each DC j. However, we might choose not to always branch on the assignment variables. In that case, we would instead have V_j^{LP} values for each DC j. Then we need to follow these modified rules:

1. If $X_j = 0$ in the current solution to the Lagrangian problem using the current multipliers π and if $LB + V_j^{LP} > UB$ then we also have $LB + V_j > UB$ since $V_j^{LP} <= V_j$. So, DC j can not be part of the optimal solution as before. Hence, we can fix $X_j = 0$.

2. If $X_j = 1$ in the current solution to the Lagrangian problem using the current multipliers π and if $LB - V_j^{LP} > UB$ then we don't necessarily have $LB - V_j > UB$ since $V_j^{LP} <= V_j$. So we need to find an upper bound value for V_j. In Algorithm 1, at step 3, we record the best partial sum that is due to the integral solution. This provides a feasible solution to the subproblem ($\mathbf{C\text{-}SP_j}$) and hence provides an upper bound, V_j^{UB} for V_j. So, if $LB - V_j^{UB} > UB$, then $LB - V_j > UB$. Hence, we can fix $X_j = 1$.

We perform the variable fixing technique at the root node using both the optimal multipliers π and the current ones.

4.2.4 Branch and Bound

At the end of the Lagrangian procedure, if the lower bound equals to the upper bound then the solution corresponding to the upper bound is optimal. If that is not the case, even after applying the variable fixing procedure, then we apply branch and bound on the location variables. At each node of the branch and bound tree, the DCs with the largest assigned demand are processed first. At every stage of this procedure, a DC is always first forced open and then closed. Branching is done in a depth-first manner. Note that branching is done only on the location variables, we do not branch on the assignment variables. Since we found the results we obtained at the root node of the branch and bound tree to be satisfactory , we only report computational results without Branch and Bound on the location variables in the next section.

5 Computational Results

In this section, we first explain the design of our experiments then outline the computational results for the CLMRP. We tested our algorithm for the CLMRP on a 15-node, a

49-node, an 88-node data and a 150-node data set. Each node represents a retail location. In all the experiments, each retail location was also a candidate DC location. The 15-node data set consists of the 15 retailers with the highest demand from the 49-node data set which is described in Daskin (1995). The 49-node data set represents the capitols of the lower 48 United States plus Washington, DC. The 88-node data set contains the 49-node data set plus the 50 largest cities in the 1990 U.S. census, and the 150-node data set contains the 150 largest cities in the 1990 U.S. census as described in Daskin (1995).

For all data sets, the mean demand was obtained by dividing the population data given in Daskin (1995) by 1000. Fixed facility location costs were obtained by dividing the facility location costs in Daskin (1995) by 100. In all the experiments, we set the unit cost of shipping from candidate DC j to retailer i, d_{ij}, to the great circle distance between these locations. The fixed ordering F_j and shipping g_j costs were set to 10 and the variable shipping cost a_j was set to 5 for all DCs. The lead time, L, and the days per year χ were set to 1. Note that although $\chi = 1$, the difference between the daily parameters and yearly parameters are realized through the weights β and θ. We have tested different values of the DC capacity. To define capacities, initially we set the capacity of the DC with the most assigned demand in the optimal solution to the LMRP to a lesser value than the assigned demand. For example, let DC j be the facility with the most assigned demand, $\hat{D}_j$ in the optimal solution to the test problem k. Then in the test problem $k+1$, the capacity of DC j is set to $\hat{D}_j/2$ to make the test problem $k+1$ more capacitated (harder).

The parameters for Lagrangian relaxation are given in Table 1. The notation $\bar{\mu}$ in Table 1 stands for the average mean demand across all retailers. We terminated the Lagrangian procedure based on the optimality gap, or the maximum number of iterations allowed or the minimum value of alpha, which ever occurs first. Times obtained for our algorithm are on a Dell Inspiron running at 1.7 GHz using Windows XP. The program was written in C++. The computational results presented does not include Branch and Bound on the X_j variables.

Parameter	*Value*
Maximum number of iterations at root node	3600
Maximum number of iterations at other nodes	1200
Number of number of iterations before halving α at root node	36
Number of number of iterations before halving α	12
Initial value of α	2
Minimum value of α	0.00000001
Minimum LB-UB gap	0.001 %
Initial value for π_i	$10\bar{\mu} + 10f_i$

Table 1: Parameters for Lagrangian relaxation procedure

The columns in the following tables are as follows:

Prob# Problem Number

#**Ret** The number of retailers in that problem.

Last Capacitated DC The DC that was capacitated last.

Max Assigned Demand The largest demand that is assigned to a single DC in that problem solution.

DC w/ MaxDemand The DC that is assigned the largest amount of demand in that problem solution.

Solution The DCs that are located in the optimal solution to that problem.

Root LB The best lower bound obtained during the Lagrangian process at the root node.

Root UB The objective value of the best feasible solution found during the Lagrangian process at the root node

Root Gap The percentage difference between the upper bound and the lower bound at the root node.

Lag Iter The total number of Lagrangian relaxation iterations performed during the algorithm.

Prob #	# Rets	Last Capacitated DC	Max Demand assigned	DC w/ Max Demand	Solution	Root LB	Root UB	% Gap	Lag Iter	Sub Iter	CPU Time	stopping reason
1	15	None	29,760,021	1	1,2,3,4,5,6,7,8,9,10,11,13,14,15	1,782,920	1,782,920	0.0000	4	0	0	gap
2	15	1	46,746,531	3	2,3,4,5,6,7,8,9,10,11,13,14,15	19,069,501	19,069,501	0.0000	29	45	1	gap
3	15	3	34,877,094	15	2,3,4,5,6,7,8,9,10,11,13,14,15	20,415,087	20,415,087	0.0000	26	57	1	gap
4	15	15	41,190,623	6	2,3,4,5,6,7,8,9,10,11,13,14,15	21,878,997	21,878,997	0.0000	24	89	1	gap
5	15	6	35,304,180	14	2,3,4,5,6,7,8,9,10,11,13,14,15	24,069,591	24,069,591	0.0000	33	233	2	gap
6	15	14	39,055,318	8	2,3,4,5,6,7,8,9,10,11,13,14,15	24,771,747	24,771,747	0.0000	21	97	1	gap
7	15	8	40,607,136	7	2,3,4,5,6,7,8,9,10,11,13,14,15	26,009,707	26,009,707	0.0000	33	334	3	gap
8	15	7	36,238,237	11	2,3,4,5,6,7,8,9,10,11,13,14,15	26,414,628	26,414,628	0.0000	31	355	3	gap
9	15	11	42,697,947	4	2,3,4,5,6,7,8,9,10,11,13,14,15	27,545,258	27,545,258	0.0000	22	233	2	gap
10	15	4	42,576,016	10	2,3,4,5,6,7,8,9,10,11,13,14,15	29,557,039	29,557,039	0.0000	22	317	3	gap
11	49	None	30,961,854	1	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,41,43,44	3,764,452	3,764,452	0.0000	41	0	2	gap
12	49	1	30,961,854	39	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	4,901,703	4,901,703	0.0000	71	165	3	gap
13	49	39	30,766,770	41	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	8,961,005	8,961,005	0.0000	111	555	7	gap
14	49	41	32,602,342	29	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	9,020,282	9,020,282	0.0000	93	555	7	gap
15	49	29	31,482,871	35	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	9,995,474	9,996,539	0.0107	1111	12,462	118	alpha
16	49	35	34,626,713	18	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	10,722,394	10,722,394	0.0000	91	816	9	gap
17	49	18	33,425,249	24	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	11,176,085	11,186,739	0.0953	1077	17,598	162	alpha
18	49	24	30,559,086	43	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	12,424,659	12,425,114	0.0037	1148	21,945	200	alpha
19	49	43	31,275,090	37	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	14,138,212	14,139,164	0.0067	1087	24,483	219	alpha
20	49	37	33,508,003	26	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,28,29,30,31,32,33,34,35,36,37,39,41,43,44	14,317,486	14,317,486	0.0000	174	3,300	30	gap

Prob #	# Rets	Last Capacitated DC	Max Demand assigned	DC w/ Max Demand	Solution	Root LB	Root UB	% Gap	Lag Iter	Sub Iter	CPU Time	stopping reason
21	88	None	8,363,974	1	1,3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67	2,741,816	2,741,843	0.0010	400	0	10	gap
22	88	1	10,038,226	72	3,4,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67,72	2,765,159	2,765,198	0.0014	1,195	3,443	50	alpha
23	88	72	10,038,226	5	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67	2,811,730	2,811,730	0.0000	354	1,974	27	gap
24	88	5	8,363,974	63	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,63,67	2,940,879	2,940,904	0.0008	355	3,144	37	gap
25	88	63	8,363,974	70	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67,70	3,034,077	3,037,185	0.1024	1,277	15,148	152	alpha
26	88	70	9,530,371	83	3,4,5,7,9,10,13,15,18,22,23,24,26,28,30,32,36,41,51,67,83	3,040,466	3,041,146	0.0223	1,202	17,576	172	alpha
27	88	83	8,363,974	61	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,61,67	3,063,857	3,064,111	0.0083	1,308	23,136	221	alpha
28	88	61	7,374,940	75	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67,75	3,074,129	3,080,628	0.2114	1,308	26,489	251	alpha
29	88	75	9,582,747	12	3,4,5,7,9,10,12,13,15,18,22,23,24,26,28,30,32,36,41,51,67	3,080,034	3,081,234	0.0390	1,152	26,767	265	alpha
30	88	12	9,582,747	82	3,4,5,7,9,10,13,15,18,22,23,24,26,28,30,32,36,41,51,67,82	3,189,575	3,189,939	0.0114	1,255	33,391	307	alpha
31	150	None	11,610,020	1	1,2,3,4,7,8,24,26,30,43,51,73,91,94,101,106	3,565,142	3,565,142	0.0000	490	0	20	gap
32	150	1	11,610,020	65	2,3,4,7,8,24,26,30,43,51,65,73,91,94,101,106	3,582,290	3,582,290	0.0000	596	2,063	40	gap
33	150	65	11,610,020	54	2,3,4,7,8,24,26,30,43,51,54,73,91,94,101,106	3,596,744	3,596,744	0.0000	493	3,343	49	gap
34	150	54	11,610,020	122	2,3,4,7,8,24,26,30,43,51,73,91,94,101,106,122	3,621,432	3,621,432	0.0000	395	4,142	55	gap
35	150	122	11,610,020	81	2,3,4,7,8,24,26,30,43,51,73,81,91,94,101,106	3,623,247	3,623,247	0.0000	355	4,725	64	gap
36	150	81	11,610,020	121	2,3,4,7,8,24,26,30,43,51,73,91,94,101,106,121	3,736,564	3,736,564	0.0000	445	12,936	137	gap
37	150	121	9,517,586	136	2,3,4,5,7,8,24,26,30,43,51,91,94,97,101,106,136	3,784,366	3,784,375	0.0002	502	11,017	118	gap
38	150	136	12,952,934	5	2,3,4,5,7,8,24,26,30,43,51,91,94,97,101,106	3,796,686	3,798,419	0.0456	1,179	26,298	269	alpha
39	150	5	9,148,158	125	2,3,4,5,7,8,24,26,30,43,51,91,94,97,101,106,125	3,867,473	3,867,545	0.0019	1,203	31,742	310	alpha
40	150	125	8,960,076	108	2,3,4,5,7,8,24,26,30,43,51,91,94,97,101,106,108	3,936,029	3,936,029	0.0000	687	26,232	248	gap

Table 2: θ=0.0001 and β=0.01

Sub Iter The total number of branch and bound iterations on the assignment variables performed during the algorithm.

CPU Time (sec.) The number of CPU seconds elapsed before the algorithm terminated.

stopping reason The reason for which the algorithm terminated; iterlimit means that algorithm terminated since the maximum number of iterations allowed was reached; gap means that the percentage optimality gap was less than 0.001 %; alpha means that the minimum value of alpha allowed was reached

Table 2 provides information on the optimal solution to 40 test problems. For these test problems, the θ and β values are set to 0.0001 and 0.01, respectively. The branch and bound on the assignment variables are applied for Lagrangian iterations greater than 10. In the first test problem the DC capacities are set to infinity, so the solution in this case is the same as for the LMRP. In the solution to the first test problem, DC number 1 is assigned the largest demand in the optimal solution. In the second test problem, the capacity of DC 1 is set to half the amount of the demand it was serving in the solution to the first test problem. The other tests problems are constructed in a similar manner. The largest optimality gap is 0.2 % and the largest CPU time is less than 5 minutes.

Table 3 provides information on the optimal solution to 40 more test problems. For these test problems, the θ and β values are set to 0.000001 and 0.001, respectively, making the fixed facility costs relatively more important than they are in the problems summarized in table 2. The branch and bound on the assignment variables are applied for iterations above 10. The largest optimality gap is 0.7 %. However, the largest CPU time is about 210 minutes. As can be seen from comparison of table 2 and table 3, the increase in the CPU time is due to the increase in the total number of branch and bound iterations on the assignment variables (see SubIter columns). In table 2, the number of open DCs in the solution is a significant portion of the total candidate DC locations. For example, for problem 28, 25 % of the 88 candidate DC locations are open in the optimal solution. As a consequence, when we are using algorithm 1 to obtain the benefit of a DC j, the set of candidate retailer locations ($|I^-|$) have a few elements. So then

Prob #	# Rets	Last Capacitated DC	Max Demand assigned	DC w/ Max Demand	Solution	Root LB	Root UB	% Gap	Lag Iter	Sub Iter	CPU Time	stopping reason
1	15	None	56,434,706	5	1,3,4,5,14	567,564	567,564	0.0000	57	0	1	gap
2	15	5	56,434,706	9	1,3,4,9,14	595,707	595,707	0.0000	85	1,154	11	gap
3	15	9	42,234,246	14	1,2,3,4,5,14	621,764	621,764	0.0000	82	2,251	20	gap
4	15	14	42,234,246	8	1,2,3,4,5,8	630,045	630,051	0.0009	101	4,207	37	gap
5	15	8	42,234,246	6	1,2,3,4,5,6	630,976	630,976	0.0000	69	3,589	33	gap
6	15	6	48,862,883	7	1,2,3,4,5,7	638,725	642,722	0.6257	1,074	68,943	618	alpha
7	15	7	31,737,068	2	1,2,3,4,5,6,8	651,488	657,981	0.9968	1,063	83,033	743	alpha
8	15	2	31,588,938	11	1,3,5,6,8,9,11	661,070	661,070	0.0000	309	23,419	197	gap
9	15	11	29,760,021	1	1,3,4,5,6,8,9	661,164	668,430	1.0990	1,082	96,989	813	alpha
10	15	1	51,863,604	3	3,4,5,6,8,9	980,063	987,298	0.7382	1,091	104,553	896	alpha
11	49	None	82,320,352	5	1,3,5,6,22	874,670	874,678	0.0010	324	0	3	gap
12	49	5	69,679,760	9	1,3,8,9,22,30	899,815	899,824	0.0009	250	16,355	139	gap
13	49	9	45,864,759	1	1,2,3,5,8,22,30	906,149	910,817	0.5152	1,092	123,303	1,054	alpha
14	49	1	45,864,759	39	2,3,5,8,22,30,39	911,818	916,486	0.5120	1,073	135,468	1,142	alpha
15	49	39	42,199,531	29	2,3,5,8,22,29,30	961,395	966,063	0.4855	1,092	148,938	1,268	alpha
16	49	29	49,612,741	41	2,3,5,6,8,22,41	971,880	976,872	0.5136	1,147	150,301	1,295	alpha
17	49	41	65,065,853	6	2,3,5,6,22,35	999,002	1,002,017	0.3017	1,159	168,188	1,437	alpha
18	49	6	51,127,810	35	2,3,5,8,22,30,35	999,716	1,005,712	0.5998	1,205	163,814	1,387	alpha
19	49	35	42,199,531	18	2,3,5,8,18,22,30	1,021,555	1,026,224	0.4571	1,190	167,031	1,419	alpha
20	49	18	40,964,531	22	2,3,5,8,22,24,29,30	1,054,126	1,060,598	0.6140	1,204	194,414	1,630	alpha
21	88	None	17,040,905	75	34,46,75	322,624	322,627	0.0008	168	0	3	gap
22	88	75	27,973,614	7	7,33	327,227	327,230	0.0009	186	8,361	71	gap
23	88	7	17,183,874	33	5,7,33	328,618	328,702	0.0255	1,117	129,302	1,099	alpha
24	88	33	17,757,801	22	5,7,22	328,807	328,808	0.0003	303	37,400	329	gap
25	88	22	17,982,733	34	5,34,46	328,863	329,024	0.0489	1,042	490,004	3,962	alpha
26	88	34	13,786,173	5	5,7,10,46	330,897	330,900	0.0010	224	79,566	696	gap
27	88	5	23,192,134	40	29,40,46	333,437	333,440	0.0009	139	61,613	553	gap
28	88	40	26,093,517	23	10,23,46	337,908	337,911	0.0009	193	87,412	765	gap
29	88	23	14,919,235	12	7,10,12,46	340,181	342,219	0.5991	1,051	1,612,263	12,714	alpha
30	88	12	13,650,734	7	7,29,46,75	344,811	344,845	0.0099	1,113	808,812	6,649	gap
31	150	None	40,657,099	16	16,135	468,645	468,645	0.0000	239	0	10	gap
32	150	16	40,779,998	43	43,135	469,569	469,599	0.0064	1,069	72,513	656	alpha
33	150	43	40,657,099	86	86,135	469,735	469,740	0.0010	430	95,129	824	gap
34	150	86	40,779,998	67	67,135	471,320	471,320	0.0000	258	105,727	862	gap
35	150	67	38,998,091	39	39,134	473,738	473,743	0.0010	297	177,718	1,457	gap
36	150	39	39,721,166	68	68,135	474,475	474,475	0.0001	343	233,836	1,912	gap
37	150	68	40,937,613	56	56,135	474,745	474,750	0.0010	372	307,282	2,485	gap
38	150	56	40,937,613	13	13,135	476,331	476,508	0.0372	1,076	849,127	6,896	alpha
39	150	13	39,721,166	23	23,135	477,309	477,314	0.0010	407	462,340	3,783	gap
40	150	23	40,657,099	95	95,135	478,615	478,615	0.0000	266	312,736	2,536	gap

Table 3: θ=0.000001 and β=0.001

the number of branch and bound iterations required to solve for the benefit of a DC j is manageable. In table 3, the number of open DCs in the solution is a very small portion of the total candidate DC locations. For example, for problem 68, only 3% of the 88 candidate DC locations are open in the optimal solution. As a consequence, when we are using algorithm 1 to obtain the benefit of a DC j, the set of candidate retailer locations ($|I^-|$) have more elements resulting in larger numbers of branch and bound iterations on the assignment variables. To reduce the number of sub-iterations (CPU time) for test problems 41-80, we apply branch and bound on the assignment variables for Lagrangian iterations above 100 and when the number of non-improving iterations before halving alpha is greater than or equal to 30. Table 4 summarizes the results for the test problems 41-80 with reduced number of sub-iterations. The largest optimality gap is 9.4 % and the largest CPU time is about 35 minutes. However, note that for the larger data sets (88 node and 150 node) the largest optimality gap is 0.9 %. We can conclude that limiting the number of sub-iterations lowers the quality of the solutions for test problems 41-60 but works well for test problems 61-80, in the sense that the quality of the solutions is not necessarily lowered but the CPU times are improved. This suggests that by limiting the number of sub-iterations for only larger data sets with relatively high fixed facility cost or low transportation cost, we can obtain solutions within less than 1 % optimality in less than 35 minutes.

As one might expect, typically the number of DCs open increases as more and more DCs have limited capacity. However the increase is not dramatic, for example, in the solution to test problem 32, there are 16 DCs open, and in the solution to test problem 40 there are 17 DCs open. The capacitated DCs appear in the solutions for some test problems. For example for test problems 13-20, the capacitated DCs appear in the solutions. However, for test problems 71-80, the capacitated DCs don't appear in the solutions.

6 Conclusion and Future Research

We have presented a new capacitated facility location model that incorporates risk pooling effects of the working and safety stock inventory costs at the distribution centers.

Prob #	# Rets	Last Capacitated DC	Capacity at the last capacitated DC	DC w/ Max Demand	Solution	Root LB	Root UB	% Gap	Lag Iter	Sub Iter	CPU Time	stopping reason
1	15	None	Uncapacitated	5	1,3,4,5,14	567,564	567,564	0.0000	57	0	0	gap
2	15	5	28,217,353	9	1,3,4,9,14	595,707	595,707	0.0000	176	0	1	gap
3	15	9	28,217,353	14	1,2,3,4,5,14	620,440	621,764	0.2134	1,098	4,651	49	alpha
4	15	14	21,117,123	8	1,2,3,4,5,8	628,879	630,051	0.1863	1,111	7,021	69	alpha
5	15	8	21,117,123	6	1,2,3,4,5,6	630,850	630,976	0.0199	1,043	7,440	73	alpha
6	15	6	21,117,123	2	1,2,3,4,6,7	634,256	647,214	2.0430	1,132	8,936	85	alpha
7	15	7	24,431,442	2	1,2,3,4,5,6,8	646,701	657,981	1.7444	1,065	9,432	87	alpha
8	15	2	15,868,534	1	1,3,4,5,6,8,9	657,597	668,430	1.6474	1,042	9,788	94	alpha
9	15	11	15,794,469	1	1,3,4,5,6,8,9	658,772	668,430	1.4661	1,105	10,766	102	alpha
10	15	1	14,880,011	3	3,4,5,6,8,9	978,834	987,298	0.8647	1,129	11,216	102	alpha
11	49	None	Uncapacitated	5	1,3,5,6,22	874,670	874,678	0.0010	324	0	3	gap
12	49	5	41,160,176	9	1,3,8,9,22,30	899,816	899,824	0.0009	311	953	12	gap
13	49	9	34,839,880	1	1,2,3,5,8,22,30	905,500	910,817	0.5872	1,143	17,725	167	alpha
14	49	1	22,932,380	39	2,3,5,8,22,30,39	911,214	916,486	0.5786	1,146	16,236	146	alpha
15	49	39	22,932,380	29	2,3,5,8,22,29,30	960,738	966,063	0.5543	1,161	18,911	168	alpha
16	49	29	21,099,766	41	2,3,5,6,8,22,41	971,456	976,872	0.5575	1,193	20,315	179	alpha
17	49	41	24,806,371	35	2,3,5,6,8,22,35	987,905	1,003,858	1.6148	1,121	17,424	157	alpha
18	49	6	32,532,927	35	2,3,5,8,22,30,35	988,361	1,005,712	1.7556	1,089	18,305	165	alpha
19	49	35	25,563,905	2	2,3,7,22,24,29,30	988,018	1,080,736	9.3843	1,119	18,568	168	alpha
20	49	18	21,099,766	2	2,3,7,22,24,29,30	988,023	1,080,736	9.3837	1,102	18,170	166	alpha
21	88	None	Uncapacitated	75	34,46,75	322,624	322,627	0.0008	168	0	3	gap
22	88	75	8,520,453	7	7,33	327,227	327,230	0.0010	160	728	9	gap
23	88	7	13,986,807	33	5,7,33	328,613	328,702	0.0271	1,152	20,059	189	alpha
24	88	33	8,591,937	22	5,7,22	328,770	328,808	0.0116	1,073	10,015	104	alpha
25	88	22	8,878,901	34	5,34,46	329,021	329,024	0.0008	186	2,220	22	gap
26	88	34	8,991,367	5	5,7,10,46	330,787	330,900	0.0342	1,135	76,030	619	alpha
27	88	5	6,893,087	40	29,40,46	333,438	333,440	0.0006	198	1,621	17	gap
28	88	40	11,596,067	23	10,23,46	337,908	337,911	0.0009	257	3,343	31	gap
29	88	23	13,046,759	12	7,10,12,46	339,437	342,402	0.8736	1,108	265,628	2,123	alpha
30	88	12	7,459,618	7	7,46,55,75	344,764	346,495	0.5020	1,039	86,123	706	alpha
31	150	None	Uncapacitated	16	16135	468,645	468,645	0.0000	239	0	9	gap
32	150	16	20,328,550	43	43135	469,598	469,599	0.0002	319	8,889	83	gap
33	150	43	20,389,999	86	86135	469,734	469,740	0.0011	1,103	94,501	796	alpha
34	150	86	20,328,550	67	67135	471,320	471,320	0.0000	317	13,742	121	gap
35	150	67	20,389,999	39	39134	473,738	473,743	0.0009	259	14,507	126	gap
36	150	39	19,499,046	68	68135	474,475	474,475	0.0000	481	64,756	533	gap
37	150	68	19,860,583	56	56135	474,750	474,750	0.0000	250	18,847	159	gap
38	150	56	20,468,807	13	13135	476,508	476,508	0.0000	292	16,661	143	gap
39	150	13	20,468,807	23	23135	477,309	477,314	0.0009	287	24,274	205	gap
40	150	23	19,860,583	95	95135	478,610	478,615	0.0010	316	21,934	186	gap

Table 4: θ=0.000001 and β=0.001

Economies of scale also exist in the transportation costs from the supplier to the DCs. The CLMRP is also flexible in the definition of capacity as it allows the DCs to order more frequently and stock less working inventory rather than opening new DCs to meet the demands of the retailers without exceeding the capacity constraints.

The CLMRP is structurally similar to the classical uncapacitated facility location model (UFLP). There is one extra term in the objective function of CLMRP than in the UFLP, which is the optimal working inventory and safety stock costs. We assumed that the DCs use a (Q,r) model with type 1 service to replenish their inventories. We have proposed a Lagrangian-relaxation based algorithm to solve this model. The algorithm we have proposed can be used to solve for models in which the DCs replenish their inventory using different policies other than (Q,r) model. The only requirement is that the resulting optimal working inventory cost and safety stock cost function is differentiable.

In this formulation of the CLMRP, we have been considering the movement and storage of a single product. A natural extension to the CLMRP would be to consider multiple commodities and solve for the assignment and order quantity variables simultaneously. Also in modeling the CLMRP, we have assumed direct shipments from the distribution centers to the assigned retailers. However, in practice, the shipments from a DC to the assigned retailers are often a traveling salesman like tour. Thus we would like to incorporate a better approximation of the shipment costs into the CLMRP (e.g., the approximations developed by Daganzo (1991)). We are working on all of these extensions to the CLMRP.

APPENDIX

In the Appendix, we show that $\bar{W}^*(D)$, the expected optimal working and safety stock cost function for a given DC is differentiable. $\bar{W}^*(D)$ is the optimal value of the following capacitated EOQ model.

$$\bar{W}^*(D) = \begin{cases} \text{Minimize} & W(Q) = (F+\beta g)\frac{D}{Q} + \beta a D + \theta\frac{hQ}{2} + \theta h z_\alpha\sqrt{\frac{LD}{\chi}} \\ \text{subject to} & Q + z_\alpha\sqrt{LD/\chi} + L\frac{D}{\chi} \le C \\ & Q > 0 \end{cases}$$

Let Q_{EOQ} denote the optimal order quantity that minimizes the objective function W(Q) in the absence of the capacity constraint. By taking the derivative of the objective function and setting it to zero, we can solve for Q_{EOQ}. We obtain the following result:

$$Q_{EOQ} = \sqrt{\frac{2(F+\beta g)D}{\theta h}}$$

If Q_{EOQ} satisfies the capacity constraint then the optimal order quantity, $Q^* = Q_{EOQ}$. Let Q_C denote the optimal order quantity when Q_{EOQ} doesn't satisfy the capacity constraint. By, rearranging the capacity constraint , we can see that:

$$Q \leq C - \left(z_\alpha \sqrt{LD/\chi} + L\frac{D}{\chi}\right)$$

Since W(Q) is a convex function,

$$Q_C = C - \left(z_\alpha \sqrt{LD/\chi} + L\frac{D}{\chi}\right)$$

Hence, $Q^* = min\left\{Q_{EOQ}, Q_C\right\}$ where $Q_{EOQ} > 0$, $Q_C > 0$. Since $D > 0$, then $Q_{EOQ} > 0$. Let $\bar{D}$ be the smallest D such that $Q_C = C - \left(z_\alpha \sqrt{LD/\chi} + L\frac{D}{\chi}\right) \leq 0$. if $D < \bar{D}$ then $Q_C > 0$. So, the domain for $\bar{W}^*(D)$ is $(0, \bar{D})$. Let $\hat{D} \in (0, \bar{D})$ be the largest D such that Q_{EOQ} satisfies the capacity constraint (Q_{EOQ} is small enough). Then, for $D \leq \hat{D}$, $Q^* = Q_{EOQ}$, and for $\bar{D} \geq D \geq \hat{D}$, $Q^* = Q_C$. Hence, $\bar{W}^*(D)$ can be rewritten as follows:

$$\bar{W}^*(D) = \begin{cases} \bar{W}^*_{EOQ}(D) = W(Q_{EOQ}) & 0 < D \leq \hat{D} \\ \bar{W}^*_C(D) = W(Q_C) & \hat{D} \leq D < \bar{D} \end{cases}$$

Note that the following can be derived:

1. $\bar{W}^*_{EOQ}(D) = \sqrt{2\theta h(F+\beta g)D} + \beta a D + \theta h z_\alpha \sqrt{\frac{LD}{\chi}}$
2. $\frac{d\bar{W}^*_{EOQ}(D)}{dD} = \frac{\sqrt{2\theta h(F+\beta g)}}{2\sqrt{D}} + \beta a + \frac{\theta h z_\alpha \sqrt{L}}{2\sqrt{\chi D}} > 0$
3. $\bar{W}^*_C(D) = \frac{(F+\beta g)D}{C - z_\alpha\sqrt{\frac{LD}{\chi}} - L\frac{D}{\chi}} + \beta a D + \frac{\theta h\left(C - z_\alpha\sqrt{\frac{LD}{\chi}} - L\frac{D}{\chi}\right)}{2} + \theta h z_\alpha \sqrt{\frac{LD}{\chi}}$
4. $\frac{d\bar{W}^*_C(D)}{dD} = \frac{(F+\beta g)\left(C - \frac{z_\alpha}{2}\sqrt{\frac{LD}{\chi}}\right)}{\left(C - z_\alpha\sqrt{\frac{LD}{\chi}} - L\frac{D}{\chi}\right)^2} + \beta a + \frac{\theta h}{2}\left(\frac{z_\alpha\sqrt{L}}{2\sqrt{\chi D}} - \frac{L}{\chi}\right) > 0$
5. $\bar{D} = \frac{\chi}{2L}\left(2C + z_\alpha^2 - z_\alpha\sqrt{4C + z_\alpha^2}\right)$

Claim: $\bar{W}^*(D)$ is a differentiable function.

Proof: $\bar{W}^*_{EOQ}(D)$ and $\bar{W}^*_C(D)$ are differentiable functions. Then for $\bar{W}^*(D)$ to be differentiable, we need to show that the following holds:

$$\frac{d\bar{W}^*_{EOQ}(D)}{dD} |_{D=\hat{D}} = \frac{d\bar{W}^*_C(D)}{dD} |_{D=\hat{D}}$$

Let $H(D) = \bar{W}^*_C(D) - \bar{W}^*_{EOQ}(D)$. Then $H(D) \geq 0 \ \forall \ D \in (0, \bar{D})$, since Q_{EOQ} is the minimizer of $W(Q)$. $H(\hat{D}) = 0$, since $\bar{W}^*_C(\hat{D}) = \bar{W}^*_{EOQ}(\hat{D})$. Since,$\hat{D}$ is the only point that $\bar{W}^*_C(D)$ and $\bar{W}^*_{EOQ}(D)$ intersect, 0 is the unique minimizer of $H(D)$. So,

$$\frac{dH(D)}{dD} |_{D=\hat{D}} = \frac{d\bar{W}^*_C(D)}{dD} |_{D=\hat{D}} - \frac{d\bar{W}^*_{EOQ}(D)}{dD} |_{D=\hat{D}} = 0$$

Thus,

$$\frac{d\bar{W}^*_{EOQ}(D)}{dD} |_{D=\hat{D}} = \frac{d\bar{W}^*_C(D)}{dD} |_{D=\hat{D}}$$

as desired.

References

[1] Akinc U., B. M. Khumawala (1977). An efficient branch and bound algorithm for the capacitated warehouse location problem. *Management Science*, **23**, 585-594.

[2] Balinski, M. (1965). Integer Programming: Methods, uses, computation. *Management Science*, **13**, 253-313.

[3] Barahona, F. and D. Jensen (1998). Plant location with minimum inventory. *Mathematical Programming* 83, 101-111.

[4] Berger, R.T., C.R. Coullard, M.S. Daskin (1998). Modeling and Solving Location-Routing Problems with Route-Length Constraints. Working Paper, Northwestern University.

[5] Berman, 0., P. Jaillet and D Simchi-Levi (1995). Location-Routing Problems with Uncertainty. *Facilities Location, Z. Drezner ed.*, Springer Verlag, 427-452.

[6] Bramel, J., D. Simchi-Levi (1997). *The Logic of Logistics*, Springer, NY.

[7] Chan, L. M. A., A. Federgruen and D. Simchi-Levi (1995). Probabalistic Analysis and Practical Algorithms for Inventory Routing Models. *Operations Research*, 46, 96-106.

[8] Daganzo, C., (1991). *Logistics Systems Analysis*, Lecture Notes in Economics and Mathematical Systems, M. Beckmann and W. Krelle eds., Springer Verlag, Berlin.

[9] Daskin, M. S. and P. C. Jones (1993) A new approach to Solving Applied Location/Allocation Problems. *Microcomputers in Civil Engineering* , 8, 409-421

[10] Daskin, M.S. (1995). *Network and Discrete Location: Models, Algorithms, and Applications.* John Wiley, NY.

[11] Daskin, M.S., C.R. Coullard, and Z.-J. M. Shen, 2001, "An Inventory-Location Model: Formulation, Solution Algorithm and Computational Results," submitted for publication to Annals of Operations Research.

[12] Drezner, Z., editor (1995). *Facility Location: A survey of Applications and Methods.* Springer, NY.

[13] Drezner, Z., H. W. Hamacher (2002) *Facility location : applications and theory* Springer, NY.

[14] Elson D.G. (1972). Site Location via Mixed-Integer Programming. *Operational Research Quarterly*, 23, 31-43.

[15] Eppen, G. (1979). Effects of Centralization on Expected Costs in a Multi-Location Newsboy Problem. *Management Science,* 25, No.5, 498-501.

[16] Erlebacher, S.J. and R.D. Meller (2000). The interaction of location and inventory in designing distribution systems. *IIE Transactions*, 32, 155-166.

[17] Federgruen, A. and P. Zipkin (1984), A combined Vehicle Routing and Inventory Allocation Problem, *Operations Research*, 32, 1019-1036.

[18] Fisher, M.L. (1981). The Lagrangian relaxation method for solving integer programming problems. *Management Science*, 27 (1), 1-18.

[19] Fisher, M.L. (1985). An applications-oriented guide to Lagrangian relaxation. *Interfaces*, 15 (2), 10-21.

[20] Garey, M.R. and D.S. Johnson (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H. Freeman, New York, 1979.

[21] Geoffrion M. and R. McBride (1978). Lagrangian relaxation applied to capacitated facility location problem. *AIIE Transaction* 10, 40-47.

[22] Graves, S.C., A.G. Rinnoy Kan, and P.H. Zipkin (1993). *Logistics of Production and Inventory.* Elsevier Science Publishers.

[23] Guignard M. and K. Spielberg (1979). A direct dual method for the mixed plant location problem. *Mathematical Programming* 17, 198-228.

[24] Hinojosa, Y., J. Puerto, F.R. Fernandez (2000). A multiperiod two-echelon multicommodity capacitated plant location problem. *European Journal of Operational Research*,123, 271-291.

[25] Hopp, W., and M.L. Spearman (1996). *Factory Physics: Foundations of Manufacturing Management*, Irwin, Chicago.

[26] Hurter, A.P. and J.S. Martinich (1989). *Facility Location and the Theory of Production.* Kluwer Academic Publishers.

[27] Jia, S., C.-P. Teo, and Z.J.M. Shen (2001). Stochastic transportation-inventory network design problem. Working Paper.

[28] Kaufman L., M.V. Eede, P. Hansen (1977). A Plant and Warehouse Location Problem. *Operational Research Quarterly*, 28, 547-554.

[29] Klincewicz, J.G. and H. Luss (1986). A Lagrangian relaxation heuristic for capacitated facility location with single-sourcing constraints. *Journal of Operational Research Society*, 37, 495-500.

[30] Krarup, J. and P.M. Pruzan (1983). The simple plant location problem-survey and synthesis. *European Journal of Operational Research*, 2, 36-81

[31] Laporte, G, P.J. Dejax (1989). Dynamic Location-Routing Problems. *Journal of Operations Research Society*, 40, No.5, 471-482.

[32] Montgomery, D.C., G.C. Runger, and N.F. Hubele, (1998). *Engineering Statistics*, John Wiley, NY.

[33] Nahmias, S. (1997). Production and Operations Management, Third Edition, Irwin,Chicago.

[34] Nauss, R.M. (1978). An improved algorithm for the capacitated facility location problem. *Operational Research Society*, 29, 1195-1202.

[35] Pirkul, H. (1987). Efficient algorithms for the capacitated concentrator location problems. *Computers and Operations Research*, 14, No.3, 197-208.

[36] Pirkul, H. and J. Vaidyanathan (1998). A multi-commodity, multi-plant, capacitated facility location problem: formulation and efficient heuristic solution. *Computers and Operations Research*, 25, 869-878.

[37] Shen, Z.-J. M. (2000). Efficient algorithms for various supply chain problems. Ph.D. Dissertation, Northwestern University, Evanston, IL.

[38] Shen, Z.-J. M., C. Coullard and M.S. Daskin, 2000, "A joint location-inventory model," forthcoming in Transportation Science.

[39] Teo, C.-P., J. 0u, and M. Goh (2001). Impact on inventory costs with consolidation of distribution centers, *IIE Transactions*, 33, 99-110.

[40] Teitz, M.B. and P. Bart (1968). Heuristic methods for estimating generalized vertex median of a weighted graph. *Operations Research*, 16, 955-961.

[41] Viswanathan, S. and Mathur, K. (1997), Integrating Routing and Inventory Decisions in One-Warehouse Multiretailer Multiproduct Distribution System, *Management Science*, 43, 294-312.

[42] Wolsey, L. A. (1998). *Integer Programming*, John Wiley, NY.

[43] Zipkin, P. H. (1997). *Foundations of Inventory Management.* Irwin.