# Lifting 2-integer knapsack inequalities

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**Abstract**

In this paper we discuss the generation of strong valid inequalities for (mixed) integer knapsack sets based on lifting of valid inequalities for basic knapsack sets with two integer variables (and one continuous variable). The description of the basic polyhedra can be made in polynomial time. We use superadditive valid functions in order to obtain sequence independent lifting.

## 1 Introduction

The idea of lifting was introduced by Gomory (1969). An important contribution was given by Wolsey (1977) when the link between superadditivity and the lifting procedure was established. The use of superadditive functions in the lifting procedure allows the introduction of several variables simultaneously which reduces the computational burden of the generation of valid inequalities. The procedure of lifting has been explored more intensively in the last few years for the generation of strong valid inequalities for (mixed) integer sets. Usually these inequalities are obtained from the lifting of facet defining inequalities of elementary polyhedra. We mention the works of Gu, Nemhauser & Savelsbergh (1999) and Gu, Nemhauser & Savelsbergh (2000) on the single node flow sets and on the use of superadditivity, the papers of Marchand & Wolsey (1999) and Marchand & Wolsey (2001) based on the lifting of the well known MIR inequalities. More recently Atamtürk has produced important work (2001*a*, 2001*b*, 2002) on the lifting of valid inequalities for mixed integer sets.

We consider the integer set $Y = \{y \in \mathbb{N}_0^{|N|} : \sum_{j\in N} a_j y_j \leq D\}$ and the mixed integer set $X = \{(y,s) \in \mathbb{N}_0^{|N|} \times \mathbb{R} : \sum_{j\in N} a_j y_j \leq D + s, s \geq 0\}$ where $a_j, j \in N$ and $D$ are positive integers. These sets may arise as aggregation of more general (mixed) integer sets or as subsets of more complex sets. First we consider the restriction of both sets obtained by setting all but two integer variables to zero. Suppose $y_1$, $y_2$ are those variables, thus the restricted sets are, respectively,

$$Y_{\leq} = \{(y_1,y_2) \in \mathbb{N}_0^2 : a_1y_1 + a_2y_2 \leq D\} \text{ and } R = \{(y_1,y_2,s) \in \mathbb{N}_0^2 \times \mathbb{R} : a_1y_1 + a_2y_2 \leq D + s, s \geq 0\}.$$

Then, for each facet defining inequality of the convex hull of the restricted sets we construct valid inequalities for $X$ and $Y$ using superadditive valid lifting functions.

In Section 2 we review some concepts on the lifting procedure. In Section 3 we summarize some results on the description of the 2-integer knapsack polyhedra. In Section 4 a polinomial superadditive valid lifting function for tight valid inequalities for the 2-integer knapsack set is given. In Section 5 we consider facet defining inequalities and characterize the lifting function for those inequalities. In Section 6 we construct several superadditive valid lifting functions for the facet defining inequalities. In Section 7 the results obtained in the previous sections are adapted to the integer knapsack set with a continuous variable. The computational experience is reported in Section 8.

## 2 Lifting and superadditivity

In this section we introduce the basic concepts of the lifting procedure applied to the generation of valid inequalities for $Y$. These concepts are well known for the binary case (see, for instance, Wolsey 1977, Gu et al. 2000) and recently have been extended for the integer case (Atamtürk 2001*b*). Consider the restriction of this set obtained by setting $y_j = 0,\ j \in N \setminus M : Y_M = \{y \in \mathbb{N}_0^{|M|} : \sum_{j \in M} a_j y_j \leq D\}$. Suppose

$$\sum_{j \in M} \alpha_j y_j \leq \alpha, \tag{2.1}$$

is a valid inequality for $Y_M$. We assume that (2.1) is tight for $Y_M$. The *lifting problem* consists of computing the *lifting coefficients* $\alpha_j$, for $j \in N \setminus M$ in order to obtain a valid inequality

$$\sum_{j \in M} \alpha_j y_j + \sum_{j \in N \setminus M} \alpha_j y_j \leq \alpha$$

for $Y$. To compute these coefficients we use the *lifting function* associated with (2.1),

$$\begin{aligned} \phi_M(z) &= \min \alpha - \sum_{j \in M} \alpha_j y_j \\ s.t. \quad & \sum_{j \in M} a_j y_j \leq D - z, \\ & y_j \geq 0, \text{ and integer, } j \in M. \end{aligned}$$

One way to lift variables is to introduce them one at a time (sequential lifting). Suppose $y_k, k \in N \setminus M$ is the first variable to be introduced. In order to obtain a valid inequality for $Y_{M \cup \{k\}}$ the lifting coefficient $\alpha_k$ must satisfy $\alpha_k \times y_k \leq \phi_M(a_k \times y_k)$ for $y_k = 1, 2, \ldots, \lfloor D/a_k \rfloor$, thus

$$\alpha_k \leq \min_{n=1,\ldots,\lfloor D/a_k \rfloor} \left\{ \frac{\phi_M(a_k \times n)}{n} \right\}.$$

For $z \in ]0, D]$, we denote by $\phi_M^I(z) = \min_{n=1,\ldots,\lfloor D/z \rfloor} \{\frac{\phi_M(z \times n)}{n}\}$ the function that gives the upper bound on the lifting coefficients. Notice that if $y_k$ is binary then $\alpha_k \leq \phi_M(a_k)$. Formally,

**Proposition 2.1** *If (2.1) is valid for $Y_M$ and $\alpha_k \leq \phi_M^I(a_k)$ then*

$$\sum_{j \in M} \alpha_j y_j + \alpha_k y_k \leq \alpha \tag{2.2}$$

*is valid for $Y_{M \cup \{k\}}$. If (2.1) defines a facet of $conv(Y_M)$ and $\alpha_k = \phi_M^I(a_k)$ then (2.2) defines a facet of $conv(Y_{M \cup \{k\}})$.*

The lifting function associated with (2.2) is

$$\phi_{M \cup \{k\}}(z) = \min_{n=0,\ldots,\lfloor D/a_k \rfloor} \{\phi_M(z + n a_k) - n\alpha_k\}. \tag{2.3}$$

If $y_k$ is binary then $n = 0, 1$.

The value of the lifting function may decrease as the set of variables is increased because $\phi_{M\cup\{k\}}(z) \leq \phi_M(z)$ (consider $n = 0$ in (2.3)). Thus the lifting coefficients may depend on the order that the variables are lifted. However there are cases where the order is not important. This happens when the lifting function has a nice property.

**Definition 2.2** *A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is superadditive on $A$ if $f(x_1) + f(x_2) \leq f(x_1 + x_2)$ for all $x_1, x_2, x_1 + x_2 \in A$.*

**Theorem 2.3** *If $\phi_M$ is superadditive on $[0, D] \subseteq \mathbb{R}$ then, for all $z \in [0, D]$*
*(i) $\phi_M(z) = \phi_{M\cup\{k\}}(z)$ for all $k \in N \setminus M$;*
*(ii) $\phi_M(z) = \phi_M^I(z)$.*

In general the lifting function is not superadditive. In that cases if we want to lift simultaneously all variables we may use other functions.

**Definition 2.4** *A function $\psi$ is said a valid lifting function if $\psi(z) \leq \phi_M(z)$ for all $z \in [0, D]$.*

**Proposition 2.5** *If $\psi$ is a superadditive valid lifting function then $\psi(z) \leq \phi_M^I(z)$ for all $z \in [0, D]$.*

Thus, if a valid lifting function is supperadditive we can use it to compute the lifting coefficients simultaneously. The following definition will be useful to compare several valid lifting functions.

**Definition 2.6** *A valid lifting function $\psi$ on $A$ is said non-dominated if there is no other valid lifting function $\chi$ such that $\psi(z) \leq \chi(z) \leq \phi_M(z)$, for all $z \in [0, D]$ and $\chi(z') > \psi(z')$ for some $z' \in [0, D]$.*

Next we consider another property of the superadditive valid lifting functions.

**Definition 2.7** *Let $E = \{z \in [0, D] : \phi_{M\cup S}(z) = \phi_M(z)\}$ for all $S \in N \setminus M, S \neq \emptyset$. If $\psi(z) = \phi_M(z)$, for all $z \in E$, $\psi$ is said a maximal superadditive valid lifting function.*

In order to prove the existence of a superadditive valid lifting function Gu et al. (2000) constructed the following function: $\zeta(z) = \min_{u\in[0,D-z]}\{\phi_M(z + u) - \phi_M(u)\}$. And, using this function, they prove the following result.

**Lemma 2.8** *If $\phi_M(z)$ is integer for all $z$ and if $\psi$ is a superadditive valid lifting function such that $\psi(z) > \phi_M(z) - 1$ for all $z$ and $\psi(z') = \phi_M(z')$ then $z' \in E$.*

# 3 Results on polyhedral 2-integer knapsack polyhedra

In this section we summarize some results on the polyhedral description of the convex hull of the 2-integer knapsack sets:

$$Y_{\leq} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1y_1 + a_2y_2 \leq D\} \text{ and } Y_{\geq} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1y_1 + a_2y_2 \geq D\}.$$

In order to obtain the extreme points of $conv(Y_{\leq})$, denoted by $(a^j, b^j)$, that maximize functions $f_1y_1 + f_2y_2$ with $\frac{f_2}{f_1} \leq \frac{a_2}{a_1}$ we can use the following algorithm given by Hirschberg & Wong (1976).

**Algorithm HW**

Step 0: $j \leftarrow 1, (a^j, b^j) \leftarrow (\left\lfloor \frac{D}{a_1} \right\rfloor, 0), k \leftarrow 1, (c^k, d^k) \leftarrow (\left\lfloor \frac{a_2}{a_1} \right\rfloor, 1), \ell \leftarrow 1, (e^\ell, f^\ell) \leftarrow (\left\lceil \frac{a_2}{a_1} \right\rceil, 1), r \leftarrow 1$

Step 1: While $a^j - c^k \geq 0$ do

Set $\gamma(a^j, b^j) \leftarrow D - a_1 a^j - a_2 b^j$, $R_{\leq}(c^k, d^k) \leftarrow -a_1 c^k + a_2 d^k$, $R_{\geq}(e^\ell, f^\ell) \leftarrow a_1 e^\ell - a_2 f^\ell$

(i) if $\gamma(a^j, b^j) \geq R_{\leq}(c^k, d^k)$ set $j \leftarrow j+1, (a^j, b^j) \leftarrow (a^{j-1}, b^{j-1}) + r(-c^k, d^k)$;

(ii) if $\gamma(a^j, b^j) < R_{\leq}(c^k, d^k)$ and $R_{\leq}(c^k, d^k) \geq R_{\leq}(e^\ell, f^\ell)$ set
$k \leftarrow k+1, (c^k, d^k) \leftarrow (c^{k-1}, d^{k-1}) + r(e^\ell, f^\ell)$;

(iii) if $\gamma(a^j, b^j) < R_{\leq}(c^k, d^k)$ and $R_{\leq}(c^k, d^k) < R_{\leq}(e^\ell, f^\ell)$ set
$\ell \leftarrow \ell+1, (e^\ell, f^\ell) \leftarrow (e^{\ell-1}, f^{\ell-1}) + r(c^k, d^k)$.

This algorithm is not polynomial and some points $(a, b)$ generated may not be extreme. In order to obtain only the extreme points in polynomial time instead of computing, in Step 1, $r = 1$ it suffices to compute $r$ as follows:

$$\text{(i) } r = min\left\{ \left\lfloor \frac{\gamma(a^j, b^j)}{R_{\leq}(c^j, d^j)} \right\rfloor, \left\lfloor \frac{a^j}{c^k} \right\rfloor \right\};$$

$$\text{(ii) } r = min\left\{ \left\lfloor \frac{R_{\leq}(c^k, d^k)}{R_{\geq}(e^\ell, f^\ell)} \right\rfloor, \left\lceil \frac{R_{\leq}(c^k, d^k) - \gamma(a^j, b^j)}{R_{\geq}(e^\ell, f^\ell)} \right\rceil \right\};$$

$$\text{(iii) } r = \left\lfloor \frac{R_{\geq}(e^\ell, f^\ell)}{R_{\leq}(c^k, d^k)} \right\rfloor.$$

Although we are interested in obtaining the extreme points in polynomial time, the non polynomial Algorithm HW has some properties that we will use later.

We denote by $k(\ell)$ the index of the pair $(c, d)$ used in (iii) of Step 1 to obtain $(e^\ell, f^\ell)$, this is, $(e^\ell, f^\ell) = (e^{\ell-1}, f^{\ell-1}) + (c^{k(\ell)}, d^{k(\ell)})$. Similarly, we denote by $\ell(k)$ the index of the pair $(e, f)$ used in (ii) of Step 1 to obtain $(c, d)$, and by $k(j)$ the index of the pair $(c, d)$ used in (i) of Step 1 to obtain to obtain $(a^j, b^j)$. In this last case $\ell(k(j))$ denote the index of the pair $(e, f)$ used to obtain $(c^{k(j)}, d^{k(j)})$. Let $n_1$ and $n_2$ denote the number of distinct pairs $(c, d)$ and $(e, f)$ generated, respectively. Now we summarize some of the properties of these coefficients.

**Lemma 3.1** *(i)* $e^\ell c^k - f^\ell d^k = 1$ *if either* $\ell = \ell(k)$ *or* $k = k(\ell)$.
*(ii)* $\frac{c^1}{d^1} \leq \cdots \leq \frac{c^{n_1}}{d^{n_1}} \leq \frac{a_2}{a_1} \leq \frac{e^{n_2}}{f^{n_2}} \leq \cdots \leq \frac{e^1}{f^1}$.
*(iii)* $\frac{c^k}{d^k}$*, for* $k = 1, \ldots, n_1$ *(*$\frac{e^\ell}{f^\ell}$*, for* $\ell = 1, \ldots, n_2$*) are the best approximations from below (from above) to* $\frac{a_2}{a_1}$ *for that size denominator.*
*(iv) (a) The set* $\{(c^1, d^1), \ldots, (c^k, d^k)\}$ *is an integral Hilbert basis for* $Cone\{(c^1, d^1), (c^k, d^k)\}$*. (b) The set* $\{(e^1, f^1), \ldots, (e^\ell, f^\ell)\}$ *is an integral Hilbert basis for* $Cone\{(e^1, f^1), (e^\ell, f^\ell)\}$*.*

The first three properties are well known since the coefficients $(c, d), (e, f)$ are those obtained by the continuous fraction method to approximate $\frac{a_2}{a_1}$ (see Schrijver 1986). Property (iv) is essentially due to (Weismantel 1996).

To obtain the extreme extreme points that maximize functions with $\frac{f_2}{f_1} > \frac{a_2}{a_1}$ it suffices to exchange $a_1$ with $a_2$. This algorithm can be easily adapted to generate the extreme points of $conv(Y_{\geq})$.

## 4 Lifting the 2-integer knapsack inequalities

In this section we consider the lifting of the 2-integer knapsack facet defining inequalities in order to obtain valid inequalities for $Y$. First we consider the restriction of $Y$ obtained by setting all but two variables to zero. W.l.o.g. we assume that those two variables are $y_1$ and $y_2$. Thus the restricted set is $Y_M = Y_{\leq} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \leq D\}$, where $M = \{1, 2\}$. If either $a_1/a_2$ or $a_2/a_1$ are integer $conv(Y_M)$ has only one non trivial facet and, in that case, we may use the well known superadditive mixed integer rounding function as it is shown in Section 5. Thus we assume that $a_1/a_2$ and $a_2/a_1$ are not integer. Consider a valid inequality

$$\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha \tag{4.1}$$

for $Y_{\leq}$. In this section we consider $\frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1}$. The case $\frac{\alpha_2}{\alpha_1} > \frac{a_2}{a_1}$ can be reduced to the previous one by exchanging $a_1$ with $a_2$.

Let $\phi(z)$, $0 \leq z \leq D$, be the lifting function associated to (4.1). For $0 \leq z \leq D$, the value of $\phi(z)$ is obtained by solving the following knapsack problem,

$$\begin{aligned} \phi(z) \quad = \quad & \min \alpha - \alpha_1 y_1 - \alpha_2 y_2 \\ \text{s.t.} \quad & a_1 y_1 + a_2 y_2 \leq D - z \\ & y_1, y_2 \in \mathbb{N}_0. \end{aligned}$$

For simplicity of notation we will omit the subscript $M$ when $M = \{1, 2\}$. In general $\phi$ is not superadditive.

**Example 4.1** *Consider the following set.*

$$\{21y_1 + 76y_2 + 3y_3 + 16y_4 \leq 1154,\ y_j \geq 0,\ \textit{and integer}, j = 1, 2, 3, 4\}.$$

*The inequality* $5y_1 + 18y_2 \leq 274$ *is valid for the restricted set with* $y_3 = y_4 = 0$. *The lifting function is:*

$$\begin{aligned} \phi(z) \quad = \quad & \min 274 - 5y_1 - 18y_2 \\ s.t. \quad & 21y_1 + 76y_2 \leq 1154 - z, \\ & y_1, y_2 \in \mathbb{N}_0. \end{aligned}$$

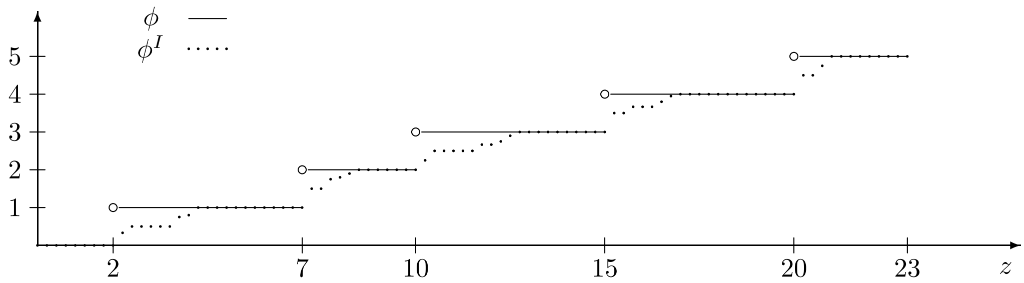


Figure 1: Non superadditive lifting function of Example 4.1.

*The graph of this function and the graph of* $\phi^I$, *for* $0 \leq z \leq 23$, *are shown in Figure 1. Considering* $z_1 = 3$ *and* $z_2 = 16$ *we have* $\phi(3) + \phi(16) = 5 > \phi(19) = 4$ *and* $\phi^I(3) + \phi^I(16) = 0.5 + 3.6667 =$

$4.1667 > \phi^I(19) = 4$. *Thus, functions* $\phi$ *and* $\phi^I$ *are not superadditive. Introducing* $y_3$ *before* $y_4$ *we obtain* $5y_1 + 18y_2 + 0.5y_3 + 3.333y_4 \leq 274$, *(*$5y_1 + 18y_2 + y_3 + 3y_4 \leq 274$, *if* $y_3, y_4$ *were binary) and introducing* $y_3$ *after* $y_4$ *we obtain,* $5y_1 + 18y_2 + 0.0833y_3 + 3.6667y_4 \leq 274$ *(*$5y_1 + 18y_2 + 0y_3 + 4y_4 \leq 274$, *in the binary case). Therefore, the lifting coefficients are sequence dependent.*

## 4.1 A polynomial superadditive valid lifting function

As $\phi$, in general, is not superadditive, we will construct superadditive valid lifting functions. The following proposition will help us to find such a function.

**Proposition 4.2** *Consider a function* $f : I = [0, D] \subseteq \mathbb{R}_0^+ \rightarrow \mathbb{R}$ *satisfying*

*(i)* $f(0) = 0$,

*(ii)* $f$ *is convex.*

*Then* $f$ *is superadditive on* $I$.

The proof is given in the Appendix.

Next we construct a function, $\psi_1$, considering the convex lower envelope of the graph of $\phi$. Given the knapsack coefficients, $a_1, a_2, D$, let us define $\gamma(a, b) = D - a_1 a - a_2 b$ and, for the inequality, $\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha$ define $\tau(a, b) = \alpha - \alpha_1 a - \alpha_2 b$. If $(a, b)$ belongs to an ordered set $E = \{(A^t, B^t), t = 1, \ldots, j\}$, then we use the notation $\gamma^t = \gamma(A^t, B^t)$ and $\tau^t = \tau(A^t, B^t)$. The function $\psi_1$ is a piecewise linear function whose graph contains the points

$$(0, 0), (\gamma^j, \tau^j), \ldots, (\gamma^1, \tau^1).$$

Although we are constructing $\psi_1$ for a general ordered set $E$, in particular, we are concerned with a subset of the set of the extreme points of *conv*($Y_{\leq}$).

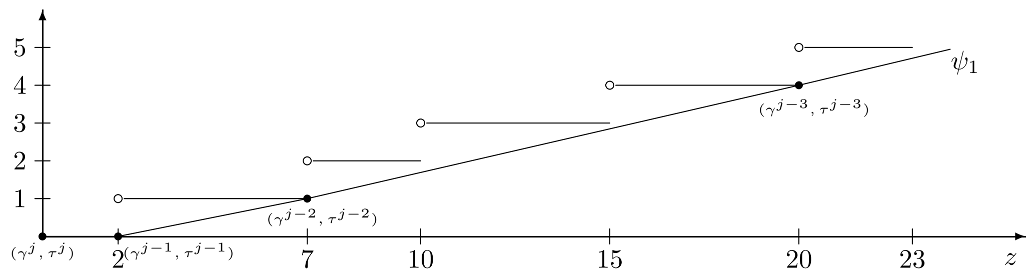


Figure 2: Function $\psi_1$ for Example 4.1.

Consider the function $\psi_1$, depending on the set $E$ (set of ordered points), $\alpha_1, \alpha_2, \alpha$ (coefficients of the valid inequality) and $a_1, a_2, D$ (coefficients of the knapsack constraint), defined by

$$\psi_1(z) = \begin{cases} 0, & 0 \leq z \leq \gamma^j, \\ \tau^t + \frac{\tau^{t-1} - \tau^t}{\gamma^{t-1} - \gamma^t}(z - \gamma^t), & \gamma^t < z \leq \gamma^{t-1},\ t = 2, \ldots, j, \\ \tau^1 + \frac{\alpha_1}{a_1}(z - \gamma^1), & \gamma^1 < z \leq D, \end{cases}$$

The following propositions are proved in the Appendix.

**Proposition 4.3** *Let $E = \{(A^t, B^t), t = 1, \ldots, j\}$ be a set of points satisfying $0 \leq A^j \leq \cdots \leq A^1$, $B^j > \cdots > B^1 \geq 0$, $\gamma^t > \gamma^{t+1}$ and $\tau^t > \tau^{t+1}$ for $t = 1, \ldots, j-1$. If $\frac{A^1 - A^2}{B^2 - B^1} < \cdots < \frac{A^{j-1} - A^j}{B^j - B^{j-1}} \leq \frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1}$, then $\psi_1$ is convex on $[0, D]$.*

We are considering inequalities (4.1) defining proper faces of $conv(Y_{\leq})$ so, containing at least one extreme point of $conv(Y_{\leq})$. In this case, the set E is the subset of extreme points $\{(a^1, b^1), \ldots, (a^j, b^j)\}$, of $conv(Y_{\leq})$ that maximizes functions $\pi_1 y_1 + \pi_2 y_2$ with $\frac{c^{k(j)}}{d^{k(j)}} \leq \frac{\pi_2}{\pi_1} \leq \frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1}$. It can be shown that $\gamma^t > \gamma^{t+1}$ and $\tau^t > \tau^{t+1}$ for $t = 1, \ldots, j-1$.

The following result ensures that $\psi_1$ is a valid supperadditive lifting function on $[0, D]$. In order to use this result notice that for $t = 1, \ldots, j-1$, $d^{k(t)} y_1 + c^{k(t)} y_2 \leq d^{k(t)} a^t + c^{k(t)} b^t$ is a valid inequality for $Y_{\leq}$ and that $b^1 = 0$. Notice that the "first" extreme point of $conv(Y_{\leq})$ is $(a^1, b^1) = (\lfloor \frac{D}{a_1} \rfloor, 0)$.

**Proposition 4.4** *Let $E, \alpha_1, \alpha_2, \alpha, a_1, a_2, D$ be the parameters satisfying the conditions of Proposition 4.3 and suppose that they satisfy, additionally, the following conditions:*

*(i) $B^1 = 0$,*

*(ii) for all $t = 2, \ldots, j$, the inequality $\alpha_1^t y_1 + \alpha_2^t y_2 \leq \alpha_1^t A^t + \alpha_2^t B^t$, where $\alpha_1^t = B^t - B^{t-1}, \alpha_2^t = A^{t-1} - A^t$, is valid for $Y_{\leq}$.*

*Then, the function $\psi_1$ is a superadditive valid lifting function on $[0, D]$.*

As the number of extreme points of $conv(Y_{\leq})$ is polynomial and, as those points can be generated in polynomial time using the polynomial version of Algorithm HW (see Hirschberg & Wong 1976), it follows that $\psi_1$ can be constructed in polynomial time.

# 5 Characterization of the lifting function for a facet defining inequality

In order to construct better superadditive valid lifting functions, this means, functions $\psi$ satisfying $\psi_1(z) \leq \psi(z) \leq \phi(z)$ for all $z \in [0, D]$ we need to characterize the lifting function $\phi$. In this section we impose the hypotheses that (4.1) defines a facet of $conv(Y_{\leq})$. Therefore, we assume the facet includes two points $(a^{j-1}, b^{j-1})$ and $(a^j, b^j)$ generated by the Algorithm HW and so we assume $\alpha_1 = d^{k(j)}$, $\alpha_2 = c^{k(j)}$ and $\alpha = d^{k(j)} a^j + c^{k(j)} b^j$. For the particular case of the facet defining inequality with coefficients $\alpha_1 = d^1 = 1$, $\alpha_2 = c^1$ we may consider the well known mixed integer rounding function (see Nemhauser & Wolsey 1988)

$$\psi(z) = \begin{cases} k, & \text{if } ka_1 < z \leq ka_1 + \gamma^{j-1}, k \in \{0, \ldots, \alpha_1\}, \\ k + \frac{z - ka_1 - \gamma^{j-1}}{a_1 - \gamma^{j-1}}, & \text{if } ka_1 + \gamma^{j-1} < z \leq (k+1)a_1, k \in \{0, \ldots, \alpha_1\}, \end{cases}$$

which is known to be superadditive. Therefore we will exclude this case from now on. Thus we assume $\frac{c^1}{d^1} < \frac{\alpha_2}{\alpha_1} = \frac{c^{k(j)}}{d^{k(j)}} \leq \frac{a_2}{a_1}$.

Figure 1 shows that $\phi$ is a stepwise function. The length of the first step is $\gamma(a^{j-1}, b^{j-1}) = D - a_1 a^{j-1} - a_2 b^{j-1} = 2$ and the following steps have length either 3 or 5. Initially, the hight of each step is increased by one unit which follows from the properties of the coefficients of the facet defining inequalities (see (i) in Lemma 3.1).

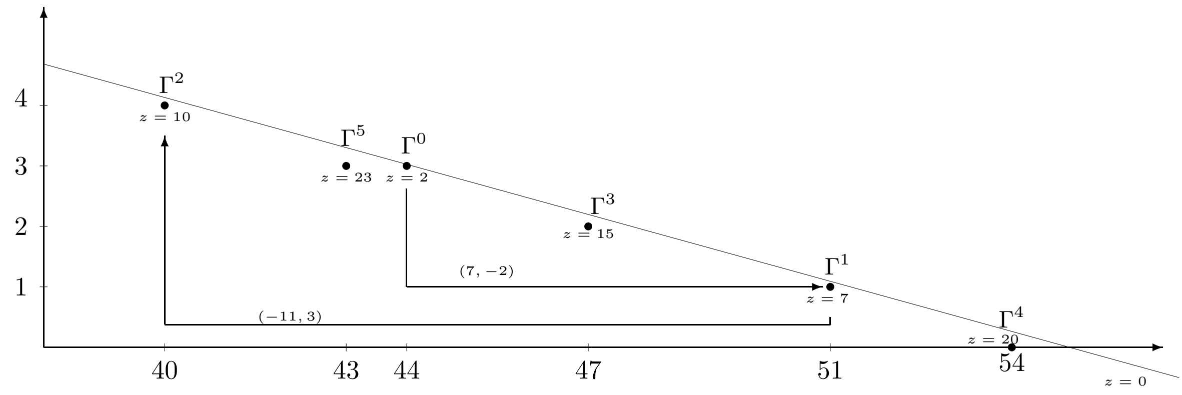


Figure 3: Computing function $\phi$ given in Example 4.1.

For simplicity we refer to solutions of $\phi(z)$ instead of solutions to the problem associated to $\phi(z)$. Figure 3 shows how the optimal solution to $\phi(z)$ can be obtained. Notice that starting from $(44, 3)$ (optimal solution to $\phi(2) = 0$) the optimal solutions to the corresponding problems when we increase $z$ can be obtained using two possible moves: $(2, -7)$ and $(-11, 3)$. This last move is only used when the first one can not be used because the second coordinate becomes negative. So, for a given integer $v$ we aim to find the solution associated to the greatest $z$ such that $\phi(z) = v$. To find this solution only these two "moves" have to be considered. Notice that after some iterations the value of the first coordinate may become negative and, in that case, we may conclude that for the corresponding $v \in \{1, \ldots, \alpha\}$ there is no $z$ such that $\phi(z) = v$.

Next we formalize these ideas and establish some results on the properties of function $\phi$. First we characterize the solutions that are obtained using these two moves and prove that these moves are cyclical where each cycle has length $\alpha_1$. We also prove that in the first cycle all the solutions obtained using these two moves are feasible. Then, in Theorem 5.11 we characterize the function $\phi$.

For $v = 1, \ldots, \alpha$, we define the set $Z(v) = \{0 \leq z \leq D : \phi(z) = v\}$. If $Z(v) \neq \emptyset$ then $Z(v)$ has a maximum. Below we show that this set is not empty for $v = 1, \ldots, \alpha_1$. We denote by $\Gamma^v = (p_1^v, p_2^v)$ the $v^{th}$ point generated using those two moves. Formally, for $v = 0, 1, \ldots \alpha - 1$,

$$\Gamma^{v+1} = (p_1^{v+1}, p_2^{v+1}) = \begin{cases} (p_1^v, p_2^v) + (c^{k(j)-1}, -d^{k(j)-1}) \text{ if } p_2^v - d^{k(j)-1} \geq 0, \\ (p_1^v, p_2^v) - (e^{\ell(k(j))}, -f^{\ell(k(j))}) \text{ if } p_2^v - d^{k(j)-1} < 0, \end{cases}$$

where $\Gamma^0 = (a^{j-1}, b^{j-1})$.

**Example 1 (Cont.).** See Figure 3. For $a_1 = 21$, $a_2 = 76$, we have $(c^1, d^1) = (3, 1)$, $(c^2, d^2) = (7, 2)$, $(c^3, d^3) = (18, 5)$ and $(e^1, f^1) = (4, 1)$, $(e^2, f^2) = (11, 3)$. Thus $\alpha_1 = d^3 = 5$, $\alpha_2 = c^3 = 18$. As $(a^{j-1}, b^{j-1}) = (44, 3), (a^j, b^j) = (26, 8)$, then $\Gamma^0 = (44, 3)$; $p_2^0 = 3 \geq d^2 = 2 \Rightarrow \Gamma^1 = (44, 3) + (7, -2) = (51, 1)$; $p_2^1 < d^2 \Rightarrow \Gamma^2 = (51, 1) + (-11, 3) = (40, 4)$; $p_2^2 > d^2 \Rightarrow \Gamma^3 = (47, 2)$; $p_2^3 > d^2 \Rightarrow \Gamma^4 = (54, 0)$; $p_2^4 < d^2 \Rightarrow \Gamma^5 = (43, 3)$.

**Lemma 5.1** $\tau(\Gamma^v) = v$, $\forall v \in \{0, \ldots, \alpha\}$.

**Proof:** We prove this result by induction. As the point $(a^{j-1}, b^{j-1})$ belongs to the facet defined by the valid inequality (4.1), we have $\tau(\Gamma^0) = 0$. Suppose that $\tau(\Gamma^v) = v$. Using Lemma 3.1, $-d^{k(j)}c^{k(j)-1} + c^{k(j)}d^{k(j)-1} = 1$. Thus, if $p_2^v - d^{k(j)-1} \geq 0$, we have $\tau(\Gamma^{v+1}) = \alpha - \alpha_1 p_1^{v+1} - \alpha_2 p_2^{v+1} = \alpha - \alpha_1 p_1^v - \alpha_2 p_2^v - \alpha_1 c^{k(j)-1} + \alpha_2 d^{k(j)-1} = \tau(\Gamma^v) - d^{k(j)}c^{k(j)-1} + c^{k(j)}d^{k(j)-1} = v + 1$. Similarly, since $d^{k(j)}e^{\ell(k(j))} - c^{k(j)}f^{\ell(k(j))} = 1$, we conclude that $\tau(\Gamma^{v+1}) = v + 1$ if $p_2^v - d^{k(j)-1} < 0$. □

**Lemma 5.2** $p_2^v < \alpha_1 = d^{k(j)}, \forall v \in \{0, \ldots, \alpha\}$.

**Proof:** Notice that $d^{k(j)} = f^{\ell(k(j))} + d^{k(j)-1}$. We use induction. For $v = 0$ we have $p_2^0 = b^{j-1} < d^{k(j)}$, otherwise $(a^{j-1}, b^{j-1})$ would not be an extreme point. Suppose that $p_2^v < f^{\ell(k(j))} + d^{k(j)-1}$. If $p_2^v < d^{k(j)-1}$ then $p_2^{v+1} = p_2^v + f^{\ell(k(j))} < d^{k(j)-1} + f^{\ell(k(j))}$. If $d^{k(j)-1} \leq p_2^v < d^{k(j)-1} + f^{\ell(k(j))}$ then $0 \leq p_2^{v+1} = p_2^v - d^{k(j)-1} < d^{k(j)-1} + f^{\ell(k(j))}$. □

Now we introduce parameters that indicate the move to take in each iteration. For $v = 1, \ldots, \alpha$ define:

$$\delta_v = \begin{cases} 1 \text{ if } \Gamma^v = \Gamma^{v-1} + (c^{k(j)-1}, -d^{k(j)-1}), \\ 0 \text{ if } \Gamma^v = \Gamma^{v-1} - (e^{\ell(j)-1}, -f^{\ell(k(j))}). \end{cases}$$

Let us denote $\Delta_p^q = \sum_{j=p}^q \delta_j$, for $q \geq p$ and $\Delta_p^q = 0$, otherwise. $\Delta_p^q$ indicates the number of moves using the direction $(c^{k(j)-1}, -d^{k(j)-1})$ to go from $\Gamma^{p-1}$ to $\Gamma^q$. The following results are useful to compute the values of these parameters.

**Lemma 5.3** $\Delta_1^v = \left\lfloor \dfrac{b^{j-1} + v \times f^{\ell(k(j))}}{\alpha_1} \right\rfloor, \forall v \in \{1 \ldots \alpha\}$.

**Proof:** Let $n = \Delta_1^v$. We know that $p_2^v = b^{j-1} - nd^{k(j)-1} + (v-n)f^{\ell(k(j))} = b^{j-1} - n(d^{k(j)-1} + f^{\ell(k(j))}) + vf^{\ell(k(j))} = b^{j-1} - nd^{k(j)} + vf^{\ell(k(j))}$. As $p_2^v \geq 0$ and, from Lemma 5.2, $p_2^v < d^{k(j)}$ we have,

$$\frac{b^{j-1} + v \times f^{\ell(k(j))}}{d^{k(j)}} - 1 < n \leq \frac{b^{j-1} + v \times f^{\ell(k(j))}}{d^{k(j)}}.$$

Thus, $n = \left\lfloor \dfrac{b^{j-1} + v \times f^{\ell(k(j))}}{d^{k(j)}} \right\rfloor$. □

Next we stat that the sequence of parameters $\delta^v$ is cyclical.

**Corollary 5.4** *Let $v = t\alpha_1 + r$ with $r \in \{1, \ldots, \alpha_1\}$ and $t \in \{1, \ldots, \lfloor \alpha/\alpha_1 \rfloor\}$. Then $\Delta_1^v = \Delta_1^r + tf^{\ell(k(j))}$.*

**Proof:** Considering in Lemma 5.3 $v = t\alpha_1 + r$ we have $\Delta_1^v = \lfloor \frac{b^{j-1} + rf^{\ell(k(j))}}{\alpha_1} + tf^{\ell(k(j))} \rfloor = \Delta_1^r + tf^{\ell(k(j))}$. □

Using Corollary 5.4 to compute the points $\Gamma^v$ we obtain the following result.

**Proposition 5.5** *Let $v = t\alpha_1 + r$ with $r \in \{1, \ldots, \alpha_1\}$ and $t \in \{1, \ldots, \lfloor \alpha/\alpha_1 \rfloor\}$. Then,*

$$\Gamma^v = \Gamma^r + t(-1, 0)$$

**Proof:** $\Gamma^v$ can be written as $\Gamma^v = \Gamma^0 + \sum_{i=1}^{v} \delta_i(d^{k(j)-1}, -c^{k(j)-1}) - \sum_{i=1}^{v}(1-\delta_i)(e^{\ell(k(j))}, -f^{\ell(k(j))})$

$$= \Gamma^0 + \sum_{i=t\alpha_1+1}^{v} \delta_i(c^{k(j)-1}, -d^{k(j)-1}) - \sum_{i=t\alpha_1+1}^{v}(1-\delta_i)(e^{\ell(k(j))}, -f^{\ell(k(j))})$$

$$+ \sum_{i=1}^{t\alpha_1} \delta_i(c^{k(j)-1}, -d^{k(j)-1}) - \sum_{i=1}^{t\alpha_1}(1-\delta_i)(e^{\ell(k(j))}, -f^{\ell(k(j))}).$$

Corollary 5.4 implies (i) $\delta_{t\alpha_1+k} = \delta_k$ for $k \in \{1, \ldots, \alpha_1\}$ and (ii) $\sum_{i=1}^{t\alpha_1}(1-\delta_i) = t\alpha_1 - \Delta_1^{t\alpha_1} = t(\alpha_1 - f^{\ell(k(j))}) = td^{k(j)-1}$. Using (i), (ii) and part (i) of Lemma 3.1, it follows that,

$$\Gamma^v = \Gamma^r + tf^{\ell(k(j))}(c^{k(j)-1}, -d^{k(j)-1}) - td^{k(j)-1}(e^{\ell(k(j))}, -f^{\ell(k(j))}) = \Gamma^r + t(-1, 0).$$

□

Next we present a technical result we will use later. The proof can be found in the Appendix.

**Lemma 5.6** $|\Delta_1^s - \Delta_{r+1}^{r+s}| \leq 1$ *for all* $s, r \in \{1, \ldots, \alpha\}$, *such that* $s + r \in \{1, \ldots, \alpha\}$.

It is important to notice that there is no guarantee that $p_1^v \geq 0$. Therefore $\Gamma^v$ may not be a feasible solution to $\phi(z)$. Next consider, for each $z \in [0, D]$, the following relaxation of the problem associated to the lifting function where $y_1$ may assume negative values:

$$\underline{\phi}(z) = \min\{\alpha - \alpha_1 y_1 - \alpha_2 y_2 : a_1 y_1 + a_2 y_2 \leq D - z,\ y_1 \in \mathbb{Z}, y_2 \in \mathbb{N}_0\}.$$

Let $z_v = \max\{0 \leq z \leq D : \underline{\phi}(z) = v\}$, for $v = 0, \ldots, \alpha$. Notice that $\underline{\phi}(\Gamma^v) = v$ so, there always exists a $z \in [0, D]$ such that $\underline{\phi}(z) = v$. For $v = 0$ we have $z_0 = \gamma(a^{j-1}, b^{j-1}) = \gamma^{j-1}$. As $\underline{\phi}$ is a non decreasing function, $z_{v+1} \geq z_v$ for all $v = 0, \ldots, \alpha - 1$. Next we explain how to obtain the optimal solution to $\underline{\phi}(z_v)$ for all $v = 1, \ldots, \alpha$. Notice that if $\underline{\phi}(z_v)$ has more than one optimal solution then $\dfrac{\alpha_2}{\alpha_1} = \dfrac{c^{k(j)}}{d^{k(j)}} = \dfrac{a_2}{a_1}$.

**Proposition 5.7** $\Gamma^v = (p_1^v, p_2^v)$ *is an optimal solution to* $\underline{\phi}(z_v)$ *for* $v = 1, \ldots, \alpha$.

**Proof:** We prove by induction. $\Gamma^0 = (a^{j-1}, b^{j-1})$ is an optimal solution to $\underline{\phi}(z_0)$. Suppose $\Gamma^v = (p_1^v, p_2^v)$ is an optimal solution to $\underline{\phi}(z_v)$ and that $(p_1, p_2)$ is an optimal solution to $\underline{\phi}(z_{v+1})$. We are assuming $\dfrac{c^1}{d^1} < \dfrac{\alpha_2}{\alpha_1} = \dfrac{c^{k(j)}}{d^{k(j)}} \leq \dfrac{a_2}{a_1}$. Next we consider two cases: (i) $p_1 \geq p_1^v$ and $p_2 \leq p_2^v$; (ii) $p_1 \leq p_1^v$ and $p_2 \geq p_2^v$. The proof that it can not occur $p_1 > p_1^v, p_2 > p_2^v$ and $p_1 < p_1^v, p_2 < p_2^v$, is trivial.

Consider case (i). Let $(a, b) = (p_1 - p_1^v, p_2^v - p_2)$. Now we prove that $(a, b) \in Cone\{(a_2, a_1), (\lfloor \frac{a_2}{a_1} \rfloor, 1)\}$. Notice that $(c^1, d^1) = (\lfloor \frac{a_2}{a_1} \rfloor, 1)$. Let $(a, b) = \lambda_1(a_2, a_1) + \lambda_2(c^1, d^1)$. We must show that $\lambda_1, \lambda_2 \geq 0$. The case $\lambda_1, \lambda_2 < 0$ can not occur because $a \geq 0, b \geq 0$. If $\lambda_1 \geq 0$ and $\lambda_2 < 0$ we obtain $z_{v+1} = D - a_1 p_1 - a_2 p_2 = D - a_1(p_1^v + a) - a_2(p_2^v - b) = D - a_1 p_1^v - a_2 p_2^v - a_1(\lambda_1 a_2 + \lambda_2 c^1) + a_2(\lambda_1 a_1 + \lambda_2 d^1) = z_v + \lambda_1(-a_1 a_2 + a_2 a_1) + \lambda_2(-a_1 c^1 + a_2 d^1) = z_v + \lambda_2(-a_1 c^1 + a_2 d^1) < z_v$ because, from (ii) of Lemma 3.1, $-a_1 c^1 + a_2 d^1 > 0$. This is absurd since $z_{v+1} \geq z_v$.

Consider $\lambda_1 < 0$ and $\lambda_2 \geq 0$. First assume $b > 0$. Notice that inequalities $\lambda_1 < 0$, $\lambda_2 \geq 0$ and $\dfrac{a_2}{a_1} > \dfrac{c^1}{d^1}$ imply $\dfrac{a}{b} = \dfrac{\lambda_1 a_2 + \lambda_2 c^1}{\lambda_1 a_1 + \lambda_2 d^1} < \dfrac{c^1}{d^1}$. Thus, since $\dfrac{a}{b} < \dfrac{c^1}{d^1} < \dfrac{\alpha_2}{\alpha_1}$ we have $\alpha_2 b > \dfrac{c^1}{d^1} b\alpha_1 > \alpha_1 a$ and as $d^1 = 1$ and all the coefficients are non negative integers, it follows that $\alpha_2 b \geq c^1 b\alpha_1 + 1 \geq \alpha_1 a + 2$. Therefore $v+1 = \alpha - \alpha_1 p_1 - \alpha_2 p_2 = \alpha - \alpha_1(p_1^v + a) - \alpha_2(p_2^v - b) = \alpha - \alpha_1 p_1^v - \alpha_2 p_2^v - \alpha_1 a + \alpha_2 b \geq v+2$ which is absurd. If $b = 0$ then $v+1 = \alpha - \alpha_1 p_1 - \alpha_2 p_2 = \alpha - \alpha_1(p_1^v + a) - \alpha_2(p_2^v - b) = v - \alpha_1 a < v$.

From Lemma 3.1, part (iv), we know that $(a, b)$ can be obtained as non negative integer combination of the vectors in $G_1 = \{(c^1, d^1), \ldots, (c^{n_1}, d^{n_1})\}$. Thus, $(a, b) = \sum_{k=1}^{n_1} \rho_k (c^k, d^k)$ where $\rho_k \geq 0$ and integer for all $k = 1, \ldots, n_1$.

Since $v+1 = \alpha - \alpha_1 p_1 - \alpha_2 p_2 = \alpha - \alpha_1 p_1^v - \alpha_2 p_2^v - \alpha_1 a + \alpha_2 b = v - \alpha_1 a + \alpha_2 b \Rightarrow -\alpha_1 a + \alpha_2 b = 1$. Then, $1 = -\alpha_1 a + \alpha_2 b = \sum_{k=1}^{n_1} \rho_k u_k$ where $u_k = -\alpha_1 c^k + \alpha_2 d^k$.

Using Lemma 3.1 it can be shown that: $u_k < 0$ for $k > k(j)$, $u_k = 0$ for $k = k(j)$, $u_k = 1$ for $k = k(j) - 1$, $u_k > 1$ for $k < k(j) - 1$ (notice that $\alpha_1 = d^{k(j)}, \alpha_2 = c^{k(j)}$). From Lemma 5.2 we have, for all $v \in \{1, \ldots, \alpha\}$, $p_2^v < d^{k(j)}$. Thus $\rho_k = 0$ for $k > k(j) - 1$, which implies $\rho_k = 1$ for $k = k(j) - 1$ and $\rho_k = 0$ for $k < k(j) - 1$. Thus $(a, b) = (c^{k(j)-1}, d^{k(j)-1})$.

The proof of case (ii) is similar. Consider $(a, b) = (p_1^v - p_1, p_2 - p_2^v)$. Again, proving that $(a, b)$ belongs to $Cone\{(\lceil \frac{a_2}{a_1} \rceil, 1), (a_2, a_1)\}$ then, from Lemma 3.1, $(a, b)$ can be written as a non negative linear combination of the vectors in $G_2 = \{(e^1, f^1), \ldots, (e^{n_2}, f^{n_2})\}$, this is, $(a, b) = \sum_{\ell=1}^{n_2} \rho_\ell (e^\ell, f^\ell)$ where $\rho_\ell \geq 0$ and integer, for all $\ell = 1, \ldots, n_2$. Similarly we conclude that $1 = \alpha_1 a - \alpha_2 b = \sum_{\ell=1}^{n_2} \rho_\ell \nu_\ell$ where $\nu_\ell = \alpha_1 e^\ell - \alpha_2 f^\ell$. Since $\frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1} \leq \frac{e^\ell}{f^\ell}$, we have $\nu_\ell > 0$ for all $\ell$ and, in particular, $\nu_\ell = 1$ for $\ell = \ell(k(j)), \ldots, \ell(k(j) + 1)$ and $\nu_\ell > 1$ for all other values of $\ell$. Thus, $\rho_{\ell^*} = 1$ where $\ell^* \in \{\ell(k(j)), \ldots, \ell(k(j) + 1)\}$ and $\rho_\ell = 0$ for $\ell \neq \ell^*$. We must prove that $\ell^* = \ell(k(j))$. Consider the solution $\Gamma^v - (e^{\ell(k(j))}, -f^{\ell(k(j))})$. Let $z' = D - a_1(p_1^v - e^{\ell(k(j))}) - a_2(p_2^v + f^{\ell(k(j))})$. As $z' > z_v$ and as $\tau(\Gamma^v - (e^{\ell(k(j))}, -f^{\ell(k(j))})) = v + 1$ it follows that $\underline{\phi}(z') = v + 1$. By definition of $z_{v+1}$ we have $z_{v+1} \geq z'$. Suppose $\rho_\ell = 1$ for $\ell \in \{\ell(k(j)) + 1, \ldots, \ell(k(j) + 1)\}$. Thus $(a, b) = (e^\ell, f^\ell)$. Noticing that, for $\ell > \ell(k(j))$, $a_1 e^\ell - a_2 f^\ell < a_1 e^{\ell(k(j))} - a_2 f^{\ell(k(j))}$, it follows that $z_{v+1} = D - a_1 p_1 - a_2 p_2 = D - a_1 p_1^v - a_2 p_2^v + a_1 e^\ell - a_2 f^\ell = z_v + a_1 e^\ell - a_2 f^\ell < z_v + a_1 e^{\ell(k(j))} - a_2 f^{\ell(k(j))} = z' \leq z_{v+1}$, which is absurd. Thus $\rho_\ell = 0$ for $\ell = \ell(k(j)) + 1, \ldots, \ell(k(j) + 1)$ and $\rho_{\ell(k(j))} = 1$.

If $p_2^v - d^{k(j)-1} < 0$ then $\Gamma^v - (-c^{k(j)-1}, d^{k(j)-1})$ is not feasible. In this case (i) can not occur because, as we saw in the proof of (i), $\Gamma^v - (-c^{k(j)-1}, d^{k(j)-1})$ is the unique point, $(p_1, p_2)$, satisfying $p_1 \geq p_1^v$, $p_2 \leq p_2^v$ and $\tau(p_1, p_2) = v + 1$. If $p_2^v - d^{k(j)-1} \geq 0$ then

$$\tau(\Gamma^v - (-c^{k(j)-1}, d^{k(j)-1})) = \tau(\Gamma^v - (e^{\ell(k(j))}, -f^{\ell(k(j))})) = v + 1.$$

Noticing that $-a_1 c^{k(j)-1} + a_2 d^{k(j)-1} > a_1 e^{\ell(k(j))} - a_2 f^{\ell(k(j))}$ we can prove, as we did above to prove $\ell^* = \ell(k(j))$, that $z_{v+1}$ is obtained at the point $\Gamma^v - (-c^{k(j)-1}, d^{k(j)-1})$. Thus $\Gamma^{v+1} = \Gamma^v + (c^{k(j)-1}, -d^{k(j)-1})$ is optimal to $\underline{\phi}(z_{v+1})$. □

Starting at $\Gamma^0 = (a^{j-1}, b^{j-1})$, we may compute the optimal solution to $\underline{\phi}(z_v)$ for all $v = 1, \ldots, \alpha$, iteratively. As the set of feasible solutions to $\phi(z)$ is a subset of the set of feasible solution to $\underline{\phi}(z)$ we have the following consequence.

**Corollary 5.8** *If $p_1^v \geq 0$ then $\Gamma^v$ is the optimal solution to $\phi(z)$ for all $z_{v-1} < z \leq z_v$ and, in this case $\phi(z) = v$.*

**Lemma 5.9** *$Z(v) \neq \emptyset$ if and only if $\Gamma^v$ is a feasible solution to $\phi(z)$, for all $z_{v-1} < z \leq z_v$.*

**Proof:** If $\Gamma^v$ is a feasible solution to $\phi(z)$, then $Z(v) \neq \emptyset$ because $\gamma(\Gamma^v) = \gamma(p_1^v, p_2^v)$ belongs to $Z(v)$. To prove the implication in the other direction suppose that $\Gamma^v$ is not a feasible solution and that $Z(v) \neq \emptyset$. Then there exists $z' \in Z(v)$ such that $\phi(z') = v$. Let $(a, b)$ be an optimal solution to $\phi(z')$. As $v = \tau(\Gamma^v) = \tau(a, b)$ then $\alpha - \alpha_1 p_1^v - \alpha_2 p_2^v = \alpha - \alpha_1 a - \alpha_2 b \Rightarrow \alpha_1(a - p_1^v) = \alpha_2(p_2^v - b) \Rightarrow \dfrac{a - p_1^v}{p_2^v - b} = \dfrac{\alpha_2}{\alpha_1} \leq \dfrac{a_2}{a_1}$. Notice that $a \geq 0 > p_1^v$, thus $b < p_2^v$. The case $\dfrac{\alpha_2}{\alpha_1} = \dfrac{a_2}{a_1}$ can not occur because as $\dfrac{\alpha_2}{\alpha_1}$ is the best approximation to $\dfrac{a_2}{a_1}$ for that size denominator, we would have $p_2^v - b \geq d^{k(j)}$, contradicting Lemma 5.2. So, $\dfrac{a - p_1^v}{p_2^v - b} < \dfrac{a_2}{a_1} \Rightarrow a_1 a - a_1 p_1^v < a_2 p_2^v - a_2 b \Rightarrow -a_1 p_1^v - a_2 p_2^v < -a_1 a - a_2 b$ $\Rightarrow D - a_1 p_1^v - a_2 p_2^v < D - a_1 a - a_2 b \Rightarrow z_v = \gamma(p_1^v, p_2^v) < \gamma(a, b)$. Setting $z = \gamma(a, b) > z_v$ we have $\underline{\phi}(z) = \phi(z) = v$, obtained at $(a, b)$, which contradicts the definition of $z_v$. $\square$

It remains to determine the value of $\phi(z)$, $z_{v-1} < z \leq z_v$, when $Z(v) = \emptyset$.

**Proposition 5.10** *Let $z_r < z \leq z_{r+1}$ with $r \in \{1, \ldots, \alpha - 1\}$. Then $\phi(z) = v$ where $v = \min\{j \in \{r + 1, \ldots, \alpha\} : p_1^j \geq 0\}$.*

**Proof:** Notice that $\phi(z)$ is integer for all $z$. For $z > z_r$ we have $\phi(z) > r$. From Lemma 5.9, $Z(k) = \emptyset$ for all $k = r + 1, \ldots, v - 1$, i.e., there is no $z \leq z_{v-1}$ such that $\phi(z) = k$, for $k = r + 1, \ldots, v - 1$, which implies $\phi(z) \geq v$, for $z > z_r$. On the other hand, as $\Gamma^v$ is a feasible solution to $\phi(z)$ with $z_r < z \leq z_{r+1}$, then $\phi(z) \leq v$. So $\phi(z) = v$. $\square$

Finally, from this discussion we can characterize function $\phi$.

**Theorem 5.11** *Let $v_k$ denote the $k^{th}$ value in $\{1, \ldots, \alpha\}$ such that $Z(v_k) \neq \emptyset$ and define $v_0 = 0$. Thus*

$$\phi(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq z_0, \\ v_{k+1}, & \text{if } z_{v_k} < z \leq z_{v_{k+1}}, k \in \{0, \ldots, n^*\}, \end{cases}$$

*where $n^*$ denotes the number of values $v \in \{1, \ldots, \alpha\}$ for which $Z(v) \neq \emptyset$.*

**Lemma 5.12** *$Z(v) \neq \emptyset$ for all $v = 0, \ldots, \alpha_1$.*

**Proof:** Suppose that there exists $v \in \{1, \ldots, \alpha_1\}$ such that $Z(v) = \emptyset$ and $Z(v - 1) \neq \emptyset$. Notice that $Z(0) \neq \emptyset$. Thus $p_2^{v-1} - d^{k(j)-1} < 0$ and $p_1^{v-1} - e^{\ell(k(j))} < 0 \Rightarrow p_1^{v-1} \leq e^{\ell(k(j))} - 1$. Since $b^j = b^{j-1} + rd^{k(j)}$ for some positive integer $r$, then $b^j \geq d^{k(j)} = d^{k(j)-1} + f^{\ell(k(j))}$. So, as $p_2^{v-1} - d^{k(j)-1} < 0$ we have $p_2^{v-1} < b^j - f^{\ell(k(j))}$. Hence $\phi(z_{v-1}) = \alpha - \alpha_1 p_1^{v-1} - \alpha_2 p_2^{v-1} = \alpha_1(a^j - p_1^{v-1}) + \alpha_2(b^j - p_2^{v-1}) > \alpha_1(a^j - e^{\ell(k(j))} + 1) + \alpha_2(b^j - b^j + f^{\ell(k(j))}) \geq \alpha_1(0 - e^{\ell(k(j))} + 1) + \alpha_2 f^{\ell(k(j))}$ $= d^{k(j)} - d^{k(j)} e^{\ell(k(j))} + c^{k(j)} f^{\ell(k(j))} = d^{k(j)} - 1$. Therefore, $v - 1 = \phi(z_{v-1}) > \alpha_1 - 1 \Rightarrow v > \alpha_1$. $\square$

Lemma 5.12 states that for all $v = 0, \ldots, \alpha_1$, $\Gamma^v$ is a feasible solution and therefore the optimal solution to $\phi(z)$.

Let $w_1 = d^{k(j)-1} a_2 - c^{k(j)-1} a_1$ and $w_2 = -f^{\ell(k(j))} a_2 + e^{\ell(k(j))} a_1$. Thus, $w_1$ is the length of the steps corresponding to moves based in the vector $(c^{k(j)-1}, -d^{k(j)-1})$ and $w_2$ is the length of the steps corresponding to moves based in the vector $(-e^{\ell(k(j))}, f^{\ell(k(j))})$. Using the notation of Algorithm HW we have $w_1 = R_{\leq}(c^{k(j)-1}, d^{k(j)-1})$ and $w_2 = R_{\geq}(e^{\ell(k(j))}, f^{\ell(k(j))})$. Hence $w_1 \geq w_2$. Notice that $w_1 - w_2 = -a_1(c^{k(j)-1} + e^{\ell(k(j))}) + a_2(d^{k(j)-1} + f^{\ell(k(j))})) = -a_1 c^{k(j)} + a_2 d^{k(j)} = R_{\leq}(c^{k(j)}, d^{k(j)})$. As $\gamma^{j-1} \geq$

$\gamma^j - a_1 c^{k(j)} + a_2 d^{k(j)}$ (observe that $(a^j, b^j) = (a^{j-1}, b^{j-1}) + r(-c^{k(j)}, d^{k(j)})$ for some integer $r > 0$) and as $\gamma^j \geq 0$ we have $\gamma^{j-1} \geq w_1 - w_2$. Next we state these conclusions as a result.

**Lemma 5.13** *(i)* $w_1 \geq w_2 \geq 0$; *(ii)* $\gamma^{j-1} \geq w_1 - w_2$.

As a corollary of Proposition 5.5 and noticing that $\alpha_1 w_2 + f^{\ell(k(j))}(w_1 - w_2) = (\alpha_1 - f^{\ell(k(j))})w_2 + f^{\ell(k(j))}w_1 = d^{k(j)-1}w_2 + f^{\ell(k(j))}w_1 = a_1$ (last equality follows from part (i) of Lemma 3.1) we obtain the following result.

**Corollary 5.14** *Let* $v = t\alpha_1 + r$ *with* $r \in \{0, \ldots, \alpha_1 - 1\}$ *and* $v \in \{1, \ldots, \alpha\}$*. Then* $z_v = \gamma^{j-1} + vw_2 + (w_1 - w_2)\Delta_1^v = \gamma^{j-1} + (t\alpha_1 + r)w_2 + (tf^{\ell(k(j))} + \Delta_1^r)(w_1 - w_2) = \gamma^{j-1} + ta_1 + rw_2 + \Delta_1^r(w_1 - w_2)$.

It can be verified that $z_{v_{n^*}} = z_\alpha = D$.

**Remark 5.15** *Although we are considering* $z \in [0, D]$*, it is important for the study in Section 7, to describe* $\phi$ *on* $]-\infty, 0[$*. In this case, for* $v \in \mathbb{N}$*, we can obtain an optimal solution,* $\Gamma^{-v} = (p_1^{-v}, p_2^{-v})$*, for* $\phi(z_{-v})$ *iteratively,*

$$\Gamma^{-v} = \Gamma^{-v+1} + \begin{cases} (e^{\ell(k(j))}, -f^{\ell(k(j))}) \text{ if } p_2^{-v+1} - f^{\ell(k(j))} \geq 0, \\ (-c^{k(j)-1}, d^{k(j)-1}) \text{ if } p_2^{-v+1} - f^{\ell(k(j))} < 0, \end{cases}$$

*where* $\Gamma^0 = (a^{j-1}, b^{j-1})$*. Again we define, for* $v \in \mathbb{N}$*,*

$$\delta_{-v} = \begin{cases} 1 \text{ if } \Gamma^{-v} = \Gamma^{-v+1} + (e^{\ell(k(j))}, -f^{\ell(k(j))}), \\ 0 \text{ if } \Gamma^{-v} = \Gamma^{-v+1} + (-c^{k(j)-1}, d^{k(j)-1}). \end{cases}$$

*For* $v \in \mathbb{N}$ *and for* $z_{-v-1} < z \leq z_{-v}$ *we have* $\phi(z) = -v$*, where* $z_{-v} = \gamma^{j-1} - vw_2 - (w_1 - w_2)\sum_{i=1}^{v} \delta_{-i}$ *and* $\Gamma^{-v}$ *is an optimal solution to* $\phi(z)$*. This characterization, for* $z < 0$*, can be proven as we proved the corresponding properties for* $\underline{\phi}$*, with* $z \geq 0$*. Notice that for all* $v \in \mathbb{N}$*,* $\{z \leq 0 : \phi(z) = -v\} \neq \emptyset$*.*

# 6 Superadditive valid lifting functions for facet defining inequalities

In this section we construct new superadditive valid lifting functions, depending on the parameters $\gamma^{j-1}, w_1, w_2, \delta_1, \ldots, \delta_\alpha$, used to characterize the lifting function in the previous section. We start by considering functions with a similar analytical expression for intervals with the same length. To do that consider the family $\mathcal{C}$ of functions on $[0, D]$ such that $\chi \in \mathcal{C}$ if and only if $\chi$ is a superadditive valid lifting function and can be written as:

$$\chi(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq z_0, \\ v + \chi_1(z - z_v), & \text{if } z_v < z \leq z_{v+1} \text{ and } \delta_{v+1} = 1, v = 0, \ldots, \alpha - 1, \\ v + \chi_2(z - z_v), & \text{if } z_v < z \leq z_{v+1} \text{ and } \delta_{v+1} = 0, v = 0, \ldots, \alpha - 1, \end{cases}$$

where, for $i \in \{1, 2\}$, $\chi_i(r) = \begin{cases} \frac{r}{k_i}, & \text{if } r \leq k_i, \\ 1, & \text{if } r > k_i, \end{cases}$ and $0 < k_1 < w_1$, $0 < k_2 < w_2$. This family will also be very useful since it can easily be extended for the knapsack inequalities with one continuous variable as we will see in Section 7.

Consider the function $\psi_2$ that belongs to $\mathcal{C}$ with $k_1 = w_1$ and $k_2 = w_2$, defined by,

$$\psi_2(z) = \begin{cases} 0, & \text{if } 0 \le z \le z_0, \\ v + \frac{z - z_v}{z_{v+1} - z_v}, & \text{if } z_v < z \le z_{v+1}, v = 0, \dots, \alpha - 1. \end{cases}$$

An example of the graph of such function is shown in Figure 4.

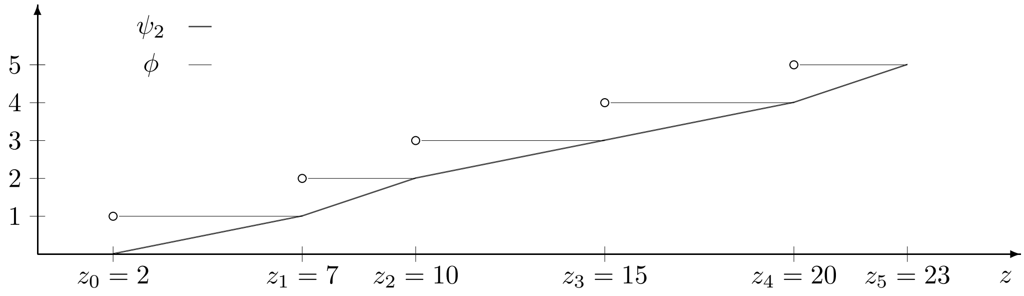


Figure 4: Example of $\psi_2$.

**Proposition 6.1** *Function $\psi_2$ is a valid supperadditive valid function on* $[0, D]$.

We omit the proof because below we construct a new function, $\psi_3$, dominating $\psi_2$ and the proof of superadditivity of $\psi_2$ is similar to the proof of superadditivity of $\psi_3$.

**Lemma 6.2** $\psi_1(z) \le \psi_2(z), \forall z \in [0, D]$.

**Proof:** As $\psi_1$ is convex on $[0, D]$ then its epigraph, $Epi(\psi_1) = \{(y, z) \in \mathbb{R} \times [0, D] : y \ge \psi_1(z)\}$, is a convex set. Proposition 4.4 ensures $(z, \phi(z)) \in Epi(\psi_1)$ for all $z \in [0, D]$. For all $z \in [z_0, D]$, the point $(z, \psi_2(z))$ can be written as linear convex combination of two points: $(z_v, \phi(z_v))$ and $(z_{v+1}, \phi(z_{v+1}))$ in $Epi(\psi_1)$, where $z_v \le z \le z_{v+1}$ (for $z \le z_0$ we have $\psi_2(z) = \phi(z)$). So $(z, \psi_2(z)) \in Epi(\psi_1)$ which implies $\psi_1(z) \le \psi_2(z), \forall z \in [0, D]$. □

Although $\psi_1$ is dominated by $\psi_2$, it is important to notice that $\psi_1(z)$ can be computed in a polynomial number of steps while $\psi_2(z)$ may require a non polynomial number of elementary operations because we need to compute $\delta_1, \dots, \delta_{\alpha_1}$. Notice also that the function $\psi_2$ requires more restricted conditions than $\psi_1$, namely, it requires that (4.1) defines a facet of $conv(Y_{\le})$.

Next we construct a better superadditive valid lifting function. Consider $\psi_3$ (see Figure 5) defined by

$$\psi_3(z) = \begin{cases} 0, & \text{if } 0 \le z \le z_0, \\ v + \frac{z - z_v}{\mu}, & \text{if } z_v < z \le z_v + \mu \text{ and } \delta_{v+1} = 0, v = 0, \dots, \alpha - 1, \\ v + 1, & \text{if } z_v + \mu < z \le z_{v+1} \text{ and } \delta_{v+1} = 0, v = 0, \dots, \alpha - 1, \\ v + \frac{z - z_v}{\lambda}, & \text{if } z_v < z \le z_v + \lambda \text{ and } \delta_{v+1} = 1, v = 0, \dots, \alpha - 1, \\ v + 1, & \text{if } z_v + \lambda < z \le z_{v+1} \text{ and } \delta_{v+1} = 1, v = 0, \dots, \alpha - 1, \end{cases}$$

where $\mu = w_1 - \gamma^{j-1}$ and $\lambda = w_1 + (w_1 - w_2) - \gamma^{j-1}$. Notice that as $\gamma^{j-1} \ge w_1 - w_2$ we have $\mu \le w_2$ and $\lambda \le w_1$. Therefore, when $\gamma^{j-1} > w_1 - w_2$, $\psi_3$ dominates $\psi_2$.

**Proposition 6.3** *The function $\psi_3$ is a superadditive valid lifting function.*

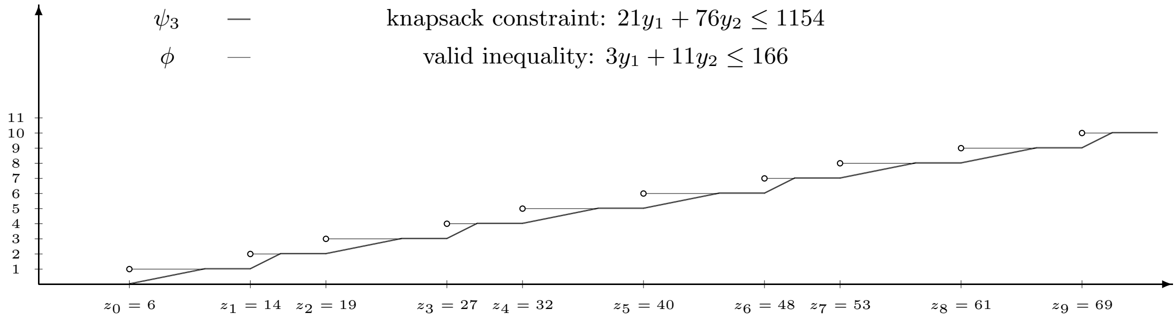


Figure 5: Example of $\psi_3$ with $\mu = 2$ and $\lambda = 5$.

The proof is left to the Appendix.

In order to find better superadditive valid lifting functions we must consider more restrictive conditions on the sequence of the values of the parameters $\delta$. Next we construct several valid lifting functions satisfying $\psi_3(z) \leq \psi_4(z) \leq \psi_5(z) \leq \psi_6(z) \leq \phi(z)$, $\forall z \in [0, D]$, and indicate sufficient conditions on the parameters $\delta$ under which superadditivity holds. In that case these functions belong to $\mathcal{C}$.

**Proposition 6.4** *If, for each $n$ and $m$ one of the following conditions hold: (i) $\Delta_1^m \geq \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m < \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$, then the function*

$$\psi_4(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq z_0, \\ v + \frac{z - z_v}{\mu}, & \text{if } z_v < z \leq z_v + \mu, v = 0, \ldots, \alpha - 1, \\ v + 1, & \text{if } z_v + \mu < z \leq z_{v+1}, v = 0, \ldots, \alpha - 1, \end{cases}$$

*where $\mu = w_1 - \gamma^{j-1}$, is a superadditive valid lifting function.*

The proof is similar to the proof of case $\delta_{n+1} = \delta_{m+1} = 0$ in Proposition 6.3. Noticing that the sequence of the parameters is cyclical it is only necessary to check conditions (i) and (ii) for $n = 1, \ldots, \alpha_1 - 1$ and $m = 1, \ldots, \alpha_1$.

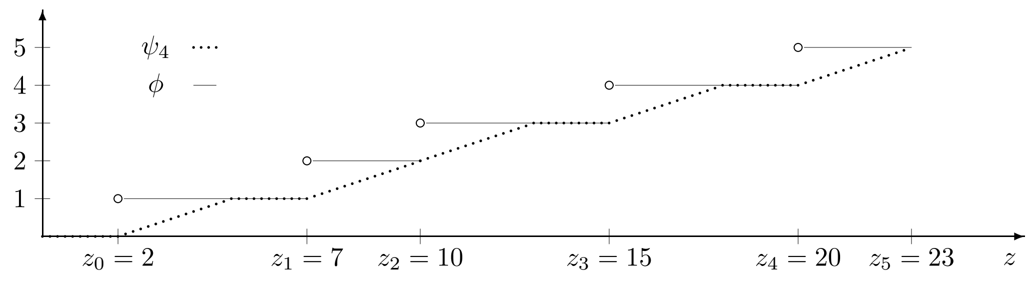


Figure 6: Example of $\psi_4$ with $\mu = 3$.

If more restrictive conditions on the sequence of the values of the parameters $\delta$ are verified we can use a better superadditive valid lifting function.

**Proposition 6.5** *If, for each $n$ and $m$, one of the following conditions hold: (i) $\Delta_1^m > \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m = \Delta_{n+1}^{n+m}$, $\delta_{n+1} = \delta_{m+1} = \delta_{n+m+1} = 1$; (iii) $\Delta_1^m = \Delta_{n+1}^{n+m}$, $\delta_{n+m+1} = 0$, then*

$$\psi_5(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq z_0, \\ v + \frac{z - z_v}{\theta}, & \text{if } z_v < z \leq z_v + \theta \text{ and } \delta_{v+1} = 0, v = 0, \ldots, \alpha - 1, \\ v + 1, & \text{if } z_v + \theta < z \leq z_{v+1} \text{ and } \delta_{v+1} = 0, v = 0, \ldots, \alpha - 1, \\ v + \frac{z - z_v}{\mu}, & \text{if } z_v < z \leq z_v + \mu \text{ and } \delta_{v+1} = 1, v = 0, \ldots, \alpha - 1, \\ v + 1, & \text{if } z_v + \mu < z \leq z_{v+1} \text{ and } \delta_{v+1} = 1, v = 0, \ldots, \alpha - 1, \end{cases}$$

*where $\mu = w_1 - \gamma^{j-1}$ and $\theta = w_2 - \gamma^{j-1}$, is a superadditive valid lifting function.*

The proof is similar to the proof of Proposition 6.3. However, it is important to notice that if $\gamma^{j-1} \geq w_2$, then it can be proven that there are $n$ and $m$ such that $\Delta_1^m < \Delta_{n+1}^{n+m}$ and hence the hypothesis of Proposition 6.5 do not hold.

Under further more restrict conditions superadditivity of $\psi_6$ holds.

**Proposition 6.6** *If, for each $n$ and $m$ one of the following conditions hold: (i) $\Delta_1^m > \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$, then*

$$\psi_6(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq z_0, \\ v + \frac{z - z_v}{\theta}, & \text{if } z_v < z \leq z_v + \theta, v = 0, \ldots, \alpha - 1, \\ v + 1, & \text{if } z_v + \theta < z \leq z_{v+1}, v = 0, \ldots, \alpha - 1, \end{cases}$$

*where $\theta = w_2 - \gamma^{j-1}$, is a superadditive valid lifting function.*

The proof is similar to the proof of case $\delta_{n+1} = \delta_{m+1} = 0$ in Proposition 6.3, so it will be omitted.

Consider $v^* = \max\{v \in \{1, \ldots, \alpha\} : \underline{\phi}(z) = \phi(z)$ for all $0 \leq z \leq z_v\}$. Next we analise some properties of these valid lifting functions on $[0, z_{v^*}]$. On $[z_{v^*}, D]$, $\phi$ may increase faster in some intervals since solutions $\Gamma^v$ with $v > z_{v^*}$ may be unfeasible.

Notice that $\psi_3$ is a superadditive function while superadditivity of $\psi_4$, $\psi_5$, $\psi_6$, requires more restrict conditions. All these four functions, when superadditivity holds, belong to $\mathcal{C}$. Next we prove that for each specific sequence of parameters $\delta$ one of these functions is the best function in $\mathcal{C}$, this means, there is no function in $\mathcal{C}$ dominating the corresponding function.

**Proposition 6.7** *Consider the conditions for superadditivity of $\psi_6$, $\psi_5$, $\psi_4$, respectively:*
*(C6) For each $n$ and $m$ one of the following conditions hold: (i) $\Delta_1^m > \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$.*
*(C5) For each $n$ and $m$ one of the following conditions hold: (i) $\Delta_1^m > \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$; (iii) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+1} = \delta_{m+1} = \delta_{n+m+1} = 1$.*
*(C4) For each $n$ and $m$ one of the following conditions hold: (i) $\Delta_1^m \geq \Delta_{n+1}^{n+m}$; (ii) $\Delta_1^m < \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$.*

*Let $\chi \in \mathcal{C}$ be a valid lifting function for $\phi$.*
*(a) If (C6) hold on $[0, z_{v^*}]$, then $\chi(z) \leq \psi_6(z)$ for all $z \in [0, z_{v^*}]$.*
*(b) If (C5) hold and (C6) do not hold on $[0, z_{v^*}]$, then $\chi(z) \leq \psi_5(z)$ for all $z \in [0, z_{v^*}]$.*
*(c) If (C4) hold and (C5) do not hold on $[0, z_{v^*}]$, then $\chi(z) \leq \psi_4(z)$ for all $z \in [0, z_{v^*}]$.*
*(d) If (C4) do not hold $[0, z_{v^*}]$, then $\chi(z) \leq \psi_3(z)$ for all $z \in [0, z_{v^*}]$.*

**Proof:** Consider case (a). Suppose that for $t = z_n + r$ and $r < \theta$, $\chi(t) > \psi_6(t)$ (for $r \geq \theta$ and $n \leq v^*$ we have $\psi_6(t) = \phi(t)$). Let $t' = \gamma^{j-1} + \theta - r$. Thus $\chi(t + t') = \chi(z_n + r + \gamma^{j-1} + \theta - r)$ $= \chi(z_n + w_2) \leq \phi(z_{n+1}) = n + 1$. On the other hand, $\chi(t + t') \geq \chi(t) + \chi(t') > \psi_6(t) + \psi_6(t')$ $= n + r/\theta + (\theta - r)/\theta = n + 1$ which is absurd.

Now we prove (b). First we prove that for all $n$ and $m$ with $\delta_{n+1} = \delta_{m+1} = 0$ it can not always occur $\Delta_1^m > \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 0$. If $\delta_1 = 0$ consider $n = m = 0$ which implies $\Delta_1^m = \Delta_{n+1}^{n+m} = 0$. If $\delta_1 = 1$ consider $m + 1 = \min\{k \in \{1, \ldots, \alpha_1\} : \delta_k = 0\}$ and $n + 1 = \max\{k \in \{1, \ldots, \alpha_1\} : \delta_k = 0\}$ thus, $\Delta_1^m > \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 1$. Therefore, as there must exist $n$ and $m$ with $\delta_{n+1} = \delta_{m+1} = 0$ such that either $\Delta_1^m = \Delta_{n+1}^{n+m}$ or $\Delta_1^m > \Delta_{n+1}^{n+m}$ and $\delta_{n+m+1} = 1$ we can prove, as we did in case case (a) considering $t' = z_m + \theta - r$, that for all $n$ such that $\delta_{n+1} = 0$, $\chi(z) \leq \psi_5(z)$, $z \in [z_n, z_{n+1}]$.

Now suppose that there is a $r$, $0 < r < \mu$, such that for all $v$ with $\delta_{v+1} = 1$ and for all $z = z_v + r$ we have $\chi(z) = \chi_1(z) > \psi_5(z)$. Case (b) implies that there are $n$ and $m$ with $n + m + 1 \leq v^*$ such that $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+1} = \delta_{m+1} = \delta_{n+m+1} = 1$. Consider $t = z_n + r$ and $t' = z_m + \mu - r$. Thus, $\chi(t + t') = \chi(z_n + z_m + r + \mu - r) = \chi(z_{n+m} + \gamma^{j-1} + \mu) = \chi(z_{n+m+1}) \leq \phi(z_{n+m+1}) = n + m + 1$ and $\chi(t + t') \geq \chi(t) + \chi(t') > \psi_5(t) + \psi_5(t') = n + r/\mu + m + (\mu - r)/\mu) = n + m + 1$, which is absurd.

Proof of (c). This case implies that there are $n$ and $m$ such that one of the following conditions hold (see Lemma 5.6):
(1) $\Delta_1^m = \Delta_{n+1}^{n+m} - 1$ and $\delta_{n+m+1} = 0$.
(2) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{m+1} = 0, \delta_{n+1} = 1$, $\delta_{n+m+1} = 1$.
(3) $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{m+1} = 1, \delta_{n+1} = 0$, $\delta_{n+m+1} = 1$.
Notice that it can not happen $\Delta_1^m = \Delta_{n+1}^{n+m}$ and $\delta_{n+1} = 0, \delta_{m+1} = 0$, $\delta_{n+m+1} = 1$ because it would imply that $\Delta_1^{m+1} = \Delta_{n+2}^{n+m+1} - 2$ which contradicts Lemma 5.6. Also notice that (1) imply (2). In order to (1) occur it is necessary that there is $m'$, $m' < m$, such that $\Delta_1^{m'} = \Delta_{n+1}^{m'}$ and $\delta_{m'+1} = 0$, $\delta_{n+m'+1} = 1$. Since it can not occur $\Delta_1^{m'} = \Delta_{n+1}^{n+m'}$ and $\delta_{n+1} = 0, \delta_{m'+1} = 0$, $\delta_{n+m'+1} = 1$ we also have $\delta_{n+1} = 0$.

Suppose that there is $r$, $0 < r < \mu$, such that for all $v$ with $\delta_{v+1} = \delta$, $\delta \in \{0, 1\}$ and for all $z = z_v + r$, we have $\chi(z) > \psi_4(z)$. Consider case (2) (the case (3) is similar) and suppose $\chi(t) > \psi_4(t)$ for $t = z_n + r$. Let $t' = z_m + \mu - r$. Thus $\chi(t + t') = \chi(z_{n+m} + \gamma^{j-1} + \mu) = \chi(z_{n+m+1}) = n + m + 1$ and $\chi(t + t') \geq \chi(t) + \chi(t') > \psi_4(t) + \psi_4(t') = n + m + 1$.

Finally consider case (d). This case implies that there is $n$ and $m$ such $\Delta_1^m = \Delta_{n+1}^{n+m} - 1$ and $\delta_{n+m+1} = 1$. And these conditions can only occur (see proof of Proposition 6.3) with $\delta_{n+1} = \delta_{m+1} = 1$. Therefore, if $\chi_1(t) > \psi_3(t)$ for $t = z_n + r$ and $\delta_{n+1} = 1$ then we obtain again the contradiction $\chi(t + t') > n + m + 1 = \chi(t + t')$ with $t' = z_m + r'$ and $r' = \lambda - r$.

Now we consider $\chi_2$. Notice that if there is $n$ and $m$ such that $\Delta_1^m = \Delta_{n+1}^{n+m} - 1$ and $\delta_{n+1} = \delta_{m+1} = \delta_{n+m+1} = 1$ then there must exist $n$ and $m'$ such that $\Delta_1^{m'} = \Delta_{n+1}^{n+m'}$ and $\delta_{n+1} = 1$, $\delta_{m'+1} = 0$, $\delta_{n+m'+1} = \delta_{n+m'+2} = 1$. In this case if $\chi_2(z) > \psi_3(z)$ for $z = z_v + r$ with $\delta_v = 0$ and $r < \mu$, then $\chi_2(t) > \psi_3(t)$ for $t = z_{m'} + k_2$, $\delta_{m'+1} = 0$ and $k_2 < \mu$. Considering $t' = z_n + \lambda$ we have, $\chi(t+t') = \chi(z_{n+m'} + \gamma^{j-1} + \lambda + k_2) = \chi(z_{n+m'} + w_1 + (w_1 - w_2) + k_2) = \chi_1(z_{n+m'+1} + (w_1 - w_2) + k_2) = n + m' + 1 + \frac{(w_1 - w_2) + k_2}{\lambda} < n + m' + 1 + \frac{(w_1 - w_2) + \mu}{\lambda} = n + m' + 2 = \chi_1(t') + \chi_2(t)$. Thus, $\chi$ is not superadditive. □

Notice that we proved stronger results than the results stated in Proposition 6.7. In fact we proved that $\psi_6$ can not be dominated by other superadditive valid lifting function. Only for the case of $\psi_3$ with

$z \in [z_n, z_{n+1}]$ and $\delta_{n+1} = 0$ we used explicitly the expression of $\chi_2$. In all other cases the proofs are valid considering $\chi_1$ and $\chi_2$ as generic functions. However, if there is $n$ and $m$ with $\delta_{n+1} = \delta_{m+1} = 0$ and $\Delta_1^{m+1} < \Delta_{n+1}^{n+m} + \delta_{n+m+1}$ then the expression of $\chi_2$ is not important to prove that $\psi_3$ can not be dominated by other valid lifting function in $\mathcal{C}$. Otherwise, if for all $n$ and $m$ with $\delta_{n+1} = \delta_{m+1} = 0$ we have $\Delta_1^{m+1} \geq \Delta_{n+1}^{n+m} + \delta_{n+m+1}$ then we can improve $\psi_3$ for $z \in [z_v, z_{v+1}]$ with $\delta_{v+1} = 0$, by setting $\psi_3(z) = v + 1 - \frac{\mu}{\lambda} + \frac{z - z_v}{\lambda}$, if $z_v \leq z < z_v + \mu$ and $\delta_{v+1} = 0, v = 0, \ldots, \alpha - 1$. The function with this modification is not in $\mathcal{C}$ and can not be useful for Section 7. However, if conditions (C4) do not hold it can be proven, as we did above, that this function dominates all other functions $\chi \in \mathcal{C}$ where $\chi_1$ and $\chi_2$ are generic functions provided that $\chi$ is a superadditive valid lifting function.

Next we analyze maximality of the superadditive valid lifting functions given above. Lemma 2.8 implies that for all $z \in [0, z_{v^*}]$ for which the value of the functions $\psi_i$, $i \in \{2, 3, 4, 5, 6\}$, coincides with the value of $\phi$ is in the maximality set $E$. Remember that if $\phi_{M \cup S}(z) = \phi_M(z)$, for all $S$, $\emptyset \neq S \subseteq N \backslash M$, then $z \in E$.

**Proposition 6.8** *$[ta_1, ta_1 + \gamma^{j-1}] \subseteq E$ for $t = 1, \ldots, \lfloor \frac{D}{a_1} \rfloor$.*

**Proof:** Consider $S$ such that $\emptyset \neq S \subseteq N \setminus M$. As $\phi_{M \cup S}(z) \leq \phi(z)$ for all $z$, we must show that the strict inequality can not occur for $z \in [ta_1, ta_1 + \gamma^{j-1}]$. Suppose that there exists $z \in [ta_1, ta_1 + \gamma^{j-1}]$ such that $\phi_{M \cup S}(z) < \phi(z) = t\alpha_1$. Let $y* \in \mathbb{N}^{|M \cup S|}$ be the optimal solution to $\phi_{M \cup S}(z)$. Then $\phi_{M \cup S}(z) = \alpha - \sum_{j \in M \cup S} \alpha_j y_j^* < t\alpha_1 = \phi(z)$ and $\sum_{M \cup S} a_j y_j^* \leq D - z \leq D - ta_1$. Setting $y_1' = y_1^* + t$ and $y_j' = y_j^*$ for $j \in S \cup (M \backslash \{1\})$ then $\sum_{j \in M \cup S} a_j y_j' \leq D$ and $\alpha - \sum_{j \in M \cup S} \alpha_j y_j' = \alpha - \sum_{M \cup S} \alpha_j y_j^* - t\alpha_1 < 0$, i.e. $\phi_{M \cup S}(0) < 0$ which is absurd because it implies that $\sum_{M \cup S} \alpha_j y_j \leq \alpha$ is not valid for $Y_{M \cup S}$. □

Noticing that $\psi_i(ka_1)$, for $i \in \{2, 3, 4, 5, 6\}$, may not assume the value $\phi(ka_1) = k\alpha_1$ we will improve these functions.

**Lemma 6.9** *Let $\psi$ be a valid superadditive lifting function such that $\psi(ka_1 + r) = k\alpha_1 + \psi(r)$ for all $k = 0, \ldots, \lfloor D/a_1 \rfloor$ and $\gamma^{j-1} \leq r < a_1$, then the function $\psi'$ defined by*

$$\psi'(z) = \begin{cases} k\alpha_1, & \text{if } z \in A, \\ \psi(z), & \text{otherwise,} \end{cases}$$

*where $A = \bigcup_{k=0,\ldots,\lfloor D/a_1 \rfloor} [ka_1, ka_1 + \gamma^{j-1}]$, is a superadditive valid lifting function.*

**Proof:** From the definition of $\psi'$ and from Proposition 6.8 it follows that $\phi(z) \geq \psi'(z) \geq \psi(z)$ for all $z \in [0, D]$. Next we prove superadditivity of $\psi'$.

Case $t_1 \in A, t_2 \in A$. Suppose $t_1 = k_1 a_1 + r_1$, $t_2 = k_2 a_1 + r_2$, with $0 \leq r_1 \leq \gamma^{j-1}$, $0 \leq r_2 \leq \gamma^{j-1}$. Then $\psi'(t_1 + t_2) \geq \psi'((k_1 + k_2)a_1) = (k_1 + k_2)\alpha_1 = \psi'(t_1) + \psi'(t_2)$.

Case $t_1 \in A, t_2 \notin A$ (similarly for $t_1 \notin A, t_2 \in A$). Let $t_1 = k_1 a_1 + r_1$, $t_2 = k_2 a_1 + r_2$, with $0 \leq r_1 \leq \gamma^{j-1}$ and $\gamma^{j-1} < r_2 < a_1$. If $r_1 + r_2 < a_1$ then $t_1 + t_2 \notin A$ and $r_1 + r_2 \notin A$ which implies $\psi'(t_1 + t_2) = \psi(t_1 + t_2) = (k_1 + k_2)\alpha_1 + \psi(r_1 + r_2) \geq k_1\alpha_1 + \psi(r_1) + k_2\alpha_1 + \psi(r_2) = k_1\alpha_1 + 0 + k_2\alpha_1 + \psi(r_2) = \psi'(t_1) + \psi'(t_2)$. If $r_1 + r_2 \geq a_1$ then $t_1 + t_2 \in A$ and $r_1 + r_2 \in A$ because $r_1 + r_2 = a_1 + r$ with $0 \leq r < \gamma^{j-1}$. Thus $\psi'(t_1 + t_2) = (k_1 + k_2 + 1)\alpha_1 \geq k_1\alpha_1 + k_2\alpha_1 + \psi(r_2) = \psi'(t_1) + \psi'(t_2)$.

Finally, if $t_1 \notin A, t_2 \notin A$ then $\psi'(t_1+t_2) \geq \psi(t_1+t_2) \geq \psi(t_1)+\psi(t_2) = \psi'(t_1)+\psi'(t_2)$. □

The fact that $\psi_i$, with $i \in \{2,3,4,5,6\}$, satisfies $\psi(ka_1+r) = k\alpha_1 + \psi(r)$ for all $k = 1,\ldots,\lfloor D/a_1 \rfloor$ and $\gamma^{j-1} \leq r < a_1$ is a consequence of Corollary 5.14. Using this procedure to improve $\psi_1$ we may obtain a non superadditive function. However, for each facet defining inequality with $k(j) > 1$, we may use $\psi'_1$ to lift all the coefficients since $\psi'_1(z) \leq \psi'_2(z)$ for all $z \in [0, D]$.

Notice that, in general, the interval $[0, z_{v^*}]$ is not too much restrictive. Remember that $z_{v^*} \geq z_{\alpha_1} = a_1 + \gamma^{j-1}$. In Example 4.1, $z_{v^*} = 850$. We will not analyze these functions on $[z_{v^*}, D]$.

**Remark 6.10** *In this section we constructed several superaddive valid lifting functions based on a non polynomial number of parameters. However, it is not clear whether is it possible to compute the coefficients using these functions in polynomial time.*

# 7 Lifting the 2-integer continuous knapsack inequalities

In this section we consider mixed integer knapsack sets of the form: $X = \{(y,s) \in \mathbb{N}_0^{|N|} \times \mathbb{R} : \sum_{j\in N} a_j y_j \leq D + s, s \geq 0\}$ where $a_j, j \in N$ and $D$ are positive integers. To generate strong valid inequalities for $X$ we restrict $X$ by setting all integer variables to zero except two of them. W.l.o.g. we assume that those two variables are $y_1$ and $y_2$. Then, using a superadditive valid lifting function, we lift each facet defining inequality of the restricted set $R = \{(y_1, y_2, s) \in \mathbb{N}_0^2 \times \mathbb{R} : a_1y_1 + a_2y_2 \leq D+s, s \geq 0\}$. The description of *conv*$(R)$ is given in (Agra & Constantino 2003). Each non trivial facet defining inequality of *conv*$(R)$,

$$\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha + \beta s. \tag{7.1}$$

belongs to one of two families. First we consider the family that is obtained from the lifting of a facet defining inequality for *conv*$(Y_{\leq})$ after $s$ has been set to zero.

**Proposition 7.1** *Consider a facet defining inequality for* $conv(Y_{\leq})$, *containing the extreme points of* $conv(Y_{\geq})$, $(a^{j-1}, b^{j-1})$ *and* $(a^j, b^j)$ *with* $(a^j, b^j) = (a^{j-1}, b^{j-1}) + r(-c^{k(j)}, d^{k(j)})$ *for some positive integer* $r$,

$$d^{k(j)} y_1 + c^{k(j)} y_2 \leq d^{k(j)} a^j + c^{k(j)} b^j \tag{7.2}$$

*then, if* $\frac{c^{k(j)}}{d^{k(j)}} \leq \frac{a_2}{a_1}$, *the following inequality defines a facet of* $conv(R)$,

$$d^{k(j)} y_1 + c^{k(j)} y_2 \leq d^{k(j)} a^j + c^{k(j)} b^j + \frac{1}{\eta^j} s \tag{7.3}$$

*where* $\eta^j = a_1\lceil (D/a_1) \rceil - D$ *if* $k = 1$ *and*

$$\eta^j = \begin{cases} -\gamma(a^{j-1}, b^{j-1}) + a_1 e^{\ell(k(j))} - a_2 f^{\ell(k(j))}, & \text{if } b^{j-1} \geq f^{\ell(k(j))}, \\ -\gamma(a^{j-1}, b^{j-1}) - a_1 c^{k(j)-1} + a_2 d^{k(j)-1}, & \text{if } b^{j-1} < f^{\ell(k(j))}, \end{cases}$$

*otherwise.*

Coefficients $(a^{j-1}, b^{j-1}), (e^{\ell(k(j))}, f^{\ell(k(j))})$ and $(c^{k(j)}, d^{k(j)})$ can be obtained using the polynomial version of Algorithm HW and $(c^{k(j)-1}, d^{k(j)-1})$ can be obtained as $(c^{k(j)-1}, d^{k(j)-1}) = (c^{k(j)}, d^{k(j)}) - r(e^{\ell(k(j))}, f^{\ell(k(j))})$ where $r = \lfloor \frac{R_{\geq}(e^{\ell(k(j))}, f^{\ell(k(j))})}{R_{\leq}(c^{k(j)}, d^{k(j)})} \rfloor$. The case $\frac{c^{k(j)}}{d^{k(j)}} \geq \frac{a_2}{a_1}$ is similar. It suffices to exchange

$a_1$ with $a_2$. Considering $w_1 = a_2 d^{k(j)-1} - a_1 c^{k(j)-1}$ and $w_2 = -a_2 f^{\ell(k(j))} + a_1 e^{\ell(k(j))}$, then $\eta^j$ can be written as

$$\eta^j = \begin{cases} w_2 - \gamma^{j-1}, & \text{if } b^{j-1} \geq f^{\ell(k(j))}, \\ w_1 - \gamma^{j-1}, & \text{if } b^{j-1} < f^{\ell(k(j))}. \end{cases}$$

Since $w_1 \geq w_2$ and $\gamma^{j-1} \geq w_1 - w_2$, we have $\eta^j \leq w_2 \leq w_1$.

The lifting function associated to (7.3) is given by:

$$\begin{aligned} \phi^c(z) \quad &= \min \quad \alpha - \alpha_1 y_1 - \alpha_2 y_2 + \beta s \\ \text{s. t.} \quad & \quad a_1 y_1 + a_2 y_2 \leq D - z + s, \\ & \quad s \geq 0, \\ & \quad y_1, y_2 \in \mathbb{N}_0. \end{aligned}$$

where $\alpha_1 = d^{k(j)}, \alpha_2 = c^{k(j)}, \alpha = d^{k(j)} a^j + c^{k(j)} b^j, \beta = \frac{1}{\eta^j}$. In the following consider $z_v, v = 0, \ldots, \alpha$, and $v_k, k = 0, \ldots, n^*$, as defined in Section 6.

**Proposition 7.2** *For $z \geq 0$,*

$$\phi^c(z) = \begin{cases} 0, & \text{if } z \leq z_0, \\ v_k + \frac{z - z_{v_k}}{\eta^j}, & \text{if } z_{v_k} \leq z < z_{v_k} + \eta^j(v_{k+1} - v_k), \text{ for } k = 0, \ldots, n^*, \\ v_{k+1}, & \text{if } z_{v_k} + \eta^j(v_{k+1} - v_k) \leq z \leq z_{v_{k+1}}, \text{ for } k = 0, \ldots, n^*, \\ \alpha + \frac{z - D}{\eta^j}, & \text{if } z > D. \end{cases}$$

**Proof:** First consider $0 \leq z \leq D$. Hence $\phi^c$ can be written as $\phi^c(z) = \min_{s>0}\{\phi(z), \phi(z-s) + \frac{1}{\eta^j} s\}$. Suppose $z_{v_k} \leq z < z_{v_k} + \eta^j(v_{k+1} - v_k)$. Considering $t = z - z_{v_k} > 0$ we have $0 \leq t < \eta^j(v_{k+1} - v_k)$, thus $\phi^c(z) \leq \phi(z-t) + \frac{t}{\eta^j} = v_k + \frac{t}{\eta^j}$. In order to prove the other direction we consider the following cases. If $0 < s < t$ then as $z - s > z_{v_k}$, it follows that $\phi(z-s) + \frac{s}{\eta^j} = v_{k+1} + \frac{s}{\eta^j} \geq v_k + \frac{t}{\eta^j}$ because, as we are assuming $t < \eta^j(v_{k+1} - v_k)$ and as $\frac{s}{\eta^j} > 0$, we have $\frac{t}{\eta^j} < \frac{s}{\eta^j} + (v_{k+1} - v_k)$. If $s > t$ and $z_{v_p} < z - s \leq z_{v_{p+1}}$ with $p \in \{0, \ldots, k-1\}$, then $s \geq z_{v_k} - z_{v_{p+1}} + t = (v_k - v_{p+1})w_2 + (w_1 - w_2)\Delta^{v_k}_{v_{p+1}} + t \geq (v_k - v_{p+1})w_2 + t \geq (v_k - v_{p+1})\eta^j + t$. Therefore $\phi(z-s) + \frac{s}{\eta^j} = v_{p+1} + \frac{s}{\eta^j} \geq v_{p+1} + \frac{(v_k - v_{p+1})\eta^j + t}{\eta^j} \geq v_{p+1} + v_k - v_{p+1} + \frac{t}{\eta^j} = v_k + \frac{t}{\eta^j}$. If $s > t$ and $z_{-p-1} < z - s \leq z_{-p}$ for $p \in \mathbb{N}_0$, then $s \geq z_{v_k} - z_{-p} + t = v_k w_2 + (w_1 - w_2)\Delta^{v_k}_1 + p w_2 + (w_1 - w_2)\sum_{i=1}^{p} \delta_{-i} + t \geq (v_k + p)w_2 + t \geq (v_k + p)\eta^j + t$. Therefore $\phi(z-s) + \frac{s}{\eta^j} = -p + \frac{s}{\eta^j} \geq -p + \frac{(v_k + p)\eta^j + t}{\eta^j} \geq v_k + \frac{t}{\eta^j}$. Hence $\phi(z-s) + \frac{s}{\eta^j} \geq \phi(z-t) + \frac{t}{\eta^j} = v_k + \frac{t}{\eta^j}$ for all $s > 0$. Now suppose $z > D$. Again, $\phi^c(z) = \min_{s > z-D}\{\phi(z - (z-D)) + \frac{1}{\eta^j}(z-D), \phi(z-s) + \frac{1}{\eta^j} s\}$. Similarly, it can be checked that for $s > z - D$, $\phi(z-s) + \frac{1}{\eta^j} s \geq \phi(z - (z-D)) + \frac{z-D}{\eta^j} = \alpha + \frac{z-D}{\eta^j}$.

The case $z_{v_k} + \eta^j(v_{k+1} - v_k) \leq z \leq z_{v_{k+1}}$, is similar to the previous one. □

It is important to notice that, in general, $\phi^c$ is not superadditive (in the example of Figure 7, $\phi^c(3) + \phi^c(3) > \phi^c(6)$). In order to extend the superadditive functions obtained in Section 4, $\psi'_i, i \in \{2, 3, 4, 5, 6\}$, for $[D, +\infty[$ we define $z_v$, for $v > \alpha, v \in \mathbb{N}$, as $z_v = \gamma^{j-1} + v w_2 + (w_2 - w_1)\sum_{i=1}^{v} \delta_i$, where, for $i > \alpha$, $\delta_i = \delta_r$ with $r = i - \alpha_1 \lfloor (i-1)/\alpha_1 \rfloor$. The proofs of superadditivity of $\psi'_i, i \in \{2, 3, 4, 5, 6\}$, are also valid replacing $v = 0, \ldots, \alpha - 1$ with $v \in \mathbb{N}_0$. Next we prove that the extended functions of $\psi'_i, i \in \{2, 3, 4\}$, are also valid considering $\phi^c$.

**Lemma 7.3** *$\psi'_4(z) \leq \phi^c(z)$, for all $z \in [0, +\infty[$.*

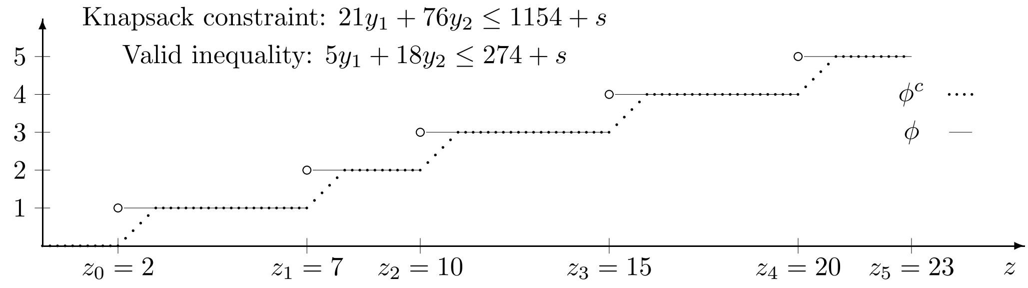


Figure 7: Example of $\phi^c$.

**Proof:** Consider the following auxiliar function:

$$\phi^a(z) = \begin{cases} v + \frac{z-z_v}{\eta^j}, & \text{if } z_v \leq z < z_v + \eta^j, v \in \mathbb{N}_0, \\ v+1, & \text{if } z_v + \eta^j \leq z \leq z_{v+1}, v \in \mathbb{N}_0. \end{cases}$$

First we prove that $\phi^a(z) \leq \phi^c(z)$ for all $z \in [0, D]$. The unique case where is not obvious that $\phi^a(z) \leq \phi^c(z)$ is the case $z_{v_k} \leq z \leq z_{v_k} + \eta^j(v_{k+1} - v_k)$. Let $v$ be such that $z_v \leq z \leq z_{v+1}$ and $r = z - z_v$. Hence, $z - z_{v_k} = z_v - z_{v_k} + r = w_2(v - v_k) + (w_1 - w_2)\Delta^v_{v_k} + r \geq w_2(v - v_k) + r \geq \eta^j(v - v_k) + r$. Therefore $\phi^c(z) \geq v_k + \frac{\eta^j(v-v_k)+r}{\eta^j} = v_k + v - v_k + \frac{r}{\eta^j} \geq \phi^a(z)$. Now consider $z > D$. Suppose $z_v < z < z_{v+1}$ with $v \geq \alpha$. Thus, $\phi^c(z) = \alpha + \frac{z-D}{\eta^j} = \alpha + \frac{z_v + (z-z_v) - z_\alpha}{\eta^j} = \alpha + \frac{z_\alpha + (v-\alpha)w_2 + (w_1-w_2)\sum_{i=\alpha+1}^{v}\delta_i + (z-z_v) - z_\alpha}{\eta^j}$ $\geq \alpha + \frac{(v-\alpha)w_2 + (z-z_v)}{\eta^j} \geq v + \frac{z-z_v}{\eta^j} \geq \phi^a(z)$.

Now we prove $\psi'_4(z) \leq \phi^a(z)$ for all $z \in [0, +\infty[$. As $\mu = w_1 - \gamma^j \geq \eta^j$ then $\psi_4(z) \leq \phi^a(z)$ for all $z \in [0, +\infty[$. It remains to prove that, for each $k \in \mathbb{N}_0$, $[ka_1, ka_1 + \gamma^{j-1}] \subseteq [z_{k\alpha_1 - 1} + \eta^j, z_{k\alpha_1}]$. Using Corollary 5.14, $z_{k\alpha_1} = ka_1 + \gamma^{j-1}$. Thus, we must show that $z_{k\alpha_1 - 1} + \eta^j \leq ka_1 \Leftrightarrow z_{k\alpha_1} - w_2 - (w_1 - w_2)\delta_{\alpha_1} + \eta^j \leq ka_1 \Leftrightarrow \gamma^{j-1} - w_2 - (w_1 - w_2)\delta_{\alpha_1} + \eta^j \leq 0 \Leftrightarrow \eta^j \leq w_2 - \gamma^{j-1} + (w_1 - w_2)\delta_{\alpha_1}$. Notice that if $\eta^j = w_1 - \gamma^{j-1}$ then $b^{j-1} < f^{k(j-1)}$ and so, as $p^2_{\alpha_1} = b^{j-1}$, it can not occur $\delta_{\alpha_1} = 0$ because it would imply $p^2_{\alpha_1} = p^2_{\alpha_1 - 1} + f^{k(j-1)} \geq f^{k(j-1)}$. Thus $\eta^j = w_1 - \gamma^{j-1}$ implies $\delta_{\alpha_1} = 1$. Using the definition of $\eta^j$, it follows that $\eta^j \leq w_2 - \gamma^{j-1} + (w_1 - w_2)\delta_{\alpha_1}$. □

Therefore we have $\psi'_1(z) \leq \psi'_2(z) \leq \psi'_3(z) \leq \psi'_4(z) \leq \phi^c(z)$ for all $z \in [0, +\infty[$. If $\eta^j = w_2 - \gamma^{j-1}$ then $\phi^a$ coincides with $\psi_6$, and, in that case, $\psi_5$ and $\psi_6$ are also valid for $\phi^c(z)$.

Now we consider the other family of facet defining inequalities for *conv*$(R)$. Setting $s = a_1y_1 + a_2y_2 - D$, generating a facet valid inequality for *conv*$(Y_\geq)$ and then introducing $s$ again, we obtain the following family of inequalities.

**Proposition 7.4** *Considering a facet defining inequality for conv*$(Y_\geq)$ *containing the extreme points of conv*$(Y_\geq)$, $(a^{j-1}, b^{j-1})$ *and* $(a^j, b^j)$, *with* $(a^j, b^j) = (a^{j-1}, b^{j-1}) + r(-e^{\ell(j)}, f^{\ell(j)})$ *for some positive integer* $r$,

$$f^{\ell(j)}y_1 + e^{\ell(j)}y_2 \geq f^{\ell(j)}a^j + e^{\ell(j)}b^j \tag{7.4}$$

*then, if* $\frac{e^{k(j)}}{f^{k(j)}} \geq \frac{a_2}{a_1}$*, the following inequality defines a facet of* $conv(R)$,

$$f^{\ell(j)}y_1 + e^{\ell(j)}y_2 + \frac{1}{\eta^j}(D + s - a_1y_1 - a_2y_2) \geq f^{\ell(j)}a^j + e^{\ell(j)}b^j \tag{7.5}$$

*where* $\eta^j = D - a_1\lfloor(D/a_1)\rfloor$ *if* $\ell(j) = 1$ *and*

$$\eta^j = \begin{cases} -\gamma^{j-1} - a_1c^{k(\ell(j))} + a_2d^{k(\ell(j))}, & \text{if } b^{j-1} \geq d^{k(\ell(j))}, \\ -\gamma^{j-1} + a_1e^{\ell(j)-1} - a_2f^{\ell(j)-1}, & \text{if } b^{j-1} < d^{k(\ell(j))}, \end{cases}$$

*where* $\gamma^{j-1} = a_1a^{j-1} + a_2b^{j-1} - D$*, otherwise.*

Noticing that inequality (7.5) can be written as

$$(a_1 - \eta^j f^{k(j)})y_1 + (a_2 - \eta^j e^{k(j)})y_2 \leq D - \eta^j(f^{\ell(j)}a^j + e^{\ell(j)}b^j) + s$$

where $a_1 - \eta^j f^{k(j)} > 0$ and $a_2 - \eta^j e^{k(j)} > 0$ (see Agra & Constantino 2003), we may use function $\psi_1$ as a superadditive valid lifting function for this case. However, Proposition 7.4 indicates another way to obtain valid inequalities for $X$. Let $x = D + s - \sum_{j\in N} a_jy_j$. Hence $(y, s) \in X$ if and only if $(y, x) \in S$ where $S = \{(y, x) \in \mathbb{N}_0^{|N|} \times \mathbb{R} : x + \sum_{j\in N} a_jy_j \geq D, x \geq 0\}$. The inequality $\beta x + \sum_{j\in N} \alpha_jy_j \geq \alpha$ is valid to $S$ if and only if $\sum_{j\in N}(\beta a_i - \alpha_j)y_j \leq \beta D - \alpha + \beta s$ is valid to $X$. We study the lifting function associated to (7.5), or equivalently, associated to

$$f^{\ell(j)}y_1 + e^{\ell(j)}y_2 + \frac{1}{\eta^j}x \geq f^{\ell(j)}a^j + e^{\ell(j)}b^j,$$

from the lifting function associated to (7.4). The lifting function associated to (7.4) is given by,

$$\begin{aligned} \varphi(z) \quad = \quad & \max \alpha - \alpha_1y_1 - \alpha_2y_2 \\ \text{s.t.} \quad & a_1y_1 + a_2y_2 \geq D - z \\ & y_1, y_2 \in \mathbb{N}_0, \end{aligned}$$

and, the corresponding continuous lifting function, is given by,

$$\begin{aligned} \varphi^c(z) \quad = \quad & \max \alpha - \alpha_1y_1 - \alpha_2y_2 - \beta x \\ \text{s.t.} \quad & x + a_1y_1 + a_2y_2 \geq D - z, \\ & x \geq 0, \\ & y_1, y_2 \in \mathbb{N}_0. \end{aligned}$$

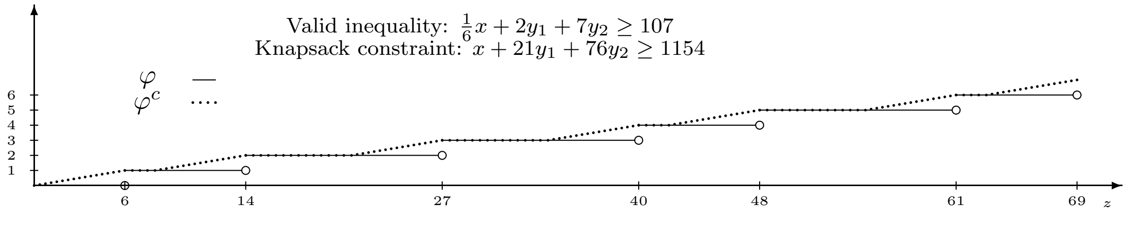


Figure 8: Functions $\varphi$, $\varphi^c$ with $\gamma^{j-1} = 7, w_1 = 13, w_2 = 8$.

Next we introduce some properties similar to those presented for function $\phi$ and $\phi^c$. The major differences are related with the fact that associated to $\varphi$ we have maximization problems and, therefore, we must use subadditive functions instead of superadditive functions. A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is subadditive on $A$ if $f$ is bounded on $A$ and $f(x_1) + f(x_2) \geq f(x_1 + x_2)$ for all $x_1, x_2, x_1 + x_2 \in A$.

We will use the same notation as we used for the lifting function $\phi$ and omit all the proofs.

**Proposition 7.5** *Let (7.4) be a valid inequality for $Y_{\geq}$. If $\omega$ is a subadditive function on* $[0, +\infty[$ *and* $\omega(z) \geq \varphi(z)$ *for all* $z \in [0, +\infty[$, *then* $\alpha_1 y_1 + \alpha_2 y_2 + \sum_{j \in N \setminus \{1,2\}} \omega(a_j) y_j \geq \alpha$ *is valid for* $\{y \in \mathbb{N}_0^{|N|} : \sum_{j \in N} a_j y_j \geq D\}$.

We call such a function $\omega$ a subadditive valid lifting function. Let us define, for $v \in \mathbb{N}$,

$$(p_1^v, p_2^v) = \begin{cases} (p_1^{v-1}, p_2^{v-1}) + (c^{k(\ell(j))}, -d^{k(\ell(j))}), \text{ if } p_2^{v-1} - d^{k(\ell(j))} \geq 0, \\ (p_1^{v-1}, p_2^{v-1}) + (-e^{\ell(j)-1}, f^{\ell(j)-1}), \text{ if } p_2^{v-1} - d^{k(\ell(j))} < 0, \end{cases}$$

where $(p_1^0, p_2^0) = (a^{j-1}, b^{j-1})$. And define, for $v \in \mathbb{N}$,

$$\delta_v = \begin{cases} 0 \text{ if } (p_1^v, p_2^v) = (p_1^{v-1}, p_2^{v-1}) + (c^{k(\ell(j))}, -d^{k(\ell(j))}), \\ 1 \text{ if } (p_1^v, p_2^v) = (p_1^{v-1}, p_2^{v-1}) + (-e^{\ell(j)-1}, f^{\ell(j)-1}), \end{cases}$$

and let $w_1 = a_1 e^{\ell(j)-1} - a_2 f^{\ell(j)-1}$ and $w_2 = -a_1 c^{k(\ell(j))} + a_2 d^{k(\ell(j))}$. Then $z_n = n w_2 + (w_1 - w_2) \sum_{i=1}^{n} \delta_i - \gamma^{j-1}$. For $k \in \mathbb{N}$ and $r \in \{1, \dots, \alpha_1\}$ we also have $\delta_{k\alpha_1 + r} = \delta_r$.

Notice that considering the upper convex envelope of the graph of $\varphi$ we could also construct a subadditive function, similar to $\psi_1$. Next we construct subadditive valid lifting functions for $\varphi$, similar to functions $\psi_i$, $i \in \{3, 4, 5, 6\}$ constructed for $\phi$, considering facet defining inequalities (7.4) with $\ell(j) > 1$. For $\ell(j) = 1$ we can use the following subadditive valid lifting function,

$$\omega(z) = \begin{cases} k + \frac{z - k a_1}{z_1}, & \text{if } k a_1 \leq z \leq k a_1 + z_1, k \in \mathbb{N}_0, \\ k + 1, & \text{if } k a_1 + z_1 < z \leq (k+1) a_1, k \in \mathbb{N}_0. \end{cases}$$

**Proposition 7.6** *Consider the lifting function $\varphi$ and consider the following functions satisfying* $\omega(z) \geq \varphi(z)$ *for all* $z \in [0, +\infty[$.

$$\omega_3(z) = \begin{cases} \frac{z}{z_1}, & \textit{if } 0 \leq z \leq z_1, \\ v, & \textit{if } z_v \leq z < z_{v+1} - \mu \textit{ and } \delta_{v+1} = 0, \textit{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\mu}, & \textit{if } z_{v+1} - \mu \leq z \leq z_{v+1} \textit{ and } \delta_{v+1} = 0, \textit{ for all } v \in \mathbb{N}, \\ v, & \textit{if } z_v \leq z < z_{v+1} - \lambda \textit{ and } \delta_{v+1} = 1, \textit{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\lambda}, & \textit{if } z_{v+1} - \lambda \leq z \leq z_{v+1} \textit{ and } \delta_{v+1} = 1, \textit{ for all } v \in \mathbb{N}, \end{cases}$$

$$\omega_4(z) = \begin{cases} \frac{z}{z_1}, & \textit{if } 0 \leq z \leq z_1, \\ v, & \textit{if } z_v < z \leq z_{v+1} - \mu, \textit{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\mu}, & \textit{if } z_{v+1} - \mu < z \leq z_{v+1}, \textit{ for all } v \in \mathbb{N}, \end{cases}$$

$$\omega_5(z) = \begin{cases} \frac{z}{z_1}, & \textit{if } 0 \leq z \leq z_1, \\ v, & \textit{if } z_v \leq z < z_{v+1} - \theta \textit{ and } \delta_{v+1} = 0, \textit{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\theta}, & \textit{if } z_{v+1} - \theta \leq z \leq z_{v+1} \textit{ and } \delta_{v+1} = 0, \textit{ for all } v \in \mathbb{N}, \\ v, & \textit{if } z_v \leq z < z_{v+1} - \mu \textit{ and } \delta_{v+1} = 1, \textit{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\mu}, & \textit{if } z_{v+1} - \mu \leq z \leq z_{v+1} \textit{ and } \delta_{v+1} = 1, \textit{ for all } v \in \mathbb{N}, \end{cases}$$

$$\omega_6(z) = \begin{cases} \frac{z}{z_1}, & \text{if } 0 \le z \le z_1, \\ v, & \text{if } z_v < z \le z_{v+1} - \theta, \text{ for all } v \in \mathbb{N}, \\ v + 1 - \frac{z_{v+1} - z}{\theta}, & \text{if } z_{v+1} - \theta < z \le z_{v+1}, \text{ for all } v \in \mathbb{N}, \end{cases}$$

*where* $\lambda = w_1 + (w_1 - w_2) - \gamma^{j-1}$, $\mu = w_1 - \gamma^{j-1}$ *and* $\theta = w_2 - \gamma^{j-1}$.
*(a) The function* $\omega_3$ *is a subadditive valid lifting function.*
*(b) If, for each* $n$ *and* $m$ *one of the following conditions hold: (i)* $\Delta_1^m \le \Delta_{n+1}^{n+m}$*; (ii)* $\Delta_1^m > \Delta_{n+1}^{n+m}$ *and* $\delta_{n+m} = 0$*, then* $\omega_4$ *is a subadditive valid lifting function.*
*(c) If, for each* $n$ *and* $m$ *one of the following conditions hold: (i)* $\Delta_1^m < \Delta_{n+1}^{n+m}$*; (ii)* $\Delta_1^m = \Delta_{n+1}^{n+m}$ *and* $\delta_{n+m} = 0$*; (iii)* $\Delta_1^m = \Delta_{n+1}^{n+m}$ *and* $\delta_n = \delta_m = \delta_{n+m} = 1$*, then* $\omega_5$ *is a subadditive valid lifting function.*
*(d) If, for each* $n$ *and* $m$ *one of the following conditions hold: (i)* $\Delta_1^m < \Delta_{n+1}^{n+m}$*; (ii)* $\Delta_1^m = \Delta_{n+1}^{n+m}$ *and* $\delta_{n+m} = 0$*, then* $\omega_6$ *is a subadditive valid lifting function.*

For facet defining inequalities (7.4) with $\ell(j) > 1$, these functions can be improved and we can use

$$\omega_i'(z) = \begin{cases} k\alpha_1, & \text{if } z \in [ka_1 - \gamma^{j-1}, ka_1], k \in \mathbb{N}, \\ \omega_i(z), & \text{otherwise,} \end{cases}$$

instead of $\omega_i$, with $i \in \{3, 4, 5, 6\}$. Functions $\omega_3', \omega_4'$ are also valid considering $\varphi^c$, and $\omega_5', \omega_6'$ are valid if $\eta^j = w_2 - \gamma^{j-1}$.

Notice that for $z > D$ we have $\varphi(z) = \varphi^c(z) = \alpha$. Hence, the subadditive valid lifting functions $\omega_i', i \in \{3, 4, 5, 6\}$, can be easily improved for $z > z_\alpha = D$, setting $\omega_i'(z) = \alpha$.

# 8 Computational tests

In this section we report some computational tests. All tests were performed on a PC, Pentium IV, 2.4Ghz with 768 Mb RAM and using the optimization package Xpress-MP, Version 14.10 with MOSEL.

We developed two types of tests, one for the pure integer single knapsack constraint problem and other for the integer single knapsack constraint problem with one continuous variable. All the instances were randomly generated for $n = 100$ variables. We tested two sets of instances where the coefficients of the objective function, $f_j$, and the coefficients of the knapsack constraint, $a_j$, were generated in the same interval and one set of instances where the intervals are different:

instance $a$ : $a_j \in [1700, 2000]$, $f_j \in [1700, 2000]$, and $D \in [200000, 300000]$,
instance $b$ : $a_j \in [7000, 10000]$, $f_j \in [7000, 10000]$, and $D \in [2000000, 5000000]$,
instance $c$ : $a_j \in [100, 200]$, $f_j \in [50, 60]$, and $D \in [2000, 5000]$.

For each instance we solve the linear relaxation without any additional inequality. Then, for each pair of variables, such that at least one has a strictly positive value in the optimal solution, we consider a restricted knapsack problem. Finally, we lift each one of the corresponding facet defining inequalities and add the inequalities violated by the linear solution to the problem. This procedure is repeated until the last linear solution satisfies all the valid inequalities generated.

In Table 1 we present the results obtained for the pure integer case. Columns OPT and LP give the optimum value and the value of the linear relaxations, respectively. Columns $\psi_1'$, $\psi_2'$, $\psi_3'$ give the value of the upper bound obtained when the lifting is done using the corresponding function and $\psi_i'$, $i \in \{4, 5, 6\}$, gives the upper bound using in each case the best of the functions available, for example, $\psi_5'$, uses the best of the functions $\psi_i'$, $i \in \{3, 4, 5\}$, that is superadditive. Columns $T$ give the CPU

| Instance | Opt | LP | $\psi'_1$ | T | $\psi'_2$ | T | $\psi'_3$ | T | $\psi'_4$ | T | $\psi'_5$ | T | $\psi'_6$ | T |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n100va1 | 242588 | 243172.4 | 242765.9 | 1.9 | 242597.4 | 1.6 | 242588 | 8.8 | 242588 | 30.2 | 242588 | 30.1 | 242588 | 30.0 |
| n100va2 | 292996 | 293401.7 | 292996.6 | 2.5 | 292996 | 9.9 | 292996 | 6.9 | 292996 | 27.3 | 292996 | 27.1 | 292996 | 27.2 |
| n100va3 | 239872 | 241064.8 | 239872 | 0.7 | 239872 | 0.9 | 239872 | 0.9 | 239872 | 2.9 | 239872 | 2.9 | 239872 | 2.9 |
| n100vb1 | 4138911 | 4143929.6 | 4142371.9 | 2.0 | 4138911 | 2.9 | 4138911 | 2.9 | 4138911 | 37.5 | 4138911 | 37.4 | 4138911 | 36.9 |
| n100vb2 | 2774589 | 2774925.1 | 2774589 | 1.4 | 2774589 | 1.7 | 2774589 | 1.7 | 2774589 | 12.3 | 2774589 | 12.3 | 2774589 | 12.4 |
| n100vb3 | 3835844 | 3837287.4 | 3835872.8 | 10.3 | 3835871.9 | 3.7 | 3835863 | 2.9 | 3835863 | 41.2 | 3835863 | 41.2 | 3835863 | 41.1 |
| n100vc1 | 2263 | 2278.88 | 2263 | 1.3 | 2263 | 1.5 | 2263 | 1.3 | 2263 | 1.3 | 2263 | 1.3 | 2263 | 1.3 |
| n100vc2 | 1275 | 1288.99 | 1275 | 0.7 | 1275 | 0.6 | 1275 | 0.8 | 1275 | 0.7 | 1275 | 0.7 | 1275 | 0.8 |
| n100vc3 | 1260 | 1318.2 | 1260 | 0.4 | 1260 | 0.6 | 1260 | 0.4 | 1260 | 0.4 | 1260 | 0.4 | 1260 | 0.4 |

Table 1: Computational results for the pure integer single knapsack constraint problem.

| Instance | LP gap | $\psi'_1$ | $\psi'_2$ | $\psi'_3$ | $\psi'_4$ | $\psi'_5$ | $\psi'_6$ |
|---|---|---|---|---|---|---|---|
| n100va1 | 0.241 | 69.6 | 98.4 | 100 | 100 | 100 | 100 |
| n100va2 | 0.138 | 99.9 | 100 | 100 | 100 | 100 | 100 |
| n100va3 | 0.497 | 100 | 100 | 100 | 100 | 100 | 100 |
| n100vb1 | 0.121 | 31.0 | 100 | 100 | 100 | 100 | 100 |
| n100vb2 | 0.012 | 100 | 100 | 100 | 100 | 100 | 100 |
| n100vb3 | 0.038 | 98.0 | 98.1 | 98.7 | 98.7 | 98.7 | 98.7 |
| n100vc1 | 0.70 | 100 | 100 | 100 | 100 | 100 | 100 |
| n100vc2 | 1.10 | 100 | 100 | 100 | 100 | 100 | 100 |
| n100vc3 | 4.52 | 100 | 100 | 100 | 100 | 100 | 100 |

Table 2: Percentage of reduction of the LP gap.

| Instance | S | Opt | LP | GAP | $\psi' + \omega'$ | GAP Red. | T |
|---|---|---|---|---|---|---|---|
| n100va1 | -10 | 242588 | 243172.4 | 0.24 | 242588 | 100 | 7.8 |
| n100va2 | -10 | 292996 | 293401.7 | 0.138 | 292996 | 100 | 7.6 |
| n100va3 | -10 | 239872 | 241064.8 | 0.497 | 239872 | 100 | 5.7 |
| n100va3 | -2 | 240568 | 241064.8 | 0.2 | 240568 | 100 | 3.8 |
| n100vb1 | -10 | 4138911 | 4143929.7 | 0.121 | 4138911 | 100 | 70.6 |
| n100vb1 | -2 | 4142124 | 4143929.7 | 0.044 | 4142124 | 100 | 35.4 |
| n100vb2 | -10 | 2774589 | 2774925.1 | 0.012 | 2774589 | 100 | 24.5 |
| n100vb3 | -10 | 3835844 | 3837287.4 | 0.038 | 3835853.55 | 99.3 | 78.5 |
| n100vc1 | -10 | 2263 | 2278.88 | 0.7 | 2263 | 100 | 2.4 |
| n100vc1 | -1 | 2263 | 2278.88 | 0.7 | 2263 | 100 | 2.3 |
| n100vc2 | -10 | 1275 | 1288.99 | 1.09 | 1275 | 100 | 1.1 |
| n100vc2 | -1 | 1276 | 1288.99 | 1.01 | 1276.13 | 99.9 | 1.1 |
| n100vc3 | -10 | 1290 | 1318.2 | 2.186 | 1290 | 100 | 0.6 |
| n100vc3 | -1 | 1317 | 1318.2 | 0.091 | 1317 | 100 | 0.6 |

Table 3: Computacional tests for the mixed integer single knapsack constraint problem.

time, in seconds, used to obtain the corresponding upper bound. The gaps are presented in Table 2. Column LP gives the liner gap: $LP = \frac{LP-OPT}{OPT} \times 100$ and, in the remaining columns, we present the percentage of reduction of the linear gap, when the corresponding superadditive function $\psi'$ is used for lifting: column value $= \frac{LPgap-\psi gap}{LPgap} \times 100$.

Notice that only in one case the upper bound obtained using $\psi'_3$ is strictly greater than the optimal value and, in this case, using $\psi'_3$ the initial linear gap was reduced by 98.7%. Using $\psi'_1$ it was possible to reduce significantly the linear gap and, as expected, the CPU time required to obtain the lower bound was, in general, lower than the corresponding CPU time using $\psi'_3$. However, only in two of the first six instances the upper bound obtained using $\psi'_1$ coincides with the optimum. Lifting with functions $\psi'_i$, $i \in \{4, 5, 6\}$, was not efective in reducing the linear gap of the unique instance where the optimal value was not obtained.

Table 3 present similar results for the knapsack constraint with one continuous variable. The column $S$ indicates the coefficient of the continuous variable $s$ on the objective function. Column LP gives the value of the linear relaxation and, in column GAP the corresponding gap is presented. Column $\psi' + \omega'$ indicates the upper bound obtained using the best valid lifting function and the reduction of the gap, in percentage, is given in column GAP Red.

Only in two instances the upper bound was greater than the optimum but the gap was always reduced at least in 99%.

# 9 Appendix

**Proof of Proposition 4.2**. Notice that $f(x_1 + x_2)$, with $x_2 > x_1$, can be written as

$$\begin{aligned} f(x_1 + x_2) &= f(x_1) + \left[\frac{x_1}{x_2} f(x_1) + (1 - \frac{x_1}{x_2}) f(x_1 + x_2) - f(x_1)\right] \frac{x_2}{x_2 - x_1} \\ &\geq f(x_1) + [f(x_2) - f(x_1)] \frac{x_2}{x_2 - x_1}. \end{aligned}$$

This inequality follows from $x_2 > 0$ and from the hypotheses that $f$ is convex (consider $\lambda = \frac{x_1}{x_2}, y_1 = x_1, y_2 = x_1 + x_2$). As $f$ is convex and $f(0) = 0$ we have $(1 - \frac{x_1}{x_2}) f(0) + \frac{x_1}{x_2} f(x_2) \geq f(x_1)$. Therefore,

$$x_1 f(x_2) \geq x_2 f(x_1)$$

$$\begin{aligned} &\Rightarrow x_2 f(x_2) - x_2 f(x_1) \geq x_2 f(x_2) - x_1 f(x_2) \\ &\Rightarrow x_2(f(x_2) - f(x_1)) \geq (x_2 - x_1) f(x_2) \\ &\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_2)}{x_2}. \end{aligned}$$

Thus,

$$\begin{aligned} f(x_1 + x_2) &\geq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1} x_2 \\ &\geq f(x_1) + \frac{f(x_2)}{x_2} \times x_2 = f(x_1) + f(x_2). \end{aligned}$$

To show that $x_1 = x_2 = x > 0$ (the case $x_1 = x_2 = 0$ is trivial) let

$$f(x + x) = f(x - \epsilon + x + \epsilon) \geq f(x - \epsilon) + f(x + \epsilon) \text{ where } x > \epsilon > 0.$$

As $f$ is convex on $[0, D]$ then $f$ is continuous on $]0, D[$, which implies $f(x + x) \geq lim_{\epsilon \to 0^+}(f(x - \epsilon) + f(x + \epsilon)) = f(x) + f(x)$. □

**Proof of Proposition 4.3**. First we show that the value of the slopes satisfy

$$\frac{\tau^j - \tau^{j-1}}{\gamma^j - \gamma^{j-1}} \leq \frac{\tau^{j-1} - \tau^{j-2}}{\gamma^{j-1} - \gamma^{j-2}} \leq \cdots \leq \frac{\tau^2 - \tau^1}{\gamma^2 - \gamma^1} \leq \frac{\alpha_1}{a_1}.$$

As $\frac{(A^{t-2} - A^{t-1})}{(B^{t-1} - B^{t-2})} < \frac{(A^{t-1} - A^t)}{(B^t - B^{t-1})} \leq \frac{(A^{j-1} - A^j)}{(B^j - B^{j-1})} \leq \frac{\alpha_2}{\alpha_1}$, for all $t = 3, \ldots, j - 1$, then

$$(A^{t-1} - A^t)(B^{t-1} - B^{t-2}) \geq (A^{t-2} - A^{t-1})(B^t - B^{t-1})$$

$$\Rightarrow (\frac{\alpha_2}{\alpha_1} - \frac{a_2}{a_1})(A^{t-1} - A^t)(B^{t-1} - B^{t-2}) \le (\frac{\alpha_2}{\alpha_1} - \frac{a_2}{a_1})(A^{t-2} - A^{t-1})(B^t - B^{t-1})$$

$$\Rightarrow (A^{t-1} - A^t)(A^{t-2} - A^{t-1}) - \frac{a_2}{a_1}(A^{t-1} - A^t)(B^{t-1} - B^{t-2}) - (A^{t-2} - A^{t-1})\frac{\alpha_2}{\alpha_1}(B^t - B^{t-1})$$
$$+\frac{\alpha_2}{\alpha_1}\frac{a_2}{a_1}(B^t - B^{t-1})(B^{t-1} - B^{t-2})$$
$$\le (A^{t-1} - A^t)(A^{t-2} - A^{t-1}) - \frac{\alpha_2}{\alpha_1}(A^{t-1} - A^t)(B^{t-1} - B^{t-2}) - (A^{t-2} - A^{t-1})\frac{a_2}{a_1}(B^t - B^{t-1})$$
$$+\frac{\alpha_2}{\alpha_1}\frac{a_2}{a_1}(B^t - B^{t-1})(B^{t-1} - B^{t-2})$$

$$\Rightarrow [(A^{t-1} - A^t) - \frac{\alpha_2}{\alpha_1}(B^t - B^{t-1})][(A^{t-2} - A^{t-1}) - \frac{a_2}{a_1}(B^{t-1} - B^{t-2})]$$
$$\le [(A^{t-1} - A^t) - \frac{a_2}{a_1}(B^t - B^{t-1})][(A^{t-2} - A^{t-1}) - \frac{\alpha_2}{\alpha_1}(B^t - B^{t-1})]$$

(Notice that $(A^{t-1} - A^t) - \frac{a_2}{a_1}(B^t - B^{t-1}) \le (A^{t-1} - A^t) - \frac{\alpha_2}{\alpha_1}(B^t - B^{t-1}) \le 0$)

$$\Rightarrow \frac{(A^{t-1} - A^t) - \frac{\alpha_2}{\alpha_1}(B^t - B^{t-1})}{(A^{t-2} - A^{t-1}) - \frac{\alpha_2}{\alpha_1}(B^{t-1} - B^{t-2})} \le \frac{(A^{t-1} - A^t) - \frac{a_2}{a_1}(B^t - B^{t-1})}{(A^{t-2} - A^{t-1}) - \frac{a_2}{a_1}(B^{t-1} - B^{t-2})}$$

$$\Rightarrow \frac{\alpha_1(A^{t-1} - A^t) - \alpha_2(B^t - B^{t-1})}{\alpha_1(A^{t-1} - A^{t-2}) - \alpha_2(B^{t-1} - B^{t-2})} \le \frac{a_1(A^{t-1} - A^t) - a_2(B^t - B^{t-1})}{a_1(A^{t-2} - A^{t-1}) - a_2(B^{t-1} - B^{t-2})}$$

$$\Rightarrow \frac{\tau^t - \tau^{t-1}}{\tau^{t-1} - \tau^{t-2}} \le \frac{\gamma^t - \gamma^{t-1}}{\gamma^{t-1} - \gamma^{t-2}}$$

$$\Rightarrow \frac{\tau^t - \tau^{t-1}}{\gamma^t - \gamma^{t-1}} \le \frac{\tau^{t-1} - \tau^{t-2}}{\gamma^{t-1} - \gamma^{t-2}}, \text{ for all } 3 \le t \le j.$$

It remains to show that

$$\frac{\tau^2 - \tau^1}{\gamma^2 - \gamma^1} \le \frac{\alpha_1}{a_1}.$$

As $\frac{A^1 - A^2}{B^2 - B^1} \le \frac{\alpha_2}{\alpha_1} \le \frac{a_2}{a_1}$ and $B^2 - B^1 \ge 0$, it follows that

$$0 \le -(A^1 - A^2) + \frac{\alpha_2}{\alpha_1}(B^2 - B^1) \le -(A^1 - A^2) + \frac{a_2}{a_1}(B^2 - B^1)$$

$$\Rightarrow 0 \le (A^2 - A^1) + \frac{\alpha_2}{\alpha_1}(B^2 - B^1) \le (A^2 - A^1) + \frac{a_2}{a_1}(B^2 - B^1)$$

$$\Rightarrow \frac{(A^2 - A^1) + \frac{\alpha_2}{\alpha_1}(B^2 - B^1)}{(A^2 - A^1) + \frac{a_2}{a_1}(B^2 - B^1)} \le 1$$

$$\Rightarrow \frac{\alpha_1(A^2 - A^1) + \alpha_2(B^2 - B^1)}{a_1(A^2 - A^1) + a_2(B^2 - B^1)} \le \frac{\alpha_1}{a_1}$$

$$\frac{\tau^2 - \tau^1}{\gamma^2 - \gamma^1} \le \frac{\alpha_1}{a_1}.$$

Since $\psi_1$ is continuous piecewise linear function, and the value of the slopes increases with $z$, the function $\psi_1$ is convex. □

**Proof of Proposition 4.4**. From Proposition 4.3 we know that $\psi_1$ is convex and by Proposition 4.2, $\psi_1$ is superadditive on $[0, D]$. It remains to show that $\psi_1(z) \leq \phi(z)$ for all $z \in [0, D]$.

For $0 \leq z \leq \gamma^j$ we have $\psi_1(z) = \phi(z) = 0$. Consider $\gamma^t < z \leq \gamma^{t-1}, t \in \{1, \dots, j\}$. Let $(a, b)$ be the optimal solution to the problem associated to $\phi(z)$. As $\alpha_1^t y_1 + \alpha_2^t y_2 \leq \alpha_1^t A^t + \alpha_2^t B^t$ is valid to $Y_{\leq}$ we can write

$$-\alpha_1^t(A^t - a) - \alpha_2^t(B^t - b) \leq 0$$

$$\Rightarrow -\alpha_1^t(A^t - a)(\frac{a_2}{a_1} - \frac{\alpha_2}{\alpha_1}) - \alpha_2^t(B^t - b)(\frac{a_2}{a_1} - \frac{\alpha_2}{\alpha_1}) \leq 0$$

$$\Rightarrow \alpha_1^t(A^t - a)(\frac{\alpha_2}{\alpha_1} - \frac{a_2}{a_1}) - \alpha_2^t(B^t - b)(\frac{a_2}{a_1} - \frac{\alpha_2}{\alpha_1}) \leq 0$$

$$\Rightarrow \frac{\alpha_2}{\alpha_1}\alpha_1^t(A^t - a) - \alpha_2^t\frac{a_2}{a_1}(B^t - b) \leq \frac{a_2}{a_1}\alpha_1^t(A^t - a) - \frac{\alpha_2}{\alpha_1}\alpha_2^t(B^t - b)$$

$$\Rightarrow -\alpha_2^t(A^t - a) + \frac{a_2}{a_1}\frac{\alpha_2}{\alpha_1}\alpha_1^t(B^t - b) + \frac{\alpha_2}{\alpha_1}\alpha_1^t(A^t - a) - \alpha_2^t\frac{a_2}{a_1}(B^t - b)$$

$$\leq -\alpha_2^t(A^t - a) + \frac{a_2}{a_1}\frac{\alpha_2}{\alpha_1}\alpha_1^t(B^t - b) + \frac{a_2}{a_1}\alpha_1^t(A^t - a) - \frac{\alpha_2}{\alpha_1}\alpha_2^t(B^t - b)$$

$$\Rightarrow (-\alpha_2^t + \frac{\alpha_2}{\alpha_1}\alpha_1^t)[(A^t - a) + \frac{a_2}{a_1}(B^t - b)] \leq (-\alpha_2^t + \frac{a_2}{a_1}\alpha_1^t)[(A^t - a) + \frac{\alpha_2}{\alpha_1}(B^t - b)]$$

$$\Rightarrow \frac{-\alpha_2^t + \frac{\alpha_2}{\alpha_1}\alpha_1^t}{-\alpha_2^t + \frac{a_2}{a_1}\alpha_1^t} \leq \frac{(A^t - a) + \frac{\alpha_2}{\alpha_1}(B^t - b)}{(A^t - a) + \frac{a_2}{a_1}(B^t - b)}$$

$$\Rightarrow \frac{\alpha_1(A^t - A^{t-1}) + \alpha_2(B^t - B^{t-1})}{a_1(A^t - A^{t-1}) + a_2(B^t - B^{t-1})} \leq \frac{\alpha_1(A^t - a) + \alpha_2(B^t - b)}{a_1(A^t - a) + a_2(B^t - b)}$$

$$\Rightarrow \frac{\tau^{t-1} - \tau^t}{\gamma^{t-1} - \gamma^t} \leq \frac{\tau(a, b) - \tau^t}{\gamma(a, b) - \gamma^t}$$

$$\Rightarrow \tau^t + \frac{\tau^{t-1} - \tau^t}{\gamma^{t-1} - \gamma^t}(\gamma(a, b) - \gamma^t) \leq \tau(a, b)$$

$$\Rightarrow \tau^t + \frac{\tau^{t-1} - \tau^t}{\gamma^{t-1} - \gamma^t}(z - \gamma^t) \leq \phi(z)$$

Notice that $-\alpha_2^t + \frac{a_2}{a_1}\alpha_1^t > 0$ and $(A^t - a) + \frac{a_2}{a_1}(B^t - b) = \frac{1}{a_1}(\gamma(a, b) - \gamma^t) > 0$, because $\gamma(a, b) \geq z$ and $(a, b)$ is a feasible solution to $\phi(z)$ and because we are assuming $\gamma^t < z \leq \gamma^{t-1}$.

Finally, consider $\gamma^1 < z \leq D$. Again, let $(a, b)$ be the optimal solution associated to $\phi(z)$. As $B^1 = 0$ we have $(\frac{\alpha_2}{\alpha_1} - \frac{a_2}{a_1})(B^1 - b) \geq 0$. Thus,

$$(A^1 - a) + \frac{\alpha_2}{\alpha_1}(B^1 - b) \geq (A^1 - a) + \frac{a_2}{a_1}(B^1 - b)$$

$$\Rightarrow \alpha_1(A^1 - a) + \alpha_2(B^1 - b) \geq \frac{\alpha_1}{a_1}(a_1(A^1 - a) + a_2(B^1 - b))$$

$$\Rightarrow \tau(a, b) - \tau^1 \geq \frac{\alpha_1}{a_1}(\gamma(a, b) - \gamma^1)$$

$$\Rightarrow \phi(z) \geq \tau^1 + \frac{\alpha_1}{a_1}(z - \gamma^1).$$

□

**Proof of Lemma 5.6**. First we prove the following results.

**Lemma 9.1** *If* $p_2^{v_1} \geq p_2^{v_2}$ *then* $\Delta_{v_1+1}^{v_1+q} \geq \Delta_{v_2+1}^{v_2+q}$ *for all* $q \in \{1, \ldots, \min\{\alpha - v_1, \alpha - v_2\}\}$.

**Proof:** The result is true for $q = 1$ because $\delta_{v_1+1} = 0$ implies $p_2^{v_1} < d^{k(j)-1} \Rightarrow p_2^{v_2} < d^{k(j)-1} \Rightarrow \delta_{v_2+1} = 0$. Suppose there exists $q > 1$ such that $\sum_{i=v_1+1}^{v_1+q-1} \delta_i = \sum_{i=v_2+1}^{v_2+q-1} \delta_i$, $\delta_{v_1+q} = 0$ and $\delta_{v_2+q} = 1$. Then $p_2^{v_1+q-1} = p_2^{v_1} - d^{k(j)-1} \sum_{i=v_1+1}^{v_1+q-1} \delta_i + f^{\ell(k(j))} \sum_{i=v_1+1}^{v_1+q-1}(1 - \delta_i) \geq p_2^{v_2} - d^{k(j)-1} \sum_{i=v_2+1}^{v_2+q-1} \delta_i + f^{\ell(k(j))} \sum_{i=v_2+1}^{v_2+q-1}(1 - \delta_i) = p_2^{v_2+q-1}$. Therefore, $\delta_{v_2+q} = 1$ implies $p_2^{v_2+q-1} \geq d^{k(j)-1}$ and hence $p_2^{v_1+q-1} \geq d^{k(j)-1}$. Thus $\delta_{v_1+q} = 1$, which contradicts the assumption $\delta_{v_1+q} = 0$. □

**Lemma 9.2** *If* $\delta_{v_1} = 0$ *and* $\delta_{v_2} = 1$ *with* $v_1 \neq v_2$ *then* $p_2^{v_1} \geq p_2^{v_2}$.

**Proof:** Suppose that $p_2^{v_1} < p_2^{v_2}$. Then $p_2^{v_1-1} + f^{\ell(k(j))} < p_2^{v_2-1} - d^{k(j)-1} \Rightarrow p_2^{v_2-1} > p_2^{v_1-1} + f^{\ell(k(j))} + d^{k(j)-1} \geq f^{\ell(k(j))} + d^{k(j)-1} = d^{k(j)}$ contradicting Lemma 5.2. □

Now we prove Lemma 5.6. We show that it can not happen $\sum_{i=1}^{s} \delta_i < \sum_{i=r+1}^{r+s} \delta_i - 1$ (the proof that $\sum_{i=1}^{s} \delta_i \leq \sum_{i=r+1}^{r+s} \delta_i + 1$ is similar). Suppose $\sum_{i=1}^{s} \delta_i < \sum_{i=r+1}^{r+s} \delta_i - 1$. Then, it must exist $1 \leq f < s$ such that $\delta_f = 0$, $\delta_{r+f} = 1$ and $\delta_i \geq \delta_{r+i}$ for all $i < f$. Thus $\sum_{i=1}^{f-1} \delta_i \geq \sum_{i=r+1}^{r+f-1} \delta_i$. From Lemma 9.2 we have $p_2^{f} \geq p_2^{r+f}$. And, from Lemma 9.1, $\sum_{i=f+1}^{s} \delta_i \geq \sum_{i=r+f+1}^{r+s} \delta_i$. Therefore $\sum_{i=1, i\neq f}^{s} \delta_i \geq \sum_{i=r+1, i\neq f}^{r+s} \delta_i$ which implies $\sum_{i=1}^{s} \delta_i \geq \sum_{i=r+1}^{r+s} \delta_i - 1$ contradicting the assumption that $\sum_{i=1}^{s} \delta_i < \sum_{i=r+1}^{r+s} \delta_i - 1$. □

**Proof of Proposition 6.3**. From the definition of $\phi$ and $\psi_3$ we have $\psi_3(z) \leq \phi(z)$, for all $z \in [0, D]$. Next we prove that $\psi_3$ is superadditive.

Consider $t_1$, $t_2 \geq 0$ such that $z_n \leq t_1 < z_{n+1}$ and $z_m \leq t_2 < z_{m+1}$, i.e. $t_1 = z_n + r_1$ and $t_2 = z_m + r_2$ with $0 \leq r_1 < z_{n+1} - z_n$ and $0 \leq r_2 < z_{m+1} - z_m$. Equivalently, using Corollary 5.14, $t_1 = \gamma^{j-1} + nw_2 + (w_1 - w_2)\Delta_1^n + r_1$ and $t_2 = \gamma^{j-1} + mw_2 + (w_1 - w_2)\Delta_1^m + r_2$. Hence,

$$t_1 + t_2 = z_{n+m} + (w_1 - w_2)(\Delta_1^m - \Delta_{n+1}^{n+m}) + \gamma^{j-1} + r_1 + r_2.$$

Consider the following cases:

**Case $\boldsymbol{\delta_{n+1} = \delta_{m+1} = 1}$.** We have,

$$\psi_3(t_1) = \begin{cases} n+1, & \text{if } r_1 \geq \lambda, \\ n + \frac{r_1}{\lambda}, & \text{if } r_1 < \lambda, \end{cases} \quad \text{and } \psi_3(t_2) = \begin{cases} m+1, & \text{if } r_2 \geq \lambda, \\ m + \frac{r_2}{\lambda}, & \text{if } r_2 < \lambda. \end{cases}$$

Lemma 5.6 implies $\Delta_1^m \geq \Delta_{n+1}^{n+m} - 1$. So $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + r_1 + r_2$.

Subcase $r_1 \geq \lambda, r_2 \geq \lambda$. Hence $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \lambda + \lambda$. By definition of $\lambda$, $\gamma^{j-1} = w_1 + (w_1 - w_2) - \lambda$ so $t_1 + t_2 \geq z_{n+m} + w_1 + \lambda$. Thus $\psi_3(t_1 + t_2) \geq n + m + 2 = \psi_3(t_1) + \psi_3(t_2)$.

Subcase $r_1 \geq \lambda, r_2 < \lambda$ (similarly, $r_1 < \lambda, r_2 \geq \lambda$). We have $\psi_3(t_1) + \psi_3(t_2) = n + m + 1 + \frac{r_2}{\lambda}$. On the other hand, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \lambda + r_2 = z_{n+m} + w_1 + r_2 \geq z_{n+m+1} + r_2$ $\Rightarrow \psi_3(t_1 + t_2) \geq \psi_3(t_1) + \psi_3(t_2)$.

Subcase $r_1 < \lambda, r_2 < \lambda$. Consider $r_1 + r_2 \geq \lambda$ (the case $r_1 + r_2 < \lambda$ is trivial). Then $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \lambda + r_1 + r_2 - \lambda = z_{n+m} + w_1 + r_1 + r_2 - \lambda \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_1 + r_2 - \lambda}{\lambda} =$

$n+m+\frac{r_1}{\lambda}+\frac{r_2}{\lambda}=\psi_3(t_1)+\psi_3(t_2)$.

**Case $\boldsymbol{\delta_{n+1}=\delta_{m+1}=0}$.** Then

$$\psi_3(t_1)=\begin{cases} n+1, & \text{if } r_1\geq\mu,\\ n+\frac{r_1}{\mu}, & \text{if } r_1<\mu,\end{cases} \quad\text{and } \psi_3(t_2)=\begin{cases} m+1, & \text{if } r_2\geq\mu,\\ m+\frac{r_2}{\mu}, & \text{if } r_2<\mu.\end{cases}$$

Subcase $\Delta_1^m=\Delta_{n+1}^{n+m}-1$. Hence $t_1+t_2=z_{m+n}+\gamma^{j-1}-(w_1-w_2)+r_1+r_2$. Lemma 5.6 with $s=m+1$ and $r=n+1$ implies $\Delta_{n+2}^{n+m+2}\leq\Delta_1^{m+1}+1$. Then, $\Delta_{n+2}^{n+m+2}=\Delta_{n+1}^{n+m}-\delta_{n+1}+\delta_{n+m+1}+\delta_{n+m+2}\leq\Delta_1^{m+1}+1=\Delta_1^m+\delta_{m+1}+1\Rightarrow\Delta_1^m+1-0+\delta_{n+m+1}+\delta_{n+m+2}\leq\Delta_1^m+0+1\Rightarrow\delta_{n+m+1}+\delta_{n+m+2}\leq 0$. Therefore $\delta_{n+m+1}=\delta_{n+m+2}=0$.

If $r_1+r_2\leq\mu$ then $\psi_3(t_1)+\psi_3(t_2)=n+m+\frac{r_1+r_2}{\mu}$. As $\gamma^{j-1}\geq w_1-w_2$ we have $t_1+t_2\geq z_{n+m}+r_1+r_2\Rightarrow\psi_3(t_1+t_2)\geq n+m+\frac{r_1+r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. If $r_1\geq\mu,\ r_2\geq\mu$ then $\psi_3(t_1)+\psi_3(t_2)=n+m+2$. On the other hand $t_1+t_2=z_{n+m}+\gamma^{j-1}-(w_1-w_2)+r_1+r_2=z_{n+m}+w_2-\mu+r_1+r_2=z_{n+m+1}+r_1+r_2-\mu\geq z_{n+m+1}+\mu\Rightarrow\psi_3(t_1+t_2)\geq n+m+2$. If $r_1\leq\mu$ and $r_2\geq\mu$ (similarly $r_1\geq\mu$ and $r_2\leq\mu$) then $t_1+t_2=z_{n+m}+\gamma^{j-1}-(w_1-w_2)+r_1+r_2\geq z_{n+m}+w_2+r_1\Rightarrow\psi_3(t_1+t_2)\geq n+m+1+\frac{r_1}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. Finally, if $r_1\leq\mu, r_2\leq\mu$ and $r_1+r_2\geq\mu$ then $t_1+t_2=z_{n+m}+\gamma^{j-1}-(w_1-w_2)+r_1+r_2=z_{n+m}+w_2+(r_1+r_2-\mu)\Rightarrow\psi_3(t_1+t_2)\geq n+m+1+\frac{r_1+r_2-\mu}{\mu}=n+m+\frac{r_1+r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$.

Subcase $\Delta_1^m=\Delta_{n+1}^{n+m}$. Thus $t_1+t_2=z_{n+m}+\gamma^{j-1}+r_1+r_2$. It can not occur $\delta_{n+m+1}=\delta_{n+m+2}=1$ because it would imply $\Delta_1^{m+1}=\Delta_1^m+\delta_{m+1}=\Delta_{n+1}^{n+m}+0=\delta_{n+1}+\Delta_{n+2}^{n+m+2}-\delta_{n+m+1}-\delta_{n+m+2}=0+\Delta_{n+2}^{n+m+2}-2$, which contradicts Lemma 5.6 with $s=m+1$ and $r=n+1$. We consider only the worst case, the case where $\psi_3$ increases slower, this means, $\delta_{n+m+1}=1$ and $\delta_{n+m+2}=0$. If $r_1\geq\mu$, $r_2\geq\mu$ then $t_1+t_2\geq z_{n+m}+\gamma^{j-1}+\mu+\mu=z_{n+m}+w_1+\mu=z_{n+m+1}+\mu\Rightarrow\psi_3(t_1+t_2)\geq n+m+2$ $=\psi_3(t_1)+\psi_3(t_2)$. If $r_1\geq\mu,\ r_2<\mu$ (similarly, $r_1<\mu,\ r_2\geq\mu$) then $t_1+t_2\geq z_{n+m}+\gamma^{j-1}+\mu+r_2=z_{n+m}+w_1+r_2\Rightarrow\psi_3(t_1+t_2)\geq n+m+1+\frac{r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. Suppose $r_1<\mu$ and $r_2<\mu$. If $r_1+r_2\geq\mu$ then $t_1+t_2=z_{n+m}+\gamma^{j-1}+\mu+r_1+r_2-\mu\geq z_{n+m+1}+r_1+r_2-\mu\Rightarrow\psi_3(t_1+t_2)\geq n+m+1+\frac{r_1+r_2-\mu}{\mu}$ $=n+\frac{r_1}{\mu}+m+\frac{r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. If $r_1+r_2<\mu$ then, as $t_1+t_2\geq z_{n+m}+(w_1-w_2)+r_1+r_2=z_{n+m}+r_1+r_2+(\lambda-\mu)$ we have $\psi_3(t_1+t_2)\geq n+m+\frac{r_1+r_2+(\lambda-\mu)}{\mu+(\lambda-\mu)}\geq n+m+\frac{r_1+r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$.

Subcase $\Delta_1^m=\Delta_{n+1}^{n+m}+1$. Hence $t_1+t_2=z_{n+m}+\gamma^{j-1}+w_1-w_2+r_1+r_2$. If $r_1\geq\mu,\ r_2\geq\mu$ then $t_1+t_2\geq z_{n+m}+\gamma^{j-1}+(w_1-w_2)+\mu+\mu=z_{n+m}+w_1+\mu+w_1-w_2\geq z_{n+m+1}+\lambda$ $\Rightarrow\psi_3(t_1+t_2)\geq n+m+2=\psi_3(t_1)+\psi_3(t_2)$. If $r_1\geq\mu,\ r_2<\mu$ (similarly, $r_1<\mu,\ r_2\geq\mu$) then $t_1+t_2\geq z_{n+m}+\gamma^{j-1}+(w_1-w_2)+\mu+r_2=z_{n+m}+w_1+w_1-w_2+r_2\Rightarrow\psi_3(t_1+t_2)\geq n+m+1+\frac{r_2+(\lambda-\mu)}{\mu+(\lambda-\mu)}\geq n+m+1+\frac{r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. Suppose $r_1<\mu$ and $r_2<\mu$. If $r_1+r_2\geq\mu$ then $t_1+t_2=z_{n+m}+\gamma^{j-1}+(w_1-w_2)+\mu+r_1+r_2-\mu\geq z_{n+m+1}+(w_1-w_2)+r_1+r_2-\mu\Rightarrow$ $\psi_3(t_1+t_2)\geq n+m+1+\frac{r_1+r_2-\mu+(w_1-w_2)}{\lambda}=n+m+\frac{r_1+r_2-\mu+(w_1-w_2)+\lambda}{\lambda}=n+m+\frac{r_1+(\lambda-\mu)+r_2+(\lambda-\mu)}{\lambda}$ $\geq n+\frac{r_1}{\mu}+m+\frac{r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$. If $r_1+r_2<\mu$ then, as $t_1+t_2\geq z_{n+m}+r_1+(\lambda-\mu)+r_2+(\lambda-\mu)$ we have $\psi_3(t_1+t_2)\geq n+m+\frac{r_1+(\lambda-\mu)}{\mu+(\lambda-\mu)}+\frac{r_2+(\lambda-\mu)}{\mu+(\lambda-\mu)}\geq n+m+\frac{r_1}{\mu}+\frac{r_2}{\mu}=\psi_3(t_1)+\psi_3(t_2)$.

**Case $\boldsymbol{\delta_{n+1}=0,\delta_{m+1}=1}$** (similar to the case $\delta_{n+1}=1,\delta_{m+1}=0$). Then

$$\psi_3(t_1)=\begin{cases} n+1, & \text{if } r_1\geq\mu,\\ n+\frac{r_1}{\mu}, & \text{if } r_1<\mu,\end{cases} \quad\text{and } \psi_3(t_2)=\begin{cases} m+1, & \text{if } r_2\geq\lambda,\\ m+\frac{r_2}{\lambda}, & \text{if } r_2<\lambda.\end{cases}$$

Subcase $\Delta_1^m=\Delta_{n+1}^{n+m}-1$. So $t_1+t_2=z_{n+m}+\gamma^{j-1}-(w_1-w_2)+r_1+r_2$. In this case we have $\Delta_1^m$ $=\Delta_{n+1}^{n+m}-1\Leftrightarrow\Delta_1^m=\delta_{n+1}+\Delta_{n+2}^{n+m}-1\Leftrightarrow\Delta_1^m=\Delta_{n+2}^{n+m+1}-\delta_{n+m+1}-1$. Lemma 5.6 with $s=m$ and $r=n+1$ implies $\delta_{n+m+1}=0$.

Case $r_1 \geq \mu$ and $r_2 \geq \lambda$. Hence $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \mu + \lambda = z_{n+m} + w_2 + \lambda = z_{n+m+1} + \lambda \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 2 = \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 < \mu$ and $r_2 < \lambda$. Suppose $r_1 + r_2 \geq \mu$. Thus, $t_1 + t_2 = z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \mu + (r_1 + r_2 - \mu) = z_{n+m+1} + r_1 + r_2 - \mu \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_1+r_2-\mu}{\lambda} = n + m + \frac{r_1+(\lambda-\mu)}{\mu+(\lambda-\mu)} + \frac{r_2}{\lambda} \geq n + m + \frac{r_1}{\mu} + \frac{r_2}{\lambda} = \psi_3(t_1) + \psi_3(t_2)$. Now suppose $r_1 + r_2 < \mu$. Thus, $t_1 + t_2 = z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + r_1 + r_2 \geq z_{n+m} + r_1 + r_2 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + \frac{r_1+r_2}{\mu} \geq \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 \geq \mu$ and $r_2 < \lambda$. Hence, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + \mu + r_2 = z_{n+m+1} + r_2 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_2}{\lambda} = \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 < \mu$ and $r_2 \geq \lambda$. Thus, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} - (w_1 - w_2) + r_1 + \lambda = z_{n+m+1} + (\lambda - \mu) + r_1 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_1+(\lambda-\mu)}{\lambda} = n + m + 1 + \frac{r_1+(\lambda-\mu)}{\mu+(\lambda-\mu)} \geq n + m + 1 + \frac{r_1}{\mu} = \psi_3(t_1) + \psi_3(t_2)$.

Subcase $\Delta_1^m \geq \Delta_{n+1}^{n+m}$. Thus $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} + r_1 + r_2$. Case $r_1 \geq \mu$ and $r_2 \geq \lambda$. Hence $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} + \mu + \lambda = z_{n+m} + w_1 + \lambda \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 2 = \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 < \mu$ and $r_2 < \lambda$. Suppose $r_1 + r_2 \geq \mu$. Thus, $t_1 + t_2 = z_{n+m} + \gamma^{j-1} + \mu + (r_1 + r_2 - \mu) = z_{n+m} + w_1 + r_1 + r_2 - \mu \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_1+r_2-\mu}{\lambda} = n + m + \frac{r_1+(\lambda-\mu)}{\mu+(\lambda-\mu)} + \frac{r_2}{\lambda} \geq n + m + \frac{r_1}{\mu} + \frac{r_2}{\lambda} = \psi_3(t_1) + \psi_3(t_2)$. Now suppose $r_1 + r_2 < \mu$. Thus, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} + r_1 + r_2 \geq z_{n+m} + (\lambda - \mu) + r_1 + r_2 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + \frac{r_1+r_2+(\lambda-\mu)}{\lambda} \geq n + m + \frac{r_1}{\mu} + \frac{r_2}{\lambda} = \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 \geq \mu$ and $r_2 < \lambda$. Hence, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} + \mu + r_2 \geq z_{n+m+1} + r_2 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_2}{\lambda} = \psi_3(t_1) + \psi_3(t_2)$. Case $r_1 < \mu$ and $r_2 \geq \lambda$. Hence, $t_1 + t_2 \geq z_{n+m} + \gamma^{j-1} + r_1 + \lambda \geq z_{n+m+1} + (\lambda - \mu) + r_1 \Rightarrow \psi_3(t_1 + t_2) \geq n + m + 1 + \frac{r_1+(\lambda-\mu)}{\mu+(\lambda-\mu)} \geq n + m + 1 + \frac{r_1}{\mu} = \psi_3(t_1) + \psi_3(t_2)$. □

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