Convex- and Monotone-Transformable Mathematical Programming Problems and a Proximal-Like Point Method

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Abstract

The problem of finding singularities of monotone vectors fields on Hadamard manifolds will be considered and solved by extending the well-known proximal point algorithm. For monotone vector fields the algorithm will generate a well defined sequence, and for monotone vector fields with singularities it will converge to a singularity. It will be also shown how tools of convex analysis on Riemannian manifolds can solve non-convex constrained problems in Euclidean spaces. To illustrate this remarkable fact examples will be given.

1 Introduction

Convexity is a sufficient but not necessary condition for many important results of mathematical programming, since there are diverse extensions of the notion of convexity bearing the same properties. E.g., the critical points of pseudo-convex and strictly quasi-convex differentiable functions are global minimizers. Moreover, it is possible to modify numerical methods to solve non-convex optimization problems. E.g., the steepest descent method with a proximal regularization [6] or with Armijo's stepsize [2] generates a sequence that, starting at any point of $\mathbb{R}^n$, converges to a minimizer of a pseudo-convex differentiable function.

It is well-known that a function is convex iff its restriction to each line segment in its domain is convex. This property inspired Ortega and Rheinboldt [10], M. Avriel [1] and others to introduce the concept of arcwise convex functions. The idea of arcwise convexity can be further extended

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to functions that are arcwise non-convex, but can be transformed to arcwise convex functions. By using the tools of Riemannian Geometry, T. Rapcsák [12] introduces a modern novel method to investigate such non-convex problems.

Inspired by T. Rapcsák and C. Udriste's geometrical viewpoint, beside some non-convex problems, we shall consider non-monotone problems too. We shall solve them by extending the proximal point algorithm.

The above mentioned non-convex and non-monotone problems are of the form

$$\min_{p\in M} f(p) \tag{1}$$

and

$$\text{Find } x \in M \text{ such that } T(x) = 0, \tag{2}$$

respectively where M is a subset of the Euclidean space $\mathbb{R}^n$, $f : M \to \mathbb{R}$ is a function and $T : \mathbb{R}^n \to \mathbb{R}^n$ a vector field.

By choosing an appropriate Riemannian metric [3] on M we shall transform problems (1) and (2) into a convex and monotone unconstrained problem on M, respectively, that can be studied by using the intrinsic geometry of M. Since there is an analogy of ideas, throughout this paper we shall use this parallel approach of optimization and singularity problems. On the meantime, note that for a gradient vector field (i.e., a vector field that is the gradient of a function with respect to the metric of M) a singularity problem is equivalent to an optimization problem, and if the gradient vector field is monotone (with respect to the metric of M [8]) it is equivalent to a convex optimization problem (with respect to the metric of M) [12]. Bearing this in mind, problem (2) can be viewed as a non-gradient extension of problem (1) considered by T. Rapcsák in [12]. The examples given for problem (1) follow the ideas of T. Rapcsák and will be presented here for the sake of parallelism between gradient (i.e., optimization problems) and non-gradient singularity problems. However, solving optimization problems of type (1) by using an extended proximal point method is a new idea in the Theory of Optimization on Riemannian manifolds.

For illustrating (1) and (2), consider the following unconstrained problems defined in the positive orthant

$$\mathbb{R}^2_{++} = \left\{p = (p_1, p_2) \in \mathbb{R}^2 \, : \, p_1, p_2 > 0\right\}.$$

Problem 1.1. In the optimization problem (1) take the function $f : \mathbb{R}^2_{++} \to \mathbb{R}$, defined by

$$f(p_1, p_2) = {p_1}^{-1} + {p_1}^{1/2} + {p_2}^{-1} + {p_2}^{1/2}.$$

Problem 1.2. In problem (2) take the vector field $X : \mathbb{R}^2_{++} \to \mathbb{R}^2$, defined by

$$X(p_1, p_2) = (-{p_1}^{\frac{1}{2}} + {p_1}^{\frac{3}{2}}, -{p_2}^{\frac{1}{2}} + {p_2}^{\frac{3}{2}}).$$

Problems 1.1 and 1.2 are not convex and monotone in the classical sense, that is the objective function f is not convex and the vector field X is not monotone, respectively.

Endowing $\mathbb{R}^2_{++}$ with the Riemannian metric $G : \mathbb{R}^2_{++} \longrightarrow S^n_{++}$, defined by

$$G(p_1, p_2) = \begin{pmatrix} {p_1}^{-2} & 0 \\ 0 & {p_2}^{-2} \end{pmatrix},$$

we obtain the Riemannian manifold $M_G = (M, G)$ which is isometric to the Euclidean space $\mathbb{R}^2$ through the isometry $\Phi : \mathbb{R}^2 \to M_G$, defined by

$$\Phi(x_1, x_2) = (e^{x_1}, e^{x_2}).$$

Consider the convex function $g : \mathbb{R}^2 \to \mathbb{R}$;

$$g(x_1, x_2) = e^{-x_1} + e^{\frac{x_1}{2}} + e^{-x_2} + e^{\frac{x_2}{2}}$$

and observe that $g(x_1, x_2) = f(\Phi(x_1, x_2))$. Proposition 1 states that the image of a convex function through an isometry is convex. Hence, the function f is convex in M_G. Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be the monotone vector field, defined by

$$Y(x) = (-e^{-\frac{1}{2}x_1} + e^{\frac{1}{2}x_1}, -e^{-\frac{1}{2}x_2} + e^{\frac{1}{2}x_2}).$$

Note that $X = d\Phi \circ Y \circ \Phi^{-1}$. Proposition 2 states that the image of a monotone vector field through an isometry is monotone. It follows that X is monotone in M_G. Summarizing, we transformed the non-convex problem 1.1 and the non-monotone problem 1.2 into convex and monotone problems, respectively.

The proximal point algorithm for finding zeroes of monotone operators T on Hilbert spaces, generates a sequence of points $\{p_k\}$ as follows:

p_{k+1} is the unique zero of the regularized operator $T + \lambda_k I$, where λ_k is a real number satisfying $0 < \lambda_k \leq \tilde{\lambda}$, for some $\tilde{\lambda} > 0$, and I is the identity operator. The idea is to solve the possibly ill-posed problem of finding zeros of T, by solving a sequence of well-posed problems (i.e., have exactly one solution when T is strongly monotone) of finding the zeros of $T + \lambda_k I$.

An extension of this problem is the following variational inequality problem: given a convex constraint set C and the monotone operator T find p_* in C such that $\langle T(p_*), p - p_* \rangle \geq 0$ for all $p \in C$. When the constraint set of the variational inequality problem is a Riemannian manifold and the operator is a monotone vector field with respect to the metric of the Riemannian manifold, the variational inequality problem becomes the problem of finding the singularities of the monotone vector field.

In the case of Hadamard manifolds we shall solve this problem by extending the proximal point algorithm as follows:

We shall generate a sequence $\{p_k\}$, where p_{k+1} is defined as the unique singularity of the regularized vector field $X + \lambda_k \operatorname{grad} \rho_{p_k}$, the sequence $\{\lambda_k\}$ is such that $0 < \lambda_k < \tilde{\lambda}$ for some $\tilde{\lambda} > 0$, the vector field $\operatorname{grad} \rho_{p_k}$ is the gradient vector field of the map $\rho_{p_k} = \frac{1}{2} d^2(., p_k)$ and d is the Riemannian distance.

2 Basics Concepts

In this section some frequently used notations, basic definitions and important properties of Riemannian manifolds are presented. They can be found in any introductory book on Riemannian Geometry, for example [3] and [13]. Throughout this paper, all manifolds are smooth, paracompact and connected and all functions and vector fields are smooth.

Given a manifold M, denote by $\mathfrak{X}(M)$ the set of vector fields over M, by T_pM the tangent space of M at p and by $\mathfrak{F}(M)$ the ring of functions over M. M can be always endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$, with corresponding norm denoted by $\|\cdot\|$, to become a *Riemannian* manifold. The length (with respect to the metric $\langle\cdot,\cdot\rangle$) of a piecewise smooth curve $c:[a,b]\to M$ is defined by $l(c)=\int_a^b\|c'(t)\|dt$. Minimizing this length functional over the set of curves $c:[a,b]\to M$ joining two arbitrary points $p,q\in M$ (i.e, c(a)=p and c(b)=q) we obtain a distance function $(p,q)\mapsto d(p,q)$ which induces the original topology of M. The metric induces a map $f\in\mathfrak{F}(M)\mapsto \text{grad } f\in\mathfrak{X}(M)$ which associates to each f its *gradient* via the rule $\langle\text{grad } f,X\rangle=df(X),\ X\in\mathfrak{X}(M)$. Let ∇ be the Levi-Civita connection associated to $(M,\langle,\rangle)$. The *hessian* of a function f is given by $Hess\, f(X,Y)=\langle\nabla_X(\text{grad}),Y\rangle=XYf-\nabla_XYf$, for all $X,Y\in\mathfrak{X}(M)$. If c is a curve joining the points p and q in M, then, for each $t\in[a,b]$, ∇ induces an isometry, relative to $\langle,\rangle$, $P(c)_t^a: T_{c(a)}M\to T_{c(t)}M$, the so-called *parallel transport* along c from $c(a)$ to $c(t)$. The inverse map of $P(c)_t^a$ is denoted by $P(c^{-1})_t^a: T_{c(t)}M\to T_{c(a)}M$. A vector field V along c is said to be *parallel* if $\nabla_{c'}V=0$. If c' itself is parallel we say that c is a *geodesic*. The geodesic equation $\nabla_{\ \gamma'}\gamma'=0$ is a second order nonlinear ordinary differential equation, and γ is determined by its position and velocity at one point. It is easy to check that $\|\gamma'(t)\|$ is constant. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*.

A Riemannian manifold is *complete* if its geodesics are defined for any values of t. Hopf-Rinow's theorem asserts that if this is the case then any pair of points in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M,d) is a complete metric space and bounded and closed subsets are compact. In this paper, all manifolds are assumed to be complete. The *exponential map* $exp_p:T_pM\to M$ is defined by $exp_xv=\gamma_v(1,x)$, where $\gamma(.)=\gamma_v(.,p)$ is the geodesic defined by it's position p and velocity v at p. It is easy to show that $exp_ptv=\gamma_v(t,p)$ for every t. A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Hadamard's Theorem [3, 13] asserts that the topological and differential structure of a Hadamard manifolds coincide with those of an Euclidean space of the same dimension. More precisely, at any point $p\in M$, the exponential map $exp_p:T_pM\to M$ is a diffeomorphism. Furthermore, Hadamard manifolds have some geometrical properties similar to some well-known geometrical properties of Euclidean spaces. From now on let H be a Hadamard manifold.

A *geodesic triangle* $\Delta(p_1p_2p_3)$ in H is the set consisting of three distinct points p_1, p_2 ,p_3 called the *vertices* and three geodesic segments γ_i joining p_{i+1} to p_{i+2} called the *sides*, where $i=1,2,3(mod\,3)$.

Theorem 1. Let $\Delta(p_1p_2p_3)$ be a geodesic triangle in H. Denote by γ_i the geodesic segment joining p_{i+1} to p_{i+2} and set the lengths $\ell_i = l(\gamma_i)$ and the angles $\theta_i = \measuredangle\left(\gamma'_{i-1}(0), -\gamma'_{i+1}(l_{i+1})\right)$, where $i = 1, 2, 3(mod\, 3)$. Then

$$\theta_1 + \theta_2 + \theta_3 \leqslant \pi, \tag{3}$$

and

$$\ell_{i+1}^2 + \ell_{i+2}^2 - 2\ell_{i+1}\ell_{i+2}cos\theta_i \leqslant \ell_i^2. \tag{4}$$

Proof. Inequalities (3) and (4) are proved in [13] Proposition 4.5, page 223. □

Let M and N be connected Riemannian manifolds and $\Phi : M \to N$ be an isometry, that is, Φ is C^∞, and for all $p \in M$ and $u, v \in T_pM$, we have

$$\langle d\Phi_p u, d\Phi_p v\rangle = \langle u, v\rangle.$$

One can verify that, when Φ is an isometry, Φ preserves the Levi-Civita connection; in particular one has that Φ preserves geodesics, that is, β is a geodesic in M iff $\gamma = \Phi \circ \beta$ is a geodesic in N, and that

$$d\Phi_{\gamma(t)}\gamma'(t) = \beta'(t).$$

Furthermore, Φ preserves the distance function, that is,

$$d(\Phi(p), \Phi(q)) = d(p, q),$$

for all $p, q \in M$.

3 Monotone Vector Fields

For the sake of completeness, we shall include in this section some results which can be found in [4, 7, 8]. Given $X \in \mathfrak{X}(M)$ and a geodesic γ in M,

$$\varphi_{(X,\gamma)}(t) = \langle X(\gamma(t)), \gamma'(t)\rangle$$

defines a real function of t. In [7] S. Z. Németh introduced the notion of *monotone vector fields* on M as follows: X is monotone if $\varphi_{(X,\gamma)}$ is monotone nondecreasing for all geodesics γ in M. In [4] a vector field X on M was called *strongly monotone* if

$$\Psi_{(X,\gamma)}(t) = \varphi_{(X,\gamma)}(t) - \lambda\|\gamma'(0)\|^2 t,$$

is a monotone nondecreasing function of t for some $\lambda > 0$ and all geodesics γ in M. It can be easily checked that the above definitions are sound, i.e., they are independent of the choice of parameter t.

In the case of $M = H$ it has been proved [4] that X is monotone (strongly monotone) iff for all p, $q \in H$ it holds that

$$\langle P(\gamma^{-1})_1^0 X(q) - X(p), {exp_p}^{-1} q\rangle \geq 0,$$

$$\Big(\langle P(\gamma^{-1})_1^0 X(q) - X(p), {exp_p}^{-1} q\rangle \geq \lambda d^2(p,q)\Big) \tag{5}$$

where $\gamma :\ [0,1] \to H$ is the geodesic joining p to q and P is the parallel transport.

Example 3.1. Take $p' \in H$. By Hadamard's Theorem the exponential map has inverse $exp_{p'}^{-1}$: $H \to T_{p'}H$, and hence $d(p,p') = \|exp_{p'}^{-1}p\|$. Therefore, the function $\rho_{p'} : H \to R$, defined by

$$\rho_{p'}(p) = \frac{1}{2}d^2(p,p'), \tag{6}$$

is smooth and its gradient can be calculated by the formula [13]

$$\text{grad}\,\rho_{p'}(p) = -exp_p^{-1}p'. \tag{7}$$

It has been proved [4] that, for all fixed $p' \in H$, the gradient vector field $\text{grad}\,\rho_{p'}(p)$ is strongly monotone.

Example 3.2. A function $f : M \to \mathbb{R}$ is called convex, strictly convex or strongly convex if its composition with each geodesic γ in M is a convex, strictly convex or strongly convex function, respectively. In [7] it was proved that if f is convex (strictly convex), then $\text{grad}\, f$ is a monotone (strictly monotone) vector field. In [4] it was proved that if f is strongly convex, then $\text{grad}\, f$ is a strongly monotone vector field.

The *differential* of X at $p \in H$ is the linear map $A_X(p) : T_pH \to T_pH$, given by $A_X(p).v = \nabla_v X(p)$. If $X = \text{grad}\, f$ then $A_X(p) = \text{Hess}\ f_p$, where $f \colon M \to R$. Note that

$$\begin{aligned}\Psi'_{(X,\gamma)}(t) &= \langle \nabla_{\gamma'(t)}X, \gamma'(t)\rangle - \lambda\|\gamma'(t)\|^2 \\ &= \langle A_X(\gamma(t)).\gamma'(t), \gamma'(t)\rangle - \lambda\|\gamma'(t)\|^2.\end{aligned}$$

Then, X is strongly monotone iff there exist $\lambda > 0$ such that

$$\langle A_X(p)v, v\rangle \geq \lambda\|v\|^2, \tag{8}$$

for all $p \in H$ and $v \in T_pH$, because X is strongly monotone iff $\Psi_{(X,\gamma)}$ is monotone nondecreasing and $\Psi_{(X,\gamma)}$ is monotone nondecreasing iff $\Psi'_{(X,\gamma)}$ is nonnegative.

Let X be a vector field on $\mathfrak{X}(H)$. Consider the map $f : H \to R$ defined by

$$f(p) = \frac{1}{2}\|X(p)\|^2. \tag{9}$$

Lemma 1. If X is strongly monotone then f, defined by (9), is coercive, i.e., for all fixed p', $\lim_{d(p',p)\to\infty} f(p) = \infty$.

Proof. Assume, on the contrary, that there exist $c > 0$ and a sequence $\{p_k\} \subset H$ such that $\lim_{k\to\infty} d(p', p_k) = \infty$ and $f(p_k) \leq c$, for all k. Let γ_k be the geodesic with $\gamma_k(0) = p'$ and $\gamma_k(1) = p_k$. Then, by (5) there exists $\lambda > 0$ such that

$$\lambda d^2(p', p_k) \leq \langle exp_{p'}^{-1} p_k, P(\gamma_k^{-1})_1^0 X(p_k) - X(p')\rangle.$$

By using the Cauchy inequality and the fact that $f(p_k) \leq c$ for all k, we get $\lambda d(p', p_k) \leq \sqrt{2c} + \|X(p')\|$, i.e., $d(p', p_k)$ is bounded, in contradiction with our assumption. □

Corollary 1. If X is strongly monotone then there exists a unique $\hat{p} \in H$ such that $X(\hat{p}) = 0$.

Proof. By Lemma 1 f, as defined in (9), is coercive. Therefore, f attains its minimum. Let $\hat{p}$ be a minimizer of f. Then,

$$0 = df_{\hat{p}} v = \langle A_X(\hat{p})v, X(\hat{p})\rangle,$$

for all $v \in T_{\hat{p}}H$. Taking $v = X(\hat{p})$ and by using (8), we get that

$$0 = \langle A_X(\hat{p})X(\hat{p}), X(\hat{p})\rangle \geq \lambda \|X(\hat{p})\|^2$$

for some $\lambda > 0$. Thus, $\|X(\hat{p})\| = 0$. The uniqueness is an immediate consequence of the definition of strong monotonicity. □

Proposition 1. Let M, N be Riemannian manifolds and $\Phi : M \to N$ an isometry. The function $f : N \to \mathbb{R}$ is convex iff $g : M \to \mathbb{R}$, defined by $g(p) = f(\Phi(p))$, is convex.

Proof. Follows from the definition of convexity and the fact that isometries preserve geodesics. □

Proposition 2. Let M and N be Riemannian manifolds, $X \in \mathfrak{X}(M)$ and $\Phi : M \to N$ an isometry. Let $Y \in \mathfrak{X}(N)$ be defined by

$$Y = d\Phi \circ X \circ \Phi^{-1}.$$

Then,

1. X is monotone iff Y is monotone;

2. X is strictly monotone iff Y is strictly monotone and

3. X is strongly monotone iff Y is strongly monotone.

Proof. We shall prove iii). The proofs of i) and ii) are similar.

Since Φ is an isometry, $\beta = \Phi^{-1} \circ \gamma$ is a geodesic in M iff γ is a geodesic in N and it holds that $\|\gamma'(t)\| = \|\beta'(t)\|$. Then, for all λ, we have

$$\begin{aligned}
\Psi_{(Y,\gamma)}(t) &= \varphi_{(Y,\gamma)}(t) - \lambda\|\gamma'(0)\|^2 t \\
&= \langle Y(\gamma(t)), \gamma'(t)\rangle - \lambda\|\gamma'(0)\|^2 t \\
&= \langle d\Phi_{\Phi^{-1}(\gamma(t))}.X(\Phi^{-1}(\gamma(t)))\,,\, \gamma'(t)\rangle - \lambda\|\gamma'(0)\|^2 t \\
&= \langle d\Phi_{\beta(t)} X(\beta(t))\,,\, d\Phi_{\beta(t)}\beta'(t)\rangle - \lambda\|\beta'(0)\|^2 t \\
&= \langle X(\beta(t))\,,\, \beta'(t)\rangle - \lambda\|\beta'(0)\|^2 t \\
&= \Psi_{(X,\beta)}(t).
\end{aligned}$$

Therefore, $\Psi_{(Y,\gamma)}$ is monotone for some λ iff $\Psi_{(X,\beta)}$ is monotone. □

4 Problems From the Geometric Viewpoint

In the Euclidean space $\mathbb{R}^n$ let $M \subset \mathbb{R}^n$, $f : M \to \mathbb{R}$ and $X : \mathbb{R}^n \to \mathbb{R}^n$. Consider the optimization problem

$$\min_{p\in M} f(p) \tag{10}$$

and the more general problem

$$\text{Find } p \in M \text{ such that } X(p) = 0. \tag{11}$$

Next, we shall give several examples for these problems which are non-convex and non-monotone, but, by choosing an appropriate metric, can be transformed into convex and monotone problems, respectively.

4.1 The Plane With Other Metrics

Consider the following unconstrained problems defined in the Euclidean plane.

Problem 4.1. In the optimization problem (10) take the Rosenbock's banana function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(p_1, p_2) = 100\big(p_2 - {p_1}^2\big)^2 + \big(1 - p_1\big)^2.$$

Problem 4.2. In problem (11) take the vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$X(p) = (-p_1^2 + p_1 + p_2, -2p_1^3 + 2p_1^2 + 2p_1p_2 - p_1).$$

Problem 4.1 is not convex in the classical sense, i.e., the objective function f is not convex, and problem 4.2 is not monotone in the classical sense, i.e., the vector field X is not monotone. Endowing $\mathbb{R}^2$ with the Riemannian metric $G : \mathbb{R}^2 \to S^n_{++}$, defined by

$$G(p_1, p_2) = \begin{pmatrix} 1 + 4p_1^2 & -2p_1 \\ -2p_1 & 1 \end{pmatrix},$$

we obtain the Riemannian manifold M_G, that is complete and of constant curvature $K = 0$. Note that the map $\Phi : \mathbb{R}^2 \to M_G$, defined by

$$\Phi(x_1, x_2) = \big(x_1, x_1^2 - x_2\big),$$

is an isometry. Now consider the convex function $g : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$g(x_1, x_2) = 100x_2^2 + (1 - x_1)^2$$

and observe that $g(x_1, x_2) = f(\Phi(x_1, x_2))$. Therefore, by Proposition 1, it follows that f is convex in M_G. Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a monotone vector field defined by $Y(x_1, x_2) = (x_1 - x_2, x_1)$. Note that $X = d\Phi \circ Y \circ \Phi^{-1}$. Therefore, by Proposition 2, X is monotone in M_G.

Problem 4.3. In the optimization problem (10) take the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(p_1, p_2) = e^{p_1}\big(cosh(p_2) - 1\big).$$

Problem 4.4. In problem (11) take the vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$X(p_1, p_2) = \big(e^{p_1}(\cosh(p_2) - 1)\,,\, e^{-p_1}\sinh(p_2)\big).$$

Problem 4.3 is not convex in the classical sense, i.e., the objective function f is not convex, and problem 4.4 is not monotone in the classical sense, i.e., the vector field X is not monotone. Endowing $\mathbb{R}^2$ with the Riemannian metric $G : \mathbb{R}^2 \to S^n_{++}$, defined by

$$G(p_1, p_2) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2p_1} \end{pmatrix}$$

we obtain the Riemannian manifold M_G, that is complete and of constant curvature $K = -1$. The Christoffel symbols are given by

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = 0,\ \Gamma^2_{12} = \Gamma^2_{21} = 1 \text{ and } \Gamma^1_{22} = -e^{2p_1}.$$

Then, for each vector field $Y(p_1, p_2) = \big(a(p_1, p_2), b(p_1, p_2)\big)$, defined on M_G, we have

$$A_Y\big(p_1, p_2\big) = \begin{pmatrix} \frac{\partial a}{\partial p_1} & \frac{\partial a}{\partial p_2} - e^{2p_1}b \\ e^{2p_1}\big(\frac{\partial b}{\partial p_1} + b\big) & e^{2p_1}\big(\frac{\partial b}{\partial p_2} + a\big) \end{pmatrix}. \tag{12}$$

The gradient vector field of f is $\operatorname{grad} f(p) = G^{-1}(p)\big(\frac{\partial f}{\partial p_1}(p), \frac{\partial f}{\partial p_2}(p)\big)$. From (12) it follows that the hessian matrix $Hess(f) = A_{\operatorname{grad}(f)}$ of f is given by

$$Hess(f) = \begin{pmatrix} e^{p_1}\big(\cosh(p_2) - 1\big) & 0 \\ & \\ 0 & e^{p_1}\cosh(p_2) + e^{3p_1}\big(\cosh(p_2) - 1\big) \end{pmatrix}.$$

Note that this matrix is positive semidefinite. Therefore, f is convex in M_G. It can be also checked that

$$A_X(p_1, p_2) = \begin{pmatrix} e^{p_1}\big(\cosh(p_2) - 1\big) & 0 \\ 0 & e^{p_1}\cosh(p_2) + e^{3p_1}(\cosh(p_2) - 1)\big) \end{pmatrix}.$$

Thus, X is monotone in M_G.

4.2 The Positive Orthant With Other Metrics

Consider the following constrained problems defined in the positive orthant.

Problem 4.5. In the optimization Problem (10) take the polynomial $f : \mathbb{R}^n_{++} \to \mathbb{R}$, defined by

$$f(p_1, ..., p_n) = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} p_j^{b_{ij}},$$

where $c_i \in \mathbb{R}_{++}$ and $b_{ij} \in \mathbb{R}$ for all i, j.

Problem 4.6. In problem (11) take the vector field $X : \mathbb{R}^n_{++} \to \mathbb{R}^n$, defined by $X(p_1, ..., p_n) = (a_1, \ldots, a_n)$, where $a_i = p_i \ln(p_1 \ldots p_i p_{i+1}^{-1} \ldots p_n^{-1})$ for all $i = 1, \ldots n$.

Problem 4.5 is not convex in the classical sense, i.e., the objective function f is not convex, and Problem 4.6 is not monotone in the classical sense, i.e., the vector field X not monotone. Endowing $\mathbb{R}^n_{++}$ with the Riemannian metric $G : \mathbb{R}^n \to S^n_{++}$, defined by

$$G = diag\big({p_1}^{-2}, {p_2}^{-2}, \ldots, {p_n}^{-2}\big), \tag{13}$$

we obtain the Riemannian manifold M_G, that is complete and of constant curvature $K = 0$. Note that the map $\Phi : \mathbb{R}^n \to M_G$, defined by

$$\Phi(x_1, \ldots, x_n) = \big(e^{x_1}, \ldots, e^{x_n}\big),$$

is an isometry. Now consider the function $g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$g(x_1, \ldots, x_n) = \sum_{i=1}^{m} c_i e^{\sum_{j=1}^{n} b_{ij} x_j}.$$

Note that this function is convex in the classical sense and that $g(x_1,\ldots,x_n) = f(\Phi(x_1,\ldots,x_n))$. Therefore, by Proposition, 1, it follows that f is convex in M_G. Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be the monotone vector field , defined by $Y(x) = Ax$, where $x = (x_1,\ldots x_n)$ and

$$A = \begin{pmatrix} 1 & -1 & \ldots & -1 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 1 & \ldots & 1 & 1 \end{pmatrix}, \tag{14}$$

Note that $Y = d\Phi \circ X \circ \Phi^{-1}$. Hence, by Proposition 2, X is monotone in M_G.

4.3 The Hypercube With Other Metric

Set

$$Q^n = \left\{(p_1,\ldots,p_n) \in \mathbb{R}^n : |p_i| < \frac{\pi}{2}, i = 1,2,\ldots n\right\}$$

and let $\psi : \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to \mathbb{R}$ be the function defined by $\psi(\tau) = \ln(\sec\tau + \tan\tau)$. Consider the following problems.

Problem 4.7. In optimization problem (10) take the function $f : Q^n \to \mathbb{R}$, defined by $f(p_1,\ldots,p_n) = \psi(p_1) + \cdots + \psi(p_n)$.

Problem 4.8. In problem (11) take the vector field $X : Q^n \to \mathbb{R}^n$, defined by $X(p_1,\ldots,p_n) = \big(a_1,\ldots,a_n\big)$, where

$$a_i = \cos(p_i)(\textstyle\sum_{j\le i}\psi(p_j) - \sum_{j>i}\psi(p_j)),$$

for all $i = 1,\ldots n$.

Problem 4.7 is not convex in the classical sense, i.e., the objective functions f is not convex, and Problem 4.8 is not monotone in the classical sense, i.e., the vector field X is not monotone. Endowing Q^n with the Riemannian metric $G : Q^n \to S^n_{++}$, defined by

$$G = diag\big(\sec^2 p_1, \sec^2 p_2, \ldots, \sec^2 p_n\big),$$

we obtain the Riemannian manifold M_G, that is complete and of constant curvature $K = 0$. Note that the map $\Phi : M_G \to \mathbb{R}^n$, defined by

$$\Phi(p_1,\ldots,p_n) = \big(\psi(p_1),\ldots,\psi(p_n)\big), \tag{15}$$

is an isometry. Now consider the convex function $g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$g(x_1,\ldots,x_n) = x_1 + \cdots + x_n.$$

Observe that $f(p_1,\ldots,p_n) = g(\Phi(p_1,\ldots,p_n))$. Therefore, by Proposition 1, it follows that f is convex in M_G. Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Y(x) = Ax$, where A is the matrix (14). Taking Φ, the isometry defined in (15), we obtain that $X = d\Phi^{-1} \circ Y \circ \Phi$. Hence, by Proposition 2, X is monotone in M_G.

4.4 The Cone of the Positive Semidefinite Matrices With Other Metric

Consider the following constraint problems defined on S^n_{++} with the Frobenius metric.

Problem 4.9. In the optimization problem (10) take the function $f : S^n_{++} \to \mathbb{R}$, defined by

$$f(X) = \big(\ln \det X\big)^2.$$

Problem 4.10. In problem (11) take the vector field $T : S^n_{++} \to S^n$, defined by

$$T(X) = 2\big(\ln \det X\big)X.$$

The Problem 4.9 is not convex in the classical sense, i.e., the objective function f is not convex, and the Problem 4.10 is not monotone in the classical sense, i.e., the vector field X is not monotone. Endowing S^n_{++} with the Riemannian metric defined by

$$\langle U, V\rangle = \mathrm{tr}\big(VX^{-1}UX^{-1}\big),$$

we obtain the Riemannian manifold, that is complete of curvature $K \leq 0$. The geodesic equation in this Riemannian manifold is given in [9] by

$$\zeta''(t) = \zeta'(t)\zeta^{-1}(t)\zeta'(t). \tag{16}$$

A function defined on S^n_{++} is convex iff for any geodesic ζ in S^n_{++}

$$Hess\, f_{\zeta(t)}\big(\zeta'(t), \zeta'(t)\big) = \mathrm{tr}\big(f''(\zeta(t))\zeta'(t), \zeta'(t)\big) + \mathrm{tr}\big(f'(\zeta(t)), \zeta''(t)\big) \geq 0, \tag{17}$$

that is, the Hessian matrix of the function f is positive semidefinite. Therefore, from equations (16), (17) and the definition of the Hessian, it follows that the function f is convex in S^n_{++} if it satisfies the condition

$$\mathrm{tr}\big(Vf''(X)V\big) + \mathrm{tr}\big(VX^{-1}Vf'(X)\big) \geq 0, \tag{18}$$

for all $X \in S^n_{++}$ and $V \in S^n$. It can be checked that f satisfies the condition (18) and $\mathrm{grad}\, f(X) = T(X)$. Hence, f is convex and T is monotone (see Example 3.2).

5 The Proximal Point Algorithm

5.1 The Proximal Point Algorithm for Optimization Problems

The proximal point algorithm for minimization of a convex function on a Hadamard manifold was studied in [5]. For a convex function $f : H \to \mathbb{R}$, the proximal point sequence for minimization of f on H is given in [5] by

$$p^{k+1} = \arg\min_{p\in H}\left\{f(p) + \frac{\lambda_k}{2}d^2(p, p^k)\right\}. \tag{19}$$

We begin this section by giving some examples of proximal iteration for the manifolds introduced in the previous section.

5.1.1 In the Space $\mathbb{R}^n$ With Other Metric

Endowing $\mathbb{R}^n$ with the metric

$$G = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1+4p_{n-1}^2 & -2p_{n-1} \\ 0 & & 0 & -2p_{n-1} & 1 \end{pmatrix},$$

we obtain the Riemannian manifold M_G. Considering $\mathbb{R}^n$ with the usual Euclidean metric, the map $\Phi : \mathbb{R}^n \to M_G$, defined by

$$\Phi(x) = (\, x_1 \,,\, x_2 \,,\, \dots \,,\, x_{n-1}, x_{n-1}^2 - x_n \,)$$

is an isometry. Then the Riemannian distance in M_G is given by

$$\begin{aligned} d^2(p,q) &= \|\Phi^{-1}(p) - \Phi^{-1}(q)\|^2 \\ &= \sum_{i=1}^{n-1} \big(p_i - q_i\big)^2 + \big(p_{n-1}^2 - p_n - q_{n-1}^2 + q_n\big)^2, \end{aligned}$$

and the proximal point iteration (19) is

$$p^{k+1} = \arg\min_{p\in\mathbb{R}^n} \left\{ f(p) + \frac{\lambda_k}{2} \sum_{i=1}^{n-1} \big(p_i - (p^k)_i\big)^2 + \big(p_{n-1}^2 - p_n - (p^k)_{n-1}^2 + (p^k)_n\big)^2 \right\}.$$

5.1.2 In the Positive Orthant With Other Metric

Endowing $\mathbb{R}^n_{++}$ with the metric defined in (13) and $\mathbb{R}^n$ with the Euclidean metric, the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n_{++}$, defined by

$$\Phi(x) = (\, e^{x_1} \,,\, e^{x_2} \,,\, \dots \,,\, e^{x_n} \,)$$

is an isometry. Then,

$$d^2(p,q) = \|\Phi^{-1}(p) - \Phi^{-1}(q)\|^2 = \sum_{i=1}^{n} \ln^2 \big(\frac{p_i}{q_i}\big),$$

and the proximal point iteration (19) is

$$p^{k+1} = \arg\min_{p\in\mathbb{R}^n{}_{++}} \left\{ f(p) + \frac{\lambda_k}{2} \sum_{i=1}^{n} \ln^2 \big(\frac{p_i}{(p^k)_i}\big) \right\}.$$

5.1.3 In the Hypercube With Other Metric

Endowing the Hypercube Q^n with the Riemannian metric defined in (15) and $\mathbb{R}^n$ with the Euclidean metric, the mapping $\Phi : Q^n \to \mathbb{R}^n$, defined by $\Phi(p) = \big(\psi(p_1), \dots , \psi(p_n)\big)$ is an isometry. Then,

$$d^2(p, q) = \|\Phi(p) - \Phi(q)\|^2 = \sum_{i=1}^{n} [\psi(q_i) - \psi(p_i)]^2,$$

and the proximal point iteration (19) is

$$p^{k+1} = \arg\min_{p \in Q^n} \left\{ f(p) + \frac{\lambda_k}{2} \sum_{i=1}^{n} [\psi((p^k)_i) - \psi(p_i)]^2 \right\}.$$

5.1.4 In the Cone of Positive Semidefinite Matrices S^n_{++} With Other Metric

The Riemannian distance in the manifold S^n_{++}, presented in Subsection 4.4, is given by

$$d^2(X, Y) = \sum_{i=1}^{n} \ln^2 \lambda_i(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}),$$

where $\lambda(A)$ denotes the eigenvalue of the symmetric matrix A (see [9]). Therefore, the proximal point iteration (19) is

$$X_{k+1} = \arg\min_{X \in S^n_{++}} \left\{ f(X) + \frac{\lambda_k}{2} \sum_{i=1}^{n} \ln^2 \lambda_i(X^{-\frac{1}{2}} X_k X^{-\frac{1}{2}}) \right\}.$$

5.2 The Proximal Point Algorithm for Singularity Problems

Let $X \in \mathfrak{X}(H)$ be a monotone vector field and $\mathcal{O}^* \subset H$ the set of singularities of X. The proximal point algorithm for finding zeroes of monotone operators was proposed by T. Rockafellar in [11]. We will extend this algorithm for finding singularities of monotone vector fields.

The *proximal point algorithm* for finding a singularity of a monotone vector field on a Hadamard manifold requires one exogenous constant $\tilde{\lambda} > 0$ and one exogenous sequence $\{\lambda_k\}$, satisfying $0 < \lambda_k < \tilde{\lambda}$, for all k. It is defined as follows: take $p_0 \in H$ and define p_{k+1} as the solution of the following equation

$$(X + \lambda_k \operatorname{grad} \rho_{p_k})(p_{k+1}) = 0, \tag{20}$$

where $\rho_{p'}$ is defined in (6). As we have already proved, $\operatorname{grad} \rho_{p_k}$ is strongly monotone. Then, $X + \lambda_k \operatorname{grad} \rho_{p_k}$ is strongly monotone, when X is monotone and $\lambda_k > 0$. Therefore, by Corollary 1, there exists a unique $p_{k+1} \in H$ such that $(X + \lambda_k \operatorname{grad} \rho_{p_k})(p_{k+1}) = 0$ and our algorithm is well defined. From now on, we will refer to the sequence $\{p_k\}$ generated by (20) as the *proximal sequence*. Note that, by (7), it holds that $\operatorname{grad} \rho_{p_k}(p_{k+1}) = -\exp^{-1}_{p_{k+1}} p_k$. Then, the equation (20) is equivalent to

$$\lambda_k \exp^{-1}_{p_{k+1}} p_k = X(p_{k+1}). \tag{21}$$

5.2.1 Convergence of the Proximal Sequence

We begin the convergence proof with an auxiliary result. First, we present the well-known concept of Fejér convergence and its application in our context.

In a complete metric space (M, d), the sequence $\{p_k\} \subset M$ is said to be *Fejér convergent* to the nonempty set $U \subset M$ when

$$d(p_{k+1}, y) \leqslant d(p_k, y) \tag{22}$$

for all $y \in U$ and $k \geqslant 0$.

Lemma 2. In a complete metric space, (M, d) if $\{p_k\} \subset M$ is Fejér convergent to a nonempty set $U \subset M$, then $\{p_k\}$ is bounded. If furthermore a cluster point p of $\{p_k\}$ belongs to U then $\lim_{k\to+\infty} p_k = p$.

Proof. Take $p \in U$. Inequality (22) implies that $d(p_k, p) \leq d(p_0, p)$, for all k. Therefore, $\{p_k\}$ is bounded. Take a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that $\lim_{k\to+\infty} p_{k_j} = p$. By (22), the sequence of positive numbers $\{d(p_k, p)\}$ is decreasing and it has a subsequence, namely $\{d(p_{k_j}, p)\}$, which converges to 0. Thus, the whole sequence converges to 0, i.e., $\lim_{k\to+\infty} d(p_k, p) = 0$, implying $\lim_{k\to+\infty} p_k = p$. □

Lemma 3. If $X \in \mathfrak{X}(H)$ is monotone and $\{p_k\}$ is the proximal sequence, then

$$d^2(p_{k+1}, p_k) + d^2(p_{k+1}, q) - \frac{2}{\lambda_k}\langle X(p_{k+1}), \exp^{-1}_{p_{k+1}} q\rangle \leq d^2(p_k, q), \tag{23}$$

for all $q \in H$.

Proof. Take $q \in H$. Consider the geodesic triangle $\Delta(qp_kp_{k+1})$. From Theorem 1 we have

$$d^2(p_{k+1}, p_k) + d^2(p_{k+1}, q) - 2d(p_{k+1}, p_k)d(p_{k+1}, q)\cos\theta \leq d^2(p_k, q),$$

where $\theta = \sphericalangle(\exp^{-1}_{p_{k+1}} p_k, \exp^{-1}_{p_{k+1}} q)$, implying that

$$d^2(p_{k+1}, p_k) + d^2(p_{k+1}, q) - 2\langle \exp^{-1}_{p_{k+1}} p_k, \exp^{-1}_{p_{k+1}} q\rangle \leq d^2(p_k, q). \tag{24}$$

The statement of the Lemma follows by using (21) in (24). □

Theorem 2. If $X \in \mathfrak{X}(H)$ is monotone, $\{p_k\}$ is the proximal sequence and $\mathcal{O}^*$ is non empty, then $\lim_{k\to+\infty} p_k = p_*$ for some $p_* \in H$.

Proof. Take $\tilde{p} \in \mathcal{O}^*$. By the monotonicity of X it follows that

$$\langle X(p_{k+1}), \exp^{-1}_{p_{k+1}} \tilde{p}\rangle \leq \langle X(\tilde{p}), \exp^{-1}_{p_{k+1}} \tilde{p}\rangle.$$

Since $X(\tilde{p}) = 0$, we have that

$$\langle X(p_{k+1}), \exp^{-1}_{p_{k+1}} \tilde{p}\rangle \leq 0. \tag{25}$$

Then, substituting q by $\tilde{p}$ in (23) and by using (25), we get

$$0 \leq d^2(p_{k+1}, p_k) \leq d^2(p_k, \tilde{p}) - d^2(p_{k+1}, \tilde{p}). \tag{26}$$

The inequality (26) implies that $\{p_k\}$ is Fejér convergent to the set $\mathcal{O}^*$ and that $\lim_{k\to\infty} d^2(p_{k+1}, p_k) = 0$. By Lemma 2, there exists a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ which converges to some $p_* \in H$. It holds that $d(p_{k_j+1}, p_{k_j}) = \| \exp^{-1}_{p_{k_j+1}} p_{k_j}\|$. Then, by (21)

$$\lambda_{k_j} d(p_{k_j+1}, p_{k_j}) = \lambda_{k_j} \| \exp^{-1}_{p_{k_j+1}} p_{k_j}\| = \|X(p_{k_j+1})\|. \tag{27}$$

Since X is continuous, $\{p_{k_j}\}$ is convergent to p_* and $0 < \lambda_k < \tilde{\lambda}$, (26) implies that $\lim_{k\to\infty} d^2(p_{k_j+1}, p_{k_j}) = 0$. Hence, (27) yields,

$$\begin{aligned}\|X(p_*)\| &= \lim_{j\to+\infty} \|X(p_{k_j})\| \\ &= \lim_{j\to+\infty} \lambda_{k_j} d(p_{k_j+1}, p_{k_j}) \\ &= 0,\end{aligned}$$

implying that $p_* \in \mathcal{O}^*$. Therefore, by Lemma 2, $\lim_{k\to\infty} p_k = p_*$ and the proof is complete. □

5.3 Invariance of the Proximal Sequence Through Isometries

Isometric manifolds bear the same properties from Riemannian geometric viewpoint, but on some of them calculus is much easier. On the other hand, in this subsection we shall show that proximal sequences are invariant through isometries. The above remarks will be exploited in the example of subsection 5.4.

Proposition 3. Let H_1, H_2 be Hadamard manifolds and $\Phi : H_1 \to H_2$ an isometry. If $\{p_k\}$ is the proximal sequence on H_1 with the starting point $p_0 \in H_1$ associated to the vector field $X \in \mathfrak{X}(H_1)$ and the sequence $\{\lambda_k\}$, then $\{\Phi(p_k)\}$ is the proximal sequence on H_2 with the starting point $\Phi(p_0)$ associated to the vector field $Y = \mathrm{d}\,\Phi \circ X \circ \Phi^{-1} \in \mathfrak{X}(H_2)$ and the sequence $\{\lambda_k\}$.

Proof. Since Φ is an isometry, the geodesics of H_1 are transformed into geodesics of H_2 such that the tangent vector of a geodesic on H_1 is transformed into the tangent vector of its transformed geodesic on H_2. Hence, we have

$$\mathrm{d}\,\Phi(p_{k+1})\left(\exp^{-1}_{p_{k+1}} p_k\right) = \exp^{-1}_{\Phi(p_{k+1})} \Phi(p_k). \tag{28}$$

The proximal sequence $\{p_k\}$ on H_1 with respect to a starting point $p_0 \in H_1$ associated to the vector field $X \in \mathfrak{X}(H_1)$ and the sequence $\{\lambda_k\}$ is given by

$$\lambda_k \exp^{-1}_{p_{k+1}} p_k = X\left(p_{k+1}\right). \tag{29}$$

Since $Y \circ \Phi = \mathrm{d}\,\Phi \circ X$, by applying $\mathrm{d}\,\Phi(p_{k+1})$ to (29) and by using (28) we obtain

$$\lambda_k \exp^{-1}_{\Phi(p_{k+1})} \Phi(p_k) = Y(\Phi(p_{k+1})).$$

Hence, $\{\Phi(p_k)\}$ is the proximal sequence on H_2 with respect to the starting point $\{\Phi(p_0)\}$ associated to the vector field $Y \in \mathfrak{X}(H_2)$ and the sequence $\{\lambda_k\}$. □

With the notations of Proposition 3 we have as follows:

Corollary 2. If the proximal sequence $\{p_k\}$ is convergent to a singularity p_* of X, then the proximal sequence $\{\Phi(p_k)\}$ is convergent to the singularity $\Phi(p_*)$ of Y.

Proof. Follows immediately from the equality $Y \circ \Phi = \mathrm{d}\,\Phi \circ X$. □

5.4 Example

Let $\mathbb{H}^n$ be the n dimensional hyperbolic space of constant sectional curvature $K = -1$. Consider the following model for $\mathbb{H}^n$:

$$M = \left\{\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1} : \; \xi_{n+1} > 0 \text{ and } \{\xi, \xi\} = -1\right\},$$

where for the vectors $\xi = (\xi_1, \ldots, , \xi_{n+1})$, $\eta = (\eta_1, \ldots, \eta_{n+1}) \in \mathbb{R}^{n+1}$,

$\{\xi, \eta\} = \xi_1\eta_1 + \ldots + \xi_n\eta_n - \xi_{n+1}\eta_{n+1}$. The metric of M is induced from the Lorentz metric $\{\,.\,,.\,\}$ of $\mathbb{R}^{n+1}$ and it will be denoted by the same symbol. Then a normalized geodesic γ_x of $\mathbb{H}^n$ starting from x ($\gamma_x(0) = x$) will have the equation

$$\gamma_x(t) = (\cosh t)x + (\sinh t)v, \tag{30}$$

where $v = \dot{\gamma}_x(0) \in T_x\mathbb{H}^n$ is the tangent unit vector of γ in the starting point. We also have

$$\{u, x\} = 0,$$

for all $u \in T_x\mathbb{H}^n$. Equation (30) implies

$$\exp tv = (\cosh t)x + (\sinh t)v,$$

for any unit vector v and

$$\exp_x^{-1} y = \operatorname{arccosh}(-\{x,y\}) \frac{y + \{x,y\}x}{\sqrt{\{x,y\}^2 - 1}}, \tag{31}$$

for all $x, y \in \mathbb{H}^n$ and $v \in T_x\mathbb{H}^n$. This model of the hyperbolic space is called the Minkowski model.

Next consider the following model for $\mathbb{H}^n$:

$$U = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

The set U is the upper half-plane of dimension n. Endowing U with the Riemannian metric defined by matrix $G = (g_{ij})$, where

$$g_{11}(x_1, \dots, x_n) = \dots = g_{nn}(x_1, \dots, x_n) = \frac{1}{x_n}, \;\; g_{ij}(x_1, \dots, x_n) = 0, \text{ if } i \neq j.$$

we obtain the upper half-plane model of the Hyperbolic space $\mathbb{H}^n$

Consider the case $n = 2$. It can be seen that the map $\Phi : M \to U$ given by the equation

$$(x_1, x_2, x_3) \mapsto \frac{2}{x_3 - x_2}(x_1, 1) \tag{32}$$

is an isometry between M and U with inverse $\Phi^{-1} : U \to M$ given by the equation

$$(x_1, x_2) \mapsto \frac{1}{4x_2}\left(4x_1, x_1^2 + x_2^2 - 4, x_1^2 + x_2^2 + 4\right).$$

By (31), if X is a vector field on M and $\{\lambda_k\}$ is an exogenous sequence, then the proximal sequence $\{p^k\}$ with respect to a starting point $p^0 \in M$, X and $\{\lambda_k\}$ is given by the recurrence

$$\operatorname{arccosh}(-\left\{p^{k+1}, p^k\right\}) \frac{p^k + \left\{p^{k+1}, p^k\right\} p^{k+1}}{\sqrt{\left\{p^{k+1}, p^k\right\}^2 - 1}} = X\left(p^{k+1}\right)$$

If $Y = \mathrm{d}\,\Phi \circ X \circ \Phi^{-1}$ is the transformed vector field of X on U with respect to Φ then $\{\Phi(p^k)\}$ is the proximal sequence with respect to the starting point $\{\Phi(p^0)\}$, to Y and $\{\lambda_k\}$. If X is monotone and has at least one singularity, then the proximal sequence $\{p^k\}$ is convergent to a singularity

p^* of X. In this case the proximal sequence $\{\Phi(p^k)\}$ is convergent to the singularity $\Phi(p^*)$ of the monotone vector field Y.

In [7] it is shown that the vector field $X(x_1, x_2, x_3) = (x_1x_3, x_2x_3, x_3^2 - 1)$ on M is strictly monotone. The only singularity of X is $(0,0,1)$. The proximal sequence $\{p^k\}$ with respect to a starting point $p^0 \in M$, X and $\{\lambda_k\}$ is given by the recurrence

$$\operatorname{arccosh}(-\left\{p^{k+1}, p^k\right\}) \frac{p^k + \left\{p^{k+1}, p^k\right\} p^{k+1}}{\sqrt{\left\{p^{k+1}, p^k\right\}^2 - 1}} = \left(p_1^{k+1} p_3^{k+1}, p_2^{k+1} p_3^{k+1}, \left(p_3^{k+1}\right)^2 - 1\right)$$

and is convergent to $(0,0,1)$. It is easy to calculate that the image of X through Φ is $Y = \frac{1}{32}\left(16x_1x_2, 2x_1^2x_2^2 - 8x_1^2 + 8x_2^2 - 32\right)$ and $\Phi(0,0,1) = (0,2)$. Y is strictly monotone on U. By (32) $\left\{\frac{2}{p_3^k - p_2^k}\left(p_1^k, 1\right)\right\}$ is the proximal sequence with respect to the starting point $\frac{2}{p_3^0 - p_2^0}\left(p_1^0, 1\right)$, Y and $\{\lambda_k\}$. It is convergent to $(0,2)$ the only singularity of Y.

6 Conclusion

We presented here a novel method of finding singularities of monotone vector fields on Hadamard manifolds by using an extension of the classical proximal point method of Rockafellar for finding zeroes of monotone operators. It is unclear yet whether Rockafellar's method can be extended to more general Riemannian methods or not.

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