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A Pivoting Procedure for a Class of Second-order Cone Programming

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Abstract

We propose a pivoting procedure for a class of Second-Order Cone Programming (SOCP) having one second-order cone. We introduce a dictionary, basic variables, nonbasic variables, and other necessary concepts to define a pivot for the class of SOCP. In a pivot, two-dimensional SOCP subproblems are solved to decide which variables should be entering to or leaving from the basis. Under a nondegeneracy assumption, we prove that the objective function value is strictly decreasing by a pivot unless the current basic solution is optimal. We also propose an algorithm using the pivoting procedure which has global convergence property.

Key words: Second-order Cone Programming, Pivot, The Simplex Method, Quadratic Programming

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1. Introduction

A second-order cone programming (SOCP) problem is the optimization problem that minimizes a linear function over an intersection of an affine set and second-order cones. SOCP is a natural extension of linear programming (LP), and a special case of the cone-linear programming or the symmetric-cone programming (see Ben-Tal and Nemirovski [3], Faybusovich [7] and Muramatsu [10]). Recently, more and more applications are proposed for SOCP. Lobo, Vandenberghe, Boyd and Lebret [8] contains some early direct applications of SOCP. Applications in more complex contexts such as the branch-and-bound method (Muramatsu and Suzuki [11]) or the robust optimization (Sasakawa and Tsuchiya [13]) are also being popular. In such applications, a sequence of SOCP problems should be solved.

To date, the interior-point method has been the only solution for SOCP. In particular, the primal-dual interior-point method is proved to be very efficient both in theory (Nesterov and Todd [12], Monteiro and Tsuchiya [9], and Tsuchiya [14]) and in practice (Andersen, Roos, and Terlaky [2], Cai and Toh [4], and Sasakawa and Tsuchiya [13]). It is not hard to solve an SOCP problem having several thousand variables in a few seconds.

In LP, a special case of SOCP, the simplex method proposed by Dantzig (see [5]) is the other major algorithm. From the theoretical point of view, it is important to inverstigate the conterpart in SOCP by itself. However, this is not the only reason why we study a pivoting algorithm for SOCP. The simplex method is particularly important if one should solve a sequence of LP problems whose problem data are closely related to each other. Because the simplex method can start from the previous optimal solution which could be near to the current optimal solution, we can expect that the simplex method terminates in few iterations. Compared to the simplex method, the interior-point methods have a difficulty to handle the so-called warm-starting point in practice. Now SOCP is used in complex contexts; from the practical point of view, it is also important to develop a simplex-method-type algorithm for SOCP.

In this paper, we study a pivoting structure of a class of SOCP to propose a simplex-method-type algorithm, and introduce dictionary, basis, basic and nonbasic variables, and other necessary concepts needed to define a pivot in SOCP.

Recall that a pivot in LP can be viewed as follows. For each nonbasic variable, we solve a one-dimensional LP problem that minimizes the corresponding dual slack times the nonbasic variable, subject to keeping the feasibility of the solution. If this one-dimensional LP has nonzero optimal solution, then we can perform a pivot moving the corresponding nonbasic variable into the basis, and the optimal solution of the subproblem tells us which basic variable is leaving from the basis. If this LP is unbounded, the original LP is also unbounded. Such a subproblem can be solved in strongly polynomial time by the minimum-ratio test. If all the LP subproblems have zero optimal solutions, then the current basic solution is optimal for the original LP.

Our pivoting procedure for a class of SOCP is quite analogous to the LP case described above. We shall solve two-dimensional SOCP subproblems to determine an entering variable. These subproblems can be solved easily; calculating an optimal solution of a subproblem corresponds to the min-ratio test in the simplex method. If one of the subproblems has a nontrivial optimal solution, then we can perform a pivot. The optimal solution gives us information on the pivot. If one of the two-dimensional SOCP subproblems is unbounded, the original SOCP is unbounded, too. If all the SOCP subproblems have trivial optimal solutions, then the current basic solution is optimal for the original SOCP. Under a nondegeneracy assumption, the objective value will strictly decrease by a pivot unless the current basic solution is optimal. We can even construct a globally convergent algorithm using this pivoting procedure.

The rest of this paper is organized as follows. In the remaining part of this chapter, we introduce the SOCP problem we deal with. In Section 2, we define dictionary for that problem. In Section 3, two types of subproblems are proposed and some properties of them are shown. Section 4 is devoted to a detailed description of the pivoting procedure. In Section 5, we prove that if we cannot do a pivot any more, then the current basic solution is optimal. Section 6 deals with a special case which is closely related to LP. In Section 7, we analyze the complexity of the subproblems. In Section 8, we propose a piviting algorithm for the class of SOCP having global convergence property.

We consider an SOCP problem having only one second-order cone:

$$\langle P\rangle \begin{cases} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{u} + u_0 \\ \text{subject to} & A\boldsymbol{x} + R\boldsymbol{u} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0},\ \begin{pmatrix} u_0 \\ \boldsymbol{u} \end{pmatrix} \in \mathcal{K}_{r+1} \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $R \in \mathbb{R}^{m\times r}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{d} \in \mathbb{R}^r$, and

$$\mathcal{K}_{r+1} = \left\{ \begin{pmatrix} u_0 \\ \boldsymbol{u} \end{pmatrix} \in \mathbb{R}^{r+1} \,\middle|\, u_0 \geq \sqrt{\sum_{j=1}^{r} u_j^2} \right\}$$

is the $r+1$ dimensional second-order cone. We assume that the coefficient matrix $[A\ R]$ is of rank m. The dual of $\langle P\rangle$ is

$$\langle D\rangle \begin{cases} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{s} + A^T\boldsymbol{y} = \boldsymbol{c} \\ & \boldsymbol{z} + R^T\boldsymbol{y} = \boldsymbol{d} \\ & \boldsymbol{s} \geq 0,\ \begin{pmatrix} 1 \\ \boldsymbol{z} \end{pmatrix} \in \mathcal{K}_{r+1}. \end{cases} \tag{2}$$

Note that the equality constraints of $\langle P\rangle$ does not contain u_0. At a glance, this restriction seems very tight, but $\langle P\rangle$ still includes several important classes of optimization problems. For example, a convex quadratic programming problem of the form

$$\langle QP\rangle \begin{cases} \text{minimize} & \boldsymbol{x}^T V \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{cases}$$

can be reformulated into $\langle P\rangle$. In fact, if we decompose $V = QQ^T$ where $Q \in R^{n\times r}$, then $\langle QP\rangle$ is equivalent with

$$\begin{cases} \text{minimize} & \theta \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & Q^T\boldsymbol{x} - \boldsymbol{\xi} = \boldsymbol{0} \\ & \begin{pmatrix} \theta \\ \boldsymbol{\xi} \end{pmatrix} \in \mathcal{K}_{r+1}, \quad \boldsymbol{x} \geq 0, \end{cases}$$

which is an SOCP problem of the form (1) with $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{d} = \boldsymbol{0}$.

Notation. For an $m\times n$ matrix A, and an index set $B \subseteq \{1,\ldots,n\}$, we denote by A_B the $m\times|B|$ matrix whose columns are the columns of A corresponding to B. For $i \in \{1,\ldots,m\}$, A_{iB} is the i-th row vector of A_B. For $N \subseteq \{1,\ldots,m\}$, A_{NB} is the $|N|\times|B|$ matrix whose rows are those of A_B corresponding to N. For $j \in B$, A_{Nj} is the j-th column vector of A_{NB}. Similarly, for a vector $\boldsymbol{x} \in R^n$, $\boldsymbol{x}_B$ is a sub-vector of $\boldsymbol{x}$ corresponding to B.

2. The Dictionary

Let us assume that we are given two index sets $B \subseteq \{1,\ldots,n\}$ and $B' \subseteq \{1,\ldots,r\}$, for which

$$G = (A_B\ R_{B'}) \in \mathbb{R}^{m\times m}$$

is invertible. Premultiplying G^{-1} to both sides of the equality condition of $\langle P\rangle$, we get

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{u}_{B'} \end{pmatrix} = G^{-1}\boldsymbol{b} - G^{-1}A_N\boldsymbol{x}_N - G^{-1}R_{N'}\boldsymbol{u}_{N'},$$

where $N = \{1,\ldots,n\}\backslash B$ and $N' = \{1,\ldots,r\}\backslash B'$. We call B and B' the *basis*, and N and N' the *nonbasis*, respectively. The matrix G is called the *basis matrix.* We introduce an auxiliary variable $\boldsymbol{v}_{N'} = \boldsymbol{u}_{N'} - \tilde{\boldsymbol{u}}_{N'}$ for a given *displacement vector* $\tilde{\boldsymbol{u}}_{N'} \in \mathbb{R}^{|N'|}$, and rewrite the above equality as

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{u}_{B'} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{x}}_B \\ \tilde{\boldsymbol{u}}_{B'} \end{pmatrix} - \begin{pmatrix} D_{BN} \\ D_{B'N} \end{pmatrix} \boldsymbol{x}_N - \begin{pmatrix} D_{BN'} \\ D_{B'N'} \end{pmatrix} \boldsymbol{v}_{N'},$$

where

$$\begin{pmatrix} \tilde{\boldsymbol{x}}_B \\ \tilde{\boldsymbol{u}}_{B'} \end{pmatrix} = G^{-1}(\boldsymbol{b} - R_{N'}\tilde{\boldsymbol{u}}_{N'}),$$
$$\begin{pmatrix} D_{BN} \\ D_{B'N} \end{pmatrix} = G^{-1}A_N$$
$$\begin{pmatrix} D_{BN'} \\ D_{B'N'} \end{pmatrix} = G^{-1}R_{N'}.$$

The objective function value of $\langle P \rangle$ denoted by θ can be written as

$$\begin{aligned} \theta &= \boldsymbol{c}_N^T \boldsymbol{x}_N + \boldsymbol{d}_{N'}^T(\tilde{\boldsymbol{u}}_{N'} + \boldsymbol{v}_{N'}) + \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} \end{pmatrix}^T \left\{ \begin{pmatrix} \tilde{\boldsymbol{x}}_B \\ \tilde{\boldsymbol{u}}_{B'} \end{pmatrix} - \begin{pmatrix} D_{BN} \\ D_{B'N} \end{pmatrix} \boldsymbol{x}_N - \begin{pmatrix} D_{BN'} \\ D_{B'N'} \end{pmatrix} \boldsymbol{v}_{N'} \right\} + u_0 \\ &= \tilde{\theta} + \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N + \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} + u_0, \end{aligned}$$

where

$$\begin{aligned} \tilde{\theta} &= \boldsymbol{c}_B^T \tilde{\boldsymbol{x}}_B + \boldsymbol{d}_{B'}^T \tilde{\boldsymbol{u}}_{B'} + \boldsymbol{d}_{N'}^T \tilde{\boldsymbol{u}}_{N'}, \\ \tilde{\boldsymbol{s}}_N &= \boldsymbol{c}_N - D_{BN}^T \boldsymbol{c}_B - D_{B'N}^T \boldsymbol{d}_{B'}, \\ \tilde{\boldsymbol{z}}_{N'} &= \boldsymbol{d}_{N'} - D_{BN'}^T \boldsymbol{c}_B - D_{B'N'}^T \boldsymbol{d}_{B'}. \end{aligned}$$

Now, we obtain a linear system that is equivalent with the equality condition of $\langle P \rangle$:

$$\begin{array}{rclclclcl} \theta &=& \tilde{\theta} &+& \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N &+& \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} &+& u_0 \\ \boldsymbol{x}_B &=& \tilde{\boldsymbol{x}}_B &-& D_{BN}\boldsymbol{x}_N &-& D_{BN'}\boldsymbol{v}_{N'} && \\ \boldsymbol{u}_{B'} &=& \tilde{\boldsymbol{u}}_{B'} &-& D_{B'N}\boldsymbol{x}_N &-& D_{B'N'}\boldsymbol{v}_{N'} && \\ \boldsymbol{u}_{N'} &=& \tilde{\boldsymbol{u}}_{N'} &&&+& \boldsymbol{v}_{N'}. && \end{array} \tag{3}$$

We call (3) a *dictionary*. Because a dictionary is determined by a basis (B, B') and a displacement vector $\tilde{\boldsymbol{u}}_{N'}$, we denote (3) by $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$. Note that any solution $(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}_{B'}, \boldsymbol{u}_{N'})$ satisfying (3) with $\boldsymbol{x}_B \geq \boldsymbol{0}$, $\boldsymbol{x}_N \geq \boldsymbol{0}$, and $u_0 \geq \|(\boldsymbol{u}_{B'}, \boldsymbol{u}_{N'})\|$ is feasible for $\langle P \rangle$, and its objective value is θ. The variables $\boldsymbol{x}_B$ and $\boldsymbol{u}_{B'}$ are called *basic variables*, while $\boldsymbol{x}_N$, $\boldsymbol{u}_{N'}$, and $\boldsymbol{v}_{N'}$ are *nonbasic*. A *basic solution* associated with the dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ is given by

$$(\tilde{\boldsymbol{x}}_B, \boldsymbol{0}, \tilde{u}_0, \tilde{\boldsymbol{u}}_{B'}, \tilde{\boldsymbol{u}}_{N'}) \tag{4}$$

where $\tilde{u}_0 = \|(\tilde{\boldsymbol{u}}_{B'}, \tilde{\boldsymbol{u}}_{N'})\|$. We say that the dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ is feasible, if its associated basic solution is feasible, or equivalently, $\tilde{\boldsymbol{x}}_B \geq \boldsymbol{0}$.

3. The Subproblems

Let us assume that we are given a feasible dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ along with the basic solution (4). From this section to Section 5, we further assume that $\tilde{\boldsymbol{x}}_B > \boldsymbol{0}$ and $\tilde{u}_0 > 0$. We will deal with the case where $\tilde{u}_0 = 0$ later in Section 6.

We consider to decrease the objective function value by increasing x_i for $i \in N$ from (4). The next basic solution should be feasible, thus we have a subproblem:

$$\langle S_i \rangle \left\{ \begin{array}{ll} \text{minimize} & \tilde{s}_i x_i + u_0 \\ \text{subject to} & \tilde{\boldsymbol{x}}_B - x_i D_{Bi} \geq \boldsymbol{0}, \ x_i \geq 0, \\ & \begin{pmatrix} u_0 \\ \tilde{\boldsymbol{u}}_{B'} - x_i D_{B'i} \\ \tilde{\boldsymbol{u}}_{N'} \end{pmatrix} \in \mathcal{K}_{r+1} \end{array} \right.$$

for $i \in N$. Similarly, for $j \in N'$ we consider

$$\langle Z_j \rangle \left\{ \begin{array}{ll} \text{minimize} & \tilde{z}_j v_j + u_0 \\ \text{subject to} & \tilde{\boldsymbol{x}}_B - v_j D_{Bj} \geq \boldsymbol{0}, \\ & \begin{pmatrix} u_0 \\ \tilde{\boldsymbol{u}}_{B'} - v_j D_{B'j} \\ \tilde{\boldsymbol{u}}_{N'} + v_j \boldsymbol{e}_j \end{pmatrix} \in \mathcal{K}_{r+1} \end{array} \right.$$

where $\boldsymbol{e}_j$ is the vector of all zero except 1 at the j-th component. These problems are two-dimensional SOCP, and can be solved easily (See Section 7). Notice that $(0, \tilde{u}_0)$ is always a feasible solution of $\langle S_i \rangle$ and $\langle Z_j \rangle$. We say that $\langle S_i \rangle$ (or $\langle Z_j \rangle$) has a trivial optimal solution when $(0, \tilde{u}_0)$ is optimal.

We prove some properties of the optimal solution of the subproblems.

Lemma 1. *If either $\langle S_i \rangle$ for some $i \in N$ or $\langle Z_j \rangle$ for some $j \in N'$ is unbounded, then $\langle P \rangle$ is unbounded.*

Proof: Assume that $\langle S_i \rangle$ is unbounded, and to the contrary, that $\langle P \rangle$ is bounded; i.e., there exists $M > 0$ such that $\tilde{\theta} + \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N + \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} + u_0 \geq -M$ for any feasible solution of $\langle P \rangle$. Since $\langle S_i \rangle$ is unbounded, there is a feasible solution (x_i^*, u_0^*) of $\langle S_i \rangle$ such that $\tilde{s}_i x_i^* + u_0^* < -M - \tilde{\theta}$. Looking at (3) and $\langle S_i \rangle$, we see that

$$(\tilde{\boldsymbol{x}}_B - x_i^* D_{Bi}, x_i^* \boldsymbol{e}_i, u_0^*, \tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i}, \tilde{\boldsymbol{u}}_{N'})$$

is feasible for $\langle P \rangle$, whose objective function value is

$$\tilde{\theta} + \tilde{s}_i x_i^* + u_0^* < -M.$$

This contradicts the definition of M.

The case where $\langle Z_j \rangle$ is unbounded can be proved similarly. We omit the details. □

Lemma 2. (1) *If $(0, \tilde{u}_0)$ is an optimal solution of $\langle S_i \rangle$, then it holds that*

$$\tilde{s}_i \geq \frac{1}{\tilde{u}_0} D_{B'i}^T \tilde{\boldsymbol{u}}_{B'}.$$

(2) *If $(0, \tilde{u}_0)$ is not an optimal solution of $\langle S_i \rangle$, and if (x_i^*, u_0^*) is an optimal solution of $\langle S_i \rangle$ with $\tilde{\boldsymbol{x}}_B - x_i^* D_{Bi} > \boldsymbol{0}$, then $D_{B'i} \neq \boldsymbol{0}$.*

Proof: Since the minimum of $\langle S_i \rangle$ is taken at $u_0 = \sqrt{\|\tilde{\boldsymbol{u}}_{B'} - x_i D_{B'i}\|^2 + \|\tilde{\boldsymbol{u}}_{N'}\|^2}$, we can rewrite $\langle S_i \rangle$ as

$$\begin{cases} \text{minimize} & \tilde{s}_i x_i + \sqrt{x_i^2 \|D_{B'i}\|^2 - 2 x_i D_{B'i}^T \tilde{\boldsymbol{u}}_{B'} + \tilde{u}_0^2} \\ \text{subject to} & x_i D_{Bi} \leq \tilde{\boldsymbol{x}}_B, \ x_i \geq 0. \end{cases}$$

Let us denote the objective function of the above by $f(x_i)$. Because $\tilde{u}_0 > 0$, we obtain

$$f'(x_i) = \tilde{s}_i + \frac{\|D_{B'i}\|^2 x_i - D_{B'i}^T \boldsymbol{u}_{B'}}{\sqrt{x_i^2 \|D_{B'i}\|^2 - 2 x_i D_{B'i}^T \tilde{\boldsymbol{u}}_{B'} + \tilde{u}_0^2}}$$

and

$$f'(0) = \tilde{s}_i - D_{B'i}^T \boldsymbol{u}_{B'} / \tilde{u}_0.$$

It is easy to show that f is convex (See Section 7). Because $\tilde{\boldsymbol{x}}_B > 0$, x_i can be positive in the feasible region of $\langle S_i \rangle$, hence $f'(0) \geq 0$ if $(0, \tilde{u}_0)$ is optimal. This proves the first statement.

To prove the second, we suppose that $D_{B'i} = \boldsymbol{0}$ and derive a contradiction. First, we see from $D_{B'i} = \boldsymbol{0}$ that $u_0^* = \tilde{u}_0$ for any optimal solution. Furthermore, since $x_i = 0$ is not an optimal solution, $f'(0) = \tilde{s}_i < 0$. This means that $\tilde{\boldsymbol{x}}_B - x_i^* D_{Bi} \geq \boldsymbol{0}$ holds with at least one inequality holding at equality, which contradicts the assumption. □

Lemma 3. *If $(0, \tilde{u}_0)$ is an optimal solution of $\langle Z_j \rangle$, then*

$$\tilde{z}_j = \frac{1}{\tilde{u}_0} \left(D_{B'j}^T \tilde{\boldsymbol{u}}_{B'} - \tilde{u}_j \right).$$

Proof: Since the minimum of $\langle Z_j \rangle$ is taken at $u_0 = \sqrt{\|\tilde{\boldsymbol{u}}_{B'} - v_j D_{B'j}\|^2 + \|\tilde{\boldsymbol{u}}_{N'} + v_j \boldsymbol{e}_j\|^2}$, we can rewrite $\langle Z_j \rangle$ as

$$\begin{cases} \text{minimize} & \tilde{z}_j v_j + \sqrt{v_j^2 (\|D_{B'j}\|^2 + 1) - 2 v_j (D_{B'j}^T \tilde{\boldsymbol{u}}_{B'} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N'}) + \tilde{u}_0^2} \\ \text{subject to} & v_j D_{Bj} \leq \tilde{\boldsymbol{x}}_B. \end{cases}$$

Let us denote the objective function of the above by $f(v_j)$. In view of $\tilde{u}_0 > 0$, we have

$$f'(v_j) = \tilde{z}_j + \frac{v_j (\|D_{B'j}\|^2 + 1) - (D_{B'j}^T \tilde{\boldsymbol{u}}_{B'} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N'})}{\sqrt{v_j^2 (\|D_{B'j}\|^2 + 1) - 2 v_j (D_{B'j}^T \tilde{\boldsymbol{u}}_{B'} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N'}) + \tilde{u}_0^2}}$$

and

$$f'(0) = \tilde{z}_j - \frac{D_{B'j}^T \boldsymbol{u}_{B'} - \tilde{u}_j}{\tilde{u}_0}.$$

Again, it is easy to show that f is convex. Because $\tilde{\boldsymbol{x}}_B > 0$, v_j could be both positive and negative on the feasible region, hence $f'(0) = 0$ if $(0, \tilde{u}_0)$ is optimal. This proves the lemma. □

4. The Pivot

In this section, assuming that we are given a feasible dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ with $\tilde{\boldsymbol{x}}_B > \boldsymbol{0}$ and $\tilde{u}_0 > 0$, we describe how to pivot when one of the subproblems has an optimal solution which is not $(0, \tilde{u}_0)$. Depending on type of the subproblem that has nontrivial optimal solution, the argument is divided into two cases; In Case I, we assume that $\langle S_i \rangle$ has a nontrivial optimal solution, and in Case II, $\langle Z_j \rangle$. Each case is further divided into two situations depending on the variable leaving from the basis. In each case, we will show that the objective function value of the new dictionary is strictly less than the previous value.

Lemma 4. *Let $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ be feasible with $\tilde{\boldsymbol{x}}_B > \boldsymbol{0}$ and $\tilde{u}_0 > 0$.*

(1) *Assume that $\langle S_i \rangle$ has a nontrivial optimal solution (x_i^*, u^*) with $x^* > 0$. Then, by pivoting using x_i, the objective value of the next basic solution becomes $\tilde{\theta} + \tilde{s}_i x_i^* + u_0^* < \tilde{\theta} + \tilde{u}_0$.*
(2) *Assume that $\langle Z_j \rangle$ has a nontrivial optimal solution (v_j^*, u_0^*) with $v_j^* \neq 0$. Then, by pivoting using v_j, the objective value of the next basic solution becomes $\tilde{\theta} + \tilde{z}_j v_j^* + u_0^* < \tilde{\theta} + \tilde{u}_0$.*

For the case where $(0, \tilde{u}_0)$ is optimal for all the subproblems, we will show in Section 5 that the current basic solution is optimal for $\langle P \rangle$.

4.1. **How to Pivot I: When $\langle S_i \rangle$ has an optimal solution $x_i^* > 0$.** In this case, we decide that x_i be the entering variable into the basis. Here, we consider two cases; (I-i) $\tilde{x}_k - x_i^* D_{ki} = 0$ for some $k \in B$, or (I-ii) $\tilde{\boldsymbol{x}}_B - x_i^* D_{Bi} > \boldsymbol{0}$.

Case (I-i) Notice that $D_{ki} > 0$. Now x_k is leaving from the basis. Since the k-th row of the dictionary is

$$x_k = \tilde{x}_k - x_i D_{ki} - \sum_{j \in N, j \neq i} D_{kj} x_j - D_{kN'} \boldsymbol{v}_{N'},$$

and $\tilde{x}_k / D_{ki} = x_i^* > 0$, we decide that x_i should be

$$x_i = x_i^* - \frac{x_k}{D_{ki}} - \sum_{j \in N, j \neq i} \frac{D_{kj}}{D_{ki}} x_j - \frac{D_{kN'}}{D_{ki}} \boldsymbol{v}_{N'}.$$

We then eliminate x_i from the right-hand side of the dictionary (3). Namely, for $l \in B$ $(l \neq k)$,

$$\begin{aligned} x_l &= \tilde{x}_l - x_i D_{li} - \sum_{j \in N, j \neq i} D_{lj} x_j - D_{lN'} \boldsymbol{v}_{N'} \\ &= \tilde{x}_l - x_i^* D_{li} + \frac{D_{li}}{D_{ki}} x_k - \sum_{j \in N, j \neq i} \left(D_{lj} - \frac{D_{li} D_{kj}}{D_{ki}} \right) x_j - \left(D_{lN'} - \frac{D_{li} D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'}. \end{aligned} \tag{5}$$

Similarly, we have

$$\begin{aligned} \boldsymbol{u}_{B'} &= \tilde{\boldsymbol{u}}_{B'} - x_i D_{B'i} - \sum_{j \in N, j \neq i} D_{B'j} x_j - D_{B'N'} \boldsymbol{v}_{N'} \\ &= \tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i} + \frac{D_{B'i}}{D_{ki}} x_k - \sum_{j \in N, j \neq i} \left(D_{B'j} - \frac{D_{kj}}{D_{ki}} D_{B'i} \right) x_j - \left(D_{B'N'} - \frac{D_{B'i} D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'}. \end{aligned}$$

The objective function becomes

$$\begin{aligned} \theta &= \tilde{\theta} + \tilde{s}_i x_i + \sum_{j \in N, j \neq i} \tilde{s}_j x_j + \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} + u_0 \\ &= \tilde{\theta} + \tilde{s}_i x_i^* - \frac{\tilde{s}_i}{D_{ki}} x_k + \sum_{j \in N, j \neq i} \left(\tilde{s}_j - \frac{\tilde{s}_i D_{kj}}{D_{ki}} \right) x_j + \left(\tilde{\boldsymbol{z}}_{N'}^T - \frac{\tilde{s}_i D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'} + u_0. \end{aligned} \tag{6}$$

Now, x_k leaves the basis, i.e., $B \leftarrow B - \{k\} + \{i\}$ and $N \leftarrow N - \{i\} + \{k\}$ are the new basis and nonbasis, respectively. If we denote the new basic solution by $(\bar{\boldsymbol{x}}_B, \mathbf{0}, \bar{u}_0, \bar{\boldsymbol{u}}_{B'}, \bar{\boldsymbol{u}}_{N'})$, we have $\bar{x}_i = x_i^* > 0$ and for $l \in B$ $(l \neq i)$,

$$\bar{x}_l = \tilde{x}_l - x_i^* D_{li} > 0,$$

which imply that the new basic solution is feasible. Since $\bar{\boldsymbol{u}}_{N'} = \tilde{\boldsymbol{u}}_{N'}$ and $\bar{\boldsymbol{u}}_{B'} = \tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i}$, we have

$$\bar{u}_0 = \|(\bar{\boldsymbol{u}}_{B'}, \bar{\boldsymbol{u}}_{N'})\| = \|(\tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i}, \tilde{\boldsymbol{u}}_{N'})\| = u_0^*,$$

and thus the objective function value of the new basic solution is

$$\tilde{\theta} + \tilde{s}_i x_i^* + u_0^* < \tilde{\theta} + \tilde{u}_0.$$

Therefore, Lemma 4 holds in this case.

Case (I-ii) Lemma 2 implies that there exists $k \in B'$ such that $D_{ki} \neq 0$. Choosing such k, we introduce a new variable v_k by

$$u_k = \tilde{u}_k - x_i^* D_{ki} + v_k. \tag{7}$$

We will pivot in x_i and out u_k. (In fact, u_k will be kept in the left-hand side, but instead, v_k will be placed in the right-hand side.) Since the k-th row of the dictionary is

$$u_k = \tilde{u}_k - x_i D_{ki} - \sum_{j \in N, j \neq i} D_{kj} x_j - D_{kN'} \boldsymbol{v}_{N'}$$

we have

$$v_k - x_i^* D_{ki} = -x_i D_{ki} - \sum_{j \in N, j \neq i} D_{kj} x_j - D_{kN'} \boldsymbol{v}_{N'},$$

which produces

$$x_i = x_i^* - \sum_{j \in N, j \neq i} (D_{kj}/D_{ki}) x_j - \frac{D_{kN'}}{D_{ki}} \boldsymbol{v}_{N'} - \frac{v_k}{D_{ki}}. \tag{8}$$

Substituting this equality from the dictionary, we have

$$\begin{aligned} \boldsymbol{x}_B &= \tilde{\boldsymbol{x}}_B - x_i D_{Bi} - \sum_{j \in N, j \neq i} D_{Bj} x_j - D_{BN'} \boldsymbol{v}_{N'} \\ &= \tilde{\boldsymbol{x}}_B - x_i^* D_{Bi} - \sum_{j \in N, j \neq i} \left(D_{Bj} - \frac{D_{Bi} D_{kj}}{D_{ki}} \right) x_j - \left(D_{BN'} - \frac{D_{Bi} D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'} + \frac{D_{Bi}}{D_{ki}} v_k, \end{aligned}$$

for $l \in B'$ $(l \neq k)$,

$$\begin{aligned} u_l &= \tilde{u}_l - x_i D_{li} - \sum_{j \in N, j \neq i} D_{lj} x_j - D_{lN'} \boldsymbol{v}_{N'} \\ &= \tilde{u}_l - x_i^* D_{li} - \sum_{j \in N, j \neq i} \left(D_{lj} - \frac{D_{li} D_{kj}}{D_{ki}} \right) x_j - \left(D_{lN'} - \frac{D_{li} D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'} + \frac{D_{li}}{D_{ki}} v_k, \end{aligned}$$

and

$$\begin{aligned} \theta &= \tilde{\theta} + \tilde{s}_i x_i + \sum_{j \in N, j \neq i} \tilde{s}_j x_j + \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} + u_0 \\ &= \tilde{\theta} + \tilde{s}_i x_i^* + \sum_{j \in N, j \neq i} \left(\tilde{s}_j - \frac{\tilde{s}_i D_{kj}}{D_{ki}} \right) x_j + \left(\tilde{\boldsymbol{z}}_{N'}^T - \frac{\tilde{s}_i D_{kN'}}{D_{ki}} \right) \boldsymbol{v}_{N'} - \frac{\tilde{s}_i}{D_{ki}} v_k + u_0. \end{aligned}$$

These equations, together with (7), compose the next dictionary. Notice that we need (7) because u_k should be kept in the left-hand side. The new basis and nonbasis are $B \leftarrow B + \{i\}$, $B' \leftarrow B' - \{k\}$,

$N \leftarrow N - \{i\}$, and $N' \leftarrow N' + \{k\}$. The basic solution is

$$
\begin{aligned}
x_j &= \begin{cases} \tilde{\boldsymbol{x}}_j - x_i^* D_{ji} & \text{if } j \in B \text{ and } j \neq i, \\ x_i^* & \text{if } j = i, \\ 0 & \text{if } j \in N, \end{cases} \\
\boldsymbol{u}_{B'} &= \tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i} \\
u_l &= \begin{cases} \tilde{u}_l & \text{if } l \in N' \text{ and } l \neq k, \\ \tilde{u}_k - x_i^* D_{ki} & \text{if } l = k, \end{cases} \\
u_0 &= \sqrt{\|\tilde{\boldsymbol{u}}_{B'} - x_i^* D_{B'i}\|^2 + (\tilde{u}_k - x_i^* D_{ki})^2 + \sum_{l \in N', l \neq k} \tilde{u}_l^2} = u_0^*,
\end{aligned} \tag{9}
$$

and its objective function value is

$$\tilde{\theta} + \tilde{s}_i x_i^* + u_0^* < \tilde{\theta} + \tilde{u}_0.$$

Again Lemma 4 holds in this case.

4.2. **How to Pivot II: When $\langle Z_j \rangle$ has an optimal solution $v_j^* \neq 0$ for some j.** We pivot in v_j (or, equivalently, u_j), in this case. Again, we consider two cases; (II-i) $\tilde{x}_k - v_j^* D_{kj} = 0$ for some $k \in B$, or (II-ii) $\tilde{\boldsymbol{x}}_B - v_j^* D_{Bj} > \mathbf{0}$.

Case (II-i) We will pivot out x_k. From

$$u_j = \tilde{u}_j + v_j \tag{10}$$

and

$$x_k = \tilde{x}_k - D_{kN}\boldsymbol{x}_N - v_j D_{kj} - \sum_{l \in N', l \neq j} D_{kl} v_l,$$

we have

$$u_j = \tilde{u}_j + v_j^* - \frac{x_k}{D_{kj}} - \frac{D_{kN}}{D_{kj}}\boldsymbol{x}_N - \sum_{l \in N', l \neq j} \frac{D_{kl}}{D_{kj}} v_l.$$

Using this equality, we can remove v_j from the dictionary. As a result, we have for $i \in B$ ($i \neq k$),

$$
\begin{aligned}
x_i &= \tilde{x}_i - D_{iN}\boldsymbol{x}_N - (u_j - \tilde{u}_j) D_{ij} - \sum_{l \in N', l \neq j} D_{il} v_l \\
&= \tilde{x}_i - v_j^* D_{ij} + \frac{D_{ij}}{D_{kj}} x_k - \left(D_{iN} - \frac{D_{ij} D_{kN}}{D_{kj}} \right) \boldsymbol{x}_N - \sum_{l \in N', l \neq j} \left(D_{il} - \frac{D_{ij} D_{kl}}{D_{kj}} \right) v_l,
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{u}_{B'} &= \tilde{\boldsymbol{u}}_{B'} - D_{B'N}\boldsymbol{x}_N - (u_j - \tilde{u}_j) D_{B'j} - \sum_{l \in N', l \neq j} D_{B'l} v_l \\
&= \tilde{\boldsymbol{u}}_{B'} - v_j^* D_{B'j} + \frac{D_{B'j}}{D_{ij}} x_k - \left(D_{B'N} - \frac{D_{B'j} D_{kN}}{D_{kj}} \right) \boldsymbol{x}_N - \sum_{l \in N', l \neq j} \left(D_{B'l} - \frac{D_{B'j} D_{kl}}{D_{kj}} \right) v_l,
\end{aligned}
$$

and

$$
\begin{aligned}
\theta &= \tilde{\theta} + \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N + \tilde{z}_j (u_j - \tilde{u}_j) + \sum_{l \in N', l \neq j} \tilde{z}_l v_l + u_0 \\
&= \tilde{\theta} + \tilde{z}_j v_j^* - \frac{\tilde{z}_j}{D_{kj}} x_k + \left(\tilde{\boldsymbol{s}}_N^T - \frac{\tilde{z}_j D_{kN}}{D_{kj}} \right) \boldsymbol{x}_N + \sum_{l \in N', l \neq j} \left(\tilde{z}_l - \frac{\tilde{z}_j D_{kl}}{D_{kj}} \right) v_l + u_0.
\end{aligned}
$$

Notice that we eliminate (10) because now u_j is a basic variable. The new basis and nonbasis are $B \leftarrow B - \{k\}$, $B' \leftarrow B + \{j\}$, $N \leftarrow N + \{k\}$, and $N' \leftarrow N' - \{j\}$. Note that the equality (10) is also removed from the dictionary. Because $\tilde{x}_k/D_{kj} = v_j^*$, the basic solution is

$$\begin{aligned}
\boldsymbol{x}_B &= \tilde{\boldsymbol{x}}_B - v_j^* D_{Bj} \geq \boldsymbol{0} \\
\boldsymbol{x}_N &= \boldsymbol{0} \\
u_l &= \begin{cases} \tilde{u}_l - v_j^* D_{lj} & \text{if } l \in B' \ (l \neq j) \\ \tilde{u}_j + v_j^* & \text{if } l = j, \\ \tilde{u}_l & \text{if } l \in N', \end{cases} \\
u_0 &= \sqrt{\sum_{l \in B', l \neq j} (\tilde{u}_l - v_j^* D_{lj})^2 + (\tilde{u}_j + v_j^*)^2 + \sum_{l \in N'} \tilde{u}_l^2} = u_0^*, \qquad (11)
\end{aligned}$$

and its objective function value is

$$\tilde{\theta} + \tilde{z}_i v_j^* + u_0^* < \tilde{\theta} + \tilde{u}_0,$$

thus Lemma 4 holds in this case.

Case (II-ii) In this case, we put $v_j \leftarrow v_j - v_j^*$, and no exchange of variables occurs in the dictionary. Namely, the new v_j is defined by

$$u_j = \tilde{u}_j + v_j^* + v_j, \qquad (12)$$

and the new dictionary is

$$\begin{array}{rcccccccc}
\theta & = & \tilde{\theta} + \tilde{z}_j v_j^* & + & \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N & + & \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} & + & u_0 \\
\boldsymbol{x}_B & = & \tilde{\boldsymbol{x}}_B - v_j^* D_{Bj} & - & D_{BN} \boldsymbol{x}_N & - & D_{BN'} \boldsymbol{v}_{N'} & & \\
\boldsymbol{u}_{B'} & = & \tilde{\boldsymbol{u}}_{B'} - v_j^* D_{B'j} & - & D_{B'N} \boldsymbol{x}_N & - & D_{B'N'} \boldsymbol{v}_{N'} & & \\
\boldsymbol{u}_{N'} & = & \tilde{\boldsymbol{u}}_{N'} + v_j^* \boldsymbol{e}_j & & & + & \boldsymbol{v}_{N'}. & &
\end{array} \qquad (13)$$

The basic solution corresponding to this dictionary is

$$(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}_{B'}, \boldsymbol{u}_{N'}) = (\tilde{\boldsymbol{x}}_B - v_j^* D_{Bj}, \boldsymbol{0}, u_0^*, \tilde{\boldsymbol{u}}_{B'} - v_j^* D_{B'j}, \tilde{\boldsymbol{u}}_{N'} + v_j^* \boldsymbol{e}_j),$$

because

$$\sqrt{\|\tilde{\boldsymbol{u}}_{B'} - v_j^* D_{B'j}\|^2 + \|\tilde{\boldsymbol{u}}_{N'} + v_j^* \boldsymbol{e}_j\|^2} = u_0^*.$$

The objective function value is

$$\tilde{\theta} + \tilde{z}_j v_j^* + u_0^* < \tilde{\theta} + \tilde{u}_0.$$

Since this is the last case, we finish the proof of Lemma 4.

5. The Optimality

Given a feasible dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ with $\tilde{\boldsymbol{x}}_B > \boldsymbol{0}$ and $\tilde{u}_0 > 0$, we solve subproblems $\langle S_i \rangle$ $(i \in N)$ and $\langle Z_j \rangle$ $(j \in N')$. As was seen in Section 4, if any of the subproblems has a nontrivial optimal solution, then we can perform a pivot, strictly decreasing the objective function value. In this section, we will show that if all the subproblems have trivial optimal solutions, then the current basic solution is optimal for $\langle P \rangle$. Therefore, a pivoting algorithm can be stated as follows.

STEP 0 A feasible dictionary is given.

STEP 1 For $i \in N$, do the following in this order:
- Solve $\langle S_i \rangle$.
- If it is unbounded, then STOP; $\langle P \rangle$ is unbounded.
- If $x_i^* > 0$, then perform a pivot as was described in Section 4 to get another feasible dictionary whose objective value is strictly less than the previous one. Go to Step 1.

STEP 2 For $j \in N'$, do the following in this order:
- Solve $\langle Z_j \rangle$.
- If it is unbounded, then STOP; $\langle P \rangle$ is unbounded.
- If $v_j^* \neq 0$, then perform a pivot as was described in Section 4 to get another feasible dictionary whose objective value is strictly less than the previous one. Go to Step 1.

STEP 3 Calculate an optimal solution according to Theorem 5.

As far as the basic solutions appeared have the property that $\tilde{\boldsymbol{x}}_B > \boldsymbol{0}$ and $\tilde{u}_0 > 0$, the above algorithm works well. In Section 6, we will amend the above algorithm to deal with the case where $\tilde{u}_0 = 0$.

Though here we assume that we first solve $\langle S_i \rangle$, this is not essential, and we can solve the subproblems in any order. In fact, in Section 8, we will consider another strategy for choosing an entering variable which guarantees the global convergence property of the algorithm.

Theorem 5. *Assume that we are given a feasible dictionary* $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ *with* $\tilde{\boldsymbol{x}}_B > \mathbf{0}$ *and* $\tilde{u}_0 > 0$. *If all the subproblems* $\langle S_i \rangle$ *for* $i \in N$ *and* $\langle Z_j \rangle$ *for* $j \in N'$ *have trivial optimal solutions, then*

$$\boldsymbol{y}^* = G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/u_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{s}^*_B \\ \boldsymbol{s}^*_N \\ \boldsymbol{z}^*_{B'} \\ \boldsymbol{z}^*_{N'} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \tilde{\boldsymbol{s}}_N \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \frac{1}{\tilde{u}_0} \begin{pmatrix} \mathbf{0} \\ D^T_{B'N}\tilde{\boldsymbol{u}}_{B'} \\ \tilde{\boldsymbol{u}}_{B'} \\ \tilde{\boldsymbol{u}}_{N'} \end{pmatrix}$$

are feasible for $\langle D \rangle$, *and satisfy the complementarity slackness condition with*

$$(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}_{B'}, \boldsymbol{u}_{N'}) = (\tilde{\boldsymbol{x}}_B, \mathbf{0}, \tilde{u}_0, \tilde{\boldsymbol{u}}_{B'}, \tilde{\boldsymbol{u}}_{N'}), \tag{14}$$

i.e., they are optimal for $\langle D \rangle$ *and* $\langle P \rangle$, *respectively.*

Proof : We put $\boldsymbol{s}^*_B = \mathbf{0}$ and $\boldsymbol{z}^*_{B'} = -\tilde{\boldsymbol{u}}_{B'}/u_0$. We show that the other parts can be calculated from the equality condition of $\langle D \rangle$. The equality constraint of $\langle D \rangle$:

$$\begin{pmatrix} \boldsymbol{s}^*_B \\ \boldsymbol{z}^*_{B'} \end{pmatrix} + G^T \boldsymbol{y}^* = \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} \end{pmatrix} \tag{15}$$

produces

$$\boldsymbol{y}^* = G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/\tilde{u}_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \boldsymbol{s}^*_N &= \boldsymbol{c}_N - A^T_N \boldsymbol{y}^* = \boldsymbol{c}_N - A^T_N G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/\tilde{u}_0 \end{pmatrix} \\ &= \boldsymbol{c}_N - (D^T_{BN}\ D^T_{B'N}) \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/\tilde{u}_0 \end{pmatrix} \\ &= \boldsymbol{c}_N - D^T_{BN}\boldsymbol{c}_B - D^T_{B'N}\boldsymbol{d}_{B'} - \frac{1}{\tilde{u}_0} D^T_{B'N}\tilde{\boldsymbol{u}}_{B'} \\ &= \tilde{\boldsymbol{s}}_N - \frac{1}{\tilde{u}_0} D^T_{B'N}\tilde{\boldsymbol{u}}_{B'} \geq \mathbf{0}, \end{aligned}$$

where the last inequality follows from Lemma 2. Similarly, we have

$$\begin{aligned} \boldsymbol{z}^*_{N'} &= \boldsymbol{d}_{N'} - R^T_{N'} \boldsymbol{y}^* = \boldsymbol{d}_{N'} - R^T_{N'} G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/\tilde{u}_0 \end{pmatrix} \\ &= \boldsymbol{d}_{N'} - (D^T_{BN'}\ D^T_{B'N'}) \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B'} + \tilde{\boldsymbol{u}}_{B'}/\tilde{u}_0 \end{pmatrix} \\ &= \boldsymbol{d}_{N'} - D^T_{BN'}\boldsymbol{c}_B - D^T_{B'N'}\boldsymbol{d}_{B'} - \frac{1}{\tilde{u}_0} D^T_{B'N'}\tilde{\boldsymbol{u}}_{B'} \\ &= \tilde{\boldsymbol{z}}_{N'} - \frac{1}{\tilde{u}_0} D^T_{B'N'}\tilde{\boldsymbol{u}}_{B'} \\ &= -\frac{\tilde{\boldsymbol{u}}_{N'}}{\tilde{u}_0}, \end{aligned}$$

where the last equality is due to Lemma 3. We also have

$$\|\boldsymbol{z}^*_{B'}\|^2 + \|\boldsymbol{z}^*_{N'}\|^2 = \frac{1}{\tilde{u}_0^2}\left(\|\tilde{\boldsymbol{u}}_{B'}\|^2 + \|\tilde{\boldsymbol{u}}_{N'}\|^2\right) = 1,$$

which means that $(1, \boldsymbol{z}^*_{B'}, \boldsymbol{z}^*_{N'}) \in \mathcal{K}_{r+1}$. Therefore, $(\boldsymbol{s}^*_B, \boldsymbol{s}^*_N, \boldsymbol{z}^*_{B'}, \boldsymbol{z}^*_{N'})$ is feasible for $\langle D \rangle$.

The complementarity condition can easily be seen by $\tilde{\boldsymbol{x}}_N = \mathbf{0}$, $\boldsymbol{s}^*_N = \mathbf{0}$, and

$$\tilde{u}_0 + \tilde{\boldsymbol{u}}^T_{B'}\boldsymbol{z}^*_{B'} + \tilde{\boldsymbol{u}}^T_{N'}\boldsymbol{z}^*_{N'} = \tilde{u}_0 - \|\tilde{\boldsymbol{u}}_{B'}\|^2/\tilde{u}_0 - \|\tilde{\boldsymbol{u}}_{N'}\|^2/\tilde{u}_0 = 0.$$

□

6. The case $\tilde{u}_0 = 0$

This section discusses the case where $\tilde{u}_0 = 0$, or equivalently, the basic solution is $(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}_{B'}, \boldsymbol{u}_{N'}) = (\tilde{\boldsymbol{x}}_B, \mathbf{0}, 0, \mathbf{0}, \mathbf{0})$. In this case, we further assume that this point is *primal nondegenerate* in the sense of [6] and [1]. Let us denote by $\mathcal{T}$ the tangent space of the cone at $(\tilde{\boldsymbol{x}}_B, \mathbf{0}, 0, \mathbf{0}, \mathbf{0})$. According to [1], a feasible point $(\tilde{\boldsymbol{x}}_B, \mathbf{0}, 0, \mathbf{0}, \mathbf{0})$ is primal nondegenerate if

$$\mathcal{T} + \ker([A_B \ A_N \ \mathbf{0} \ R_{B'} \ R_{N'}]) = \mathbb{R}^{n+1+r}.$$

Since the dimension of $\ker([A_B \ A_N \ \mathbf{0} \ R_{B'} \ R_{N'}])$ is $n+1+r-m$, we must have $|B| = m$ and $\tilde{\boldsymbol{x}}_B > \mathbf{0}$. This implies that B' is empty and $N' = \{1, \ldots, r\}$. The dictionary is then written as

$$\begin{array}{rcccccccc} \theta & = & \tilde{\theta} & + & \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N & + & \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{v}_{N'} & + & u_0 \\ \boldsymbol{x}_B & = & \tilde{\boldsymbol{x}}_B & - & D_{BN} \boldsymbol{x}_N & - & D_{BN'} \boldsymbol{v}_{N'} & & \\ \boldsymbol{u}_{N'} & = & & & & & \boldsymbol{v}_{N'}. & & \end{array} \tag{16}$$

Because $A_B \tilde{\boldsymbol{x}}_B = \boldsymbol{b}$, we see that the LP problem obtained by putting $(u_0, \boldsymbol{u}) = \mathbf{0}$ in (1):

$$\langle LP \rangle \left\{ \begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{array} \right.$$

has a feasible solution $(\tilde{\boldsymbol{x}}_B, \mathbf{0})$. In fact,

$$\begin{array}{rcccc} \theta & = & \tilde{\theta} & + & \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N \\ \boldsymbol{x}_B & = & \tilde{\boldsymbol{x}}_B & - & D_{BN} \boldsymbol{x}_N \end{array}$$

is a feasible dictionary for $\langle LP \rangle$.

Now we solve $\langle LP \rangle$; we take a negative component of $\tilde{\boldsymbol{s}}$ for an entering variable, and choose a leaving variable in B according to the standard rule of the simplex method for LP. Once we determine the entering and leaving variables, we perform a pivot on (16) as was described in Case (I-i) of Section 4. Notice that while B and N will be changed through this pivot, $N' = \{1, \ldots, r\}$ is not changed.

Continuing this process, we will find an optimal solution for $\langle LP \rangle$, or a certificate for unboundedness of $\langle LP \rangle$. If $\langle LP \rangle$ is unbounded, $\langle P \rangle$ is unbounded too. If an optimal solution of $\langle P \rangle$ is found, then it holds that $\tilde{\boldsymbol{s}}_N \geq \mathbf{0}$. Therefore in the following, we assume that $\tilde{\boldsymbol{s}}_N \geq \mathbf{0}$ in (16) without loss of generality.

Theorem 6. *If $\tilde{\boldsymbol{s}}_N \geq \mathbf{0}$ and $\|\tilde{\boldsymbol{z}}_{N'}\| \leq 1$ in (16), then $(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}) = (\tilde{\boldsymbol{x}}_B, \mathbf{0}, 0, \mathbf{0})$ is optimal for $\langle P \rangle$.*

Proof: Let $(\boldsymbol{x}, u_0, \boldsymbol{u})$ be any feasible solution of $\langle P \rangle$. Due to (16), the objective function of $(\boldsymbol{x}, u_0, \boldsymbol{u})$ can be estimated as

$$\begin{array}{rl} & \tilde{\theta} + \tilde{\boldsymbol{s}}_N^T \boldsymbol{x}_N + \tilde{\boldsymbol{z}}_{N'}^T \boldsymbol{u}_{N'} + u_0 \\ \geq & \tilde{\theta} - \|\tilde{\boldsymbol{z}}_{N'}\| \|\boldsymbol{u}_{N'}\| + u_0 \\ \geq & \tilde{\theta} - \|\boldsymbol{u}_{N'}\| + u_0 \geq \tilde{\theta}. \end{array}$$

Now we see that $(\tilde{\boldsymbol{x}}_B, \mathbf{0}, 0, \mathbf{0})$ is optimal for $\langle P \rangle$ because its objective value is $\tilde{\theta}$. □

Next we assume that $\|\tilde{\boldsymbol{z}}_{N'}\| > 1$. We consider to move from the basic solution in the direction $-\tilde{\boldsymbol{z}}_{N'}$. As a result, we obtain a subproblem:

$$\langle Z_{N'} \rangle \left\{ \begin{array}{ll} \text{minimize} & -\lambda \|\tilde{\boldsymbol{z}}_{N'}\|^2 + u_0 \\ \text{subject to} & \tilde{\boldsymbol{x}}_B + \lambda D_{BN'} \tilde{\boldsymbol{z}}_{N'} \geq \mathbf{0} \\ & u_0 \geq \|\lambda \tilde{\boldsymbol{z}}_{N'}\|. \end{array} \right.$$

This subproblem can be solved easily by the min-ratio test.

Lemma 7. *The optimal value of $\langle Z_{N'} \rangle$ is negative. Furthermore, if $\langle Z_{N'} \rangle$ is unbounded, $\langle P \rangle$ is unbounded.*

Proof: Because $\tilde{\boldsymbol{x}}_B > \mathbf{0}$, $\lambda > 0$ is feasible if it is sufficiently small. For such λ, the objective value is

$$u_0 - \lambda \|\tilde{\boldsymbol{z}}_{N'}\|^2 = \lambda \|\tilde{\boldsymbol{z}}_{N'}\| - \lambda \|\tilde{\boldsymbol{z}}_{N'}\|^2 < 0$$

because $\|\tilde{\boldsymbol{z}}_{N'}\| > 1$.

When $\langle Z_{N'} \rangle$ is unbounded, consider a solution for $\lambda > 0$,

$$(\boldsymbol{x}_B, \boldsymbol{x}_N, u_0, \boldsymbol{u}) = (\tilde{\boldsymbol{x}}_B + \lambda D_{BN'} \tilde{\boldsymbol{z}}_{N'}, \boldsymbol{0}, \lambda \|\tilde{\boldsymbol{z}}_{N'}\|, -\lambda \tilde{\boldsymbol{z}}_{N'}).$$

This solution is feasible for $\langle P \rangle$, and its objective value goes to $-\infty$ as $\lambda \to \infty$. □

Finally, we assume that $\langle Z_{N'} \rangle$ is bounded, and that $\bar{\lambda} > 0$ is the optimal solution of $\langle Z_{N'} \rangle$. Let $i \in B$ be the index active at the optimal solution. Then there exists an index $j \in N'$ such that $D_{ij} \tilde{z}_j \neq 0$. We will exchange i and j to make a pivot. For this purpose, we rewrite the $\boldsymbol{x}_B$-part of (16) as

$$\boldsymbol{x}_B = \tilde{\boldsymbol{x}}_B + \bar{\lambda} D_{BN'} \tilde{\boldsymbol{z}}_{N'} - D_{BN} \boldsymbol{x}_N - D_{BN'} (\boldsymbol{v}_{N'} + \bar{\lambda} \tilde{\boldsymbol{z}}_{N'}). \tag{17}$$

We substitute $\boldsymbol{v}'_{N'} = \boldsymbol{v}_{N'} + \bar{\lambda} \tilde{\boldsymbol{z}}_{N'}$. Since $\tilde{x}_i + \bar{\lambda} D_{iN'} \tilde{\boldsymbol{z}}_{N'} = 0$, the i-th row of (17) becomes

$$x_i = -D_{iN} \boldsymbol{x}_N - D_{ij} v'_j - \sum_{k \in N', k \neq j} D_{ik} v'_k,$$

which produces

$$u_j = v_j = v'_j - \bar{\lambda} \tilde{z}_j = -\bar{\lambda} \tilde{z}_j - \frac{x_i}{D_{ij}} - \frac{D_{iN} \boldsymbol{x}_N}{D_{ij}} - \sum_{k \in N', k \neq j} \frac{D_{ik}}{D_{ij}} v'_k.$$

Substituting v_j in (16) by the above relation, we obtain a new dictionary. In fact, we have for $k \in B - \{i\}$,

$$x_k = \tilde{x}_k + \bar{\lambda} D_{kN'} \tilde{\boldsymbol{z}}_{N'} - (D_{kN} - \frac{D_{kj} D_{iN}}{D_{ij}}) \boldsymbol{x}_N + \frac{D_{kj}}{D_{ij}} x_i - \sum_{l \in N', l \neq j} \left(D_{kl} - \frac{D_{kj} D_{il}}{D_{ij}} \right) v'_l$$

and

$$\theta = \tilde{\theta} - \bar{\lambda} \|\tilde{\boldsymbol{z}}_{N'}\|^2 + \left(\tilde{\boldsymbol{s}}_N^T - \frac{\tilde{z}_j D_{iN}}{D_{ij}} \right) \boldsymbol{x}_N - \frac{\tilde{z}_j}{D_{ij}} x_i + \sum_{l \neq j, l \in N'} \left(\tilde{z}_l - \frac{\tilde{z}_j D_{il}}{D_{ij}} \right) v'_l + u_0.$$

Since $\bar{\lambda}$ is feasible for $\langle Z_{N'} \rangle$, $\tilde{x}_k + \bar{\lambda} D_{kN'} \tilde{\boldsymbol{z}}_{N'} \geq 0$, thus the basic solution is feasible. The objective value of the new basic solution $\tilde{\theta} - \bar{\lambda} \|\tilde{\boldsymbol{z}}_{N'}\|^2 + \bar{\lambda} \|\tilde{\boldsymbol{z}}_{N'}\|$ is less than $\tilde{\theta}$ because $\|\tilde{\boldsymbol{z}}_{N'}\| > 1$. Furthermore, since $\tilde{u}_j = -\bar{\lambda} \tilde{z}_j \neq 0$, we can continue the pivoting procedure described in Section 4. We will never meet a dictionary having $\tilde{u}_0 = 0$ again because the following dictionaries have objective values less than that of the optimal value of $\langle LP \rangle$.

7. Solving Subproblems

In this section, we discuss how to solve $\langle S_i \rangle$ and $\langle Z_j \rangle$. Note that the first linear inequality constraint of $\langle S_i \rangle$ implies that

$$0 \leq x_i \leq \bar{x},$$

where $\bar{x} = \sup\{\, x \mid \tilde{\boldsymbol{x}}_B - x D_{Bi} \geq 0 \,\}$. The value $\bar{x}$ can easily be calculated in $O(|B|)$ basic arithmetic operations. The only difference of $\langle Z_j \rangle$ from $\langle S_i \rangle$ is that there is no obvious lower bound of v_j, and the argument for solving $\langle Z_j \rangle$ is basically the same as that of $\langle S_i \rangle$, thus we concentrate our discussion on solving $\langle S_i \rangle$. To simplify the notation, we omit the subscript and consider to solve:

$$\langle S \rangle \left\{ \begin{array}{ll} \text{minimize} & sx + u \\ \text{subject to} & 0 \leq x \leq \bar{x} \\ & \sqrt{g(x)} \leq u \end{array} \right.$$

where

$$g(x) = ax^2 - 2bx + c \leq u^2,$$

$a = \|D_{B'i}\|^2$, $b = D_{B'i}^T \tilde{\boldsymbol{u}}_{B'}$, and $c = \|\tilde{\boldsymbol{u}}_{B'}\|^2 + \|\tilde{\boldsymbol{u}}_{N'}\|^2 = \tilde{u}_0^2 > 0$.

Theorem 8. *$\langle S \rangle$ is solvable by a constant number of the basic four operations of arithmetic and square root.*

Proof : Because the discriminant of g is non-positive, $g(x) \geq 0$ for any x. In the following, we assume that the discriminant is negative. This is true if $\tilde{\boldsymbol{u}}_{N'} \neq \boldsymbol{0}$. For the case $\tilde{\boldsymbol{u}}_{N'} = \boldsymbol{0}$, we have to take care of the case $g(x) = 0$, but the treatment is straightforward, thus we omit to describe this case.

Since the optimal solution of $\langle S\rangle$ is always taken at $u = \sqrt{g(x)}$, we can substitute the inequality constraints by the equality. As a result, $\langle S\rangle$ becomes

$$\begin{cases} \text{minimize} & h(x) = sx + \sqrt{g(x)} \\ \text{subject to} & 0 \le x \le \bar{x}. \end{cases} \tag{18}$$

It is easy to check

$$h(0) = \sqrt{c} > 0 \tag{19}$$

$$h'(x) = s + \frac{g'(x)}{2\sqrt{g(x)}} \tag{20}$$

$$h'(0) = s - \frac{b}{\sqrt{c}} \tag{21}$$

$$h''(x) = -g(x)^{-3/2}(b^2 - ac) > 0. \tag{22}$$

The inequality (22) implies that $h'(x)$ is increasing. Therefore, if $h'(0) \ge 0$, then $x = 0$ is the optimal solution.

If $h'(0) < 0$, then 0 is not an optimal solution, and we should check two points: $\bar{x}$ and the solution of $h'(x) = 0$. Let $\hat{x}$ be the solution of $h'(x) = 0$. Squaring the both sides of

$$g'(\hat{x}) = -2s\sqrt{g(\hat{x})}, \tag{23}$$

we obtain

$$4s^2 g(\hat{x}) = (g'(\hat{x}))^2 = 4(a\hat{x} - b)^2,$$

or

$$a(a - s^2)\hat{x}^2 - 2b(a - s^2)\hat{x} + b^2 - s^2 c = 0. \tag{24}$$

By a direct calculation, the discriminant of the above is turned out to be $-s^2(a - s^2)(b^2 - ac)$. When this is negative, (24) has no solutions, and $\bar{x}$ achieves minimum of (18). Otherwise, let $\hat{x}$ be a solution of (24) satisfying $a\hat{x} - b \le 0$. Such $\hat{x}$ satisfies (23). The optimal solution of $\langle S\rangle$ is then $\hat{x}$ if $\hat{x} \le \bar{x}$, and $\bar{x}$ otherwise. □

8. A Globally Convergent Algorithm

In this section, we consider another strategy for pivoting. We solve all the subproblems first, and then choose the subproblem having the least optimal value for entering variable. In view of Lemma 4, this strategy picks up a pivot that achieves maximum decrease of the objective function.

Maximum Decrease Strategy (MDS)

STEP 0 A feasible and nondegenerate dictionary $\mathcal{D}(B, B'; \tilde{\boldsymbol{u}}_{N'})$ is given.
STEP 1 Let δ_i be the optimal value of $\langle S_i\rangle$ $(i \in N)$. Let $\hat{\imath} = \operatorname{argmin} \delta_i$.
STEP 2 Let δ'_j be the optimal value of $\langle Z_j\rangle$ $(j \in N')$. Let $\hat{\jmath} = \operatorname{argmin} \delta'_j$
STEP 3 If $\delta_{\hat{\imath}} = -\infty$ or $\delta'_{\hat{\jmath}} = -\infty$, then STOP. $\langle P\rangle$ is unbounded.
STEP 4 If $\delta_{\hat{\imath}} = \tilde{u}_0$ and $\delta_{\hat{\jmath}} = \tilde{u}_0$, then calculate optimal solution according to Theorem 5.
STEP 5 If $\delta_{\hat{\imath}} \le \delta_{\hat{\jmath}}$, then pivot using $\hat{\imath}$ as an entering variable.
Else, pivot using $\hat{\jmath}$ as an entering variable.
STEP 6 Go to Step 1.

Assume that the sequence of dictionary $\left\{ \mathcal{D}(B_k, B'_k; \tilde{\boldsymbol{u}}^k_{N'_k}) \mid k = 0, 1, 2, \dots \right\}$ generated by MDS is infinite. (If this sequence is finite, the optimality of the final dictionary follows from Theorem 5.) Then, there exists a basis (B_*, B'_*) which appears in $\{ (B_k, B'_k) \mid k = 0, 1, 2, \dots \}$ infinitely many times. Let L be the set of natural numbers satisfying $(B_k, B'_k) = (B_*, B'_*)$. Because $\|\tilde{\boldsymbol{u}}^k_{N'_*}\|$ $(k \in L)$ is bounded by the initial objective value, we can choose an accumulation point $\tilde{\boldsymbol{u}}^*_{N'_*}$ of $\{ \tilde{\boldsymbol{u}}^k_{N'_*} \mid k \in L \}$. Let L' be a subsequence of L converging to $\tilde{\boldsymbol{u}}^*_{N'_*}$, i.e.,

$$\lim_{k \in L'} \tilde{\boldsymbol{u}}^k_{N'_*} = \tilde{\boldsymbol{u}}^*_{N'_*}.$$

Theorem 9. *Assume that $\langle D\rangle$ is feasible. Let $(\tilde{\boldsymbol{x}}_{B_*}, \boldsymbol{0}, \tilde{u}^*_0, \tilde{\boldsymbol{u}}^*_{B'_*}, \tilde{\boldsymbol{u}}^*_{N'_*})$ be the basic solution associated with $\mathcal{D}(B_*, B'_*; \tilde{\boldsymbol{u}}^*_{N'_*})$. If $\tilde{\boldsymbol{x}}_{B_*} > \boldsymbol{0}$ and $\tilde{u}^*_0 > 0$, then the basic solution is optimal for $\langle P\rangle$.*

Proof : Let us denote that objective function value of the basic solution corresponding to $\mathcal{D}(B_k, B'_k; \tilde{\boldsymbol{u}}^k_{N'_k})$ by θ^k. Since $\langle D\rangle$ is feasible, $\{\theta^k\}$ is bounded below. This and the fact that $\theta^k > \theta^{k+1}$ for all k imply the existence of $\theta^* = \lim \theta^k$ and that $\theta^{k+1} - \theta^k \to 0$.

We denote the subproblems of $\mathcal{D}(B_*, B'_*; \tilde{\boldsymbol{u}}^*_{N'_*})$ by $\langle S^*_i\rangle$ and $\langle Z^*_j\rangle$, and those of $\mathcal{D}(B_*, B'_*; \tilde{\boldsymbol{u}}^k_{N'_*})$ for $k \in L'$ by $\langle S^k_i\rangle$ and $\langle Z^k_j\rangle$, respectively. We suppose that one of the subproblems of $\langle S^*_i\rangle$ and $\langle Z^*_j\rangle$ has a nontrivial optimal solution, and derive a contradiction.

First, we assume that $\langle S^*_i\rangle$ has an optimal solution other than $(0, \tilde{u}^*_0)$. As was seen in Section 7, $\langle S^*_i\rangle$ can be rewritten as

$$\langle S^*_i\rangle \begin{cases} \text{minimize} & \tilde{s}_i x_i + \sqrt{\|\tilde{\boldsymbol{u}}^*_{B'_*} - x_i D_{B'_* i}\|^2 + \|\tilde{\boldsymbol{u}}^*_{N'_*}\|^2} \\ \text{subject to} & 0 \le x_i \le \bar{x}^* \end{cases}$$

with $\bar{x}^* = \min\{\, \tilde{x}^*_j / D_{ji} \mid D_{ji} > 0,\ j \in B_* \,\}$, and $\langle S^k_i\rangle$ as

$$\langle S^k_i\rangle \begin{cases} \text{minimize} & \tilde{s}_i x_i + \sqrt{\|\tilde{\boldsymbol{u}}^k_{B'_*} - x_i D_{B'_* i}\|^2 + \|\tilde{\boldsymbol{u}}^k_{N'_*}\|^2} \\ \text{subject to} & 0 \le x_i \le \bar{x}^k \end{cases}$$

with $\bar{x}^k = \min\{\, \tilde{x}^k_j / D_{ji} \mid D_{ji} > 0,\ j \in B_* \,\}$. Notice that D and $\tilde{s}_i$ are unchanged due to the structure of dictionary; only $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{x}}_{B_*}$ are changing. Since $\tilde{\boldsymbol{x}}_B$ and $\tilde{\boldsymbol{u}}_{B'}$ are affine functions of $\tilde{\boldsymbol{u}}_{N'}$, we have

$$\lim_{k \in L'} \tilde{\boldsymbol{x}}^k_{B_*} = \tilde{\boldsymbol{x}}^*_{B_*}, \ \lim_{k \in L'} \tilde{\boldsymbol{u}}^k_{B'_*} = \tilde{\boldsymbol{u}}^*_{B'_*}, \ \lim_{k \in L'} \bar{x}^k = \bar{x}^*.$$

Let x^*_i be the optimal solution of $\langle S^*_i\rangle$. By assumption, it holds that

$$\tilde{s}_i x^*_i + \sqrt{\|\tilde{\boldsymbol{u}}^*_{B'_*} - x^*_i D_{B'_* i}\|^2 + \|\tilde{\boldsymbol{u}}^*_{N'_*}\|^2} < \tilde{u}^*_0.$$

Since the objective function of $\langle S^*_i\rangle$ is continuous, the objective value is also less than $\tilde{u}^*_0$ at δx^*_i for some $\delta \in (0,1)$. Let us define $\epsilon > 0$ by

$$\tilde{u}^*_0 - \epsilon = \delta \tilde{s}_i x^*_i + \sqrt{\|\tilde{\boldsymbol{u}}^*_{B'_*} - \delta x^*_i D_{B'_* i}\|^2 + \|\tilde{\boldsymbol{u}}^*_{N'_*}\|^2}.$$

If $k \in L'$ is sufficiently large, then $\bar{x}^k > \delta x^*_i$, which implies that δx^*_i is feasible for $\langle S^k_i\rangle$. Let us denote the objective function value of $\langle S^k_i\rangle$ at δx^*_i by $\rho(\tilde{\boldsymbol{u}}^k_{B'_*}, \tilde{\boldsymbol{u}}^k_{N'_*})$. Since $\rho(\tilde{\boldsymbol{u}}^k_{B'_*}, \tilde{\boldsymbol{u}}^k_{N'_*})$ is continuous, if holds that

$$\tilde{u}^*_0 - \frac{3\epsilon}{2} < \rho(\tilde{\boldsymbol{u}}^k_{B'_*}, \tilde{\boldsymbol{u}}^k_{N'_*}) < \tilde{u}^*_0 - \frac{\epsilon}{2}$$

for sufficiently large $k \in L'$, which implies that the optimal value of $\langle S^k_i\rangle$ is less than $\tilde{u}^*_0 - \epsilon/2$. Under MDS, this means that $\theta^{k+1} - \theta^k \le -\epsilon/2$ for any sufficiently large $k \in L'$, which is a contradiction. Therefore, for all $i \in N_*$, the optimal solution of $\langle S^*_i\rangle$ is $(0, \tilde{u}^*_0)$.

The case of $\langle Z^*_j\rangle$ can be proved similarly. We omit the details. □

9. Concluding Remarks

We have defined a pivoting procedure for a class of SOCP. This pivoting procedure has a remarkable resemblance to the pivot in LP. Assuming the existence of initial feasible dictionary and the nondegeneracy assumption, we showed that the objective function value strictly decreases by this pivoting procedure, and if no further improvement is possible, the current basic solution is optimal. An initial feasible dictionary can be derived by finding $(\boldsymbol{x}, \boldsymbol{u})$ such that

$$A\boldsymbol{x} + R\boldsymbol{u} = b, \ \boldsymbol{x} \ge \boldsymbol{0}.$$

Such solution is easily obtained by solving the corresponding Phase-I LP problem.

Though we deal with an SOCP problem having only one second-order cone, extension to multiple cones is relatively straightforward if we use the result of this paper as a building block.

Recalling the simplex method for LP and its related subjects, we should say that there are a lot of problems to be solved in the pivoting algorithm for SOCP. Degeneracy resolution, the dual simplex algorithm, making the problem class wider are all interesting topics. Finally, it is important to implement the pivoting algorithm for SOCP to check its efficiency in practice.

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References

[1] F. Alizadeh and D. Goldfarb : "Second-order cone programming", Mathematical Programming, Series B 95, pp. 3–51, (2003).

[2] E. D. Andersen, C. Roos and T. Terlaky : "On Implementing a Primal-Dual Interior-Point Method for Conic Quadratic Optimization", Technical Report (2000).

[3] A. Ben-Tal and A. Nemirovski : *Lectures on Modern Convex Optimization.* SIAM, Philadelphia (2001)

[4] Z. Cai and K. C. Toh : "Solving Second Order Cone Programming via the augmented systems", Working paper, (2003).

[5] G. Dantzig : *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, USA (1963).

[6] L. Faybusovich : "Linear systems in Jordan algebras and primal-dual interior point algorithms", *Journal of computational and Applied Mathematics*, Vol. 86, pp. 149–175, (1997).

[7] L. Faybusovich : "Jordan algebras, symmetric cones, and interior point methods", Technical report, Department of Mathematics, University of Notre Dame, Notre Dame, IN, USA (1995).

[8] M. S. Lobo, L. Vandenbeghe, S. Boyd, and H. Lebret : "Second-order cone programming", *Linear Algebra and Applications*, Vol. 284, pp. 193–228, (1998)

[9] R. D. C. Monteiro, and T. Tsuchiya : "Polynomial Convergence of Primal-Dual Algorithms for the Second-Order Cone Program Based on the MZ-Family of Directions", Mathematical Programming, Vol. 88, pp. 61–83 (2000).

[10] M. Muramatsu : "On a Commutative Class of Search Directions for Linear Programming over Symmetric Cones", *Journal of Optimization Theory and Applications*, Vol. 112, pp. 595–625 (2002).

[11] M. Muramatsu and T. Suzuki : "A New Second-Order Cone Programming Relaxation for MAX-CUT problems", Journal of Operations Research of Japan, Vol. 46, No. 2, pp.164-177, 2003.

[12] Y. E. Nesterov, and M. J. Todd : "Primal-Dual Interior-Point Methods for Self-Scaled Cones", SIAM Journal on Optimization, Vol. 8, pp. 324–364 (1998).

[13] T. Sasakawa and T. Tsuchiya : "Optimal Magnetic Shield Design with Second-order Cone Programming", SIAM Journal on Scientific Computing, Vol. 24, pp.1930–1950 (2003).

[14] T. Tsuchiya : "A Convergence Analysis of the Scaling-Invariant Primal-Dual Path-Following Algorithms for Second-Order Cone Programming", Optimization Methods and Software, Vol.11/12, pp.141–182 (1999).