Valid inequalities based on simple mixed-integer sets

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Abstract

In this paper we use facets of simple mixed-integer sets with three variables to derive a parametric family of valid inequalities for general mixed-integer sets. We call these inequalities *two-step MIR inequalities* as they can be derived by applying the simple mixed-integer rounding (MIR) principle of Wolsey (1998) twice. The two-step MIR inequalities define facets of the master cyclic group polyhedron of Gomory (1969). In addition, they dominate the strong fractional cuts of Letchford and Lodi (2002).

1 Introduction

An important technique in generating cutting planes for integer programs is to work with a single implied constraint, and to derive valid inequalities for the integral vectors satisfying the constraint. These valid inequalities can be used as cutting planes for the original integer program. Facets of simple polyhedral sets can be used to obtain such valid inequalities.

In his book, Wolsey[18] studies the following simple polyhedral set with two variables

$$Q^1 = \{v \in R,\ z \in Z\ :\ v + z\ \geq\ b,\ \ v \geq 0\}$$

and derives the mixed-integer rounding (MIR) inequality using the only non-trivial facet of Q^1. The well-known Gomory mixed-integer cut (GMIC) can be derived using this inequality, see Marchand and Wolsey[16].

In this paper we study the following simple polyhedral set with three variables

$$Q^2 = \{v \in R,\ y, z \in Z\ :\ v + \alpha y + z\ \geq\ \beta,\ \ v, y \geq 0\},$$

to derive what we call the "two-step MIR inequality". We use this inequality to generate cutting planes for general integer programs. In addition, we use the two-step MIR inequality to derive some valid inequalities described in the literature; specifically a sub-class of the *two-slope* inequalities of Gomory and Johnson [12, 13], and the *strong fractional cuts* of Letchford and Lodi[15].

More precisely, we use facets of Q^2 to generate valid inequalities for the set

$$Y\ =\ \Big\{x \in Z^{|J|}, z \in Z\ :\ \sum_{j \in J} a_j\, x_j + z = b,\ \ x \geq 0\Big\}. \tag{1}$$

Y can be viewed as a relaxation of an arbitrary equation with non-negative variables (divide through by the coefficient of z), and also of a row of a simplex tableau for an integer program with non-negative variables, where z is a basic variable. Thus, the valid inequalites we derive for Y can be added as cuttting planes to an integer program, in a manner analogous to the GMIC. We are motivated by the importance of the GMIC, which is now routinely used in mixed integer programming software and is considered one of the most useful classes of cutting planes, see [6]. We assume all inequalities and equations have rational coefficients.

The set Y is closely related to the *master cyclic group polyhedron* of Gomory[10]:

$$P(n, r) = conv\{w \in Z^{n-1}\ :\ \sum_{i=1}^{n-1}(i/n)w_i + z = r/n,\ \ w_i \geq 0,\ z \in Z\}, \tag{2}$$

where $n, r \in Z$ and $n > r > 0$, and for a set $S \subseteq R^n$, $conv(S)$ denotes the convex hull of vectors in S. As discussed in [10], Y can be viewed as a face of $P(n, r)$ for some n and r. Therefore, facets of $P(n, r)$ yield valid inequalities for general integer programs. The polyhedral structure of the master cyclic group polyhedron is also studied in Gomory and Johnson [11, 12], and more recently in [9], [14], [3], and [13]. Also see [1] for an introduction to $P(n, r)$. In recent work, Evans [9] and Gomory, Johnson and Evans [14] identify, based on empirical studies, some "important" facets of $P(n, r)$ for small n. In this paper we show that MIR and two-step MIR inequalities define facets of $P(n, r)$ and subsume the most "important" facets described in [14] and [9].

The structure of the paper is as follows. In the remainder of this section we review the MIR inequality, and discuss how it can be applied to Y. In Section 2, we study Q^2 and introduce the *two-step MIR inequality.* In Section 3 we use this inequality to derive valid inequalities for Y. In Section 4 we study the limiting behavior of these inequalities and derive valid inequalities that dominate the strong fractional cut of Letchford and Lodi [15]. In Section 5 we apply cuts derived from Q^2 to obtain facets of $P(n, r)$. We conclude with a discussion on applying the two-step MIR inequality to general mixed-integer programs.

1.1 MIR Inequalities

The MIR procedure provides a unifying framework to derive valid inequalities for mixed-integer sets. See [16] for examples. For the sake of completeness, we discuss the basic idea behind these inequalities.

Given a valid inequality

$$v + z \geq b$$

for a mixed-integer set $X \subset \{(v, z) : v \in R_+,\ z \in Z\}$, it is easy to see that the MIR inequality

$$v \geq \hat{b}\,(\lceil b \rceil - z)$$

where $\hat{b} = b - \lfloor b \rfloor$ is also valid for X, see [17]. If $\hat{b} \neq 0$, this inequality can also be written as $v/\hat{b} + z \geq \lceil b \rceil$.

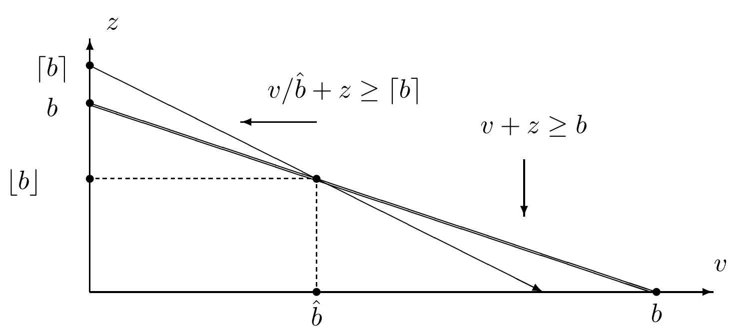

Figure 1: MIR inequality.

As seen in Figure 1, the point $(\bar{v}, \bar{z}) = (0, b)$ violates the MIR inequality when it is fractional. We also note that the MIR inequality is the only non-trivial facet defining inequality for the set

$$Q^1 = \left\{ v \in R,\ z \in Z \ :\ v + z \ \geq \ b, \quad v \geq 0 \right\}$$

and

$$conv(Q^1) = \left\{ v, z \in R \ :\ v + z \ \geq \ b, \quad v \geq \hat{b}(\lceil b \rceil - z), \quad v \geq 0 \right\}.$$

We would like to emphasize that the variable z in Q^1 is not restricted to be non-negative.

For mixed-integer sets with several variables, it is possible to generalize this idea as follows: Let $ax \geq b$ be a valid inequality for $X \subset R^n$ and let $a^1x + a^2x \geq b$ be a relaxation of $ax \geq b$. If $a^1x \geq 0$ and $a^2x \in Z$ for all $x \in X$, then, we can treat a^1x as a continuous variable and a^2x as an integral variable, to write the valid inequality $(a^1x) \geq \hat{b}(\lceil b \rceil - a^2x)$ where $\hat{b} = b - \lfloor b \rfloor$. In the rest of this paper we refer to this procedure as the *MIR procedure*.

1.2 Scaled MIR inequalities for Y

Recall the set Y where the variable z is integral. For any $t \in Z$ it is possible to define a relaxation of Y by letting z take on $1/t$-integral values as follows:

$$Y^t \;=\; \Big\{x \in Z^{|J|}, z \in \frac{1}{t}Z \;:\; \sum_{j\in J} a_j\, x_j + z = b, \qquad x \geq 0\Big\}.$$

Clearly $Y \subseteq Y^t$, and therefore, any valid inequality for Y^t is also valid for Y. By substituting $z^t = tz$, we obtain

$$\bar{Y^t} \;=\; \Big\{x \in Z^{|J|}, z^t \in Z \;:\; \sum_{j\in J} ta_j\, x_j + z^t = tb, \qquad x \geq 0\Big\},$$

which has the same form as Y.

For $c \in R$ and $t \in Z$, let $\hat{c}^t = tc - \lfloor tc \rfloor$. For $S^t \subseteq J$ we next relax the equation defining $\bar{Y^t}$ and rearrange the terms to obtain

$$\Big(\sum_{j\in S^t} \hat{a}^t_j\, x_j\Big) + \Big(\sum_{j\in S^t} \lfloor ta_j \rfloor\; x_j + \sum_{j\in J\setminus S^t} \lceil ta_j \rceil\; x_j + z^t\Big) \geq tb.$$

Note that the first part of this inequality is non-negative, and the second part is integral for all $(x, z) \in \bar{Y^t}$. Therefore the associated MIR inequality

$$\sum_{j\in S^t} \hat{a}^t_j\, x_j \geq \hat{b}^t(\lceil tb \rceil - \sum_{j\in S^t} \lfloor ta_j \rfloor\; x_j - \sum_{j\in J\setminus S^t} \lceil ta_j \rceil\; x_j - tz) \tag{3}$$

where z^t is replaced by tz, is valid for Y^t.

If $\hat{b}^t \neq 0$ (i.e., $tb \notin Z$), we can rearrange the terms of (3) and substitute for z to obtain:

$$\sum_{j\in S^t} \frac{\hat{a}^t_j}{\hat{b}^t}\, x_j + \sum_{j\in J\setminus S^t} \frac{1-\hat{a}^t_j}{1-\hat{b}^t}\, x_j \;\geq\; 1 \tag{4}$$

Clearly, $(\hat{a}^t_j/\hat{b}^t) < (1-\hat{a}^t_j)/(1-\hat{b}^t)$ if and only if $\hat{a}^t_j < \hat{b}^t$, and therefore $S^t = \{j \in J \;:\; \hat{a}^t_j < \hat{b}^t\}$ gives the strongest inequality of the form (4). We call inequality (4), or (3), *the t-scaled MIR inequality*, or *the t-MIR cut* for Y.

When applied to a row of the simplex tableau, the 1-scaled MIR inequality gives the well-known *Gomory mixed-integer cut*. When $t > 1$, the resulting inequality is referred to as the *k-cut* in Cornuéjols, Li and Vandenbussche [7].

We next define the t-scaled MIR function and show that there are only a small number of t-scaled MIR inequalities for Y.

Definition 1 *For a non-zero integer t, the t-scaled MIR function with parameter b is defined as:*

$$f^{t,b}(v) = \begin{cases} \hat{v}^t/\hat{b}^t & \text{if } \hat{v}^t < \hat{b}^t, \\ (1-\hat{v}^t)/(1-\hat{b}^t) & \text{if } \hat{v}^t \geq \hat{b}^t, \end{cases}$$

where $\hat{v}^t = tv - \lfloor tv \rfloor$ and $\hat{b}^t = tb - \lfloor tb \rfloor$.

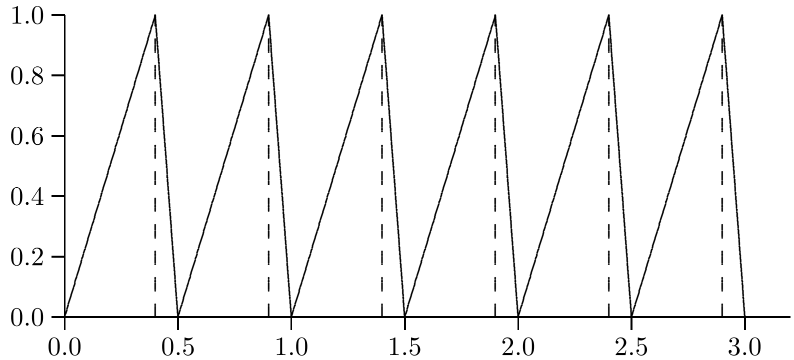

Figure 2: $f^{2,0.4}(v)$ for $v \in [0,3]$

Using this definition, the t-scaled MIR inequality (4) becomes $\sum_{j\in J} f^{t,b}(a_j)\, x_j \geq 1$. In Figure 2, we show the 2-scaled MIR function with parameter 0.4; it has a value of 1 when $\hat{v} = 0.4$ or 0.9.

Lemma 2 *Assume the equation defining Y is rational, and let n be the smallest integer such that $nb \in Z$ and $na_j \in Z$ for all $j \in J$. Then, there are at most $\lfloor n/2 \rfloor$ distinct t-MIR cuts for Y. In particular,*

a. If $t < 0$, then the t-MIR cut is the same as the $(-t)$-MIR cut, and,

b. if $t > n$, then the t-MIR cut is the same as the $(t-n)$-MIR cut, and,

c. if $n > t > \lfloor n/2 \rfloor$ then the t-MIR cut is the same as the $(n-t)$-MIR cut.

Proof. It is easy to see that (i) $f^{t,b}(v) = f^{-t,b}(v)$ for all $v \in R$, implying (a), and (ii) $f^{t,b}(v) = f^{t-n,b}(v)$ whenever v is an integral multiple of $1/n$, implying (b). From (i) and (ii) it follows that $f^{t,b}(v) = f^{n-t,b}(v)$ for v an integral multiple of $1/n$, implying (c). Finally, $t = 0$ or $t = n$ does not lead to a valid inequality, because $\hat{b}^t = 0$, therefore the only values of t that can give distinct t-MIR cuts are $1, 2, \ldots, \lfloor n/2 \rfloor$. ∎

We note that property (a) of Lemma 2 has also been observed by Cornuéjols, Li and Vandenbussche [7]. We also note that Example 1 in [7] deals with the set

$$Y \;=\; \Big\{x \in Z^2, z \in Z \;:\; 1.4x_1 + 0.1x_2 + z = 4.3, \qquad x \geq 0\Big\}$$

which has at most 5 distinct t-MIR cuts due to Lemma 2 because $n = 10$. This explains why the scaled cuts for t and $10-t$ turn out to be identical in [7]. Also note that $f^{2,0.4}(v)$ shown in Figure 2 is identical to $f^{2,0.9}(v)$ since $2 \times 0.4 - \lfloor 2 \times 0.4 \rfloor = 2 \times 0.9 - \lfloor 2 \times 0.9 \rfloor$.

2 A simple polyhedral set

In this section we study simple mixed-integer sets with three variables. We first look at the mixed-integer set

$$Q^{2+} = \big\{v \in R,\ y, z \in Z \;:\; v + \alpha y + z \;\geq\; \beta,\ \ v, y, z \geq 0\big\},$$

when $\alpha, \beta \in R$ and satisfy $1 > \beta > \alpha > 0$, and $\lceil \beta/\alpha \rceil > \beta/\alpha$. Note that variable z is required to be non-negative. In a recent study Agra and Constantino [2] study the polyhedral structure of Q^{2+} when β is an arbitrary positive number. They describe an algorithmic approach that enumerates all of the (polynomially-many) facets of the set in polynomial-time. Under our restrictions on α and β, it is possible to describe the convex hull of Q^{2+} explicitly. As we discuss later, the resulting inequalities are valid for the general case under mild conditions. First we show that the following MIR based inequalities are facet defining for Q^{2+}.

Lemma 3 *The following inequalities are valid and facet defining for* Q^{2+}*:*

$$v + \alpha y + \beta z \quad \geq \quad \beta, \tag{5}$$

$$\big(1/(\beta - \alpha \lfloor \beta/\alpha \rfloor)\big) v + y + \lceil \beta/\alpha \rceil z \quad \geq \quad \lceil \beta/\alpha \rceil . \tag{6}$$

Proof. Note that inequality (5) can be obtained by treating $v + \alpha y$ as a continuous variable and writing the MIR inequality based on $(v + \alpha y) + z \;\geq\; \beta$. To obtain inequality (6), we start with inequality (5), divide it by α and relax the resulting inequality as follows:

$$v/\alpha + y + \lceil \beta/\alpha \rceil z \quad \geq \quad \beta/\alpha \tag{7}$$

Writing the MIR inequality where v/α is treated as a continuous variable and $y + \lceil \beta/\alpha \rceil z$ is treated as an integer variable gives inequality (6).

To see that the inequalities are facet defining, consider the following distinct points:

$$p_1 = (0, 0, 1), \quad p_2 = (0, \lceil \beta/\alpha \rceil, 0), \quad p_3 = (\beta - \alpha \lfloor \beta/\alpha \rfloor, \lfloor \beta/\alpha \rfloor, 0), \quad p_4 = (\beta, 0, 0)$$

and note that (i) $p_1, p_3, p_4 \in Q^{2+}$ are affinely independent and satisfy inequality (5) with equality, (ii) $p_1, p_2, p_3 \in Q^{2+}$ are affinely independent and satisfy inequality (6) with equality. ∎

We next show that inequality (5) and inequality (6) are sufficient to obtain the convex hull of Q^{2+}. We call inequality (6) *the two-step MIR inequality.*

Lemma 4 $conv(Q^{2+}) = T$*, where*

$$T \;=\; \big\{v, y, z \in R \;:\; v, y, z \textit{ satisfy } (5),\; (6), \quad v, y, z \geq 0\big\}.$$

Proof. Clearly $conv(Q^{2+}) \subseteq T$. Note that inequality (5) is stronger than the original inequality, $v + \alpha y + z \;\geq\; \beta$ and therefore all integral points in T are contained in $conv(Q^{2+})$. We next show that all extreme points of T are integral.

Since $T \subset R^3$ is defined by five inequalities, three of which are non-negativity inequalities, any extreme point of T has to satisfy at least one non-negativity inequality as equality. Since $(0, 0, 0) \notin T$, we need to consider the following two cases for an extreme point $\bar{p} = (\bar{v}, \bar{y}, \bar{z})$ of T:

Case 1: If $\bar{p}$ satisfies two of the non-negativity inequalities as equality, one of inequality (5), or inequality (6) also has to hold as equality. If inequality (6) holds as equality, $\bar{p}$ is integral. If, on the other hand, inequality (5) holds as equality, $\bar{p}$ is integral provided that $\bar{v} > 0$, or $\bar{z} > 0$. On the other hand, if $\bar{y} > 0$, that is, $\bar{p} = (0, \beta/\alpha, 0)$, then it is easy to see that $\bar{p}$ violates inequality (6).

Case 2: If $\bar{p}$ satisfies only one non-negativity inequality as equality, both the inequalities (5) and (6) have to be satisfied as equalities. In this case, if $\bar{v} = 0$ or $\bar{y} = 0$, we obtain $\bar{p} = (0, 0, 1)$

(which falls into the case considered above). If, on the other hand, $\bar{z} = 0$ then we obtain $\bar{p} = (\beta - \alpha \lfloor \beta/\alpha \rfloor, \lfloor \beta/\alpha \rfloor, 0)$, which is integral. ∎

As shown in Figure 3, points p_1, p_2, p_3, p_4 are the only extreme points of $conv(Q^{2+})$, and the unit vectors give the only extreme directions of Q^{2+}.

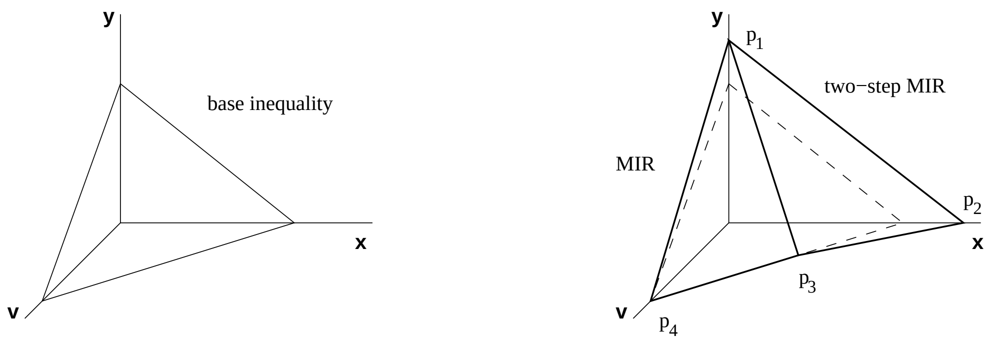

Figure 3: Facets of Q^{2+}

Let Q^2 be a relaxation of Q^{2+} obtained by allowing the z variable to assume negative values. More precisely,

$$Q^2 = \{v \in R,\ y, z \in Z\ :\ v + \alpha y + z\ \geq\ \beta,\ \ v, y \geq 0\}.$$

and remember that $\beta, \alpha \in R$ satisfy $1 > \beta > \alpha > 0$, and $\lceil \beta/\alpha \rceil > \beta/\alpha$.

Even though β is required to be less than 1 in Q^2, the fact that z can take on negative values makes the set fairly general. Some of the complexity of the facial structure of $conv(Q^{2+})$ when β is large can be captured by studying the set Q^2. We next show that inequalities (5) and (6) are facet defining for Q^2 under mild conditions. They are not necessarily sufficient to describe the convex hull.

Lemma 5 *Inequality (5) is facet defining for Q^2. In addition, inequality (6) is facet defining for Q^2 if $1/\alpha \geq \lceil \beta/\alpha \rceil$.*

Proof. Since $p_1, p_2, p_3, p_4 \in Q^{2+} \subseteq Q^2$, we only need to show validity of inequality (5) and (6). Inequality (5) is valid since it is derived in the proof of Lemma 3 without assuming that z is non-negative.

To see that inequality (6) is valid, notice that the following inequalities are valid for Q^2:

$$\begin{aligned} (1/\alpha)v\ +\ y +\ \ (\beta/\alpha)z\ \ &\geq\ \beta/\alpha, \\ (1/\alpha)v\ +\ y +\ \ (1/\alpha)z\ \ &\geq\ \beta/\alpha, \end{aligned}$$

and therefore, for any $\gamma \in R$ satisfying $1/\alpha \geq \gamma \geq \beta/\alpha$, the following inequality

$$(1/\alpha)v\ +\ y +\ \ \gamma z\ \ \geq\ \beta/\alpha$$

is also valid since it can be obtained as a convex combination of valid inequalities. Since $1/\alpha \geq \lceil \beta/\alpha \rceil$ by assumption, inequality (7) is valid for Q^2, and therefore so is inequality (6). ∎

When $1/\alpha = \lceil \beta/\alpha \rceil$, inequality (7) is the same as $(1/\alpha)v + y + (1/\alpha)z \geq \beta/\alpha$, and hence inequality (6) is an MIR inequality obtained after scaling $v + \alpha y + z \geq \beta$ by $1/\alpha$, which is an integer. We state this observation more formally in Lemma 10.

We would like to note that inequality (6) leads to strong inequalities even when $\beta > 1$. In addition, it dominates inequalities obtained by applying the MIR procedure twice in a straight forward manner. In particular, consider the following mixed-integer set

$$Q^3 = \big\{ v \in R,\ y, z \in Z\ :\ v + ay + z\ \geq\ b,\quad v, y, z \geq 0 \big\}.$$

where $b = \lfloor b \rfloor + \hat{b} > 1$ with $b \notin Z$ and $\hat{b} > a > 0$. Furthermore, assume that $1/a \geq \lceil \hat{b}/a \rceil > \hat{b}/a$.

When generating MIR based valid inequalities for the set Q^3, one possibility is to treat $v + ay$ as a continuous variable and z as an integral variable and apply the MIR procedure to obtain $v + ay + \hat{b}z\ \geq\ \hat{b}\lceil b \rceil$ which can then be relaxed to

$$(1/a)v + y + \left\lceil \hat{b}/a \right\rceil z\ \geq\ \hat{b}\lceil b \rceil / a \tag{8}$$

so that the MIR procedure can be applied again by treating $(1/a)v$ as a continuous variable and $(y + \lceil \hat{b}/a \rceil z)$ as an integral variable.

Another possibility is to view the initial inequality as $v + ay + (z - \lfloor b \rfloor)\ \geq\ \hat{b}$ and obtain $v + ay + \hat{b}(z - \lfloor b \rfloor)\ \geq\ \hat{b}$ as the first MIR inequality which then leads to

$$(1/a)v + y + \left\lceil \hat{b}/a \right\rceil (z - \lfloor b \rfloor)\ \geq\ \hat{b}/a, \tag{9}$$

via convex combinations, for the second MIR step.

Notice that inequality (9) is strictly stronger than inequality (8) since $\hat{b}\lceil b \rceil/a = \hat{b}/a + \hat{b}\lfloor b \rfloor/a < \hat{b}/a + \lceil \hat{b}/a \rceil \lfloor b \rfloor$. Therefore the MIR procedure based on inequality (9) gives a stronger valid inequality. We illustrate this in the subsequent example and in Figure 4.

Example 6 *Consider the set* Q^3 *with* $a = 0.4$ *and* $b = 1.7$ *and note that* $1/a = 2.5 > \lceil \hat{b}/a \rceil = \lceil 0.7/0.4 \rceil = 2$. *In this case, inequality (8) becomes* $2.5v + y + 2z \geq 3.5$ *whereas inequality (9) is* $2.5v + y + 2z \geq 3.75$. *The second step MIR inequality based on inequality (8) is*

$$2.5v + 0.5y + z \geq 2.$$

which is weaker than the second step MIR inequality based on inequality (9):

$$2.5v + 0.75y + 1.5z \geq 3 \tag{10}$$

We also note that the second step MIR inequality (10) is facet defining for Q^3. *For this instance, the following affinely independent points satisfy it as equality:* $p_1 = (0, 0, 2)$, $p_2 = (0, 2, 1)$, *and* $p_3 = (0.3, 1, 1)$. ∎

In Figure 4, we plot four inequalities for the example above, by showing the boundaries of their feasible regions: the original inequality; the MIR inequality based on z being integral; inequality (8), shown by a dotted line; and inequality (9), shown by a dashed line. We also display the points p_1, p_2 and p_3. As all inequalities have the term $2.5v + y$ in common, we can depict them in the plane, the horizontal axis representing $2.5v + y$ and the vertical axis standing for z values.

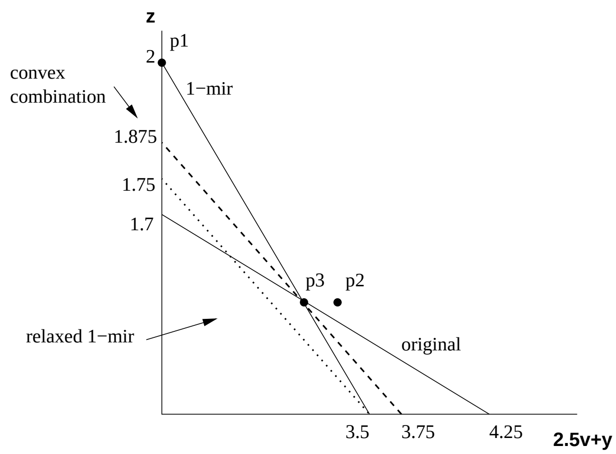

Figure 4: Different relaxations of MIRs

3 Two-step MIR inequalities

In this section we describe how to use the three variable set Q^2 and the two-step MIR inequality (6) to generate valid inequalities for the set Y. What we call the *two-step MIR procedure* is a generalization of the MIR procedure. Let $ax \geq b$ be a valid inequality for a set X. The first step of the procedure is to relax this inequality to obtain a valid inequality of the form

$$a^1x + \alpha\, a^2x + a^3x \geq b$$

where (i) α satisfies $1/\alpha \geq \lceil \hat{b}/\alpha \rceil$ and $\hat{b} > \alpha > 0$, (ii) $a^1x, a^2x \geq 0$, for all $x \in X$ and (iii) $a^2x, a^3x \in Z$ for all $x \in X$. Notice that a^1x can be treated as the non-negative continuous variable v in Q^2; a^2x can be treated as the non-negative integer variable y and $(a^3x - \lfloor b \rfloor)$ can be treated as the integer variable z. The second step of the procedure is to use the two-step MIR inequality for Q^2 to derive the following valid inequality for X:

$$(a^1x) \geq \big(\hat{b} - \alpha\lfloor \hat{b}/\alpha \rfloor\big)\big(\lceil \hat{b}/\alpha \rceil - \big(a^2x\big) - \lceil \hat{b}/\alpha \rceil (a^3x - \lfloor b \rfloor)\big).$$

We next apply this procedure to the set

$$Y \;=\; \Big\{x \in Z^{|J|}, z \in Z \;:\; \sum_{j\in J} a_j\, x_j + z = b, \qquad x \geq 0.\Big\}$$

The equality defining Y can also be written as

$$\sum_{j\in J} \hat{a}_j\, x_j + w \quad = \quad \hat{b}, \tag{11}$$

where $w = \sum_{j\in J} \lfloor a_j \rfloor\; x_j + z - \lfloor b \rfloor$ and $\hat{a}_j = a_j - \lfloor a_j \rfloor$, $\hat{b} = b - \lfloor b \rfloor$. Note that w is integral if x and y are integral.

Let $\alpha \in R$ be such that $\hat{b} > \alpha > 0$, and $1/\alpha \geq \lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$. In addition, assume that $\lceil \hat{b}/\alpha \rceil = 2$. Let $\{J_1, J_2, J_3, J_4\}$ be a partition of the index set J such that (i) if $j \in J_2$, then $\hat{a}_j < \alpha$, and (ii)

if $j \in J_3$, then $\hat{a}_j \geq \alpha$. We can now relax (11) to obtain

$$\sum_{j \in J_1} \hat{a}_j\, x_j + \sum_{j \in J_2} \alpha\, x_j + \sum_{j \in J_3} (\alpha + (\hat{a}_j - \alpha))\, x_j + \sum_{j \in J_4} x_j + w \quad \geq \quad \hat{b}.$$

Rearranging the terms leads to

$$\Big(\sum_{j \in J_1} \hat{a}_j\, x_j + \sum_{j \in J_3} (\hat{a}_j - \alpha)\, x_j \Big) + \alpha \Big(\sum_{j \in J_2 \cup J_3} x_j \Big) + \Big(\sum_{j \in J_4} x_j + w \Big) \quad \geq \quad \hat{b},$$

which resembles the set Q^2 since the first term is non-negative, the second term gives positive integral multiples of α, and the last term is integral. Since we chose α to satisfy $1/\alpha \geq \lceil \hat{b}/\alpha \rceil = 2$, if we define $\rho = \hat{b} - \alpha$, we obtain the following valid inequality for Y based on inequality (6):

$$\sum_{j \in J_1} \hat{a}_j\, x_j + \sum_{j \in J_3} (\hat{a}_j - \alpha)\, x_j \quad \geq \quad \rho \Big(2 - \sum_{j \in J_2 \cup J_3} x_j - 2 \big(\sum_{j \in J_4} x_j + w \big) \Big).$$

Substituting for w and rearranging terms, we obtain $\sum_{j \in J} \gamma_j x_j + 2\,\rho\, z \geq 2\,\rho\, \lceil b \rceil$, where

$$\gamma_j = 2\,\rho\, \lfloor a_j \rfloor + \begin{cases} \hat{a}_j & \text{if } j \in J_1 \\ \rho & \text{if } j \in J_2 \\ (\hat{a}_j - \alpha) + \rho & \text{if } j \in J_3 \\ 2\,\rho & \text{if } j \in J_4. \end{cases}$$

Note that for a fixed α, the strongest inequality of form (12) can be obtained by partitioning the index set J into $J_1^*, J_2^*, J_3^*, J_4^*$ by inspection as follows:

$$J_1^* = \{ j \in J \,:\, \hat{a}_j < \rho \}, \qquad J_2^* = \{ j \in J \,:\, \rho \leq \hat{a}_j < \alpha \},$$
$$J_3^* = \{ j \in J \,:\, \alpha \leq \hat{a}_j < \hat{b} \}, \qquad J_4^* = \{ j \in J \,:\, \hat{a}_j \geq \hat{b} \}.$$

We present an example to illustrate how these inequalities can be applied.

Example 7 *Consider the equation* $1.2x_1 + 3.35x_2 + 2.5x_3 + 0.8x_4 + x_5 = 4.7$ *with all variables non-negative and integral. As in (11), we rewrite this as* $0.2x_1 + 0.35x_2 + 0.5x_3 + 0.8x_4 + w = 0.7$*, where* $w = x_1 + 3x_2 + 2x_3 + x_5 - 4$*. Here* $\hat{b} = 0.7$*; let* $\alpha = 0.4$*. Then* $1/\alpha = 2.5 > \lceil \hat{b}/\alpha \rceil = 2$*, and* $\rho = 0.3$*. Since* x_2 *and* x_4 *are non-negative, the inequality*

$$(0.2x_1 + 0.1x_3) + 0.4(x_2 + x_3) + (x_4 + w) \geq 0.7$$

is a relaxation of the previous equation. Of the three terms in brackets, the first two are non-negative, and the last two are integral; we can thus apply inequality (6) to obtain

$$\frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{2}{3}x_3 + x_4 + w \geq 1$$

as a cutting plane. The w *variable can be eliminated by subtracting the second equation from the above, to get* $(4/9)x_1 + (1/2)x_2 + (5/9)x_3 + (2/3)x_4 \geq 1$.

We next generalize this procedure to the case when $\tau = \lceil \hat{b}/\alpha \rceil \geq 2$. We define $\rho = \hat{b} - \alpha \lfloor \hat{b}/\alpha \rfloor$. Let k_j, l_j be integers such that $k_j \leq \lfloor \hat{a}_j/\alpha \rfloor$, and $l_j \geq \lceil \hat{a}_j/\alpha \rceil$, for $j \in J$.

Let J_0, J_1 and J_2 be sets which form a partition of J. We can relax (11) to obtain

$$\sum_{j\in J_1} (k_j\alpha + (\hat{a}_j - k_j\alpha))\, x_j + \sum_{j\in J_2} l_j\alpha\, x_j + \sum_{j\in J_0} x_j + w \quad \geq \quad \hat{b},$$

which can be rewritten as

$$\sum_{j\in J_1} (\hat{a}_j - k_j\alpha)\, x_j + \alpha\Big(\sum_{j\in J_1} k_j\, x_j + \sum_{j\in J_2} l_j\, x_j\Big) + \Big(\sum_{j\in J_0} x_j + w\Big) \quad \geq \quad \hat{b},$$

Applying inequality (6) and substituting for w leads to inequality

$$\sum_{j\in J} \gamma_j x_j + \rho\,\tau\, z \geq \rho\,\tau\,\lceil b\rceil, \tag{12}$$

where

$$\gamma_j = \rho\,\tau\,\lfloor a_j\rfloor + \begin{cases} \rho\,\tau & \text{if } j\in J_0 \\ k_j\rho\ + \hat{a}_j - k_j\alpha & \text{if } j\in J_1 \\ l_j\rho & \text{if } j\in J_2 \end{cases}$$

By inspection, the strongest inequality of this form is obtained by setting $k_j = k_j^* = \lfloor \hat{a}_j/\alpha\rfloor$ and $l_j = l_j^* = \lceil \hat{a}_j/\alpha\rceil$ for $j \in J$, and letting

$$J_0 = \{j\in J\ :\ \hat{a}_j \geq \hat{b}\},$$

and

$$J_1 = \{j \in J\setminus J_0\ :\ \hat{a}_j - k_j^*\alpha < \rho\}, \qquad J_2 = \{j\in J\setminus J_0\ :\ \hat{a}_j - k_j^*\alpha \geq \rho\}.$$

In other words,

$$\gamma_j = \rho\,\tau\,\lfloor a_j\rfloor + \min\{\rho\,\tau\ ,\ k_j^*\rho\ + \hat{a}_j - k_j^*\alpha\ ,\ l_j^*\rho\ \}.$$

We next define the two-step MIR function and formally state that the two-step MIR inequality is valid for Y.

Definition 8 *Let $b, \alpha \in R$ be such that $\hat{b} > \alpha > 0$, and $1/\alpha \geq \lceil \hat{b}/\alpha\rceil > \hat{b}/\alpha$. Define $\rho = \hat{b} - \alpha\lfloor \hat{b}/\alpha\rfloor$, and $\tau = \lceil \hat{b}/\alpha\rceil$. The* two-step MIR function *for a right-hand side b with parameter α is defined by*

$$g^{b,\alpha}(v) = \begin{cases} \dfrac{\hat{v}(1-\rho\tau) - k(v)(\alpha-\rho)}{\rho\tau(1-\hat{b})} & \text{if } \hat{v} - k(v)\alpha < \rho \\[2ex] \dfrac{k(v) + 1 - \tau\hat{v}}{\tau(1-\hat{b})} & \text{if } \hat{v} - k(v)\alpha \geq \rho, \end{cases}$$

where $k(v) = \min\{\lceil \hat{v}/\alpha\rceil, \tau\} - 1$, and $\hat{v} = v - \lfloor v\rfloor$ and $\hat{b} = b - \lfloor b\rfloor$.

Lemma 9 *For any $\alpha \in R$ that satisfies the conditions in Definition 8, the* two-step MIR inequality *for right-hand side b with parameter α*

$$\sum_{j\in J} g^{b,\alpha}(a_j)\, x_j \geq 1 \tag{13}$$

is valid for Y.

Proof. Substituting for z in inequality (12) and dividing it by $\rho\tau(1-\hat{b})$ leads to the desired inequality. ■

Observe that $g^{b,\alpha}(v)$ is a piecewise linear function with two distinct slopes, $(1-\rho\tau)/(\rho\tau(1-\hat{b}))$ and $-1/(1-\hat{b})$. Also, $1 \geq g^{b,\alpha}(v) \geq 0$, for all $v \in R$, see Figure 5. In [12, 13], Gomory and Johnson describe a family of piecewise linear "two-slope" functions, containing $g^{b,\alpha}(v)$, which yields valid inequalities for Y. Lemma 9 shows that, of the two-slope functions in [12], the ones of the form $g^{b,\alpha}(v)$ can be derived from the the two-step MIR inequalities.

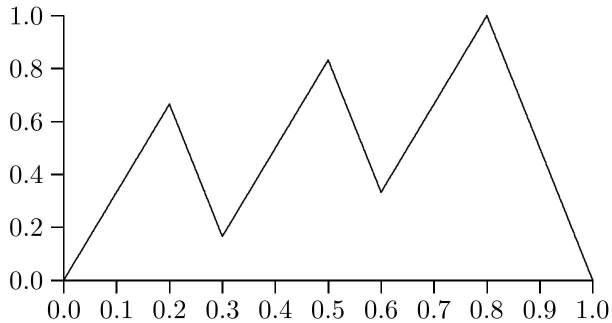

Figure 5: $g^{b,\alpha}(v)$ when $\hat{b} = 0.8, \alpha = 0.3$ and $1/\alpha > \left\lceil \hat{b}/\alpha \right\rceil = 3$.

We have already mentioned in the discussion of the set Q^2 that if $1/\alpha = \lceil \hat{b}/\alpha \rceil$, the inequality (6) reduces to a (scaled) MIR inequality, and it is not surprising that (6) applied to Y yields the t-MIR cuts for Y in (4).

Lemma 10 *Let $\alpha \in R$ satisfy the conditions in Definition 8. If $1/\alpha = \lceil \hat{b}/\alpha \rceil$, then the two-step MIR inequality (13) is precisely the $(1/\alpha)$-scaled MIR cut for Y.*

Proof. The function $g^{b,\alpha}(v)$ is identical to $f^{t,b}(v)$, with $t = 1/\alpha$. ■

It is possible to write t-scaled two-step MIR inequalities for Y by scaling the initial equality by $t \in Z$ as discussed in Section 1.2. For the sake of completeness, we next present the t-scaled two-step MIR inequalities for Y. As before, we use $\hat{a}_j^t$ and $\hat{b}^t$ to denote $ta_j - \lfloor ta_j \rfloor$, and $tb - \lfloor tb \rfloor$, respectively. Note that $1 > \hat{a}_j^t, \hat{b}^t \geq 0$ even when $t < 0$.

Lemma 11 *Let $t \in Z$ and $\alpha \in R$ be such that $\hat{b}^t > \alpha > 0$, and $1/\alpha \geq \lceil \hat{b}^t/\alpha \rceil > \hat{b}^t/\alpha$. Then, the t-scaled two-step MIR inequality $\sum_{j \in J} g^{\hat{b}^t,\alpha}(\hat{a}_j^t)x_j \geq 1$ (or, equivalently, $\sum_{j\in J} g^{tb,\alpha}(ta_j)x_j \geq 1$) is valid for Y.* ■

4 Extended two-step MIR inequalities

In this section we derive new valid inequalities by considering the limiting behavior of the two-step MIR function $g^{b,\alpha}$. For a given right-hand side b, the set of admissible values of α that give valid inequalities for Y is not very clear from Definition 8. Let $b \in R$ be fixed and let $\mathcal{I}$ denote the set of values of α that satisfy the conditions of Definition 8. Furthermore, let $\mathcal{I} = \cup_{d=2}^{\infty} \mathcal{I}^d$, where $\mathcal{I}^d$ is the set of values of α such that $\lceil \hat{b}/\alpha \rceil = d$. For $\alpha \in \mathcal{I}^d$ we have

a. $\lceil \hat{b}/\alpha \rceil = d \Rightarrow \alpha \in [\hat{b}/d,\ \hat{b}/(d-1))$,

b. $1/\alpha \geq \lceil \hat{b}/\alpha \rceil \Rightarrow \alpha \leq 1/d$,

c. $\lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha \Rightarrow \alpha > \hat{b}/d$.

Combining these, it is easy to see that

$$\mathcal{I}^d = (\hat{b}/d,\ 1/d] \cap (\hat{b}/d,\ \hat{b}/(d-1)).$$

Notice that $\mathcal{I}^d \neq \emptyset$ for all $d \geq 2$. Also, $\alpha = 1/d$ is an admissible choice if and only if $1/d < \hat{b}/(d-1)$. For any $d \geq 2$, we have,

$$\frac{1}{d} < \frac{\hat{b}}{d-1} \Leftrightarrow \frac{d-1}{d} < \hat{b} \Leftrightarrow 1 - \hat{b} < \frac{1}{d} \Leftrightarrow d < \frac{1}{1-\hat{b}}.$$

Let $\bar{d}$ be the largest integer strictly less than $1/(1-\hat{b})$, that is, $\bar{d} = \left\lceil 1/(1-\hat{b}) \right\rceil - 1$. Then, $\alpha = 1/d$ is an admissible choice in Lemma 9 and Lemma 10 only when $2 \leq d \leq \bar{d}$. Also,

$$\mathcal{I}^d = \begin{cases} (\hat{b}/d, 1/d] & \text{if } 2 \leq d \leq \bar{d} \\ (\hat{b}/d, \hat{b}/(d-1)) & \text{if } d > \bar{d}. \end{cases} \tag{14}$$

Let $\mathcal{E} = \{\hat{b}/d \,:\, d \geq \bar{d}, d \in Z\}$ and notice that

$$\cup_{d=\bar{d}}^{\infty} \mathcal{I}^d = (0, 1/\bar{d}] \setminus \mathcal{E}.$$

Next, we derive *extended two-step MIR inequalities* for Y, by considering the limiting behavior of the function $g^{b,\alpha}$ when α tends to a point p in the set $\mathcal{E}$. We separately consider the case when α converges to p from below and from above, as the limits are different. We use $\epsilon \to 0^+$ to denote that ϵ takes only positive values as it tends to 0.

Definition 12 *Let $b \in R$ be such that $\hat{b} > 0$. Let d be an integer such that $d \geq \left\lceil 1/(1-\hat{b}) \right\rceil - 1$. The* extended two-step MIR function *for a right-hand side b with parameter d is defined by*

$$h^{b,d}(v) = \begin{cases} \dfrac{\hat{v}}{\hat{b}} & \textit{if } v \textit{ is an integral multiple of } \hat{b}/d \textit{ and } \hat{v} < \hat{b}, \\[2ex] \dfrac{l(v) + 1 - (d+1)\hat{v}}{(d+1)(1-\hat{b})} & \textit{otherwise}, \end{cases}$$

where $l(v) = \min\{\lfloor d\hat{v}/\hat{b} \rfloor, d\}$.

See Figure 6 for the extended two-step MIR function for a right-hand side 0.6 with parameter 3.

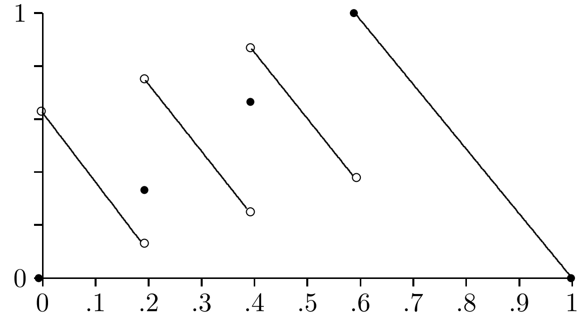

Figure 6: A plot of function $h^{0.6,3}(v)$ for $v \in [0,1]$

Lemma 13 *For any integer d satisfying the conditions of Definition 12, and for any $v \in R$,*

$$\lim_{\epsilon \to 0^+} g^{b,(\hat{b}/d)-\epsilon}(v) = h^{b,d}(v).$$

Proof. Define $\alpha = \hat{b}/d - \epsilon$. The condition that ϵ tends to 0 from above is equivalent to the condition that α tends to $\hat{b}/d$ from below. Now, for $\epsilon > 0$,

$$\frac{\hat{b}}{\hat{b}/d - \epsilon} = d + \delta_1, \;\; \frac{\hat{v}}{\hat{b}/d - \epsilon} = \frac{d\hat{v}}{\hat{b}} + \delta_2,$$

for some positive δ_1 and δ_2. Therefore, there is a fixed number $\bar{\epsilon} > 0$, such that

$$0 < \epsilon < \bar{\epsilon} \Rightarrow \; \left\lceil \hat{b}/\alpha \right\rceil = d + 1, \;\; \lceil \hat{v}/\alpha \rceil = \left\lfloor d\hat{v}/\hat{b} \right\rfloor + 1. \tag{15}$$

Let ρ, τ and $k(v)$ be defined as in Definition 8 in terms of b and α. Then, as α tends to $\hat{b}/d$ from below,

$$(i)\, \rho \to 0, \;\; (ii)\, \tau \to d + 1, \;\; (iii)\, k(v) \to l(v). \tag{16}$$

We consider three cases for the point $\hat{v} \in [0, 1)$.
Case 1: $\hat{v}$ is not an integral multiple of $\hat{b}/d$. Then $\hat{v} - k(v)\alpha$ tends to a positive number, but ρ tends to zero. Therefore, once α crosses a threshold $\bar{\alpha}$ close enough to $\hat{b}/d$, $\hat{v} - k(v)\alpha \geq \rho$ and the value of $g^{b,\alpha}(v)$ is given by the second case in Definition 8. Using the limits in (16), we see that the lemma holds for Case 1.
Case 2: $\hat{v}$ is an integral multiple of $\hat{b}/d$ and $\hat{v} < \hat{b}$. Let $\hat{v} = t\hat{b}/d$ for some integer t, where $t < d$. Using the $\bar{\epsilon}$ defined in (15), we have

$$\begin{aligned} 0 < \epsilon < \bar{\epsilon} \Rightarrow \quad \hat{v} - k(v)\alpha = (t\hat{b}/d) - t(\hat{b}/d - \bar{\epsilon}) = &\quad t\epsilon, \\ \rho = b - \left\lfloor \hat{b}/\alpha \right\rfloor \alpha = b - d(\hat{b}/d - \epsilon) = &\quad d\epsilon. \end{aligned} \tag{17}$$

Therefore $\hat{v} - k(v)\alpha < \rho$ for $\epsilon < \bar{\epsilon}$, and $g^{b,\alpha}(v)$ is given by the first case in Definition 8 when α is close enough to $\hat{b}/d$. That is,

$$g^{b,\alpha}(v) = \frac{\hat{v}(1 - \rho\tau) - k(v)(\alpha - \rho)}{\rho\tau(1 - \hat{b})}.$$

First, we rewrite the right-hand side of the above expression as

$$\frac{k(v) - \tau\hat{v} + (\hat{v} - k(v)\alpha)/\rho}{\tau(1-\hat{b})},$$

by dividing the numerator and denominator by ρ. From (17), we know that $(\hat{v} - k(v)\alpha)/\rho = t/d = \hat{v}/\hat{b}$ as ϵ tends to 0. This combined with (16) shows that

$$\lim_{\epsilon\to 0^+} \frac{\hat{v}(1-\rho\tau) - k(v)(\alpha - \rho)}{\rho\tau(1-\hat{b})} = \frac{l(v) + (\hat{v}/\hat{b}) - (d+1)\hat{v}}{(d+1)(1-\hat{b})}. \tag{18}$$

Using the fact that $l(v) = t = d\hat{v}/\hat{b}$, the numerator of the right-hand side of (18) becomes $(d+1)(1-\hat{b})(\hat{v}/\hat{b})$. Dividing out the common factors, the right-hand side becomes simply $\hat{v}/\hat{b}$.
Case 3: $\hat{v}$ is an integral multiple of $\hat{b}/d$ and $\hat{v} \geq \hat{b}$. Let $\hat{v} = t(\hat{b}/d)$ where t is an integer. We argued in the previous case that for sufficiently small ϵ, $\hat{v} - k(v)\alpha$ becomes $t\epsilon$, and ρ becomes $d\epsilon$. As $t \geq d$, this means that $g^{b,\alpha}(v)$ is given by the second case in Definition 8, and we can argue, as in Case 1, that the lemma holds. ∎

To complete the discussion on the limiting behavior of $g^{b,\alpha}$, we note that for any integer $d \geq \bar{d}$,

$$\lim_{\epsilon\to 0^+} g^{b,(\hat{b}/d)+\epsilon}(v) = f^{1,b}(v),$$

for any $v \in R$. In other words, the two-step MIR function converges to the (1-scaled) MIR function when α in $g^{b,\alpha}$ converges to a point in $\mathcal{E}$ from above.

Let $\{a_i\} \subseteq R^n$ be a sequence converging to the vector a, and let $\{b_i\} \subseteq R$ be a sequence of numbers converging to b, such that $a_i^T x \leq b_i$ is valid for a polyhedron P, for all $i > 0$. Then it is easy to show that $a^T x \leq b$ is also valid for P. Combining this observation with Lemma 13 we can show the following result.

Lemma 14 *If d is a positive integer that satisfies the conditions in Definition 12, the extended two-step MIR inequality*

$$\sum_{j\in J} h^{b,d}(a_j)\, x_j \geq 1$$

is valid for Y.

Proof. Assume d is an integer satisfying the conditions in Definition 12. Let $\{\epsilon_i\}$ be a sequence of positive numbers converging to zero. Also assume that $\hat{b}/d - \epsilon_i > \hat{b}/(d+1)$ for all positive integers i. From (14), it follows that $\hat{b}/d - \epsilon_i$ is a valid choice for α in Definition 8. Then, we know from Lemma 9 that $\sum_{j\in J} g^{b,\hat{b}/d-\epsilon_i}(\hat{a}_j)x_j \geq 1$ is a valid inequality for Y and $conv(Y)$ for all ϵ_i. The discussion on the limiting behavior of inequalities above and Lemma 13 imply the result. ∎

Corollary 15 *Let t be an arbitrary integer, and let d be a positive integer that satisfies the conditions in Definition 12 with b replaced by tb. Then the following inequality, called the t-scaled extended two-step MIR inequality, is valid for Y:*

$$\sum_{j\in J} h^{tb,d}(ta_j)\, x_j \geq 1 \tag{19}$$

4.1 The strong fractional cut of Letchford and Lodi

Consider the set

$$Y^+ \;=\; \Big\{x \in Z^{|J|}, z \in Z \;:\; z + \sum_{j\in J} a_j\, x_j = b, \quad x \geq 0\ , z \geq 0\Big\},$$

which is a restriction of the set Y obtained by requiring the z variable to be non-negative. In a recent paper, Letchford and Lodi [15] present a valid inequality for Y^+ which they call the *strong fractional cut.* Their inequality dominates the so-called *Gomory fractional cut*

$$\sum_{i\in J} \hat{a}_i\, x_i \geq \hat{b}. \tag{20}$$

It is well known that the GMIC also dominates inequality (20) and Letchford and Lodi [15] state that their inequality neither dominates nor is dominated by the GMIC. In this section we show that their inequality is dominated by the extended two-step MIR inequalities.

For convenience, we first present their main result in our notation.

Theorem 16 *(Letchford and Lodi [15]) Suppose $\hat{b} > 0$ and let $k \geq 1$ be the unique integer such that $\frac{1}{k+1} \leq \hat{b} < \frac{1}{k}$. Partition J into classes $Q_0, \ldots, Q_k$ as follows. Let $Q_0 = \{i \in J : \hat{a}_i \leq \hat{b}\}$ and, for $p = 1, \ldots, k$, let $Q_p = \{i \in J : \hat{b} + (p-1)(1-\hat{b})/k < \hat{a}_i \leq \hat{b} + p(1-\hat{b})/k\}$. Then the following* strong fractional cut*,*

$$\sum_{i\in Q_0} \hat{a}_i x_i + \sum_{p=1}^{k} \sum_{i\in Q_p} (\hat{a}_i - p/(k+1)) x_i \geq \hat{b}. \tag{21}$$

is valid for Y. ∎

Notice that inequality (21) can also be written as

$$\sum_{i\in J} \left(\hat{a}_i - \frac{1}{k+1} \max\{0, \left\lceil \frac{k(\hat{a}_i - \hat{b})}{1-\hat{b}} \right\rceil\} \right) x_i \geq \hat{b} \tag{22}$$

since for all $p = 1, \ldots, k$, we have

$$i \in Q_p \Leftrightarrow \frac{k(\hat{a}_i - \hat{b})}{1-\hat{b}} \in (p-1, p] \Leftrightarrow p = \left\lceil \frac{k(\hat{a}_i - \hat{b})}{1-\hat{b}} \right\rceil.$$

Also notice that

$$\frac{1}{k+1} \leq \hat{b} < \frac{1}{k} \;\Leftrightarrow\; k+1 \geq \frac{1}{\hat{b}} > k \;\Leftrightarrow\; k = \left\lceil \frac{1}{\hat{b}} \right\rceil - 1. \tag{23}$$

We now show that the (-1)-scaled extended 2-step MIR inequality with parameter k dominates inequality (22). First consider a relaxation of $h^{b,d}$ defined by:

$$\tilde{h}^{b,d}(v) = \frac{l(v) + 1 - (d+1)\hat{v}}{(d+1)(1-\hat{b})},\ \text{where } l(v) = \min\{\left\lfloor d\hat{v}/\hat{b} \right\rfloor, d\},\ \forall v \in R.$$

Notice that $h^{b,d}(v) = \tilde{h}^{b,d}(v)$ for all v unless $\hat{v}$ is an integer multiple of $\hat{b}/d$ and $\hat{v} < \hat{b}$. When $\hat{v}$ is an integer multiple of $\hat{b}/d$ and $\hat{v} < \hat{b}$, then $h^{b,d}(v) = \hat{v}/\hat{b}$ which is shown to be equal to the right-hand side of (18) in Case 2 of the proof of Lemma 13. This expression is almost identical to $\tilde{h}^{b,d}(v)$, except that 1 in the numerator is replaced by $\hat{v}/\hat{b} < 1$, and therefore $h^{b,d}(v) < \tilde{h}^{b,d}(v)$ for these points.

Now, multiply the equation defining the set Y^+ by -1 and apply $\tilde{h}^{-b,k}$, where k is given by (23), to get the valid inequality $\sum_{j\in J} \tilde{h}^{-b,k}(-a_j)x_j \geq 1$. As $(-a_j) - \lfloor -a_j \rfloor = 1 - \hat{a}_j$ and $(-b) - \lfloor -b \rfloor = 1 - \hat{b}$, this is:

$$\sum_{i\in J} \frac{\min\{k, \left\lfloor \frac{k(1-\hat{a}_i)}{1-\hat{b}} \right\rfloor\} + 1 - (k+1)(1-\hat{a}_i)}{(k+1)\hat{b}} x_i \geq 1$$

By simple algebraic transformations, the above inequality is the same as the inequalities

$$\sum_{i\in J} \frac{1}{k+1}\left(\min\{0, \left\lfloor \frac{k(1-\hat{a}_i)}{1-\hat{b}} \right\rfloor - k\} + k + 1 - (k+1)(1-\hat{a}_i)\right) x_i \geq \hat{b},$$

$$\sum_{i\in J} \left(\frac{1}{k+1}\min\{0, \left\lfloor \frac{k(1-\hat{a}_i) - k(1-\hat{b})}{1-\hat{b}} \right\rfloor\} + \hat{a}_i\right) x_i \geq \hat{b},$$

$$\sum_{i\in J} \left(\hat{a}_i - \frac{1}{k+1}\max\{0, \left\lceil \frac{k(\hat{a}_i - \hat{b})}{1-\hat{b}} \right\rceil\}\right) x_i \geq \hat{b},$$

and is therefore the same as inequality (22).

Recall $\bar{d}$ in Definition 12. Since we are working with the equation defining Y^+ scaled by -1, k in (23) is actually equal to $\bar{d}$. This implies the following result:

Proposition 17 *The strong fractional cut (21) of Theorem 16 is dominated by the (-1)-scaled extended two-step MIR inequality (19). Furthermore, inequality (21) is valid for all* $k \geq \left\lceil \frac{1}{\hat{b}} \right\rceil - 1$.

5 Gomory's cyclic group polyhedra

In this section, we apply the (scaled) MIR and two-step MIR procedures to master cyclic group polyhedra and show that the resulting valid inequalities are facet defining. Gomory introduced cyclic group polyhedra via group relaxations, see [10]. A group relaxation of an integer programming problem is associated with a basic solution of its linear programming relaxation, and is obtained by relaxing the non-negativity constraints on the basic variables. A single master cyclic group polyhedron encapsulates information about group relaxations of many different integer programs.

The master cyclic group polyhedron

$$P(n,r) = conv\{w \in Z^{n-1} \;:\; \sum_{i=1}^{n-1}(i/n)w_i + z = r/n, \quad w_i \geq 0, \; z \in Z\},$$

where $n, r \in Z$ and $n > r > 0$, is closely related to the set Y. A given Y can be viewed as a face (after projecting out the z variable) of some $P(\bar{n}, \bar{r})$, where $\bar{n}$ is a positive integer such that all coefficients of $\sum_{j \in J} a_j x_j + y = b$ become integral when multiplied by $\bar{n}$, and $\bar{r} = \bar{n}\hat{b}$, see Gomory [10]. What we call $P(n, r)$ is defined using a modular equation in [10], and is called $P(C_n, r)$ where C_n stands for the cyclic group of integers modulo n.

Therefore facets of $P(n, r)$ yield cutting planes for Y in the following way:

$$\sum_{i=1}^{n-1} \eta_i w_i \geq 1 \text{ is a facet of } P(n, r) \Rightarrow \sum_{k=1}^{|J|} \eta_{n\hat{a}_k} x_k \geq 1 \text{ is valid for } Y \tag{24}$$

where we define $\eta_0 = 0$. Because of (24), knowledge of one master polyhedron can be used to generate cutting planes for many different integer programs. This is the central reason why master polyhedra are interesting.

Gomory provided an elegant characterization of the nontrivial facets of $P(n, r)$.

Theorem 18 *(Gomory [10]) If $r \neq 0$, then $\sum_{j=1}^{n-1} \eta_j w_j \geq 1$ is a nontrivial facet of $P(n, r)$ if and only if $\eta = (\eta_j)$ is an extreme point of the inequality system*

$$\eta_i + \eta_j \geq \eta_{(i+j) \bmod n} \quad \forall i, j \in \{1, \ldots, n-1\}, \tag{25}$$

$$\eta_i + \eta_j = \eta_r \quad \forall i, j \text{ such that } r = (i + j) \bmod n, \tag{26}$$

$$\eta_j \geq 0 \quad \forall j \in \{1, \ldots, n-1\}, \tag{27}$$

$$\eta_r = 1. \tag{28}$$

5.1 Scaled MIR facets

In this section we apply t-MIR cuts to the cyclic group polyhedron $P(n, r)$. For a positive integer n and integers t and i, define

$$\mu_n^t(i) = ti \bmod n$$

(here, $k \bmod n$ stands for $k - n\lfloor k/n \rfloor$ which lies between 0 and $n - 1$). Therefore the t-scaled MIR inequality (4), when applied to $P(n, r)$, becomes

$$\sum_{\mu_n^t(i) < \mu_n^t(r)} \frac{\mu_n^t(i)}{\mu_n^t(r)} x_i + \sum_{\mu_n^t(i) \geq \mu_n^t(r)} \frac{n - \mu_n^t(i)}{n - \mu_n^t(r)} x_i \geq 1. \tag{29}$$

Gomory [10] showed that the 1-MIR cut in (29) is a facet of $P(n, r)$. We show that for every non-zero integer t, such that tr is not a multiple of n, the t-scaled MIR cut (29) is a facet of $P(n, r)$.

For this, we need two results of Gomory on the facets of $P(n, r)$. The first is expressed in terms of automorphisms of C_n, the cyclic group of integers modulo n. In this context, an automorphism ϕ is a bijection from the set $\{0, 1, \ldots, n-1\}$ to itself with the property that $\phi((a + b) \bmod n) = (\phi(a) + \phi(b)) \bmod n$. It is well-known that if t is coprime with n (n and t have no common divisors), then ϕ defined by $\phi(i) = \mu_n^t(i)$ is an automorphism, and every automorphism arises this way. Also, the inverse of an automorphism is also an automorphism; if $\phi(i) = \mu_n^t(i)$, then $\phi^{-1}(i) = \mu_n^u(i)$ where u satisfies $tu \equiv 1 \pmod n$ (such a u exists as t and n are coprime). The next result is a restatement of Theorem 14 in Gomory[10] in a form convenient for us.

Theorem 19 *(Gomory [10]) Let r be an integer such that $0 < r < n$. Let ϕ be an automorphism defined by $\phi(i) = \mu_n^t(i)$, and let $s = \phi(r)$. If $\sum_i \eta_i x_i \geq 1$ is a non-trivial facet of $P(n,s)$, then $\sum_i \eta_{\phi(i)} x_i \geq 1$ is a facet of $P(n,r)$.*

Theorem 19 above expresses the fact that $P(n,r)$ and $P(n,\phi(r))$ are essentially identical polytopes when ϕ is an automorphism. The facets of one polytope correspond to the facets of the other via a permutation of facet coefficients. Also observe that scaling the defining equation of $P(n,r)$ by t, where t and n are coprime, corresponds to mapping $P(n,r)$ to $P(n,\mu_n^t(r))$.

The next result describes how to get facets of $P(n,r)$ from facets of $P(m,s)$ where m is a divisor of n, and s is an appropriate integer with $0 < s < m$.

Theorem 20 *(Gomory [10]) Let m be a divisor of n but not of r, where $0 < r < n$, and let $s = r \bmod m$. If $\sum_{i=1}^{m-1} \eta_i x_i \geq 1$ is a non-trivial facet of $P(m,s)$, define $\eta' \in R^{n-1}$ by $\eta'_i = \eta_{i \bmod m}$, where η_0 is defined to be zero. Then $\sum_{i=1}^{n-1} \eta'_i x_i \geq 1$ is a facet of $P(n,r)$.*

Using Theorems 19 and 20, we now show that the t-scaled MIR inequality (29) is facet defining for the cyclic group polytope.

Theorem 21 *For every integer $t = 1, 2, \ldots, \lfloor n/2 \rfloor$, such that tr is not an integral multiple of n, the t-MIR cut defines a facet of $P(n,r)$.*

Proof. We consider two cases.

Case 1: Let t and n be coprime. Let $\phi(i) = \mu_n^t(i)$, and let $s = \phi(r)$. Let $\eta^T y \geq 1$ represent the 1-MIR cut for $P(n,s)$; it is a facet of $P(n,s)$. Applying Theorem 19 with ϕ defined as above, we see that $\sum_i \eta_{\phi(i)} x_i \geq 1$ defines a facet of $P(n,r)$. From the definition of η and ϕ, it follows that $\sum_i \eta_{\phi(i)} x_i \geq 1$ is the same as the t-MIR cut in (29).

Case 2: Let t and n have common divisors. Let $d = gcd(n,t)$ and assume tr is not divisible by n. Let $m = n/d, v = t/d$, and let $s = r \bmod m$. Now, v and m are coprime. Also, $s \neq 0$; otherwise $m|r \Rightarrow n|dr$ and therefore $n|tr$. Let $\eta^T x \geq 1$ be the v-MIR cut for $P(m,s)$. We know from Case 1 that it is a facet of $P(m,s)$. We use Theorem 20 to map $\eta^T x \geq 1$ to a facet of $P(n,r)$ which is precisely the t-MIR cut for $P(n,r)$. Observe that

$$\mu_m^v(i \bmod m) = (v(i \bmod m)) \bmod m = vi \bmod m = \mu_m^v(i).$$

Therefore $\mu_m^v(r) = \mu_m^v(s)$. Applying Theorem 20 to $\eta^T x \geq 1$ (m, n, r and s satisfy the conditions of the theorem), we see that

$$\sum_{\mu_m^v(i) < \mu_m^v(r)} \frac{\mu_m^v(i)}{\mu_m^v(r)} x_i + \sum_{\mu_m^v(i) \geq \mu_m^v(r)} \frac{m - \mu_m^v(i)}{m - \mu_m^v(r)} x_i \geq 1 \tag{30}$$

is a facet of $P(n,r)$. Now, $d\mu_m^v(i) = d(vi \bmod m) = dvi \bmod dm = ti \bmod n = \mu_n^t(i)$. Hence,

$$\frac{\mu_m^v(i)}{\mu_m^v(r)} = \frac{d\mu_m^v(i)}{d\mu_m^v(r)} = \frac{\mu_n^t(i)}{\mu_n^t(r)}, \text{ and } \frac{m - \mu_m^v(i)}{m - \mu_m^v(r)} = \frac{d(m - \mu_m^v(i))}{d(m - \mu_m^v(r))} = \frac{n - \mu_n^t(i)}{n - \mu_n^t(r)}\ .$$

Therefore (30) is precisely the t-MIR cut for $P(n, r)$. ■

From the proof of Theorem 21, the t-scaled MIR can be viewed as a combination of: (a) scaling the defining equation of $P(n, r)$, (b) mapping the scaled equation to the defining equation of $P(n, \mu_n^t(r))$, if t and n are coprime, or to the defining equation of $P(m, \mu_m^v(s))$ otherwise, (c) writing down a facet of $P(n, \mu_n^t(r))$ or $P(m, \mu_m^v(s))$, and (d) mapping the resulting inequality to an inequality for $P(n, r)$.

Gomory, Johnson and Evans [14] present results based on a *shooting experiment* of Gomory and identify the "important" facets of the cyclic group polyhedra for small n (i.e., $n \leq 20$). In particular, they present coefficients of 13 important facet defining inequalities for $P(10, 9)$, $P(20, 19)$, $P(15, 14)$, and $P(13, 12)$. Intriguingly, all of these facets are scaled MIR facets. In Table 5.1, we list these inequalities with the corresponding master polyhedra, and their scaling parameter.

		Relative importance of facet			
Polyhedron	Reference in [14]	1	2	3	4
P(10,9)	Table 2	5-MIR	2-MIR	4-MIR	-
P(20,19)	Table 3	10-MIR	5-MIR	4-MIR	-
P(15,14)	Figures 5-8	5-MIR	3-MIR	6-MIR	1-MIR
P(13,12)	Figures 9-11	1-MIR	2-MIR	6-MIR	-

Table 1: Important group facets are scaled MIR

5.2 Two-step MIR facets

When applied to $P(n, r)$, the two-step MIR inequalities yield a wide range of facets. In [3], the authors present a class of facets of $P(n, r)$ which they call "2slope" facets. We next show that these 2slope facets are $(n-1)$-scaled (or (-1)-scaled) two-step MIR inequalities for appropriate choices of α. Initially we consider 1-scaled inequalities and present a result that generalizes Theorem 3.5 of [3]. We refer to the facets in Theorem 22 as *general two-slope* facets, in keeping with the notation in [3].

Theorem 22 *Let $\Delta \in Z^+$ be such that $r > \Delta > 0$. The two-step MIR inequality*

$$\sum_{i=1}^{n-1} g^{r/n, \Delta/n}(i/n)\, x_i \geq 1$$

defines a facet of $P(n, r)$ provided that $n > \Delta \lceil r/\Delta \rceil > r$. We call the corresponding facet the 1-scaled two-step MIR facet of $P(n, r)$ with parameter Δ.

Proof. For $i \in I = \{1, 2, \ldots, n-1\}$ let $\eta_i = g^{r/n, \Delta/n}(i/n)$ so that the two-step MIR inequality can be written as $\sum_{i=1}^{n-1} \eta_i x_i \geq 1$.

First note that $\alpha = \Delta/n$ satisfies the conditions of Lemma 9: (i) $r > \Delta > 0 \Rightarrow \hat{b} = r/n > \alpha > 0$, and (ii) $n > \Delta \lceil r/\Delta \rceil > r \Rightarrow n/\Delta > \lceil r/\Delta \rceil > r/\Delta \Rightarrow 1/\alpha > \lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$. Therefore, the inequality is valid for $P(n,r)$. To prove that it is facet defining, we show that η is an extreme point of the set defined by (25)-(28) in Theorem 18.

Let $\Omega = r - \Delta\lfloor r/\Delta\rfloor$, $k(i) = \min\{\lceil i/\Delta\rceil, \lceil r/\Delta\rceil\} - 1$ and $p(i) = i - k(i)\Delta$. It is easy to check that for $i \in I \setminus \{1\}$, (i) if $p(i) \le \Omega$ then $\eta_{i-1} + \eta_1 = \eta_i$; on the other hand, (ii) if $p(i) > \Omega$ then $\eta_i + \eta_{n-1} = \eta_{i-1}$. These $n-2$ equations together with $\eta_r = 1$ form a set of $n-1$ linearly independent equations from (25)-(28). Combined with the fact that η satisfies (25)-(28), this implies that η is indeed extreme and therefore the two-step MIR inequality is indeed facet defining for $P(n,r)$. ■

Figure 7 shows two distinct two-step MIR facets for $P(10,7)$ obtained by letting $\Delta = 3$ and $\Delta = 4$. The solid line in Figure 7(a) represents the piecewise linear function $g^{0.7,0.4}(v)$ and the line in Figure 7(b) represents the function $g^{0.7,0.3}(v)$. The marked points denote the coefficients of the corresponding facet-defining inequalities $\sum_{i=1}^{9} g^{0.7,0.4}(i/10)x_i \ge 1$ and $\sum_{i=1}^{9} g^{0.7,0.3}(i/10)x_i \ge 1$, respectively.

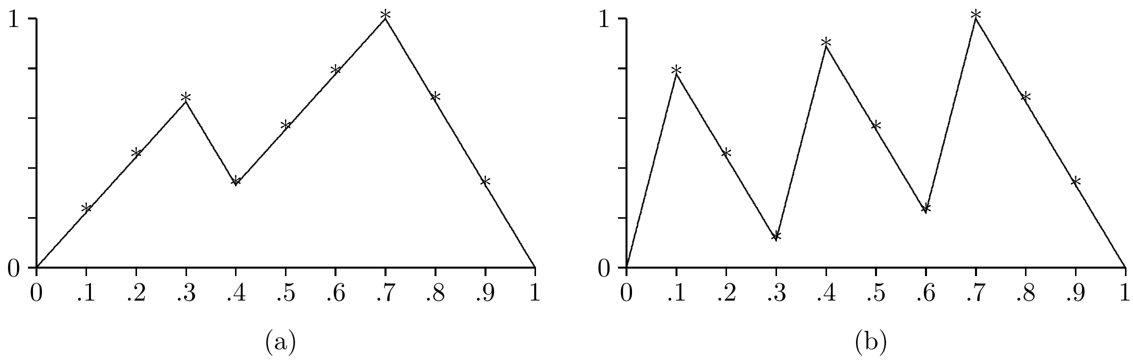

Figure 7: Two different two-step MIR facets for $P(10,7)$.

We next show that t-scaled two-step MIR inequalities define facets of $P(n,r)$ under some conditions. Remember that $\mu_n^t(i) = ti \bmod n$ for integers i.

Theorem 23 *Let t, n, r be integers such that $0 < r < n$ and tr is not divisible by n. Let $\Delta \in Z^+$ be such that $\Delta = tk \bmod n$ for some integer k. The t-scaled two-step MIR inequality*

$$\sum_{i=1}^{n} g^{tr/n,\Delta/n}(ti/n)\, x_i \ge 1$$

defines a facet of $P(n,r)$ provided that $\mu_n^t(r) > \Delta > 0$, and $n > \Delta\lceil \mu_n^t(r)/\Delta\rceil > \mu_n^t(r)$.

Proof. As in the proof of Theorem 21, we consider two cases.
Case 1: Let t and n be coprime. Define an isomorphism ϕ by $\phi(i) = \mu_n^t(i)$, and let $s = \phi(r)$. Observe that Δ satisfies the conditions in Theorem 22, when r in Theorem 22 is replaced by s. Let $\eta^T x \ge 1$ stand for the 1-scaled two-step MIR facet of $P(n,s)$ with parameter Δ. We know from Theorem 19 that $\sum_i \eta_{\phi(i)} x_i \ge 1$ defines a facet of $P(n,r)$. It is easy to see that the two-step MIR function satisfies

$$g^{b+p,\alpha}(v+q) = g^{b,\alpha}(v) \text{ for all integers } p, q, \tag{31}$$

and therefore $\sum_i \eta_{\phi(i)} x_i \ge 1$ is precisely the t-scaled two-step MIR inequality in the theorem.

Case 2: Let t and n have common divisors, and let $d = gcd(n, t)$. Define $r' = r \bmod (n/d)$. Note that $r' \neq 0$, otherwise tr is divisible by n. Define $\Delta' = \Delta/d$, and let $s = \mu_n^t(r)/d$. The conditions of the theorem imply that

$$s > \Delta' > 0, \text{ and } n/d > \Delta' \lfloor s/\Delta' \rfloor > s.$$

We also have

$$\mu_{n/d}^{t/d}(r') = (tr/d) \bmod (n/d) = \mu_n^t(r)/d = s.$$

As t/d and n/d are coprime, we know from Case 1 that the (t/d)-scaled two-step MIR inequality with parameter Δ' defines a facet of $P(n/d, r')$; denote the inequality by $\sum_{i=1}^{n/d-1} \eta_i x_i \geq 1$. Theorem 20 implies that $\sum_{i=1}^{n-1} \eta_{i \bmod (n/d)}\, x_i \geq 1$ is a facet of $P(n, r)$. Using the definition of η_i, Δ', r' and s, we see that

$$\eta_{i \bmod (n/d)} = g^{\frac{tr'/d}{n/d}, \frac{\Delta/d}{n/d}} \left((t/d) \frac{i \bmod (n/d)}{n/d} \right).$$

The right-hand side of the above expression is just $g^{tr/n, \Delta/n}(ti/n)$. This follows from (31) and the fact that for any integer j, $((tj/d) \bmod (n/d))/(n/d) = (tj/n) \bmod 1$ which is just tj/n plus some integer. The theorem follows. ∎

In Figure 8 we display the coefficients of a facet of $P(10, 9)$ obtained by setting $\Delta = 4$ and $t = 3$ in Theorem 23. We also plot the underlying the 3-scaled two-step MIR function.

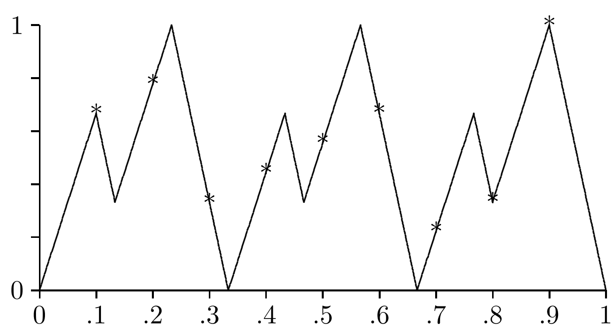

Figure 8: 3-scaled facet of $P(10, 9)$.

Since $n - 1$ and n never have common divisors besides 1, the first case in the proof of Theorem 23 applies to $(n-1)$-scaled two-step MIR inequalities for $P(n, r)$, which are the same as (-1)-scaled two-step MIR inequalities. Since $\mu_n^{n-1}(i) = n - i$, and $\mu_n^{n-1}(r) = n - r$, the coefficient of x_i in the inequality is $g^{(n-r)/n, \Delta/n}((n - i)/n)$. Based on this observation, it is not difficult to verify the following.

Corollary 24 *The 2slope facets of $P(n, r)$ in [3, Theorem 3.4] with parameter d are $(n-1)$-scaled two-step MIR facets of $P(n, r)$ with parameter $\Delta = n - d$ and $\tau = \lceil (n - r)/\Delta \rceil = 2$.*

5.3 Mixed-integer case

We also note that, there is a one-to-one correspondence between the facets of $P(n, r)$ and the facets of the mixed-integer set

$$P'(n,r) = \{v^+, v^- \in R,\ w \in Z^{n-1}\ :\ v^+ - v^- + \sum_{i=1}^{n-1} (i/n)\, w_i + z = r/n,\quad v^+, v^-, w \geq 0, z \in Z\}$$

obtained by including two continuous variables in $P(n, r)$.

As discussed in Gomory and Johnson [11, 12], the facial structure of $P'(n, r)$ is essentially the same as the facial structure of $P(n, r)$. More precisely, every non-trivial facet of $P'(n, r)$ is related to a non-trivial facet of $P(n, r)$ in the following way:

$$\sum_{i=1}^{n-1} \eta_i w_i \geq \eta_r \text{ is a facet of } P(n,r) \Leftrightarrow \eta_1 v^+ + \eta_{n-1} v^- + \sum_{i=1}^{n-1} \eta_i w_i \geq \eta_r \text{ is a facet of } P'(n,r).$$

Based on this observation, scaled MIR and two-step MIR inequalities yield facets of $P'(n, r)$ under the conditions described in Theorems 21, 22 and 23.

6 Concluding Remarks

In this paper we derived the two-step MIR inequalities (13) and showed how these inequalities generalize both the 2slope facets of $P(n, r)$ in [3] and the strong fractional cuts in [15]. In deriving these inequalities, we considered only one of the many facets of set Q^2, and applied this facet to Y. It is somewhat surprising that such a seemingly narrow procedure results in such general inequalities. We do not know whether other facets of Q^2 can be combined with relaxations of Y to obtain interesting inequalities.

It is possible to generalize the two-step MIR inequalities when the the set Y contains continuous variables in addition to integral variables. More precisely, it is easy to show that if,

$$W \;=\; \Big\{ v \in R^{|I|}, x \in Z^{|J|}\ :\ \sum_{j \in I} c_j v_j + \sum_{j \in J} a_j\, x_j \geq b,\quad x, v \geq 0 \Big\},$$

then

$$\sum_{j \in I} \max\{0, c_j\} v_j + \sum_{j \in J} \gamma_j x_j \geq \rho\, \tau\, \lceil b \rceil,$$

where ρ, τ and γ are defined as in inequality (12) is valid for W.

An important question is whether these inequalities are useful for solving general mixed-integer programs. The two-step MIR inequalities have a number of features in common with the GMIC and these similarities suggest the possibility. We are planning to investigate this in the near future.

Another interesting question is whether or not the so-called "3slope facets" of $P(n, r)$ in [3] can be derived from simple mixed-integer sets. So far, we have not been able to answer this question.

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