MULTIPROCESSOR SCHEDULING UNDER PRECEDENCE CONSTRAINTS: POLYHEDRAL RESULTS

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ABSTRACT. We consider the problem of scheduling a set of tasks related by precedence constraints to a set of processors, so as to minimize their makespan. Each task has to be assigned to a unique processor and no preemption is allowed. A new integer programming formulation of the problem is given and strong valid inequalities are derived. A subset of the inequalities in this formulation has a strong combinatorial structure, which we use to define the *polytope of partitions into linear orders*. The facial structure of this polytope is investigated and facet defining inequalities are presented which may be helpful to tighten the integer programming formulation of other variants of multiprocessor scheduling problems.

1. INTRODUCTION

Let $N = \{1, \ldots, n\}$ be a set of partially ordered tasks, $M = \{1, \ldots, m\}$ a set of processors (or machines), and $G = (N, A)$ an acyclic directed precedence graph associated with the set of tasks [1, 4], such that $(i, j) \in A$ if and only if task i must be executed before task j. Each task has to be assigned to exactly one processor, in which it is entirely executed without preemption. For each task $j \in N$ and each processor $k \in M$, we denote by d_{jk} the total processing time of task j in case it is assigned to processor k.

The problem of multiprocessor scheduling under precedence constraints (MSPC) consists in finding an assignment of the tasks in N to the processors in M minimizing the makespan, i.e. the maximum completion time among all tasks in N. The minimization of the makespan on two uniform processors (problem $Q2 \mid\mid C_{max}$ in the notation of [14]) is already NP-hard [6, 7].

An application of this problem arises in the context of scheduling tasks of parallel programs. Parallel programs can be represented as a set of interrelated tasks which are sequential units. In a heterogeneous multiprocessor system, we not only have to determine how many, but also which processors should be allocated to an application and which tasks will be assigned to each processor. Greedy algorithms for processor assignment of parallel applications modeled by task precedence graphs in heterogeneous multiprocessor architectures were proposed by Menascé and Porto [16], while Porto and Ribeiro [18, 19] studied sequential and parallel algorithms based on the tabu search metaheuristic. Porto et al. [17] presented a detailed analysis of the solutions obtained by this parallel tabu search algorithm, using a broad set of test instances corresponding to real-size and realistic problems and showing that it leads to much better solutions than the greedy algorithm.

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Maculan et al. [15] proposed a new formulation with a polynomial number of 0-1 variables for MSPC, improving a previous formulation based on the discretization of the schedule horizon into unit time-periods [2]. However, even small problems are not amenable to be exactly solved by branch-and-bound or branch-and-cut algorithms based on this formulation.

In this paper, we first describe in Section 2 a new formulation for the problem of multiprocessor scheduling under precedence constraints. A subset of the inequalities in this formulation has a strong combinatorial structure, which we use to define the *polytope of partitions into linear orders*. The facial structure of this polytope (which is a relaxation and a projection of the original polytope) is investigated in Section 3 and facet defining inequalities are presented which may be helpful to tighten the integer programming formulation of other variants of multiprocessor scheduling problems. Further valid inequalities for MSPC are derived in Section 4. Concluding remarks are made in the last section.

2. Problem Formulation

In this section, we present a new formulation for the problem of multiprocessor scheduling under precedence constraints. This formulation reveals as part of it the polytope of *partition in linear orderings*. Given the directed acyclic precedence graph $G = (N, A)$, we define the following sets for every task $j \in N$:

- $P_j = \{i \in N :$ there exists a path in G from i to $j\}$, i.e., P_j is the set of predecessors of task j;
- $\Gamma_j = \{i \in N : (i,j) \in A\}$, i.e., Γ_j is the set of immediate predecessors of task j;
- $Q_j = \{i \in N :$ there exists a path in G from j to $i\}$, i.e., Q_j is the set of sucessors of task j; and
- $R_j = \{i \in N :$ there is no path in G from i to j or from j to $i\}$.

This new formulation makes use of two types of $0-1$ variables:

$$y_{jk} = \begin{cases} 1, & \text{if task } j \text{ is scheduled to processor } k, \\ 0, & \text{otherwise,} \end{cases}$$

for all $j \in N, k \in M$, and

$$z_{ij} = \begin{cases} 1, & \text{if task } i \text{ is scheduled before task } j \text{ in the same processor,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $(i,j) \in N \times N, i \in R_j$. Moreover, we denote by e_j the starting time of the execution of each task $j \in N$. The problem of multiprocessor scheduling under precedence constraints (MSPC) may be formulated as follows:

minimize $C_{\max}$

subject to:

$$\text{(1)} \qquad \sum_{k=0}^{m-1} y_{jk} = 1 \qquad \forall j \in N,$$

$$\text{(2)} \qquad z_{ij} + z_{ji} + y_{ik} - y_{jk} \leq 1 \qquad \forall j \in N, \forall i \in R_j, \forall k \in M,$$

$$\text{(3)} \qquad z_{ij} + y_{ik} - y_{jk} \leq 1 \qquad \forall j \in N, \forall i \in P_j, \forall k \in M,$$

$$\text{(4)} \qquad y_{ik} + y_{jk} - z_{ij} - z_{ji} \leq 1 \qquad \forall j \in N, \forall i \in R_j, \forall k \in M,$$

$$\text{(5)} \qquad y_{ik} + y_{jk} - z_{ij} \leq 1 \qquad \forall j \in N, \forall i \in P_j, \forall k \in M,$$

$$\text{(6)} \qquad e_i - e_j + \sum_{k=0}^{m-1} d_{ik} \cdot y_{ik} \leq 0 \qquad \forall j \in N, \forall i \in \Gamma_j,$$

$$\text{(7)} \qquad e_j - C_{\max} + \sum_{k=0}^{m-1} d_{jk} \cdot y_{jk} \leq 0 \qquad \forall j \in N,$$

$$\text{(8)} \qquad e_i - e_j + \sum_{k=0}^{m-1} d_{ik} \cdot y_{ik} \leq \mu_{ij} \cdot (1 - z_{ij}) \qquad \forall j \in N, \forall i \in R_j,$$

$$\text{(9)} \qquad e_j \geq 0 \qquad \forall j \in N,$$

$$\text{(10)} \qquad y_{jk} \in \{0,1\} \qquad \forall (j,k) \in N \times M, \text{ and}$$

$$\text{(11)} \qquad z_{ij} \in \{0,1\} \qquad \forall j \in N, \forall i \in R_j \cup P_j.$$

Equations (1) ensure that each task is processed and assigned to exactly one processor. Inequalities (6) express the precedence constraints: no task may be started unless all its predecessors have already completed their execution. Inequalities (8) define the sequence of starting times of the tasks assigned to the same processor, ensuring that no overlap occurs. The constant μ_{ij} is such that if tasks i and j are not executed in the same processor in that order, then inequality (8) is always satisfied. Though knowing the smaller possible value of this constant is as difficult as solving the original problem itself, some good approximations can be obtained. One such a good estimation is $\widehat{\mu}_{ij} = \overline{f}_i - \underline{e}_j$ ($\widehat{\mu}_{ij} \leq \mu_{ij}$), where $\overline{f}_i$ is an overestimate of the latest time task i could finish to be processed, and $\underline{e}_j$ is an underestimate of the least time at which j could start to be processed. The tighter these estimations are, the tighter inequality (8) will be. Inequalities (7) define the makespan.

The correct relation between the z and y variables are assured by inequalities (2)–(5). Since the formulation with the variables defined earlier seem to be a novelty in the scheduling literature, we now discuss these inequalities in detail. First we consider the inequalities (2) and (4) for fixed tasks $i, j \in N$ and a machine $k \in M$. In this case, we have that $i \in R_j$ and $j \in R_i$. Therefore, inequality (2) can also be written for when the roles of i and j are interchanged. Table 1 summarizes the outcomes of inequalities (2) and (4) for the four possible combinations of the values of y_{ik} and y_{jk}. In this table, inequality (2) is denoted by (2)' when written for $j \in R_i$.

We cannot determine the exact values of z_{ij} and z_{ji} in the first row of Table 1. However, due to constraints (1), there must be a machine $\ell \neq k$ for which at least one of the variables $y_{i\ell}$ or $y_{j\ell}$ is equal to 1. Thus, for machine ℓ, one of the cases

y_{ik}	y_{jk}	(4)	(2)	(2)'
0	0	$z_{ij} + z_{ji} \geq -1$	$z_{ij} + z_{ji} \leq 1$	$z_{ij} + z_{ji} \leq 1$
0	1	$z_{ij} + z_{ji} \geq 0$	$z_{ij} + z_{ji} \leq 2$	$z_{ij} + z_{ji} \leq 0$
1	0	$z_{ij} + z_{ji} \geq 0$	$z_{ij} + z_{ji} \leq 0$	$z_{ij} + z_{ji} \leq 2$
1	1	$z_{ij} + z_{ji} \geq 1$	$z_{ij} + z_{ji} \leq 1$	$z_{ij} + z_{ji} \leq 1$

TABLE 1. Relation between z and y variables when $i \in R_j$.

in the three remaining rows of Table 1 must hold which forces the z variables to assume the correct values. In the second (third) row of Table 1, the condition on the fifth (fourth) column forces both z_{ij} and z_{ji} to be 0, which is correct since, in both cases, i and j are not assigned to the same machine. Finally, in the last row of the table, the three conditions force either z_{ij} or z_{ji} to be set to 1, which is correct since both tasks are assigned to machine k and one must precede the other.

We now investigate the relation between variables y and z when $i \in P_j$. For fixed values of i, j and k, we are left only with the two inequalities (3) and (5), since j is clearly not in P_i. As for the previous case, we build Table 2 in which the outcome of these inequalities are given for all possible values of y_{ik} and y_{jk}.

y_{ik}	y_{jk}	(5)	(3)
0	0	$z_{ij} \geq -1$	$z_{ij} \leq 1$
0	1	$z_{ij} \geq 0$	$z_{ij} \leq 2$
1	0	$z_{ij} \geq 0$	$z_{ij} \leq 0$
1	1	$z_{ij} \geq 1$	$z_{ij} \leq 1$

TABLE 2. Relation between z and y variables when $i \in P_j$.

As for the previous table, the conditions in the first row of Table 2 are inconclusive with respect to the value of z_{ij}. Once again, this is not a difficulty since, like before, when $i \in R_j$, there exists another machine for which we are in one of the cases expressed in the remaining rows of the table. Now, notice that the second row is also inconclusive. However, from equation (1), there exists another machine $\ell \neq k$ for which the third row (with k replaced by ℓ) must hold. The last column of the third row forces z_{ij} to be equal 0. This is in accordance with the definition of the z variables, since i and j are not in the same machine. Finally, the last row of Table 2 imposes that $z_{ij} = 1$ which is correct since, in this situation, i precedes j in machine k.

Inequalities (2) to (5) determine the relation between variables z and y. Inequalities (2) and (3) ensure that if task i is processed in processor k but task j is not, then both z_{ij} and z_{ji} are equal to 0. In case both tasks i and j are processed in the same processor, these inequalities become redundant. Inequalities (4) and (5) ensure that if tasks i and j are both processed in machine k, then z_{ij} or z_{ji} is forced to be equal to 1. In case task i is processed in processor k and task j is not, then the inequalities become redundant.

Equations (1) and inequalities (2) to (5) deserve a more exhaustive analysis, because they represent an underlying common structure to different multiple processor scheduling problems. This study is carried out in the next section.

The previous formulation of MSPC proposed by Maculan et al. [15] has n^2m variables. The new formulation described in this work has only $n^2 + nm$ variables.

While the latter has $O(n^3m)$ constraints, the new formulation has only $O(n^2m)$ constraints. The new formulation is more compact in terms of both the number of variables and the number of constraints.

3. The Polytopes of Partitions into Linear Orderings

In this section, we consider the problem of partitioning a directed graph in linear orders (PLO). Given a directed graph $D = (N, E)$ and a weight $w_a \in \mathbb{R}$ associate to each arc $a \in E$, a feasible solution to PLO is a set of subgraphs partitioning the vertices of D, each of them defining a linear order on its vertices. An optimal solution is one in which the sum of the weights of the arcs in all subgraphs is minimized.

In Section 2 we have presented a new formulation for MSPC. A subset of the inequalities in this model defines a linear order for the set of tasks assigned to each processor. To tighten this model, we investigate the facial structure of the polytope of partitions in linear orders. Most of the results presented here hold for the particular case where the graph is complete. This assumption simplifies several proofs and, as we show later, it does not compromises the usefulness of the results for the MSPC.

The PLO polytope is closely related to the clique partitioning polytope studied in [11, 12] and some order polytopes such as linear order [8] and partial order [13]. In Subsection 3.1 we compare the PLO polytope with some other well studied order polytopes. In Subsection 3.2 we present two alternative characterizations of the PLO polytope. Valid and facet defining inequalities for the PLO polytope are presented in Subsection 3.3 and in Section 4.

3.1. **Order polytopes.**

Several order polytopes have been extensively studied recently. In this section we summarize the relations between the PLO polytope and some well-known order polytopes ranging from the most inclusive (i.e., *acyclic subgraph*), to the most restrictive (i.e., *linear order*). We first give some basic definitions to describe the order relations.

Given a binary relation E over a set N ($E \subseteq N \times N$), the following properties are defined:

- Symmetry (Sym): if $(i, j) \in E$ then $(j, i) \in E$, $\forall\ i, j \in N$.
- Antisymmetry (Asym): if $(i, j) \in E$ and $(j, i) \in E$ then $i = j$, $\forall\ i, j \in N$.
- Transitivity (Trans): if $(h, i) \in E$ and $(i, j) \in E$ then $(h, j) \in E$, $\forall\ h, i, j \in N$.
- Comparability (Comp): $(i, j) \in E$ or $(j, i) \in E$, $\forall\ i, j \in N$.
- Weak comparability (Wcomp): $(h, i) \in E$ and $(h, j) \in E$, or, $(i, h) \in E$ and $(j, h) \in E$, then $(i, j) \in E$ or $(j, i) \in E$, $\forall\ h, i, j \in N$.

Table 3 relates some well-known types of order relations and the properties given above.

Notice that every binary relation $E \subseteq N \times N$ can be represented by a directed graph $D = (N, E)$ whose vertices are the elements in N and, for each pair $\{i, j\} \in N \times N$, the arc (i, j) exists if and only if $(i, j) \in E$. Throughout this text we sometimes describe a binary relation via its corresponding subgraph.

Assume that $D_n = (N, E)$ is the complete graph on $n = |N|$ vertices and one is interested in the set X of subgraphs of D_n that represent one type of order relation in N. For example, X could be the set of subgraphs which are PLOs of

	Asym	Trans	Comp	Wcomp
Preorder (Pre)		yes		
Complete Preorder (CP)		yes	yes	
Partial Order (PO)	yes	yes		
Linear Order (LO)	yes	yes	yes	
Partial Linear Order (PLO)	yes	yes		yes

TABLE 3. Properties of order relations.

D_n. Let $P_X \subset \{0,1\}^{|E|}$ denote the convex hull of all incidence vectors of subgraphs in X. From the polyhedral point of view, the goal is to find linear inequalities that define facets of P_X. This sort of investigation was conducted earlier for the orders listed in Table 3. Besides them, there are two other sets of subgraphs which were studied in the literature and are closely related with the PLOs. One is the set of acyclic subgraphs of D_n and the other is the set of subgraphs which correspond to an *interval order* of N. Table 4 lists some of the references found in the literature where the facial structure of the polytopes associated to the problems cited above are studied.

Subgraphs of $D_n = (N, E)$	References
Preorder (Pre)	[13]
Complete Preorder (CP)	[13]
Partial Order (PO)	[13]
Linear Order (LO)	[10]
Acyclic (AC)	[3, 8, 9]
Interval Order (IO)	[20]

TABLE 4. Properties of order relations.

The Venn diagram in Figure 1 describes the relations among the polytopes in Table 4. Some graphs are depicted to illustrate elements in different regions of the diagram. Empty regions are indicated by the symbol "$\emptyset$". From Figure 1 and the discussion following the formulation in Section 2, one can see the importance of investigating the facial structure of the PLO polytope. Though the PLO polytope can inherit all the strong valid inequalities from the polytopes in which it is included, it is different from other previously studied polytopes. Thus, there exist strong valid inequalities for the PLO which are not even valid for the other polytopes and which can help in strengthening integer formulations for a large variety of scheduling problems.

3.2. **Characterizations of the PLO polytope.** In this subsection we describe two integer linear programming formulations for the PLO problem, and present some elementary facts about the associated polyhedra. In the discussion that follows, we consider an analogy between the solutions of the scheduling problem described in Section 1 and the PLO solutions for the complete digraph D_n. This analogy is given in Table 5 and is used whenever it simplifies the presentation of the text.

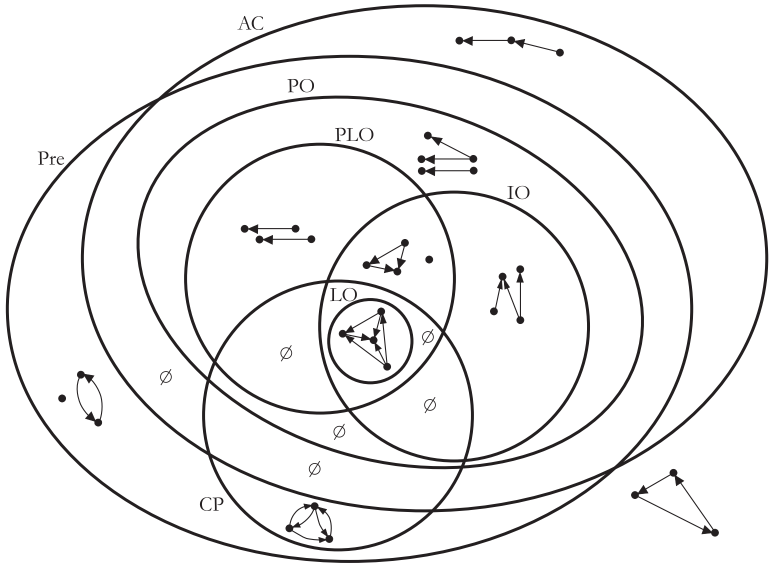

FIGURE 1. Venn diagram for order polytopes.

Scheduling problem	PLO
Tasks	vertices
Sequence of tasks processed in a processor	a linear order subgraph
Precedences within the processor	arcs in one linear order subgraph

TABLE 5. Correspondence between scheduling and PLO solutions.

Before we continue a few more notation is introduced. If H is a graph, let H^* and H^T denote the transitive closure and the transpose of H respectively. Thus, in order to define the problem of partition into linear orders corresponding to the MSCP with precedence graph $G = (N, A)$, we consider the graph D given by

$$D = (N, E) = D_n \setminus (G^*)^T,$$

where D_n is the complete digraph on n vertices. A partition into linear orders is a collection of subgraphs forming a partition of the vertices (tasks) of N such that each subgraph is a linear order of its vertices (tasks). According to the analogy in Table 5, the set of vertices induced by a subgraph of the collection corresponds to the set of tasks that are assigned to a processor of the scheduling problem. Moreover, these tasks are scheduled in the processor following to linear order provided by this subgraph.

Notice that, from the definition of D, all the precedence constraints described by G are respected by any partition of D into linear orders. However, to simplify the forthcoming proofs, we assume that the arc set of the precedence graph G is empty so that D is complete, i.e., $D = D_n$. In practice, assuming that D is complete is not too restrictive since arbitrary large weights can always be assigned to the forbidden arcs avoiding their presence in any optimal solution. There is a natural way to associate a polytope with a given instance of the partition into linear orders problem such that every vertex of the polytope corresponds to a feasible solution and vice versa. To this end, a partition into linear orders can be represented by an incidence vector $z \in \mathbb{B}^{n(n-1)}$ whose coordinates are indexed by all possible ordered pairs (i, j) of distinct vertices of D. Thus, $z_{ij} = 1$ means that both vertices i and j belong to the same subgraph of the partition and that j is preceded by i in the linear order induced by this subgraph. This representation yields to the definition of $P_{PLO}(D)$ as the convex hull of all incidence vectors z representing a partition into linear orders of the vertices of D. So, if L is the set of all vectors z in $\mathbb{B}^{n(n-1)}$ which are incidence vectors of partitions of D into linear orders, then $P_{PLO}(D) \equiv conv(L)$.

Now suppose that we define a new set of variables $y = (y_{jk}) \in \mathbb{B}^{np}$ that relate the vertices of N with the subsets of the partition of N to which they are assigned. This leads to a second representation of the partition into linear orders polytope, where the incidence vectors of the partitions are defined on both sets of variables y and z. The latter yields to the definition of $P^{y_p}_{PLO}(D)$ as the convex hull of all incidence vectors (y, z) representing a partition of the vertices of D in at most p linear orders. Thus, if L_y is the set of all vectors (y, z) in $\mathbb{B}^{n(n+p-1)}$ which are incidence vectors of partitions of D into linear orders, then $P^{y_p}_{PLO}(D) \equiv conv(L_y)$.

Theorem 3.1. *For a digraph $D = (N, E)$ with $|N| \geq 3$, a correct formulation for the partition of D into linear orders is given by the linear system*

$$z_{hi} + z_{ij} - z_{hj} + z_{ih} + z_{ji} - z_{jh} \leq 1, \qquad (h, i, j) \in N^3,\ h \neq i \neq j \neq h, \tag{12}$$

$$z_{hi} + z_{ij} + z_{jh} - z_{ih} - z_{ji} - z_{hj} \leq 1, \qquad (h, i, j) \in N^3,\ h \neq i \neq j \neq h, \tag{13}$$

where $z \in \mathbb{B}^{n(n-1)}$.

We call inequalities (12) the *double triangle* or *dt inequalities*, while inequalities (13) are called *double cycle* or *dC inequalities.* Support graphs of these inequalities are given in Figure 2. The following convention is used throughout this text to draw the support graph of valid inequalities: full lines represent arcs with positive coefficients and dashed lines represent those with negative coefficients. Moreover, lines with no arrows represent the pair of opposites arcs. Inequalities are always in the "$\leq$" form.

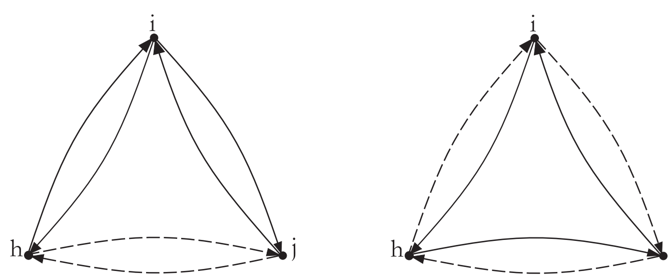

FIGURE 2. Support graphs of dT and dC Inequalities.

Proof: The proof is divided into two parts. In part (a) we show that every incidence vector z of a PLO satisfies inequalities (12) and (13). In part (b) we show that every $0-1$ vector in z-space satisfying inequalities (12) and (13) is an incidence vector of a PLO.

Part (a). If z is the incidence vector of a PLO then for every pair of tasks $i, j \in N$ we have that $z_{ij} + z_{ji} \leq 1$ so, if there is an inequality in (12) that is violated by z then we must have $z_{hi} + z_{ih} = 1$ and $z_{ij} + z_{ji} = 1$. However, in this case necessarily $z_{hj} + z_{jh} = 1$. The six possible cases are depicted in Figure 3. Notice that, due

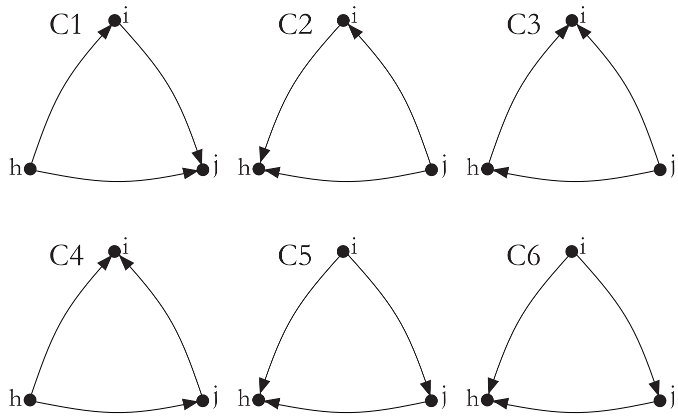

FIGURE 3. Six possible acyclic triangles.

to transitivity, the direction of the arc between h and j is mandatory in cases C1 and C2, while cases C3-C6 leave the direction free. However, weak comparability requires the existence of an arc between h and j all cases.

Concerning inequality (13), since z is the incidence vector of a PLO, $z_{hi} + z_{ij} + z_{jh} \leq 2$. Otherwise the left hand side would sum 3 and there would be a cycle. But transitivity implies that if two of these arcs are in the solution one of the arcs (i, h), (j, i) or (h, j) should be also in the PLO and the result follows.

Part (b). Let z be an incident vector of a spanning subgraph D' of D that satisfies (12) and (13). We show that D' is a PLO of D. Assume that z partitions the underlying graph of D into $p \leq n$ connected components.

First notice that the inequality $z_{ij} + z_{ji} \leq 1$, $\forall\ i \neq j \in N$, is obtained by adding the two inequalities of type (12): $z_{ij} + z_{ji} + z_{hi} + z_{ih} - z_{jh} - z_{hj} \leq 1$ and $z_{ij} + z_{ji} - z_{hi} - z_{ih} + z_{jh} + z_{hj} \leq 1$.

Secondly the transitivity inequality $z_{hi} + z_{ij} - z_{hj} \leq 1$ can be obtained by adding the following inequalities of type (12) and of type (13): $z_{hi} + z_{ih} + z_{ij} + z_{ji} - z_{hj} - z_{jh} \leq 1$ and $z_{hi} - z_{ih} + z_{ij} - z_{ji} - z_{hj} + z_{jh} \leq 1$. We now prove two facts that will complete the proof of part (b).

Fact A: The underlying graph of each component induced by z in D is a clique.

This result is obvious when the connected component has one or two vertices. If it has three vertices, inequalities (12) ensure that there must be three arcs joining the vertices. A simple induction on the number of vertices in the component can now show that we must have a clique. Suppose that there are vertices i and j, in the same component, such that no arc connects them. As they are part of the same connected component, there is a path between them. Let such path passes through the vertices $(i = u_1, \ldots, u_h = j)$. As u_1 is connected with u_2 and u_2 is connected with u_3, by applying inequality (12) we find that u_1 is connected with u_3. By induction, one can see that if u_1 is connected with u_{h-1} and u_{h-1} is connected with u_h then u_1 is connected with u_h. This implies that vertices i

and j are connected by an arc. Therefore as any pair of vertices is connected, the component is a clique.

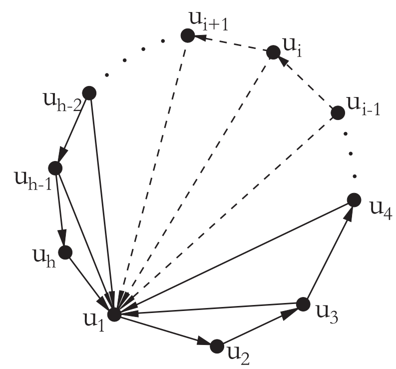

FIGURE 4. Cycle $(u_1, u_2, \ldots, u_{h-1}, u_h)$

Fact B: The subgraph induced by z in D is acyclic.

Suppose there exists a cycle C in the subgraph D' represented by z. Let the cycle be $C = (u_1, u_2, \ldots, u_h)$ (see Figure 4). From the transitivity inequality for vertices u_{h-1}, u_h, and u_1, the arc (u_{h-1}, u_1) belongs to D'. If we consider the cycle $(u_1, u_2, \ldots, u_{h-1}, u_1)$ and use the same reasoning again, we get a smaller cycle (of $h-2$ vertices) that must be in D'. Continuing in this way, we end up by concluding that the cycle (u_1, u_2, u_3, u_1) is in C. But this is impossible, since otherwise there would be a dC inequality (13) violated by z. This completes the proof of fact B.

The two facts imply that z partitions D into a complete acyclic digraph or, in other words, it partitions D into linear orders. This completes the proof. $\square$

Theorem 3.2. *For a digraph $D = (N, E)$ with $|N| = n \geq 3$, a correct formulation for the partition of D into at most m linear orders is given by the following system:*

$$\sum_{k=0}^{m-1} y_{jk} = 1, \qquad j \in N, \tag{14}$$

$$z_{ij} + z_{ji} + y_{ik} - y_{jk} \leq 1, \qquad (i,j,k) \in N^2 \times M, i \neq j, \tag{15}$$

$$y_{ik} + y_{jk} - z_{ij} - z_{ji} \leq 1, \qquad (i,j,k) \in N^2 \times M, i \neq j, \tag{16}$$

$$z_{hi} + z_{ij} + z_{jh} - z_{ih} - z_{ji} - z_{hj} \leq 1, \qquad (h,i,j) \in N^3,\ h \neq i \neq j \neq h. \tag{17}$$

where $z \in \mathbb{B}^{n(n-1)}$ and $y \in \mathbb{B}^{nm}$.

Proof: Once again the proof, is divided in two parts: (a) every incident vector (y, z) of a PLO satisfies inequalities (14) to (17), (b) every $0-1$ vector in y, z-space satisfying inequalities (14) to (17) is an incidence vector of a PLO.

Part (a). Let (y, z) be an incident vector of a PLO. Equations (14) hold, since each task should be assigned to exactly one processor.

For every pair of tasks i and j, $z_{ij} + z_{ji} \leq 1$. Thus, inequality (15) could be violated only if $z_{ij} = 1$ or $z_{ji} = 1$, $y_{ik} = 1$ and $y_{jk} = 0$. However, this is impossible because $z_{ij} = 1$ or $z_{ji} = 1$ would imply that both i and j belong to the same subset of the partition of N and, therefore, if $y_{ik} = 1$ then $y_{jk} = 1$. Thus (15) holds.

The only way inequality (16) could be violated would be if $y_{ik} = y_{jk} = 1$. But, the latter condition would imply that $z_{ij} = 1$ or $z_{ji} = 1$ which satisfies inequality (16).

In respect of inequality (17), a similar argument as the one used for inequality (13) in the proof of Theorem 3.1 holds and this completes the proof of (a).

Part (b). First notice that inequalities $z_{ij} + z_{ji} \leq 1$ can be obtained by adding $z_{ij} + z_{ji} + y_{ik} - y_{jk} \leq 1$ to $z_{ij} + z_{ji} + y_{jk} - y_{ik} \leq 1$ and dividing the resulting inequality by two.

Let N_k be the set of vertices for which $y_{ik} = 1$. The subgraph induced by (y, z) in D whose vertex set is N_k is complete, because if y_{ik} and y_{jk} are both set to one, then inequality (16) forces either z_{ij} or z_{ji} to be one.

We show that the graph D' induced by (y, z) in D is acyclic. Since every connected component of D' is complete, cycles can only go through vertices belonging to the same component. Therefore, we restrict ourselves to prove that for any connected component of D', inequalities (14) to (16) imply inequalities (12). Then, by noticing that (13) and (17) coincide, we apply part (b) of Theorem 3.1 to complete the proof.

Consider three tasks i, j and h in N and assume they all belong to the same component N_k of D'. By adding inequalities $z_{ih} + z_{hi} \leq 1$, $z_{hj} + z_{jh} \leq 1$ and (16) we get

$$z_{hi} + z_{ih} + z_{hj} + z_{jh} - z_{ij} - z_{ji} \leq 3 - y_{jk} - y_{ik}. \tag{18}$$

Since tasks i and j are in N_k, we have that $y_{jk} = y_{ik} = 1$. Thus, for processor k, (18) coincides with (12) and by repeating the steps in the proof of part (b) of Theorem 3.1 one can complete the proof. □

Theorem 3.3. *$dim(P_{PLO}(D_n)) = n(n-1)$.*

Proof: We exhibit $n(n-1)+1$ affinely independent points in the polytope $P_{PLO}(D_n)$. Since D_n is a complete digraph, the incidence vector of every single pair of vertices is in $P_{PLO}(D_n)$. Together with the null vector they form a set of $n(n-1)+1$ affinely independent points in $P_{PLO}(D_n)$. Moreover, because the polytope is embedded in $\mathbb{R}^{n(n-1)}$, $dim(P_{PLO}(D_n)) \leq n(n-1)$. Thus, we conclude that $dim(P_{PLO}(D_n)) = n(n-1)$. □

Theorem 3.4. *Let p be the maximum size of a feasible partition into linear orders of D_n. If $p = n$ then $dim(P_{PLO}^{y_p}(D_n)) = 2n(n-1)$.*

Proof: Let $\gamma y + \pi z = \pi_0$ be a hyperplane containing $P_{PLO}^{y_p}(D_n)$. Then, $\gamma y + \pi z = \pi_0$ is a linear combination of equations (14). We construct successive feasible solutions (y^i, z^i) and by means of simple algebraic operations deduce the general form of $\gamma y + \pi z = \pi_0$.

Let (y^1, z^1) be a feasible solution where $y^1_{ii} = 1$, for all $i \in N$, $y^1_{ij} = 0$, for all $i \neq j$, $i, j \in N$. Clearly $z^1 = 0$. Let (y^2, z^2) be another feasible solution such that $y^2_{hh} = 1$, for all $h \in N \setminus \{i, j\}$, $y^2_{ij} = y^2_{ji} = 1, y^2_{uv} = 0$ otherwise. Once again we have $z^2 = 0$. Subtracting $\gamma y^2 + \pi z^2 = \pi_0$ from $\gamma y^1 + \pi z^1 = \pi_0$, we get

$$\gamma_{ii} + \gamma_{jj} = \gamma_{ij} + \gamma_{ji}. \tag{19}$$

Let (y^3, z^3) be such that $y^3_{ii} = y^3_{ji} = z^3_{ij} = 1, y_{hh} = 1$, for all $h \in N \setminus \{i, j\}, y^3_{uv} = z^3_{rs} = 0$ otherwise. Let (y^4, z^4) be such that $y^4_{jj} = y^4_{ij} = z^4_{ij} = 1, y^4_{hh} = 1$, for all $h \in N \setminus \{i,j\}, y^4_{uv} = z^4_{rs} = 0$ otherwise. Subtracting $\gamma y^4 + \pi z^4 = \pi_0$ from $\gamma y^3 + \pi z^3 = \pi_0$, we get $\gamma_{ii} + \gamma_{ji} = \gamma_{ij} + \gamma_{jj}$. Subtracting the last equation from (19), we get

$$\gamma_{ji} = \gamma_{jj}. \tag{20}$$

Operating with (y^1, z^1), (y^3, z^3) and (20) yields to the following equations:

$$\sum_{h \in N, h \neq j, h \neq i} \gamma_{hh} + \gamma_{ii} + \gamma_{jj} = \pi_0, \tag{21}$$

$$\sum_{h \in N, h \neq j, h \neq i} \gamma_{hh} + \gamma_{ii} + \gamma_{ji} + \pi_{ij} = \pi_0. \tag{22}$$

Subtracting and applying identity (20), we conclude that $\pi_{ij} = 0$, for all $i \neq j \in N$. This shows that all π_{ij} are 0. Moreover, by defining $\gamma_{jk} \equiv \alpha_j$ for all $k \in \{1, \dots, p\}$, we obtain from (y^1, z^1) that $\pi_0 = \sum_{j \in N} \alpha_j$. Hence, $P^{y_p}_{PLO}(D_n) \subset \{(y,z) : (y,z) \in \mathbb{R}^{2n(n-1)}$ such that $\gamma y + \pi z = \pi_0\}$, with $(\gamma y + \pi z = \pi_0) = \sum_{i=1}^n \alpha_i (\sum_{k=1}^p y_{ik} = 1)$, which implies that $dim(P^{y_p}_{PLO}(D_n)) = 2n(n-1)$. □

Once we have established the dimension of the PLO polytope, we are now ready to study the strength of some valid inequalities. The forthcoming sections are devoted to prove that some of these inequalities define facets of the polytope. This shows, at least theoretically, that they are good candidates to tightening the integer formulation of the scheduling problem.

3.3. **Valid inequalities and facets of PLO polytopes.** In this subsection we investigate the facial structure of the PLO polyhedra. Initially we consider the inequalities that are part of the integer programming formulation of the PLO in the z-space to see whether they define facets of $P_{PLO}(D_n)$. Next, we search for new inequalities that are not part of the formulation and that define facets of $P_{PLO}(D_n)$. Besides, we also investigate the facial structure of the polytope $P^{y_p}_{PLO}(D_n)$ and present a lifting result which allows us to obtain facets for the latter polytope from some of the inequalities defining facets of $P_{PLO}(D_n)$.

The proofs establishing facet properties use either the *direct* or the *indirect* construction method described in basic texts on polyhedral combinatorics (cf. [5]).

3.3.1 The polytope $P_{PLO}(D_n)$
Trivial inequalities

A natural question while studying polyhedra associated to combinatorial optimization problems formulated as a $0-1$ integer program is whether or not the trivial inequalities of the form $x \geq 0$ and $x \leq 1$ define facets. This question is answered below.

Theorem 3.5. *The nonnegativity constraints $z_{ij} \geq 0$ define facets of $P_{PLO}(D_n)$.*

Proof: Let $a = (i,j) \in E$. Then $z_{ij} = 0$ is satisfied by the zero vector and all unit vectors $z^{\{a'\}}, a' \in E, a' \neq a$. These $|E|$ vectors are incidence vectors of PLOs and are affinely independent. □

Theorem 3.6. *The upper bound constraints $z_{ij} \leq 1$ do not define facets of $P_{PLO}(D_n)$.*

Proof: Let $a = (i, j) \in E$ and let (h, i, j) be a triangle. Then, the sum of the following facet-defining inequalities $z_{hi} + z_{ih} + z_{ij} + z_{ji} - z_{hj} - z_{jh} \leq 1$, $-z_{hi} - z_{ih} + z_{ij} + z_{ji} + z_{hj} + z_{jh} \leq 1$, $-z_{ji} \leq 0$, $-z_{ji} \leq 0$ leads to $z_{ij} \leq 1$ and hence this inequality does not define a facet of $P_{PLO}(D)$. □

Double triangle inequalities

Theorem 3.7. *Inequalities $z_{hi} + z_{ij} - z_{jh} + z_{ih} + z_{ji} - z_{hj} \leq 1$ define facets of $P_{PLO}(D_n)$.*

Proof: We look for $n(n-1)$ affinely independent points in $P_{PLO}(D_n)$ satisfying (12) at equality. For the sake of clarity, we divide the coordinates of the z variable in four sets: z^+ is the set of four coordinates that appear in (12) with positive coefficients; z^- are the two coordinates that appear in (12) with negative coefficients; z^ϕ is the set of coordinates $(h, v), (v, h), (j, v), (v, j)$ with $v \neq i$, or (u, v) with $u \notin \{h, i, j\}$ and $v \notin \{h, i, j\}$ that are not present in (12) and z^ψ be the set of coordinates that are not present in (12) and represent arcs of the form (i, v) or (v, i) with $v \notin \{h, j\}$.

The block matrix below represents the $n(n-1)$ affinely independent points in $P_{PLO}(D_n)$. The incidence vectors of feasible solutions are given in the columns of matrix T. All blocks of the diagonal are identity matrices of dimension corresponding to the number of z^+, z^-, z^ϕ, and z^ψ variables in that order.

$$T = \begin{bmatrix} I_4 & B_1 & B_2 & B_3 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_{n(n-3)} & B_4 \\ 0 & 0 & 0 & I_{2(n-3)} \end{bmatrix}$$

Matrix B_1 has two ones in each column in order to balance the presence of the negative element of the diagonal of the identity I_2. Matrix B_2 has one 1 in each column in order to satisfy (12) at equality. This can always be achieved because the coefficient of the identity $I_{n(n-3)}$ represents an arc with at most one vertex in $\{h, j\}$. Suppose that $z_{hv} = 1$ with $v \in N \setminus \{h, i, j\}$. Then, we can define $z_{ij} = 1$ and all other coordinates z_{uv} of z to be zero, so that (12) is satisfied at equality. Matrices B_3 and B_4 have one 1 in each column. Identity $I_{2(n-3)}$ has ones of the form $z_{iv} = 1$ ($z_{vi} = 1$) with $v \in N \setminus \{h, i, j\}$. To achieve the equality in (12) with a PLO, we need to consider an arc $(j, v), ((v, j))$ (matrix B_4) and one arc (i, j) (matrix B_3).

Since matrix T is upper triangular, it is not singular. Then, we have $n(n-1)$ affinely independent points satisfying (12) at equality. □

Double cycle inequalities

Theorem 3.8. *Let i, j and k be three distinct vertices in of D_n. Then, inequalities $z_{hi} + z_{ij} + z_{jh} - z_{ih} - z_{ji} - z_{hj} \leq 1$ define facets of $P_{PLO}(D_n)$.*

Proof: We look for a set of $n(n-1)$ affinely independent points in $P_{PLO}(D_n)$ satisfying (17) at equality. The incidence vectors of feasible solutions are given in the columns of matrix T below. Without loss of generality, we will consider arcs $(h, i), (i, j)$ and (j, h) to index the first three components of each point. Thus the first three columns are easily built by observing that each of these arcs alone is a PLO. Now given any other arc a, we can always choose one of the arcs $(h,i),(i,j),(j,h)$ so as to construct a PLO satisfying (17) at equality. These arcs are represented as elements of matrix B.

$$T = \begin{bmatrix} I_3 & B \\ 0 & I_{n(n-1)-3} \end{bmatrix}$$

Matrix T is upper triangular, hence it is not singular. Thus, there are $n(n-1)$ affinely independent points satisfying (17) at equality. □

The lifting theorem stated below enables us to extend facet defining results to spaces of higher dimension. It shows that, under certain conditions, every inequality that defines a facet of $P_{PLO}(D_n)$ also defines a facet of $P_{PLO}(D_h)$ for $h > n$.

Theorem 3.9. *Let $\pi \in \mathbb{R}^{h(h-1)}$ and $\pi_0 \in \mathbb{R}$. Suppose that the inequality $\pi z \leq \pi_0$ defines a nontrivial facet of $P_{PLO}(D_h)$. For an arbitrary integer $n > h$, let $\pi' \in \mathbb{R}^{n(n-1)}$ be such that $\pi'_i = \pi_i$ for all $i \in \{1, \ldots, h(h-1)\}$ and $\pi'_i = 0$ otherwise. Moreover, assume that condition $(\star)$ below holds:*

- $(\star)$ *$D_h = (N_h, E_h)$ has a partition into linear orders P whose incidence vector z^p satisfies $\pi z^p = \pi_0$ and P divides N_h into subsets $W_1, \ldots, W_p$ with $W_i = \{v\}$ for some $v \in N_h$ and some $i \in \{1, \ldots, p\}$.*

Then, for $z' \in \mathbb{R}^{n(n-1)}$, the inequality $\pi' z' \leq \pi_0$ defines a facet of $P_{PLO}(D_n)$.

Proof: We show that the given inequality defines a facet of $P_{PLO}(D_{h+1})$. The statement of the theorem then follows by induction, since condition $(\star)$ remains satisfied in $D_{h+1} = (N_{h+1}, E_{h+1})$. Let $N_h = \{1, \ldots, h\}$, $N_{h+1} = N_h \cup \{h+1\}$, and $\pi = (\pi_{ij})_{(i,j)\in E_h}$, $\overline{\pi} = (\overline{\pi}_{ij})_{(i,j)\in E_{h+1}}$, where $\overline{\pi}_{ij} = \pi_{ij}$ for $(i,j) \in E_h$ and $\overline{\pi}_{ij} = 0$ for $(i,j) \in E_{h+1} \setminus E_h$. The validity of $\pi z \leq \pi_0$ for $P_{PLO}(D_{h+1})$ is obvious.

Since $\pi z \leq \pi_0$ defines a nontrivial facet of $P_{PLO}(D_h)$ then $\pi_0 > 0$. Thus, there are $|E_h|$ feasible partitions with arc sets $P_1, \ldots, P_{|E_h|}$ whose incidence vectors are linearly independent and satisfy $\pi z \leq \pi_0$ with equality. Clearly each P_i also induces a linear order of D_{h+1} in which the new vertex is isolated. Thus, if $z'(P_i)$ denotes the incidence vector of P_i in $\mathbb{R}^{h(h+1)}$, we have that $\pi' z'(P_i) = \pi_0$.

Thus, let T be the nonsingular $|E_h| \times |E_h|$ matrix whose columns are the incidence vectors of the P_i's. We may assume that the rows and columns of T are arranged in such way that: *(a)* the last $2(h-1)$ rows correspond to the arcs $(i,v),(v,i)$ with $i \in N_h \setminus \{v\}$, where v is the special vertex verifying condition $(\star)$, and *(b)* the lower right corner $2(h-1) \times 2(h-1)$ submatrix, denoted by T', is nonsingular. Notice that such arrangement of the columns of T exists, otherwise T would be singular.

From the $2(h-1)$ linear order partitions $P_{|E_h|-2h+3}, \ldots, P_{|E_h|}$, whose incidence vectors are the last $2(h-1)$ columns of T, we construct $2(h-1)$ linear order partitions of D_{h+1} as follows. For $i \in \{|E_h| - 2h+3, \ldots, |E_h|\}$, let $(Y_i, E_h(Y_i))$ be the linear order of P_i containing v, i.e., $v \in Y_i$. If $|Y_i| < 2$ then v would be a singleton and, in this case, T' would contain a null column which imply that T' is singular, a contradiction. Hence $|Y_i| \geq 2$ holds. Now, let us define the sets Q_i as follows:

$$Q_i = P_i \cup \{(j,h+1) : (j,v) \in E_h(Y_i)\} \cup \{(h+1,j) : (v,j) \in E_h(Y_i)\} \cup \{(v,h+1)\}.$$

Then $\pi z^{Q_i} = \pi_0$ holds by construction. Finally, let P be the particular linear order described in condition ($\star$) and define the sets: $Q_{v,h+1} = P \cup \{(v, h+1)\}$ and $Q_{h+1,v} = P \cup \{(h+1, v)\}$. It is easy to check that $\pi z^{Q_{v,h+1}} = \pi z^{Q_{h+1,v}} = \pi_0$.

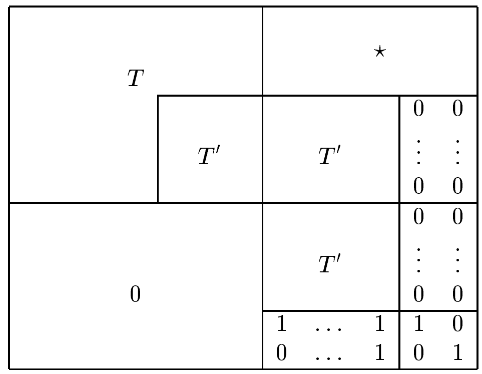

FIGURE 5. Matrix $\overline{T}$ (proof of Theorem 3.9).

Assume that $\overline{T}$ is the $|E_{h+1}| \times |E_{h+1}|$ matrix whose rows are the incidence vectors (in $\mathbb{B}^{E_{h+1}}$) of the linear order partitions $P_1, \ldots, P_{|E_h|}, Q_{|E_h|-2h+3}, \ldots, Q_{|E_h|}, Q_{v,h+1}$ and $Q_{h+1,v}$. Then $\overline{T}$ can be put into the form shown in Figure 5, where T and T are nonsingular. Obviously, $\overline{T}$ is nonsingular. Thus, there are $|E_{h+1}|$ PLOs in D_{h+1} whose incidence vectors satisfy $\pi' z' \leq \pi_0$ and are linearly independent. This implies that $\pi' z' \leq \pi_0$ defines a facet of $P_{PLO}(D_{h+1})$. $\square$

Double simplex inequalities

This set of inequalities is a generalization of the dT inequalities in (12). Let $S = \{v_0\}$ and $T = \{v_1, v_2, \ldots, v_h\}$ be two disjoint sets of vertices of D_n. We define the double simplex inequality associated with S and T as follows:

$$z(\delta(S,T)) - z(E(T)) \leq 1. \tag{23}$$

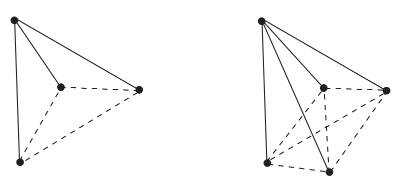

FIGURE 6. Double simplex inequalities, dS_3 and dS_4

Theorem 3.10. *The double simplex inequality (23) defines a facet of $P_{PLO}(D_n)$.*

Proof: We prove the validity of (23) by induction on $|T|$ and applying the Chvàtal–Gomory procedure. For $|T| = 2$ the result is immediate, since in this case we have a dT inequality. So, assume that $|T| = t \geq 3$. By the induction hypothesis, for every $v \in T$ the double simplex $(S, T \setminus \{v\})$-inequality given by $z(\delta(S, T \setminus \{v\})) - z(E(T \setminus \{v\})) \leq 1$ is valid for $P_{PLO}(D_n)$. Adding up these inequalities for all $v \in T$

we obtain that $(t-1)z(\delta(S,T)) - (t-2)(z(E(T)) \leq t$. Since $-(z(E(T)) \leq 0$ is also valid for $P_{PLO}(D_n)$, by adding these two inequalities we get $(t-1)(z(\delta(S,T)) - z(E(T))) \leq t$. Hence, since $(t-1) > 1$, $z(\delta(S,T)) - z(E(T)) \leq \frac{t}{t-1}$, which implies that $z(\delta(S,T)) - z(E(T)) \leq \lfloor \frac{t}{t-1} \rfloor = 1$. Thus inequality (23) is valid for $P_{PLO}(D_n)$.

We look for $\dim(P_{PLO}(D_n))$ affinely independent points in $P_{PLO}(D_n)$ satisfying (23) with equality to prove that this inequality is facet defining.

For the sake of clarity, we will separate the coordinates of the z variable in four sets: z^+ the set of $2h$ coordinates that appear in (23) with positive coefficients; z^- the set of $h(h-1)$ coordinates that appear in (23) with negative coefficients; z^ϕ the set of $(n+h-2)(n-h-1)$ coordinates that are not present in (23), and represent arcs with one vertex in T and the other in $N \setminus (S \cup T)$ or both vertices in $N \setminus (S \cup T)$, and finally z^ψ represents the set of $2(n-h-1)$ coordinates that are not present in (23), and represent arcs with one vertex in S and the other in $N \setminus (S \cup T)$.

The block matrix Q below represents the $n(n-1)$ affinely independent points in $P_{PLO}(D_n)$ and satisfying (23) at equality. The incidence vectors of feasible solutions are given as columns of matrix Q. All blocks of the diagonal are identity matrices of dimension corresponding to the number of z^+, z^-, z^ϕ, and z^ψ variables in that order.

$$Q = \begin{bmatrix} I_{2h} & B_1 & B_2 & B_3 \\ 0 & I_{h(h-1)} & 0 & 0 \\ 0 & 0 & I_{(n+h-2)(n-h-1)} & B_4 \\ 0 & 0 & 0 & I_{2(n-h-1)} \end{bmatrix}$$

Matrix B_1 has two ones in each column in order to balance the presence of the negative element of the diagonal of the identity $I_{h(h-1)}$. Matrix B_2 has a single one in each column in order to satisfy (23) with equality. This can always be achieved because a coefficient of the identity $I_{(n+h-2)(n-h-1)}$ represents an arc with exactly one vertex in T. Suppose that the $z_{ij} = 1$ with $i \in T$ and $j \in N \setminus (S \cup T)$. Then it is clear that we can define $z_{i'v} = 1$ with $i' \in T$, $i' \neq i$ and all other coordinates z_{ij} set to zero, so as to obtain $z(\delta(S,T)) - z(E(T)) = 1$.

Matrices B_3 and B_4 have a single one in each column. Identity $I_{2(n-h-1)}$ has ones of the form $z_{iv} = 1$ or $z_{vi} = 1$ with $i \in N \setminus (S \cup T)$. So as to achieve the equality in (23) with a PLO, we need to consider an arc (j, v) from T to S (matrix B_3) and another arc (i, j) from T to $N \setminus (S \cup T)$ (matrix B_4).

Matrix Q is upper triangular, hence it is not singular. Thus, we have $n(n-1)$ affinely independent points satisfying (23) with equality. $\square$

2-Partition inequalities

Here we introduce a class of facet defining inequalities that further generalizes the class of double triangle (12) and double simplex (23) inequalities.

Let S and T be subsets of N such that $S \cap T = \emptyset$ and $(S \cup T) \subseteq N$. We define the 2-partition inequality induced by S and T, or (S,T)-inequality for short, as:

$$z(\delta(S,T)) - z(E(S)) - z(E(T)) \leq \min\{|S|, |T|\}. \tag{24}$$

The support graph of a 2-partition inequality with $S = \{1, 2, 3\}$ and $T = \{4, 5, 6, 7\}$ is shown in Figure 7. Note that, if $|S| = 1$ and $|T| = 2$, the corresponding (S, T)-inequality is a dT inequality. Also, if $|S| = 1$ and $|T| > 2$ the (S, T)-inequality corresponds to a double simplex inequality.

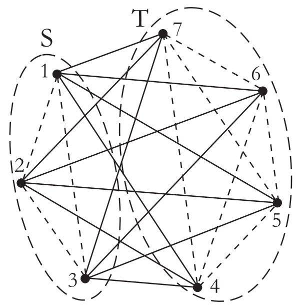

FIGURE 7. 2-partition inequality

Theorem 3.11. *For every $n \geq 3$ and every two nonempty disjoint subsets S, T of N the corresponding 2-partition inequality (24) is valid for $P_{PLO}(D_n)$. It defines a facet if and only if $|S| \neq |T|$.*

Proof: Assume without loss of generality that $|S| \leq |T|$. We prove the validity of (24) by induction on $|S| + |T|$ applying the Chvàtal–Gomory procedure. Let $|S| = 1$ and $|T| \geq 1$. For $|T| = 2$ the result is immediate since in this case we have a dT inequality. For $|T| > 2$ the result follows since the (S, T) inequality becomes a double simplex inequality.

Now, let $|S| = s \geq 2, |T| = t \geq 2, |S| + |T| = h$, and suppose that (24) is valid for $|S| + |T| \leq h - 1$. For every $v \in S$, consider the $(S \setminus \{v\}, T)$-inequality,

$$z(\delta(S \setminus \{v\}, T)) - z(E(S \setminus \{v\})) - z(E(T)) \leq s - 1, \tag{25}$$

and for every $v \in T$ consider the $(S, T \setminus \{v\})$-inequality,

$$z(\delta(S, T \setminus \{v\})) - z(E(S)) - z(E(T \setminus \{v\})) \leq \min\{s, t - 1\}. \tag{26}$$

By induction hypothesis, all these inequalities are valid for $P_{PLO}(D_n)$. Adding up the inequalities (25) for every $v \in S$ and (26) for every $v \in T$ we obtain

$$(s + t - 2)(z(\delta(S, T)) - z(E(S)) - z(E(T))) \leq s(s - 1) + t(\min\{s, t - 1\}). \tag{27}$$

If $|S| < |T|$, then (27) yields

$$z(\delta(S, T)) - z(E(S)) - z(E(T)) \leq \lfloor \frac{s(s + t - 1)}{s + t - 2} \rfloor = \lfloor s + \frac{s}{s + t - 2} \rfloor = |S|.$$

If $|S| = |T|$, i.e., $s = t$, then (27) can be written as

$$(2s - 2)(z(\delta(S, T)) - z(E(S)) - z(E(T))) \leq s(2s - 2),$$

which implies that $z(\delta(S, T)) - z(E(S)) - z(E(T)) \leq |S|$.

So, inequality (24) is valid for $P_{PLO}(D_n)$. When $|S| = |T|$ the above proof shows that the inequality (24) can be obtained by nonnegative linear combinations of other valid inequalities, and therefore it does not define a facet of $P_{PLO}(D_n)$.

Now assume that $s = |S| < |T|$. We first prove that (24) defines a facet of $P_{PLO}(D_h)$ when $h = |S| + |T|$.

Let $F = \{z \in P_{PLO}(D_n) : z(\delta(S,T)) - z(E(S)) - z(E(T)) = s\}$ be the face defined by inequality (24) in $P_{PLO}(D_h)$, and let $F' = \{z \in P_{PLO}(D_h) : \pi z = \pi_0\}$ be a generic face of $P_{PLO}(D_h)$ such that $F \subseteq F'$.

Notice that: (a) F is a proper face of $P_{PLO}(D_h)$, since $z = 0$ belongs to $P_{PLO}(D_h) \setminus F$; (b) F is nonempty since if we match each vertex in S with a distinct vertex in T and take one of the arcs joining each of those pairings, we obtain a linear order partition whose incidence vectors lies in F and (c) inequality (24) is valid for $P_{PLO}(D_h)$.

Therefore, if we prove that $\pi z \le \pi_0$ is a scalar multiple of $z(\delta(S,T)) - z(E(S)) - z(E(T)) \le s$, we can conclude that (24) defines a facet of $P_{PLO}(D_h)$. We use the following notation: indices $i_1, \dots, i_s$ represent an arbitrary order of the elements of S and indices $j_1, \dots, j_s, j_{s+1}, \dots, j_t$ represent an arbitrary order of the elements T.

Let z^1 and z^2 be two points of $P_{PLO}(D_h)$ representing the following set of arcs, both of size s: $P_1 = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1}), (i_s, j_s)\}$ and $P_2 = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1}), (j_s, i_s)\}$. It is clear that P_1 and P_2 both represent partitions in linear orders and that $z^1, z^2 \in F \subseteq F'$. Therefore, we conclude that $\pi z^1 = \pi z^2$ and

$$\pi_{i_s j_s} = \pi_{j_s i_s},\ \forall\ i_s \in S,\ \forall\ j_s \in T. \tag{28}$$

Hence, the coefficients of opposite arcs in $\delta(S,T)$ are equal.

Let z^3 be a point of $P_{PLO}(D_h)$ representing the PLO of size s with the following set of arcs: $P_3 = \{(i_1, j_1), \dots, (i_{s-1}, j_{s-1}), (i_s, j_{s+1})\}$. Since $z^1, z^3 \in F \subseteq F'$, it follows that $\pi z^1 = \pi z^3$ and we get that $\pi_{i_s j_s} = \pi_{i_s j_{s+1}}$, for all $i_s \in S$ and for all $j_s, j_{s+1} \in T$. As indices denote an arbitrary order of elements of S and T, we can conclude that, for a fixed vertex $i \in S$,

$$\pi_{ij} = \pi_{ij'},\ \forall\ j, j' \in T. \tag{29}$$

Let z^4 be a point of $P_{PLO}(D_h)$ representing the set $P_4 = P_1 \cup \{(i_s, j_{s+1}), (j_s, j_{s+1})\}$. As both $z^1, z^4 \in F \subseteq F'$ we have that $\pi z^1 = \pi z^4$. Thus,

$$\pi_{i_s j_{s+1}} + \pi_{j_s j_{s+1}} = 0,\ \forall\ i_s \in S,\ \forall\ j_s, j_{s+1} \in T. \tag{30}$$

Let z^5 be a point of $P_{PLO}(D_h)$ representing the set $P_5 = P_1 \cup \{(i_s, j_{s+1}), (j_{s+1}, j_s)\}$. Clearly $z^1, z^5 \in F \subseteq F'$ are incidence vectors of PLOs. Therefore, $\pi z^1 = \pi z^5$. Thus,

$$\pi_{i_s j_{s+1}} + \pi_{j_{s+1} j_s} = 0,\ \forall\ i_s \in S,\ \forall\ j_s, j_{s+1} \in T, \tag{31}$$

and from (30) and (31) we get

$$\pi_{jj'} = \pi_{j'j},\ \forall\ j, j' \in T. \tag{32}$$

Let z^6 and z^7 be two points of $P_{PLO}(D_h)$ representing the following sets of arcs $P_6 = \{(i_1, j_1), \dots, (i_{s-2}, j_{s-2}), (i_{s-1}, j_s), (i_s, j_{s-1})\}$ and $P_7 = P_6 \cup \{(i_{s-1}, j_{s+1}), (j_s, j_{s+1})\}$. Since $z^6, z^7 \in F \subseteq F'$, it follows that $\pi z^6 = \pi z^7$ therefore

$$\pi_{i_{s-1} j_{s+1}} + \pi_{j_s j_{s+1}} = 0,\ \forall\ i_{s-1} \in S,\ \forall\ j_s, j_{s+1} \in T. \tag{33}$$

Hence, from (30) and (33) we get $\pi_{i_s j_{s+1}} = \pi_{i_{s-1} j_{s+1}}$. As indices denote an arbitrary order of elements of S and T, we can conclude that given a fixed vertex $j \in T$

$$\pi_{ij} = \pi_{i'j},\ \forall\ i, i' \in S. \tag{34}$$

Thus, from (29), (34) and (28), we get

$$\pi_{ij} = \pi_{ji} = \lambda,\ \forall\ i \in S,\ \forall j \in T. \tag{35}$$

From (31), (32) and (35), we get

$$\pi_{jj'} = -\lambda, \ \forall\, j, j' \in T. \tag{36}$$

Let z^8 be a point of $P_{PLO}(D_n)$ representing the arc set $P_8 = P_1 \cup \{(i_{s-1}, j_s), (i_s, j_{s-1}), (i_{s-1}, i_s), (j_{s-1}, j_s)\}$. Since $z^1, z^8 \in F \subseteq F'$, we have that $\pi z^1 = \pi z^8$. Hence $\pi_{i_{s-1}j_s} + \pi_{i_s j_{s-1}} + \pi_{i_{s-1}i_s} + \pi_{j_{s-1}j_s} = 0$. Combining the last equation with (35) and (36), we obtain $\pi_{ii'} = -\lambda$, $\forall\, i, i' \in S$, and therefore (24) defines a facet of $P_{PLO}(D_h)$.

For any vertex $v \in T$ there exists a matching $\mathcal{M} \subseteq \delta(S,T)$ of size s, not covering v. Since $\mathcal{M}$ is a PLO whose incidence vector lies on F and v a vertex satisfying condition $(\star)$, Theorem 3.9 holds and therefore the (S,T)-inequality defines a facet of $P_{PLO}(D_n)$, for all $n \geq h$. This completes the proof. $\square$

2-Chorded cycle inequalities

Given digraph D_n, let $(N(C), C)$ be a directed cycle in D_n and C^2 the set of 2-chords of C with the same direction of C. In other words, if arcs (h,i) and (i,j) belongs to C then arc (h,j) belongs to C^2. Then, the inequality

$$z(C) - z(C^2) \leq \lfloor \frac{|C|}{2} \rfloor, \tag{37}$$

is the 2-chorded cycle inequality induced by C. Figure 8 shows a 7-cycle and its set of 2-chords.

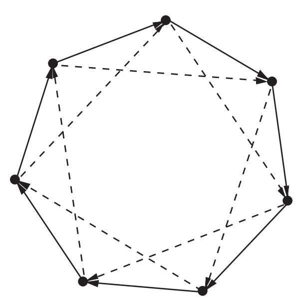

FIGURE 8. 2-chorded cycle inequality

Theorem 3.12. *Let $(N(C), C)$ be a cycle in D_n with $|C| \geq 5$ and let C^2 be the set of 2-chords of C. Then, the 2-chorded cycle inequality (37) induced by C is valid for $P_{PLO}(D_n)$ and defines a facet of $P_{PLO}(D_n)$ if and only if $|C|$ is odd.*

Proof: We start by showing the validity of the inequality. Suppose without loss of generality that $C = \langle (1,2), (2,3), \ldots, (2h, 2h+1), (2h+1, 1) \rangle$ so $C^2 = \{(i, i+2) \bmod (2h+1) : 1 \leq i \leq 2h+1\}$ is the set of 2-chords of C. For each arc (h,j) of C^2, we consider the two arcs (h,i) and (i,j) in C such that they induce the following triangle inequality

$$z_{hi} + z_{ij} - z_{hj} \leq 1. \tag{38}$$

Adding $|C|$ inequalities of the form (38), one for each arc of C^2, and the valid inequality $-z(C^2) \leq 0$ we obtain, after dividing the resulting inequality by 2, that

$$z(C) - z(C^2) \leq \frac{|C|}{2}. \tag{39}$$

Since the left hand side of (39) is integer in every vertex of $P_{PLO}(D_n)$, by applying Chvàtal-Gomory procedure we can round the right hand side while keeping validity. On the other hand, it is clear that when $|C|$ is even, (39) can be obtained as a linear combination of valid inequalities of the form (38) and then it does not define a facet of $P_{PLO}(D_n)$.

Now, we show that the inequality defines a facet of $P_{PLO}(D_{2h+1})$, then by using Theorem 3.9 we extend the result to $n > 2h+1$. Thus, let us suppose that $|C| = 2h+1,\ h \geq 2$. Moreover, assume that all indices are computed modulo $2h+1$.

Let $F = \{z \in P_{PLO}(D_{2h+1}) : z(C) - z(C^2) = h\}$ be the face defined by inequality (37) in $P_{PLO}(D_{2h+1})$ and $F' = \{z \in P_{PLO}(D_{2h+1}) : \pi z = \pi_0\}$ be a generic face of $P_{PLO}(D_{2h+1})$, such that $F \subseteq F'$. Notice that: (a) the feasible solution $z = 0$ belongs to $P_{PLO}(D_{2h+1}) \setminus F$, hence F is proper and (b) F is not empty since the arc set $\{(2,3),(4,5),\ldots,(2h,2h+1)\}$ is a PLO and its incidence vector satisfies (37) at equality.

We prove that $\pi z \leq \pi_0$ is a scalar multiple of $z(C) - z(C^2) \leq h$. Then, we conclude that (37) defines a facet of $P_{PLO}(D_{2h+1})$.

For each $i \in N$ let P_i be the set of arcs of size h given by $P_i = \{(i+1,i+2),(i+3,i+4),\ldots,(i+2h-1,i+2h)\}$. Notice that P_i defines a partition into linear orders in which the vertex i is a singleton. Moreover, $z^{P_i} \in F \subseteq F'$ for all $i \in N$. Therefore,

$$\pi z^{P_1} = \pi z^{P_2} = \ldots = \pi z^{P_{2h+1}} = \pi_0. \tag{40}$$

Since $P_i \triangle P_{i+2} = \{(i,i+1),(i+1,i+2)\}$ and $\pi z^{P_i} = \pi z^{P_{i+2}} = \pi_0$, we conclude that $\pi_{i,i+1} = \pi_{i+1,i+2}$. Thus, for all arcs in C we have the same coefficient, i.e.:

$$\exists\ \lambda \in \mathbb{R}_+ \text{ such that } \pi_e = \lambda\ \forall\ e \in C. \tag{41}$$

Let $Q_i = P_i \cup \{(i,i+1),(i,i+2)\}$, for all $i \in N$. Since $z^{Q_i} \in F \subseteq F'$, for all $i \in N$, $\pi z^{P_i} = \pi z^{Q_i}$. Hence, $\pi_{i,i+1} + \pi_{i,i+2} = 0$ which together with (41) and $\pi_{i,i+2} = -\lambda$, implies that there exists $\lambda \in \mathbb{R}_+$ such that $\pi_f = -\lambda$, for all $f \in C^2$. From (40) and (41), we conclude that $\pi_0 = h\lambda$.

Let $R_i = P_i \cup \{(i+1,i),(i+2,i)\}$, for all $i \in N$ (see Figure 9 (a)). Since $z^{R_i} \in F \subseteq F'$, for all $i \in N$, $\pi z^{P_i} = \pi z^{R_i}$ and hence

$$\pi_{i+1,i} + \pi_{i+2,i} = 0. \tag{42}$$

Let $S_i = P_{i+2} \cup \{(i+2,i+1),(i+2,i)\}$, for all $i \in N$ (see Figure 9 (b)). Since $z^{S_i} \in F \subseteq F'$, for all $i \in N$, $\pi z^{P_i} = \pi z^{S_i}$, we have that

$$\pi_{i+2,i+1} + \pi_{i+2,i} = 0. \tag{43}$$

From (42) and (43), we conclude that $\pi_{i+2,i+1} = \pi_{i+1,i}$ and, therefore,

$$\exists\ \alpha \in \mathbb{R}_+ \text{ such that } \pi_{\overline{e}} = \alpha,\ \forall\ \overline{e} \in \overline{C}. \tag{44}$$

Let $T_i = P_{i+1} \cup \{(i+2,i+1),(i+3,i+1)\}$, for all $i \in N$ (see Figure 9 (c)). But, $z^{T_i} \in F \subseteq F'$ for all $i \in N$, $\pi z^{P_i} = \pi z^{T_i}$. Hence, $\pi_{i+2,i+1} + \pi_{i+3,i+1} = 0$ and, from

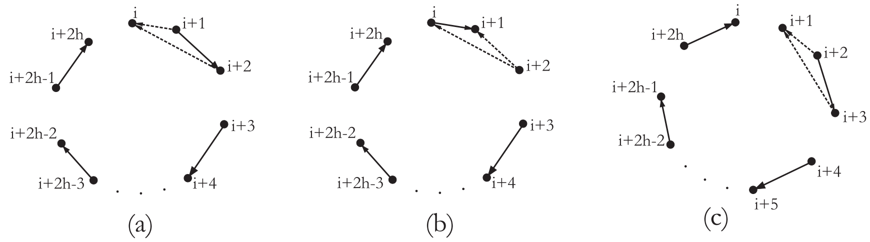

FIGURE 9. R_i, S_i and T_i

(43), we conclude that $-\pi_{i+2,i+1} = \pi_{i+2,i} = \pi_{i+3,i+1}$. Therefore,

$$\exists\ \alpha \in \mathbb{R}_+ \text{ such that } \pi_{\overline{f}} = -\alpha,\ \forall\ \overline{f} \in \overline{C}^2. \tag{45}$$

At this point, we split the proof in two parts. We first consider the case $h = 2$. Then, we investigate the case $h \geq 3$. For $h = 2$, i.e. $|C| = 5$, consider the partition into linear orders induced by the arc set $\{(1,2),(3,4),(3,2),(4,2),(3,1),(1,4)\}$. The incidence vector of this arc set lies in F. Moreover, since the first two arcs are in C, the third arc is in $\overline{C}$ and the three remaining arcs are in $\overline{C}^2$, the previous results imply that $2\lambda + \alpha - 3\alpha = 2\lambda$. Thus, we conclude that $\alpha = 0$ and the proof for $h = 2$ is complete.

Suppose now that $h \geq 3$ and assume that $j \in \{3, 5, \ldots, 2h-3\}$. Let $U_i^1 = P_i \cup \{(i, i+j), (i, i+j+1)\}$, for all $i \in N$. Since $z^{U_i^1} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{U_i^1}$ and

$$\pi_{i,i+j} + \pi_{i,i+j+1} = 0. \tag{46}$$

Let $U_i^2 = P_i \cup \{(i+j, i), (i, i+j+1)\}$, for all $i \in N$. Since $z^{U_i^2} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{U_i^2}$ and

$$\pi_{i+j,i} + \pi_{i,i+j+1} = 0. \tag{47}$$

Let $U_i^3 = P_i \cup \{(i+j, i), (i+j+1, i)\}$, for all $i \in N$. Since $z^{U_i^3} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{U_i^3}$ and

$$\pi_{i+j,i} + \pi_{i+j+1,i} = 0. \tag{48}$$

From (46) and (47), we conclude that $\pi_{i,i+j} = \pi_{i+j,i}$. From (47) and (48), we conclude that $\pi_{i,i+j+1} = \pi_{i+j+1,i}$. Then,

$$\pi_{i,i+j} = \pi_{i+j,i} = -\pi_{i,i+j+1} = -\pi_{i+j+1,i}. \tag{49}$$

Let $W_i = P_{i-1} \cup \{(i+j+1, i), (i+j+1, i+1), (i+j+2, i), (i+j+2, i+1)\}$, for all $i \in N$ (see Figure 10). Since $z^{W_i} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{W_i}$, hence

$$\pi_{i+j+1,i} + \pi_{i+j+1,i+1} + \pi_{i+j+2,i} + \pi_{i+j+2,i+1} = 0. \tag{50}$$

From (48) and (50), we conclude that

$$\pi_{i+j+1,i} = -\pi_{i+j+2,i}. \tag{51}$$

Now, consider the linear order partition induced by the arc set $K_i = P_{i-1} \cup \{(i+3, i), (i+3, i+1), (i+2, i+1), (i+2, i)\}$ whose incidence vector is in F. Since $(i+3, i+1)$ and $(i+2, i)$ are both in $\overline{C}^2$ and $(i+2, i+1)$ is in $\overline{C}$, from (44) and

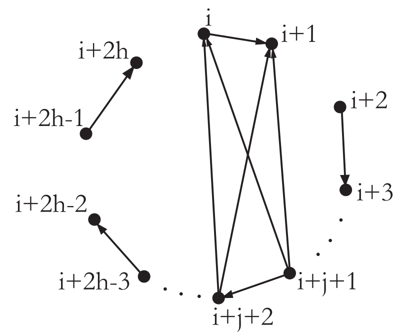

FIGURE 10. W_i

(45) we obtain that $\pi_{i+3,i} = \alpha$. Moreover, from (49) and (51), we can extend this result to

$$\alpha = \pi_{i+3,i} = \pi_{i,i+3} = -\pi_{i+4,i} = \ldots == -\pi_{i+2h-2,i} = -\pi_{i,i+2h-2}. \tag{52}$$

Now, consider the PLO whose incidence vector belongs to F and with arc set given by $P_{i+2h-3} \cup \{(i, i+2h-2), (i, i+2h-1), (i-1, i+2h-2), (i-1, i+2h-1)\}$. Since $(i, i+2h-1)$ and $(i-1, i+2h-2)$ are in $\overline{C}^2$, and $(i-1, i+2h-1)$ is in $\overline{C}$, we obtain $\pi_{i,i+2h-2} - 2\alpha + \alpha = 0 \Rightarrow \pi_{i,i+2h-2} = \alpha$. Comparing the last equation with (52), we conclude that $\alpha = 0$.

Finally, to see that the double cycle inequality also defines a facet of $P_{PLO}(D_n)$ for all $n > 2h+1$, observe that every node i together with the PLO P_i satisfies the condition (L) of Theorem 3.9. The result follows. □

Double 2-chorded cycle inequalities

Let $(N(C), C)$ be a cycle in D_n and C^2 in the set of 2-chords of C. The sets formed by the inverse arcs of C and C^2 are denoted as $\overline{C}$ and $\overline{C^2}$, respectively. Figure 11 shows a double 7-cycle and its set of 2-chords. The inequality

$$z(C) + z(\overline{C}) - z(C^2) - z(\overline{C^2}) \leq \lfloor \frac{|C|}{2} \rfloor \tag{53}$$

is the double 2-chorded cycle inequality induced by C.

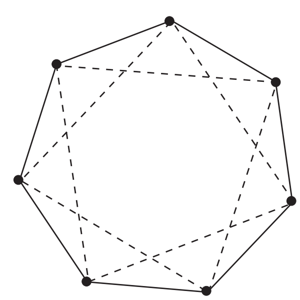

FIGURE 11. Double 2-chorded cycle inequality

Theorem 3.13. *If $(N(C), C)$ is a cycle in D_n with $|C| \geq 5$, then the double 2-chorded cycle inequality (53) induced by C is valid for $P_{PLO}(D_n)$. It defines a facet of $P_{PLO}(D_n)$ if and only if $|C|$ is odd.*

Proof: Let $C = \langle (1,2), (2,3), \ldots, (|C|-1, |C|) \rangle$. We first prove the validity of the (53). Given three consecutive vertices $i-1, i, i+1$ in C, they induce the following double triangle inequality (12)

$$z_{i-1,i} + z_{i,i+1} - z_{i-1,i+1} + z_{i,i-1} + z_{i+1,i} - z_{i+1,i-1} \leq 1.$$

Adding $|C|$ inequalities of this form, one for each triple of consecutive vertices of C, we get $2z(C) + 2z(\overline{C}) - z(C^2) - z(\overline{C^2}) \leq |C|$.

If we add to the last inequality the valid inequalities $-z(C^2) \leq 0$ and $-z(\overline{C^2}) \leq 0$ and dividing the resulting inequality by 2, we obtain

$$z(C) + z(\overline{C}) - z(C^2) - z(\overline{C^2}) \leq \frac{|C|}{2}. \tag{54}$$

The left-hand side of (54) is integer for every vertex of $P_{PLO}(D_n)$. Thus, we can apply the Chvàtal–Gomory procedure and round the right hand side of the above inequality. On the other hand, it is clear that, for $|C|$ even, inequality (54) can be obtained as a linear combination of other valid inequalities and therefore it does not define a facet of $P_{PLO}(D_n)$.

Now, suppose that $|C| = 2h+1$ for $h \geq 2$. Moreover, let $C^2 = \{(i, i+2) \bmod (2h+1) : 1 \leq i \leq 2h+1\}$ be the set of 2-chords of C. As in the previous proof, assume that all index computations are done modulo $2h+1$.

Let $F = \{z \in P_{PLO}(D_{2h+1}) : z(C) + z(\overline{C}) - z(C^2) - z(\overline{C^2}) = h\}$ be the face defined by inequality (53) in $P_{PLO}(D_{2h+1})$. Let $F' = \{z \in P_{PLO}(D_{2h+1}) : \pi z = \pi_0\}$ be a generic face of $P_{PLO}(D_{2h+1})$ such that $F \subseteq F'$. Notice that: (a) the feasible solution $z = 0$ belongs to $P_{PLO}(D_{2h+1}) \setminus F$, so F is proper, and (b) F is not empty since the arc set $\{(1,2), (3,4), \ldots, (2h, 2h+1)\}$ is a linear order partition whose incidence vector is in F.

We prove that $\pi z \leq \pi_0$ is a scalar multiple of $z(C) + z(\overline{C}) - z(C^2) - z(\overline{C^2}) \leq h$ to conclude that (53) defines a facet of $P_{PLO}(D_{2h+1})$.

For each $i \in N$ let P_i be the arc set given by $P_i = \{(i+1, i+2), (i+3, i+4), \ldots, (i+2h-1, i+2h)\}$. Notice that $|P_i| = h$ and its incidence vector lies in F. Then, $\pi z^{P_1} = \pi z^{P_2} = \ldots = \pi z^{P_{2h+1}} = \pi_0$.

Now, since $P_i \triangle P_{i+2} = \{(i, i+1), (i+1, i+2)\}$ and $z^{P_i} \in F \subseteq F'$ for all $i \in N$, $\pi z^{P_i} = \pi z^{P_{i+2}}$. Therefore, $\pi_{i,i+1} = \pi_{i+1,i+2}$, i.e. all arcs in C have the same coefficient, or

$$\exists\ \lambda \in \mathbb{R}_+ \text{ such that } \pi_e = \lambda,\ \forall\ e \in C. \tag{55}$$

For all $i \in N$, let $Q_i = P_i \cup \{(i, i+1), (i, i+2)\}$. Since $z^{Q_i} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{Q_i}$. Then, $\pi_{i,i+1} + \pi_{i,i+2} = 0$ which, together with (55) and $\pi_{i,i+2} = -\lambda$, implies that

$$\exists\ \lambda \in \mathbb{R}_+ \text{ such that } \pi_f = -\lambda,\ \forall\ f \in C^2. \tag{56}$$

Let $R_i = P_i \cup \{(i+1, i), (i, i+2)\}$, for all $i \in N$. Since $z^{R_i} \in F \subseteq F'$, $\pi z^{P_i} = \pi z^{R_i}$. Then, $\pi_{i+1,i} + \pi_{i,i+2} = 0$. From (56) we conclude that $\pi_{i+1,i} = \lambda$ or, more generally, that $\pi_{\overline{e}} = \lambda$, for all $\overline{e} \in \overline{C}$.

Let $S_i = P_i \cup \{(i+1, i), (i+2, i)\}$, $\forall\ i \in N$. Since $z^{S_i} \in F \subseteq F'$, $\pi z^{R_i} = \pi z^{S_i}$ and hence $\pi_{i+2,i} = \pi_{i,i+2} = -\lambda$. In general, we can write that $\pi_{\overline{f}} = -\lambda$, for all $\overline{f} \in \overline{C^2}$.

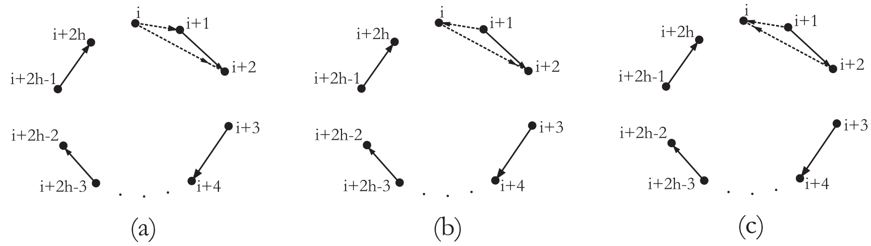

FIGURE 12. Q_i, R_i and S_i

Since for $(C \cup \overline{C} \cup C^2 \cup \overline{C^2}) = D_5$, the results above are enough to prove that (53) defines a facet of $P_{PLO}(D_5)$. To prove that it also defines a facet of $P_{PLO}(D_n)$ for all $n > 2h+1$, observe that every node i together with the PLO P_i satisfies the condition (L) of Theorem 3.9. So, from now on, assume that $h \geq 3$. We show that $\pi_g = 0$ for all $g \in E \setminus (C \cup \overline{C} \cup C^2 \cup \overline{C^2})$. To this end, assume that $j \in \{3, 5, \ldots, 2h-3\}$.

Let $U_i^1 = P_i \cup \{(i, i+j), (i, i+j+1)\}$, for all $i \in N$. Clearly $z^{U_i^1} \in F \subseteq F'$, therefore $\pi z^{P_i} = \pi z^{U_i^1}$ and

$$\pi_{i,i+j} + \pi_{i,i+j+1} = 0. \tag{57}$$

Let $U_i^2 = P_i \cup \{(i+j, i), (i, i+j+1)\}$, for all $i \in N$. Clearly $z^{U_i^2} \in F \subseteq F'$. Therefore $\pi z^{P_i} = \pi z^{U_i^2}$ and

$$\pi_{i+j,i} + \pi_{i,i+j+1} = 0. \tag{58}$$

Let $U_i^3 = P_i \cup \{(i+j, i), (i+j+1, i)\}$, for all $i \in N$. Clearly $z^{U_i^3} \in F \subseteq F'$. Thus $\pi z^{P_i} = \pi z^{U_i^3}$ and

$$\pi_{i+j,i} + \pi_{i+j+1,i} = 0. \tag{59}$$

From (57) and (58), we conclude that $\pi_{i,i+j} = \pi_{i+j,i}$. From (58) and (59), we conclude that $\pi_{i,i+j+1} = \pi_{i+j+1,i}$. Then,

$$\pi_{i,i+j} = \pi_{i+j,i} = -\pi_{i,i+j+1} = -\pi_{i+j+1,i}. \tag{60}$$

Let $W_i = P_{i-1} \cup \{(i+j+1, i), (i+j+1, i+1), (i+j+2, i), (i+j+2, i+1)\}$, for all $i \in N$ (see Figure 13). Clearly $z^{W_i} \in F \subseteq F'$, therefore $\pi z^{P_i} = \pi z^{W_i}$ and we obtain

$$\pi_{i+j+1,i} + \pi_{i+j+1,i+1} + \pi_{i+j+2,i} + \pi_{i+j+2,i+1} = 0. \tag{61}$$

Equations (60) and (61) imply that $\pi_{i+j+1,i} = -\pi_{i+j+2,i}$. This last equation joint with (60), allows us to write that $\alpha_i = \pi_{i+3,i} = \pi_{i,i+3} = -\pi_{i,i+4} = -\pi_{i+4,i} = \ldots = -\pi_{i+2h-2,i} = -\pi_{i,i+2h-2}$, for some constant α_i and for all $i \in \{1, \ldots, 2h+1\}$,. More generally, for $k \in \{3, \ldots, 2h-2\}$,

$$\pi_{i+k,i} = \pi_{i,i+k} = (-1)^{k+1}\alpha_i. \tag{62}$$

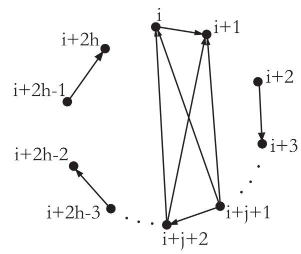

FIGURE 13. W_i

Now, define $i' = i + 2h - 2 - \ell$ and $k = 3 + \ell$ for ℓ ranging from 0 to $2h - 5$. If we rewrite equation (62) for i' and k, we get

$$\begin{aligned}
\pi_{i'+k,i'} = \pi_{i',i'+k} &= (-1)^{k+1}\alpha_{i'} \quad \Rightarrow \\
\pi_{i+2h-2-\ell+3+\ell,i+2h-2-\ell} = \pi_{i+2h-2-\ell,i} &= (-1)^{4+\ell}\alpha_{i'} \quad \Rightarrow \\
\pi_{i+2h+1,i+2h-2-\ell} &= (-1)^{\ell}\alpha_{i+2h-2-\ell} \quad \Rightarrow \\
\pi_{i,i+2h-2-\ell} &= (-1)^{\ell}\alpha_{i+2h-2-\ell}.
\end{aligned}$$

Now, if we let $k' = 2h - 2 - \ell$, the equation above can be rewritten as

$$\pi_{i+k',i} = \pi_{i,i+k'} = (-1)^{k'}\alpha_{i+k'}, \tag{63}$$

with k' ranging from 3 to $2h-2$. Notice here that we have used the fact that ℓ and k' have the same parity, so $(-1)^{\ell} = (-1)^{k'}$.

By comparing (62) and (63), we get that

$$(-1)^{k+1}\alpha_i = (-1)^k\alpha_{i+k} \quad \Rightarrow \quad -\alpha_i = \alpha_{i+k}. \tag{64}$$

Rewriting (64) for $i+2h-5$ and $k = 3$, we obtain $-\alpha_{i+2h-5} = \alpha_{i+2h-2}$. Repeating the computation above for $k = 2h-2$ and $k = 2h-5$, we obtain $-\alpha_i = \alpha_{i+2h-2}$ and $-\alpha_i = \alpha_{i+2h-5}$.

The last three equations imply that $\alpha_i = 0$ for all $i \in \{1, \dots, 2h+1\}$. This completes the proof for $P_{PLO}(D_n)$. To prove that the inequality also defines a facet of $P_{PLO}(D_n)$ for all $n > 2h + 1$, observe that every node i together with the PLO P_i satisfies the condition (L) of Theorem 3.9. $\square$

At a first glance, it may be intriguing that both inequalities (37) and (53) define facets of $P_{PLO}(D_n)$. However, for three distinct vertices i, j and k, it is easy to find points of the polytope that satisfy at equality the double 2-chorded cycle but not the 2-chorded cycle inequality. To see this, it is enough to consider the incidence vectors of the feasible solutions obtained by reversing the arcs of the sets P_i defined in the proofs of Theorems 3.12 and 3.13. The inverse statement is also easy to show. It follows from the fact that the incidence vectors of the arc sets K_i defined in the proof of Theorem 3.12 do not satisfy inequality (53) at equality.

3.3.2 The polytope $P^{y_p}_{PLO}(D_n)$
Facets in (y, z)-space

We now investigate the facial structure of $P^{y_p}_{PLO}(D_n)$. Initially the inequalities in the linear system of Theorem 3.2 are considered. Then, a lifting theorem is given which shows that under certain assumptions a facet defining inequality of $P_{PLO}(D_n)$ also defines a facet of $P^{y_p}_{PLO}(D_n)$ for some values of p. This last theorem allow us to extend some of the facet results seen before to the PLO polytope in the (y, z)-space.

Consider the inequality (15). Using the equation (14), the inequality can be written in the following form:

$$z_{ij} + z_{ji} + (1 - y_{ik}) - y_{jk} = z_{ij} + z_{ji} + \sum_{\ell=0,\ell\neq k}^{m-1} y_{i\ell} - y_{jk} \leq 0.$$

A closer inspection of the above inequality suggests that it belongs to a larger class of inequalities which takes the following general form:

$$z_{ij} + z_{ji} + \sum_{k\in M_1} y_{ik} - \sum_{k\notin M_1} y_{jk} \leq 0, \tag{65}$$

where M_1 is a nonempty and proper subset of M. The following result holds.

Theorem 3.14. *For every pair of vertices (i, j) in $N \times N$ and for every nonempty and proper subset M_1 of M, the inequality (65) defines a facet of $P^{y_n}_{PLO}(D_n)$.*

Proof: The validity of the inequality when $z_{ij} + z_{ji} = 0$ is trivial to check. On the other hand, if $z_{ij} + z_{ji} = 1$, there exists $\ell \in M$ such that $y_{i\ell} = y_{j\ell} = 1$. Therefore, exactly one of the two summations in the left hand side of the inequality has value one. This completes the proof of validity.

Now, let $F = \{(z, y) \in P^{y_n}_{PLO}(D_n) : z_{ij} + z_{ji} + \sum_{k\in M_1} y_{ik} - \sum_{k\notin M_1} y_{jk} = 0\}$ be the face defined by inequality (65) in $P^{y_n}_{PLO}(D_n)$, and let $F' = \{(z, y) \in P^{y_n}_{PLO}(D_n) : \pi z + \beta y = \pi_0\}$ be a generic face of $P^{y_n}_{PLO}(D_n)$ such that $F \subseteq F'$.

In the remainder of the proof a feasible solution is represented by a list of ordered sets of vertices in the form $\{u_1, u_2, \ldots, u_p\}_k$ meaning that vertex u_i precedes vertex u_j for all $1 \leq i < j \leq p$ and all these vertices are assigned to machine k. Moreover, the list may contain an element of the form (W, M') for $W \subset N$ and $M' \subset M$ which denotes an arbitrary assignment of the vertices of W to the machines in M' in such a way that at most one vertex is assigned to each machine.

Notice that: (a) F is a proper face of $P^{y_n}_{PLO}(D_n)$, since it does not contain the incidence vector of the feasible solution given by $(\{i\}_\ell, \{j\}_k, (N \setminus \{i, j\}, M \setminus \{\ell, k\}))$ for any given $\ell \in M_1$ and $k \notin M_1$; and (b) F is nonempty since it contains the incidence vector of the feasible solution $(\{i, j\}_\ell, (N \setminus \{i, j\}, M \setminus \{\ell\}))$ for any $\ell \in M_1$. Thus, since inequality (65) was shown to be valid, it defines a facet of $P^{y_n}_{PLO}(D_n)$ if there exist scalars $\lambda > 0$ and α_u, for $u = 1 \ldots n$, such that

$$(\pi z + \beta y = \pi_0) = \lambda(z_{ij} + z_{ji} + \sum_{k\in M_1} y_{ik} - \sum_{k\notin M_1} y_{jk} = 0) + \sum_{u=1}^{n} \alpha_u (\sum_{k=1}^{n} y_{uk} = 1).$$

Case 1: Given $u \in N \setminus \{i, j\}$, $\ell \notin M_1$ and two distinct elements b and c in $M \setminus \ell$, the incidence vectors of $(\{i, j\}_\ell, \{u\}_b, \{\ \}_c, (N \setminus \{i, j, u\}, M \setminus \{\ell, b, c\}))$ and $(\{i, j\}_\ell, \{\ \}_b, \{u\}_c, (N \setminus \{i, j, u\}, M \setminus \{\ell, b, c\}))$ are in F. Therefore, comparing the values of $\pi z + \beta y$ for these incidence vectors, yields $\beta_{ub} = \alpha_u$ for all $b \in M$ and all $u \in N \setminus \{i, j\}$.

Case 2: Given two distinct vertices u and v in $N \setminus \{i, j\}$, $\ell \notin M_1$ and two distinct elements a and b in $M \setminus \ell$, the incidence vectors of $(\{i, j\}_\ell, \{u\}_a, \{v\}_b, (N \setminus$

$\{i,j,u,v\}, M\setminus\{\ell,a,b\}))$, $(\{i,j\}_\ell, \{u,v\}_a, \{\ \}_b, (N\setminus\{i,j,u,v\}, M\setminus\{\ell,a,b\}))$ and $(\{i,j\}_\ell, \{v,u\}_a, \{\ \}_b, (N\setminus\{i,j,u,v\}, M\setminus\{\ell,a,b\}))$ are in F. Comparing the values of $\pi z+\beta y$ for these incidence vectors and using case 1, yields $\pi_{uv}=\pi_{vu}=0$ for all u and v in $N\setminus\{i,j\}$ with $u\neq v$.

Case 3: Given a vertex u in $N\setminus\{i,j\}$, $\ell\notin M_1$, $\ell'\in M_1$ and a in $M\setminus\{\ell,\ell'\}$, the incidence vectors of

$$\begin{gathered}(\{i\}_\ell, \{j\}_{\ell'}, \{u\}_a, (N\setminus\{i,j,u\}, M\setminus\{\ell,\ell',a\})),\\(\{i\}_\ell, \{j,u\}_{\ell'}, \{\ \}_a, (N\setminus\{i,j,u\}, M\setminus\{\ell,\ell',a\})),\\(\{i\}_\ell, \{u,j\}_{\ell'}, \{\ \}_a, (N\setminus\{i,j,u\}, M\setminus\{\ell,\ell',a\})),\\(\{i,u\}_\ell, \{j\}_{\ell'}, \{\ \}_a, (N\setminus\{i,j,u\}, M\setminus\{\ell,\ell',a\}))\end{gathered}$$

and $(\{u,i\}_\ell, \{j\}_{\ell'}, \{\ \}_a, (N\setminus\{i,j,u\}, M\setminus\{\ell,\ell',a\}))$ are in F. Comparing the values of $\pi z+\beta y$ for these incidence vectors and using case 1, yields $\pi_{iu}=\pi_{ui}=\pi_{ju}=\pi_{uj}=0$ for all u in $N\setminus\{i,j\}$.

Thus, from cases 2 and 3, all the coefficients of π but π_{ij} and π_{ji} are null.

Case 4: Given $\ell\notin M_1$, the incidence vectors of $(\{i,j\}_\ell, (N\setminus\{i,j\}, M\setminus\{\ell\})$ and $(\{j,i\}_\ell, (N\setminus\{i,j\}, M\setminus\{\ell\})$ are in F. Comparing the values of $\pi z+\beta y$ for these incidence vectors yields $\pi_{ij}=\pi_{ji}=\lambda$.

Case 5: Given two distinct elements b and c not in M_1, two distinct elements d and e in M_1, a in $M\setminus\{b,c,d,e\}$ and two different vertices u and v in $N\setminus\{i,j\}$, the incidence vectors of $(\{u,v\}_a, \{i\}_b, \{\ \}_c, \{j\}_d, (N\setminus\{i,j,v,u\}, M\setminus\{a,b,c,d\}))$ and $(\{u,v\}_a, \{\ \}_b, \{i\}_c, \{j\}_d, (N\setminus\{i,j,v,u\}, M\setminus\{a,b,c,d\}))$, are in F. Comparing the values of $\pi z+\beta y$ for these incidence vectors $\beta_{ib}=\alpha_i$ for all $b\notin M_1$.

Moreover, the incidence vectors of $(\{u,v\}_a, \{i\}_b, \{j\}_d, \{\ \}_e, (N\setminus\{i,j,v,u\}, M\setminus\{a,b,d,e\}))$ and $(\{u,v\}_a, \{i\}_b, \{\ \}_d, \{j\}_e, (N\setminus\{i,j,v,u\}, M\setminus\{a,b,d,e\}))$, are in F. This implies that $\beta_{jd}=\alpha_j$ for all $d\in M_1$.

From the previous results and considering the first feasible point of F presented in case 5, we obtain that $\pi_0=\sum_{u=1}^n\alpha_u$.

Case 6: Consider again the first solution in case 1. From the results derived in cases 1, 4 and 5, we obtain that

$$\lambda+\alpha_i+\beta_{j\ell}+\sum_{u\neq i\neq j}\alpha_u=\pi_0=\sum_{u\neq j}\alpha_u+\alpha_j.$$

Hence, we conclude that $\beta_{j\ell}=-\lambda+\alpha_j$ for all ℓ in M_1.

Case 7: Let ℓ be in M_1. Then, the incidence vector of $(\{i,j\}_\ell, (N\setminus\{i,j\}, M\setminus\{\ell\}))$ is in F. Computing the value of $\pi z+\beta y$ for this vector and using the results of cases 1, 4 and 5, leads to

$$\lambda+\beta_{i\ell}+\alpha_j+\sum_{u\neq i\neq j}\alpha_u=\sum_u\alpha_u.$$

Thus, $\beta_{i\ell}=-\lambda+\alpha_i$ for all ℓ in M_1 and the proof is complete. $\square$

Consider now inequality (16). Using equation (14), the former inequality can be written as

$$
\begin{aligned}
y_{ik} + (y_{jk} - 1) - z_{ij} - z_{ji} &\leq 0 \implies \\
y_{ik} - \sum_{\ell \in M, \ell \neq k} y_{j\ell} - z_{ij} - z_{ji} &\leq 0,
\end{aligned} \tag{66}
$$

for which the following result holds.

Theorem 3.15. *For every pair of vertices (i,j) in $N \times N$ and for every k in M, the inequality (66) defines a facet of $P_{PLO}^{y_n}(D_n)$.*

Proof: The validity of the inequality when $z_{ij} + z_{ji} = 1$ is obvious. On the other hand, if $z_{ij} + z_{ji} = 0$, the vertices i and j must be on different machines. If vertex i is not in k, the left hand-side of the inequality is negative while, if i is in k then j is not and therefore the left-hand side is null. This completes the proof of validity.

Now, let $F = \{(z,y) \in P_{PLO}^{y_n}(D_n) : y_{ik} - \sum_{\ell \in M, \ell \neq k} y_{j\ell} - z_{ij} - z_{ji} = 0\}$ be the face defined by inequality (66) in $P_{PLO}^{y_n}(D_n)$, and let $F' = \{(z,y) \in P_{PLO}^{y_n}(D_n) : \pi z + \beta y = \pi_0\}$ be a generic face of $P_{PLO}^{y_n}(D_n)$ such that $F \subseteq F'$.

We use the same notation as in the proof of Theorem 3.14. Notice that: (a) F is a proper face of $P_{PLO}^{y_n}(D_n)$, since it does not contain the incidence vector of the feasible solution given by $(\{i,j\}_\ell, (N \setminus \{i,j\}, M \setminus \{\ell\}))$ for any given $\ell \neq k$; and (b) F is nonempty since it contains the incidence vector of the feasible solution $(\{i\}_\ell, \{j\}_k, (N \setminus \{i,j\}, M \setminus \{\ell, k\}))$ for any $\ell \neq k$. Thus, since inequality (66) was shown to be valid, it defines a facet of $P_{PLO}^{y_n}(D_n)$ if there exist scalars $\lambda > 0$ and α_u, for $u = 1 \ldots n$, such that

$$
(\pi z + \beta y = \pi_0) = \lambda(y_{ik} - \sum_{\ell \in M, \ell \neq k} y_{j\ell} - z_{ij} - z_{ji} = 0) + \sum_{u=1}^{n} \alpha_u (\sum_{k=1}^{n} y_{uk} = 1).
$$

Case 1: Given a vertex u in $N \setminus \{i,j\}$ and any two distinct elements a and b of $M \setminus \{k\}$, the incidence vectors of the following feasible solutions are in F: $(\{i,j\}_k, \{u\}_a, \{\,\}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$, $(\{i,j\}_k, \{\,\}_a, \{u\}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$, $(\{i,j\}_b, \{u\}_k, \{\,\}_a, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$, and $(\{i,j\}_b, \{\,\}_k, \{\}_a, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$, Therefore, by comparing the values of $\pi z + \beta y$ for these incidence vectors, we conclude that $\beta_{ua} = \alpha_u$ for all $u \in N \setminus \{i,j\}$ and all $a \in M \setminus \{k\}$.

Case 2: Given two distinct vertices u and v in $N \setminus \{i,j\}$ and any two distinct elements a and b of $M \setminus \{k\}$, the incidence vectors of the following feasible solutions are in F: $(\{i,j\}_k, \{u\}_a, \{v\}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$, and $(\{i,j\}_k, \{u,v\}_a, \{\}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$. Thus, using case 1 and by comparing the values of $\pi z + \beta y$ for the previous incidence vectors, one concludes that $\pi_{uv} = 0$ for all u and v in $N \setminus \{i,j\}$.

Case 3: Given a vertex u in $N \setminus \{i,j\}$ and two distinct elements a and b in $M \setminus k$, the incidence vectors of the following feasible solutions are in F:

$$(\{i\}_k, \{j\}_a, \{u\}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\})),$$
$$(\{i,u\}_k, \{j\}_a, \{\ \}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\})),$$
$$(\{u,i\}_k, \{j\}_a, \{\ \}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\})),$$
$$(\{i\}_k, \{j,u\}_a, \{\ \}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\})),$$

and $(\{i\}_k, \{u,j\}_a, \{\ \}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\}))$. Comparing the values of $\pi z + \beta y$ for the first two solutions, we conclude that $\pi_{iu} = 0$. Analogously, the second and third solutions imply that $\pi_{iu} = \pi_{ui}$. Finally, the first and the fourth solutions imply that $\pi_{ju} = 0$ while the fourth and the fifth solution imply that $\pi_{ju} = \pi_{uj}$. Summing up, we have that $\pi_{iu} = \pi_{ui} = \pi_{ju} = \pi_{uj} = 0$ for all $u \in N \setminus \{i,j\}$.

From cases 2 and 3, we have that all the coefficients of π are null except those for edges (i,j) and (j,i).

Case 4: Consider again the vertex u and the elements a and b of M as defined in case 1. The incidence vectors of the solutions $(\{i,j\}_k, \{u\}_a, \{\ \}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$ and $(\{j,i\}_k, \{u\}_a, \{\ \}_b, (N \setminus \{i,j,u\}, M \setminus \{a,b,k\}))$. are in F. Comparing the values of $\pi z + \beta y$ for these vectors, we can conclude that $\pi_{ij} = \pi_{ji} = -\lambda$.

Case 5: Let a and b be two distinct elements in $M \setminus \{k\}$. The following incidence vectors of feasible solutions are in F: $(\{j\}_k, \{i\}_a, \{\ \}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\}))$ and $(\{j\}_k, \{\ \}_a, \{i\}_b, (N \setminus \{i,j\}, M \setminus \{a,b,k\}))$. Therefore, comparing the values of $\pi z + \beta y$ for these vectors, we can conclude that $\beta_{ia} = \alpha_i$ for all $a \neq k$ and $a \in M$.

Case 6: Consider again the first solution of case 5. Computing the value of $\pi z + \beta y$ for this vector and using the results in cases 1 and 5, we obtain that: $\alpha_i + \alpha_j + \sum_{u \in M, u \neq i \neq j} \alpha_u = \sum_u \alpha_u = \pi_0$.

Case 7: Consider again the first feasible solution given in case 2. The incidence vector of this solution is in F. By computing the value of $\pi z + \beta y$ for this vector and using the results obtained for the previous cases, we obtain that:

$$-\lambda + \beta_{ik} + \alpha_j + \sum_{u \in M, u \neq i \neq j} \alpha_u = \sum_{u \in M} \alpha_u.$$

Therefore, we have that $\beta_{ik} = \lambda + \alpha_i$.

Now consider again the first feasible solution given in case 3 whose incidence vector is in F. By computing the value of $\pi z + \beta y$ for this vector and using the results obtained for the previous cases, we obtain that:

$$\lambda + \alpha_i + \beta_{ja} + \sum_{u \in M, u \neq i \neq j} \alpha_u = \sum_{u \in M} \alpha_u.$$

Therefore, we have that $\beta_{ja} = -\lambda + \alpha_j$ for all $a \in M$ with $a \neq k$. This completes the proof □

Some of the facet defining inequalities found in the previous subsections for $P_{PLO}(D_n)$ also define facets for $P_{PLO}^{y_p}(D_n)$. This can be shown by means of Theorem 3.16 which gives some necessary conditions under which the lifting operation for the higher dimensional space is possible. The lemma below is useful to prove this result.

Lemma 3.1. *Let F be a face of $P^{y_p}_{PLO}(D_n)$ that is contained in the hyperplane $\gamma y + \pi z = \pi_0$. Then $\gamma_{jk} = \alpha_j$ for all $k \in M$ provided the following condition is satisfied:*

(X) *for all $k \in M$ there exists a point (y^*, z^*) in F for which j is the only task assigned to processor k and there exists a processor $\ell \in M \setminus \{k\}$ for which no task is assigned. Moreover, if task j is assigned to processor ℓ instead of processor k another point in F is obtained.*

Proof: This result can be easily obtained by applying the *indirect* construction method described in [5]. □

Theorem 3.16. *Let $\beta z \leq \beta_0$ a facet defining inequality for $P_{PLO}(D_n)$. Suppose that for every vertex j in D_n property (X) holds with respect to the hyperplane $\beta z = \beta_0$. Then $\beta z \leq \beta_0$ is facet defining for $P^{y_n}_{PLO}(D_n)$.*

Proof: Let $F = \{(y, z) \in P^{y_n}_{PLO}(D_n) : \beta z = \beta_0\}$ be a face of $P^{y_n}_{PLO}(D_n)$ and $F' = \{(y, z) \in P^{y_n}_{PLO}(D_n) : \gamma y + \pi z \leq \pi_0\}$ be a generic face of $P^{y_n}_{PLO}(D_n)$ such that $F \subseteq F'$. If we prove that $\gamma y + \pi z \leq \pi_0$ is a positive scalar multiple of $\beta z \leq \beta_0$, plus a linear combination of equations (14), the proof will be completed.

Since property (X) holds, by Lemma 3.1 we have that $\gamma_{jk} = \alpha_j$ for all $k \in M$. Thus, the contribution to the left hand side of $\gamma y + \pi z \leq \pi_0$ of the assignment of task j to a processor is always the same, which proves the relations of the coefficients for the y variables. Now, given any two integer points in F, say (y_1, z_1) and (y_2, z_2), since $F \subseteq F'$ we have that $\gamma y_1 + \pi z_1 = \gamma y_2 + \pi z_2$. From previous results on the coefficients of the vector γ, this leads to $\pi z_1 = \pi z_2$. Thus, using the same affinely independent points as in the proof that $\beta z \leq \beta_0$ is facet defining for $P_{PLO}(D_n)$, we complete the proof that the inequality also defines a facet of $P^{y_p}_{PLO}(D_n)$. □

For the 2-chorded and the double 2-chorded cycle inequalities in (37) and (53), it is easy to check that all points presented in the proofs of Theorems 3.12 and 3.13 satisfy the conditions of Theorem 3.16 when $p \geq h + 3$. Since validity of both inequalities for $P^{y_p}_{PLO}(D_n)$ is obvious, the next two results hold.

Theorem 3.17. *Let $(N(C), C)$ be a cycle in D_n with length at least 5 and let C^2 be the set of 2-chords of C. Then, the 2-chorded cycle inequality $z(C) - z(C^2) \leq \lfloor \frac{|C|}{2} \rfloor$ induced by C is valid for $P^{y_p}_{PLO}(D_n)$ and defines a facet of $P^{y_p}_{PLO}(D_n)$ if $|C|$ is odd and $p \geq \frac{|C|-1}{2} + 3$.*

Theorem 3.18. *Let $(N(C), C)$ be a cycle in D_n with length at least 5. Let C^2 be the set of 2-chords of C, and $\overline{C}$ and $\overline{C^2}$ the inverse sets of C and C^2. Then, the double 2-chorded cycle inequality $z(C) + z(\overline{C}) - z(C^2) - z(\overline{C^2}) \leq \lfloor \frac{|C|}{2} \rfloor$ induced by C is valid for $P^{y_p}_{PLO}(D_n)$. Moreover, it defines a facet of $P^{y_p}_{PLO}(D_n)$ if $|C|$ is odd and $p \geq \frac{|C|-1}{2} + 3$.*

For the 2-partition inequality in Theorem 3.11, we cannot apply Theorem 3.16 because there are no roots of this inequality in which a vertex in S forms a singleton. Nevertheless, we show below, this inequality also defines a facet of $P^{y_n}_{PLO}(D_n)$.

Theorem 3.19. *For every $n \geq 3$ and every two nonempty disjoint subsets S, T of N with $|S| < |T|$ and $|S| + |T| < |N| - 1$. The corresponding 2-partition inequality (24) defines a facet $P^{y_n}_{PLO}(D_n)$.*

Proof: First notice that validity is immediate from Theorem 3.11.

Now, let $F = \{(z, y) \in P^{y_n}_{PLO}(D_n) : z(\delta(S,T)) - z(E(S)) - z(E(T)) = |S|\}$ be the face defined by inequality (24) in $P^{y_n}_{PLO}(D_n)$, and let $F' = \{(z, y) \in P^{y_n}_{PLO}(D_n) : \pi z + \beta y = \pi_0\}$ be a generic face of $P^{y_n}_{PLO}(D_n)$ such that $F \subseteq F'$.

Again we use the same notation as in the proof of Theorem 3.14. Moreover, let S be the set $\{i_1, i_2, \ldots, i_s\}$ and let T be the set $\{j_1, j_2, \ldots, j_t\}$.

Notice that: (a) F is a proper face of $P^{y_n}_{PLO}(D_n)$, since it does not contain the incidence vector of the feasible solution obtained by assigning exactly one vertex to each machine and (b) F is nonempty since it contains the incidence vector of the feasible solution given by $(\{i_1, j_1\}_1, \{i_2, j_2\}_2, \ldots, \{i_s, j_s\}_s, \{j_{s+1}\}_{s+1}, \ldots, \{j_t\}_t, (N \setminus (N \cup T), M \setminus \{1, \ldots, t\}))$. Thus, since inequality (24) was shown to be valid, it defines a facet of $P^{y_n}_{PLO}(D_n)$ if there exist scalars $\lambda > 0$ and α_u, for $u = 1 \ldots n$, such that

$$(\pi z + \beta y = \pi_0) = \lambda(z(\delta(S,T)) - z(E(S)) - z(E(T)) = |S|) + \sum_{u=1}^{n} \alpha_u (\sum_{k=1}^{n} y_{uk} = 1).$$

Consider the feasible solutions below:

$$W_1 = (\{i_1, j_1\}_1, \{i_2, j_2\}_2, \ldots, \{i_s, j_s\}_s, \{j_{s+1}\}_{s+1}\}, \ldots, \{j_t\}_t, \{u\}_{t+1}, \{\ \}_{t+2}, \\ (N \setminus (N \cup T \cup \{u\}), M \setminus \{1, \ldots, t+2\}))$$

$$W_2 = (\{i_1, j_1\}_1, \{i_2, j_2\}_2, \ldots, \{i_s, j_s\}_s, \{j_{s+1}\}_{s+1}\}, \ldots, \{j_t\}_t, \{\ \}_{t+1}, \{u\}_{t+2}, \\ (N \setminus (N \cup T \cup \{u\}), M \setminus \{1, \ldots, t+2\}))$$

The incidence vectors of these solutions, say $(z, y)^1$ and $(z, y)^2$ are in F and, therefore, in F'. By comparing $(\pi, \beta)(z, y)^1$ with $(\pi, \beta)(z, y)^2$ we obtain that for all $u \notin S \cup T$, $\beta_{u,t+1} = \beta_{u,t+2} = \alpha_u$ for some constant α_u. Since the indices can be permuted to obtain solutions that are symmetrical to W_1 and W_2, the general conclusion is that $\beta_{u,k} = \alpha_u$ for all $k \in M$.

Moreover, since $|T| > |S|$ we can interchange the roles of any vertex of T with that of u. By repeating the same argument as above, one reaches the conclusion that, for all $a \in M$, $\beta_{j_a,k} = \alpha_{j_a}$ for some constant α_{j_a}.

Now consider the solutions given by

$$W_3 = (\{\ \}_1, \{i_2, j_2\}_2, \ldots, \{i_s, j_s\}_s, \{j_{s+1}\}_{s+1}\}, \ldots, \{j_t\}_t, \{i_1, j_1\}_{t+1}, \\ (N \setminus (N \cup T), M \setminus \{1, \ldots, t+1\}))$$

$$W_4 = (\{i_1, j_1\}_1, \{i_2, j_2\}_2, \ldots, \{i_s, j_s\}_s, \{j_{s+1}\}_{s+1}\}, \ldots, \{j_t\}_t, \{\ \}_{t+1}, \\ (N \setminus (N \cup T), M \setminus \{1, \ldots, t+1\}))$$

Let $(z, y)^3$ and $(z, y)^4$ be the incidence vectors of these solutions. Since they are both in F, by hypothesis they also belong to F'. Thus, comparing the values of $(\pi, \beta)(z, y)^3$ with $(\pi, \beta)(z, y)^4$ and using the results obtained so far, one concludes that $\beta_{i_1,1} = \beta_{i_1,t+1}$. However, since the indexes of the machines can be permuted arbitrarily, one can get in that $\beta_{i_1,k} = \alpha_{i_1}$ for all $kinN$ and for some constant α_{i_1}. Moreover, this reasoning can be repeated for all vertices in S which then leads to the general conclusion that, for all $u \in S$, $\beta_{u,k} = \alpha_u$ for all $k \in N$ and for some constant α_u.

The results attained so far, show that for any feasible point (z, y) in F, the contribution of βy to the left-hand side of $\pi z+\beta y=\pi_0$ is always given by $\sum_{u\in N}\alpha_u$. This means that with respect to inequality (24), the machine to which the vertices are assigned is irrelevant. Therefore, the relationships among the coefficients of the vectors π, β and π_0 that still have to be established to prove that the inequality (24) defines a facet can be obtained using exactly the same points that are given in the proof of Theorem 3.11 □

4. Further valid inequalities for MSPC

The inequalities described in the previous section involve only the z and y variables. However, the formulation of MSPC in Section 2 also contains the e variables, which are meant to represent the starting times of the jobs. Therefore, they are closely related to the makespan. To strengthen the integer programming model, we study inequalities which establish strong links between the e variables and the y and z variables.

4.1. **Successors' inequalities.** Let j be a task and k be a processor. These inequalities say that the sum of the processing times of all the tasks in the successor set of j that are assigned to machine k plus the completion time of task j cannot exceed the makespan. They can be viewed as liftings of inequalities (7) and are written as follows:

$$e_j+\sum_{k=0}^{m-1} d_{jk}y_{jk}+\sum_{i\in Q_j} d_{ik}y_{ik}\le C_{\max},\quad \text{for all } (j,k)\in N\times M. \tag{67}$$

If the precedence graph has a unique sink represented by task $n-1$, the *successors' inequality* above can be strengthened. Notice that, in this case, the task $n-1$ will not be processed until all other tasks have been completed. Therefore, task $n-1$ is never processed in parallel with other tasks and its processing time can be added to the left-hand side of (67). This leads to the new inequality below:

$$e_j+\sum_{k=0}^{m-1} d_{jk}y_{jk}+\sum_{i\in Q_j} d_{ik}y_{ik}+\sum_{h=0,h\neq k}^{m-1} d_{n-1,h}y_{n-1,h}\le C_{\max},\quad \text{for all } (j,k)\in N\times M. \tag{68}$$

4.2. **Predecessors' inequalities.** The aim of these inequalities is to take into account the total processing time of the tasks scheduled to processor k when computing the starting time of task j. If j is also scheduled to processor k, its starting time is at least as large as the sum of the processing times of the tasks that have been scheduled earlier in processor k. This class of inequalities is given by:

$$\sum_{i\in P_j} d_{ik}(y_{jk}+y_{ik}-1)+\sum_{i\in R_j} d_{ik}(y_{jk}+y_{ik}-z_{ji}-1)\le e_j,\quad \text{for all } (j,k)\in N\times M \tag{69}$$

The validity of inequality (69) is now discussed.

Assume that $i\in P_j$ and both tasks i and j are processed in processor k. In this case, the value of the first summation of (69) is d_{ik} while in any other case this value is nonpositive. We now turn our attention to the second summation. When i is in R_j, if task j is not executed in processor k, then $y_{jk}=0$ and $y_{jk}+y_{ik}-z_{ji}-1\le 0$

which makes the inequality redundant. On the other hand if task j is executed in processor k, then $y_{jk} = 1$ and three situations may occur:

1. If i is executed in processor k and prior to j, then $y_{jk} + y_{ik} - z_{ji} - 1 = 1$ and the contribution of task i to the summation is d_{ik}
2. If i is executed in processor k and later than j,then $y_{jk} + y_{ik} - z_{ji} - 1 = 0$ and the task i does not contribute to the summation.
3. If i is not executed in processor k, then $y_{jk} + y_{ik} - z_{ji} - 1 = 0$ and the contribution of task i to the summation is null.

The arguments above ensure that the inequality is valid.

In analogy to what happened to the previous inequalities, a strengthen is possible if the task precedence graph has a unique source. In this situation, the task corresponding to the source vertex, say task 0, is not executed simultaneously with other tasks. Thus, whichever the processor executing task 0, its processing time can be added to the left-hand side of inequality (69) which yields the stronger inequality below:

$$\begin{aligned}(70)\quad & \sum_{h=0,h\neq k}^{m-1} d_{0h} y_{0h} + \sum_{i \in P_j} d_{ik}(y_{jk} + y_{ik} - 1) + \\ & \qquad \sum_{i \in R_j} d_{ik}(y_{jk} + y_{ik} - z_{ji} - 1) \quad \leq e_j, \text{ for all } (j,k) \in N \times M.\end{aligned}$$

5. Concluding Remarks

We proposed a new integer programming formulation for the problem of multiprocessor scheduling under precedence constraints. This formulation has much fewer integer variables than those which appeared earlier in the literature.

A subset of the constraints in this formulation has a strong combinatorial structure which defines the set of characteristic vectors of the partitions of a graph into linear orders. A polyhedral investigation of the convex hull of these vectors yielded several results on facet defining inequalities for this new polytope. Further research in this direction will be helpful to strengthen the integer programming formulations of a large variety of multiprocessor scheduling problems.

We have also designed and implemented a branch-and-cut algorithm based upon families of strong valid inequalities presented in this paper. Computational experiments on a set of real-life instances have shown that the algorithm is capable to solve many instances to optimality. Most of the duality gaps for these instances have been reduced to less than 10%. The structure of the branch-and-cut algorithm and detailed computational results will be reported elsewhere in a forthcoming paper.

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