

Stable Matchings for A Generalized Marriage Problem

By

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Abstract

We show, that a simple generalization of the Deferred Acceptance Procedure with men proposing due to Gale and Shapley (1962), yields outcomes for a generalized marriage problem, which are necessarily stable. We also show, that any outcome of this procedure is Weakly Pareto Optimal for Men, i.e. there is no other outcome which all men prefer to an outcome of this procedure. In a final concluding section of this paper, we consider the problem of choosing a set of multi-party contracts, where each coalition of agents has a non-empty finite set of feasible contracts to choose from. We call such problems, generalized contract choice problems. The model we propose is a generalization of the model due to Shapley and Scarf (1974) called the housing market. We are able to show with the help of a three agent example, that there exists a generalized contract choice problem, which does not admit any stable outcome.

Key Words and phrases: stable outcomes, matchings, pay-offs, generalized marriage problem, contract choice problem.

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1. Introduction :

A salient feature of many markets is to match one agent with another. This is particularly true, in the case of assigning tasks to individuals where each task is under the supervision of an individual, and where the set of supervisors and the set of workers are disjoint. Such markets are usually studied with the help of “two sided matching models” introduced by Gale and Shapley (1962) called the marriage problem. However, not all matching problems where disjoint pairs are required to form, is dichotomous. The problem of forming disjoint pairs out of a given set of agents is what Gale and Shapley(1962) called a room-mates problem. The marriage problem is indeed a special case of their room-mates problem. The solution concept proposed by Gale and Shapley (1962), called a stable matching, requires that there should not exist two agents, who prefer each other, to the individual they have been paired with. It was shown in Gale and Shapley (1962), in a framework where every agent has preference defined by a linear order over the entire set of agents, that a room-mates problem may not admit any stable matching although a marriage problem always does. Indeed, given a marriage

problem, there is always a stable matching which no man considers inferior to any other stable matching, and there is always a stable matching that no woman considers inferior to any other stable matching. The first is called an M – optimal stable matching (i.e. stable matching optimal for men) and the second one a W – optimal stable matching (i.e. stable matching optimal for women). An overview of the considerable literature on marriage problems that has evolved out of the work of Gale and Shapley (1962), is available in Roth and Sotomayor (1990). Lahiri (2002) contains alternative simpler proofs of some existing results and some new conclusions for two-sided matching problems. While the body of literature on existence of stable matchings for room-mates problems is quite enormous, there has been very little investigation of the properties of such matchings, if and when they do exist. Lahiri (2003) attempts to fill this gap by analyzing the co-operative theory of stable matchings for room-mates problems.

Eriksson and Karlander (1998) considers an interesting common generalization of the marriage model due to Gale and Shapley (1962) and the assignment model of Shapley and Shubik (1972). They propose a model of a two-sided matching model, where a pair of agents each on a different side of the market produce a good which is either divided among them in a fixed proportion which is exogenously specified for the pair (ρ : in which case the pair is said to be rigid) or is divided arbitrarily among them (ρ : in which case the pair is said to be flexible). They propose the concept of a stable outcome and prove the existence of one, when the good is to be distributed is available in indivisible units.

In this paper, we propose a generalization of the model due to Eriksson and Karlander (1998) in the context of room-mates problems. We allow each pair of agents a non-empty finite set of real valued allocations of a good to choose from. Each agent is assumed to prefer more of the good to less of it. If each pair of agents is provided singletons to choose from, then we have the room-mates problem of Gale and Shapley (1962). On the other hand, if the set of agents can be divided into two disjoint sets, with one set being the set of men and the other the set of women, with no pair of the same sex being able to obtain an allocation which is at least as good as an allocation that could be obtained by them remaining single or by forming a pair with a member of the opposite sex, then we obtain a generalization of the marriage problem due to Gale and Shapley (1962). We say that an outcome(i.e. a bijection of order two from the set of agents into itself along with an allocation which is feasible for every matched pair) is stable, if there is no pair of agents who can obtain an allocation that both prefer to the ones they have been assigned. We show, that a simple generalization of the Deferred Acceptance Procedure with men proposing due to Gale and Shapley (1962), yields outcomes for the generalized marriage problem, which are necessarily stable. The main difference between the procedure we define and the Deferred Acceptance Procedure, is that a man can propose to the same woman several times. We also show, that any outcome of this procedure is Weakly Pareto Optimal for Men, i.e. there is no other outcome which all men prefer to an outcome of this procedure. This result is an extension to our framework, of a similar result due to Roth and Sotomayor (1990).

As in Sotomayor (1996), it is possible to provide a non-constructive proof of the existence of a stable outcome, in the framework of a generalized marriage problem. Such a proof is essentially non-algorithmic although as Sotomayor (1996) shows, is much simpler than its procedural counterpart. A consequence of such a proof is the absence of an explicit "design" for a stable outcome.

An immediate extension of our model is one, where each pair can choose from a non-empty finite set of contracts, on which each has preferences represented by a weak-order (i.e. a reflexive, complete and transitive binary relation). The entire analysis of this paper can be replicated in this more general framework. However, we confine our "story", to the allocation of a single individual object among pairs of agents, in order to endow the model with a simple yet tangible structure.

In a final concluding section of this paper, we consider the problem of choosing a set of multi-party contracts, where each coalition of agents has a non-empty finite set of feasible contracts to choose from. We call such problems, generalized contract choice problems. The model we propose is a generalization of the model due to Shapley and Scarf (1974) called the housing market. Shapley and Scarf (1974), considers a private ownership economy, where each individual owns exactly one object and what is sought is the existence of an allocation in the core of the economy. In the context of our generalized contract choice model, the Shapley-Scarf housing market corresponds to a situation, where each individual assigns a monetary worth to each object, and a feasible pay-off vector for a coalition, is the set of utility vectors available to the coalition, when it re-allocates objects within itself, without any one in the coalition retaining his initial endowment, unless the coalition is a singleton. Roth and Postelwaite (1977) used Gale's *Top Trading Cycle Algorithm* to show that if preferences are strict, then there exists a unique competitive equilibrium allocation, which is also the unique core allocation, for a Shapley-Scarf housing market. However, we are able to show with the help of a three agent example, that there exists a generalized contract choice problem, which does not admit any stable outcome.

2. The Model :

Let X be a non-empty finite subset of \mathbb{N} (the set of natural numbers), denoting the set of participating agents.

Let \mathfrak{R} denote the set of real numbers and \mathfrak{R}_+ the set of non-negative real numbers.

A generalized room-mates problem G is an ordered pair $\langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$ such that for all $a, b \in X$: (i) there exists a non-empty finite set $F(a, b)$ of \mathfrak{R}^2 satisfying $F(b,a) = \{(x,y) / (y,x) \in F(a,b)\}$; (ii) $F(a,a) = \{(0,0)\}$.

For $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$ and $a, b \in X$, $F(a,b)$ is the set of all feasible yields to a and b , such that the division (x,y) in $F(a,b)$ assigns an amount x to a and y to b .

Given a room-mates problem $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$, a matching for G is a bijection μ from X to itself such that for all $a \in X$: $\mu(\mu(a)) = a$.

Since the identity function on X is a matching every generalized room-mates problem admits at least one matching.

A pay-off function is a function $v : X \rightarrow \mathfrak{R}_+$.

An outcome for a generalized room-mates problem $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$ is a pair (μ, v) , where μ is a matching for G and v is a pay-off function such that (i) for all $a \in X$: $v(a) \geq 0$; (ii) for all $a \in X$: $(v(a), v(\mu(a))) \in F(a, \mu(a))$.

The pair (μ, v) , where μ is the identity function on X and $v(a) = 0$ for all $a \in X$, is an outcome for every generalized room-mates problem. Hence the set of outcomes is always non-empty.

Given an outcome (μ, v) for a generalized room-mates problem $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$, a pair $(a,b) \in X \times X$ is said to block (μ, v) if there exists $(x,y) \in F(a,b)$: $x > v(a)$ and $y > v(b)$.

An outcome (μ, v) for a generalized room-mates problem $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$ is said to be stable if it does not admit any blocking pair.

A generalized room-mates problem $G = \langle X, (F(a,b))_{(a,b) \in X \times X} \rangle$ is called a generalized marriage problem if there exists two non-empty disjoint subsets M and W of X , and a positive real number r such that (i) $M \cup W = X$; (ii) for all $m, m' \in M$, and $w, w' \in W$: $F(m, m') = F(w, w') = \{(-r, -r)\}$; (iii) for all $m \in M$, and $w \in W$: $(x, y) \in F(m, w)$ implies $x > -r$ and $y > -r$.

We represent a generalized marriage problem as $\langle (M, W), (F(a,b))_{a,b \in M \cup W} \rangle$. M is called the set of men and W the set of women.

3. Existence of Stable Outcomes for the Generalized Marriage Problem:

Theorem 1: Every generalized marriage problem admits a stable outcome.

Proof: Let $G = \langle (M, W), (F(a,b))_{(a,b) \in (M \cup W) \times (M \cup W)} \rangle$ be a given generalized marriage problem, and let $m \in M$ and $w \in W$. Let $W^*(m) = \bigcup_{w' \in W} (\{w'\} \times F(m, w'))$

$\cup \{(m, (0,0))\}$ and $M^*(w) = \bigcup_{m' \in M} (\{m'\} \times F(w, m')) \cup \{(w, (0,0))\}$.

m has preferences defined by a binary relation \geq_m over $W^*(m)$ satisfying the following property: for all $(a, (x,y)), (b, (x', y')) \in W^*(m)$: $(a, (x,y)) \geq_m (b, (x', y'))$ if and only if $x \geq x'$. Similarly, w has preferences defined by a binary relation \geq_w over $M^*(w)$ satisfying the following property: for all $(a, (x,y)), (b, (x', y')) \in M^*(w)$: $(a, (x,y)) \geq_w (b, (x', y'))$ if and only if $x \geq x'$.

Let $>_m$ denote the asymmetric part of \geq_m and $>_w$ denote the asymmetric part of \geq_w .

Let $W^{**}(m) = \{(w', (x,y)) \in W^*(m) / (w', (x,y)) >_m (m, (0,0))\}$ and $M^{**}(w) = \{(m', (x,y)) \in M^*(w) / (m', (x,y)) >_w (w, (0,0))\}$

Given $m \in M$ and an element A of $W^{**}(m)$, let $A|_W = w'$, where $(w', (x,y)) = A$.

Given $w \in W$ and an element A of $M^{**}(w)$, let $A|_M = m'$, where $(m', (x,y)) = A$.

Given a subset S of $\bigcup_{m \in M} W^*(m) \cup \bigcup_{w \in W} M^*(w)$, $m \in M$ and $w \in W$, let $U(m, S) =$

$\{(a, (x', y')) \in S \cap W^*(m) / \text{there does not exist } (b, (x'', y'')) \in S: (b, (x'', y'')) >_m (a, (x', y'))\}$ and $U(w, S) = \{(a, (x', y')) \in S \cap M^*(w) / \text{there does not exist } (b, (x'', y'')) \in S: (b, (x'', y'')) >_w (a, (x', y'))\}$.

Let $M^1 = \{m \in M / W^{**}(m) \neq \emptyset\}$. For $m \in M^1$, let $P^1(m) \in U(m, W^{**}(m))$ where $(w, (x, y)) = P^1(m)$ implies 'm' proposes to 'w' the division in $F(m, w)$ where 'm' gets 'x' and 'w' gets y. Each $m \in M^1$ proposes to the woman $P^1(m)|_w$. For $w \in \{P^1(m) / m \in M^1\}$, let $R^1(w) = \{(m, (y, x)) / (w, (x, y)) = P^1(m)\}$, $R_+^1(w) = R^1(w) \cap M^{**}(w)$ and $E^1(w)$ be any element of $U(w, R_+^1(w))$. Each 'w' receiving a proposal, rejects all proposals in $R^1(w) \setminus \{E^1(w)\}$. The proposal $E^1(w)$ is kept engaged by 'w'. Only those men who are not kept engaged at this step, are allowed to propose at the subsequent stage.

Suppose that the procedure continues to a stage 'k', $k \geq 1$, with $M^k, P^k(m)$ for $m \in M^k, R^k(w), R_+^k(w)$ and $E^k(w)$ for $w \in \{P^k(m) / m \in M^k\}$ having been defined.

The procedure stops if $M^{k+1} = \{m \in M^1 / \text{all the proposals made by 'm' at the previous step were rejected and } W^{**}(m) \setminus \bigcup_{j=1}^k \{P^j(m)\} \neq \emptyset\} = \emptyset$. If $M^{k+1} \neq \emptyset$, then for

$m \in M^{k+1}$, let $P^{k+1}(m) \in U(m, W^{**}(m) \setminus \bigcup_{j=1}^k P^j(m))$. Each $m \in M^{k+1}$ proposes to the woman in $P^{k+1}(m)|_w$. If $(w, (x, y)) = P^{k+1}(m)$, then 'm' proposes to 'w' the division where 'm' gets 'x' and 'w' gets y. For $w \in \bigcup_{m \in M^{k+1}} \{P^{k+1}(m)|_w\}$, let $R^{k+1}(w) = \{(m, (y, x)) / (w, (x, y)) = P^{k+1}(m)\}$ and $R_+^{k+1}(w) = R^{k+1}(w) \cap M^{**}(w)$. Let $E^{k+1}(w)$ be any element of $U(w, E^k(w) \cup R_+^{k+1}(w))$. The proposal $E^{k+1}(w)$ is kept engaged by 'w' at this step. The remaining proposals in $\{E^k(w)\} \cup R^{k+1}(w)$ are rejected.

Since $M \cup W$ is finite, there exists a stage K when $M^K = \emptyset$. At this stage every $m \in M^1$ is either engaged to some woman or has been rejected by every woman in $W^{**}(m)$. Further, every woman $w \in W$ for whom $M^{**}(w) \neq \emptyset$ has either not received any proposal or is engaged to a man.

Define an outcome (μ, v) as follows: for all $a \in \{m \in M / W^{**}(m) = \emptyset\} \cup \{w \in W / M^{**}(w) = \emptyset\}$, let $\mu(a) = a$ and $v(a) = 0$. For all $w \in W$, who never received a proposal or rejected each and every that she received, let $\mu(w) = w$ and $v(w) = 0$. For all $m \in M$, who have been rejected by every woman he has proposed to let $\mu(m) = m$ and $v(m) = 0$. The remaining women are the ones who are engaged at stage K . If $E^K(w) = (m, (y, x))$, then let $(\mu(w), v(w)) = (m, y)$ and $(\mu(m), v(m)) = (w, x)$.

Suppose there exists a pair $(m, w) \in M \times W$ such that (m, w) blocks (μ, v) . Thus, there exists $(x, y) \in F(m, w)$ such that $x > v(m)$ and $y > v(w)$. Thus, $(w, (x, y)) >_m (\mu(m), (v(m), v(\mu(m))))$. Thus, 'm' must have proposed $(w, (x, y))$ to 'w' and was rejected by 'w' in favor of some other proposal before he proposed $(\mu(m), (v(m), v(\mu(m))))$ to $\mu(m)$. Since \geq_w is transitive, it must be the case that $(\mu(w), (v(w), v(\mu(w)))) \geq_w (m, (y, x))$. This contradicts $y > v(w)$ and proves the theorem. Q.E.D.

Let O be the set of outcomes of the procedure defined in the proof of Theorem 1. Clearly O though non-empty and finite can admit more than one element.

An immediate consequence of the procedure, used in the proof of Theorem 1, is the following result.

Proposition 1: **Weak Pareto Optimality for Men:** Let $(\mu^*, v^*) \in O$. Then, **there does not exist any outcome which every man prefers to (μ^*, v^*) .**

Proof: If $\#M > \#W$, then there is no way in which the proposition can be falsified, since in every matching some man must be without a woman. On the other hand, every woman who is single at (μ^*, v^*) , continues to remain so at any other matching, where all men are better off. This is because, according to the procedure defined in Theorem 1, a woman who is single, either rejected all the proposals she received preferring to remain single, or every man considers his outcome at (μ^*, v^*) to be at least as good as any allocation that is feasible when he is paired with this woman. Hence, we can assume that μ^* maps M onto W , and in particular $\#M = \#W$.

Towards a contradiction suppose there is an outcome (μ, v) such that $v(m) > v^*(m)$ for all $m \in M$. This in particular implies that μ maps M onto W . Let m^* be the man whose proposal was accepted at the last stage of the procedure defined in Theorem 1. Let w^* be the woman who accepted his offer. If $(w^*, v(m^*), v(w^*))$ was the only offer that w^* had received, then $(w^*, v(m), v(w^*))$ could not have been preferred to $(\mu^*(m), v(m), v(\mu^*(m)))$ by any $m \neq m^*$. Thus, there could be no man to whom $(w^*, v(m), v(w^*))$ could be assigned under (μ, v) leading to an improvement for him over (μ^*, v^*) . Thus, there must have been some other proposal (w^*, x, y) made by an $m \neq m^*$, which was rejected by w^* in favor of $(m^*, v(w^*), v(m^*))$. Hence, m is assigned no woman under the μ^* , contradicting that μ^* maps M onto W . This proves the proposition. Q.E.D.

4. Conclusion :

In this concluding section, we consider a possible generalization of the Shapley-Scarf (1974) model, which permits any nonempty subset of agents to choose from a non-empty finite set of contracts.

As before, let X be a non-empty finite subset of \mathbb{N} (the set of natural numbers), denoting the set of participating agents. We assume that each agent prefers more money to less.

Let $[X]$ denote the set of all non-empty subsets of X . Members of $[X]$ are called coalitions.

Given $S \in [X]$, let $C(S) = \{\mu / \mu \text{ is a bijection on } X \text{ with } \mu(S) = S\}$ and $C^0(S) = \{\mu \in C(S) / T \text{ is a non-empty proper subset of } S \text{ implies } \mu(T) \neq T\}$.

A generalized contract choice problem G is an ordered pair $\langle X, (F(S))_{S \in [X]} \rangle$ such that for all $S \in [X]$: (i) $F(S)$ is a non-empty finite subset of \mathfrak{R}^S ; (ii) $F(\{a\}) = \{0\}$.

For $G = \langle X, (F(a,b))_{S \in [X]} \rangle$ and $S \in [X]$, $F(S)$ is the set of all feasible allocations of money for agents in S .

A generalized contract choice problem $G = \langle X, (F(S))_{S \in [X]} \rangle$ is said to be a Shapley-Scarf housing market if for all $a \in X$ there exists a function $u^a : X \rightarrow \mathfrak{R}$ satisfying the following property: for all $S \in [X]$, $F(S) = \{x \in \mathfrak{R}^S / \text{for some } \mu \in C^0(S), x(a) = u^a(\mu(a)) \text{ for all } a \in S\}$.

The requirement that a Shapley-Scarf housing market $G = \langle X, (F(S))_{S \in [X]} \rangle$ is a contract choice problem, implies that $F(\{a\}) = \{0\}$ for all $a \in S$. Thus, $F(\{a\}) = \{u^a(a)\}$ implies, $u^a(a) = 0$ for all $a \in S$.

Given a generalized contract choice problem $G = \langle X, (F(S))_{S \in [X]} \rangle$, a coalition structure for G is a partition of X .

A pay-off function is a function $v : X \rightarrow \mathfrak{R}_+$. If v is a pay-off function and $S \in [X]$, then $v|_S$ denotes the restriction of v to the set S .

An outcome for a generalized contract choice problem $G = \langle X, (F(S))_{S \in [X]} \rangle$ is a pair (f, v) , where f is a coalition structure for G and v is a pay-off function such that (i) for all $a \in X$: $v(a) \geq 0$; (ii) for all $S \in f$: $v|_S \in F(S)$.

The pair (f, v) , where $f = \{\{a\} / a \in X\}$ and $v(a) = 0$ for all $a \in X$, is an outcome for every generalized contract choice problem. Hence the set of outcomes is always non-empty.

Given an outcome (f, v) for a generalized contract choice problem $G = \langle X, (F(S))_{S \in [X]} \rangle$, a coalition $S \in [X]$ is said to block (f, v) if there exists $x \in F(S)$: $x(a) > v(a)$ for all $a \in S$.

An outcome (f, v) for a generalized contract choice problem $G = \langle X, (F(S))_{S \in [X]} \rangle$ is said to belong to the core of G , if it does not admit any blocking coalition.

The following result due to Roth and Postelwaite [1977] is well known:

If G is a Shapley-Scarf housing market, then there exists at least one outcome belonging to the core of G .

However for a generalized contract choice problem the result need not hold, as the following example reveals.

Example: Let $X = \{1,2,3\}$. Let $F(\{1,2\}) = \{(4(1),3(2))\}$, $F(\{1,3\}) = \{(3(1),4(3))\}$, $F(\{2,3\}) = \{(4(2),3(3))\}$, $F(\{1,2,3\}) = \{(2,1,2), (1,2,1)\}$, where for $k \in \mathbb{Z}$, $a \in X$, $k(a)$ denotes agent 'a' gets 'k' units of money.

The outcome (f, v) where $f = \{\{a\} / a \in X\}$ and $v(a) = 0$ for all $a \in X$ is blocked by $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$.

The outcome (f, v) where $f = \{\{1,2\}, \{3\}\}$ and $(v(1), v(2), v(3)) = (4, 3, 0)$ is blocked by the coalition $\{2,3\}$.

The outcome (f, v) where $f = \{\{1,3\}, \{2\}\}$ and $(v(1), v(2), v(3)) = (3, 0, 4)$ is blocked by the coalition $\{1,2\}$.

The outcome (f, v) where $f = \{\{2,3\}, \{1\}\}$ and $(v(1), v(2), v(3)) = (0, 4, 3)$ is blocked by the coalition $\{1,3\}$.

The outcome (f, v) where $f = \{\{1,2, 3\}\}$ and $(v(1), v(2), v(3)) = (2, 1, 2)$ is blocked by the coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$.

The outcome (f, v) where $f = \{\{1,2, 3\}\}$ and $(v(1), v(2), v(3)) = (1, 2, 1)$ is blocked by the coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$.

Hence there does not exist any outcome which belongs to the core of G .

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