On the optimal parameter of a self-concordant barrier over a symmetric cone

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Abstract

The properties of the barrier $F(x) = -log(det(x))$, defined over the cone of squares of a Euclidean Jordan algebra, are analyzed using pure algebraic techniques. Furthermore, relating the Carathéodory number of a symmetric cone with the rank of an underlying Euclidean Jordan algebra, conclusions about the optimal parameter of F are suitably obtained. Namely, in a more direct and suitable way than the one presented by Osman Güler and Levent Tunçel (*Characterization of the barrier parameter of homogeneous convex cones*, Mathematical Programming, 81 (1998): 55-76), it is proved that the Carathéodory number of the cone of squares of a Euclidean Jordan algebra is equal to the rank of the algebra. Then, taking into account the result obtained in the same paper where it is stated that the Carathéodory number of a symmetric cone Q is the optimal parameter of a self-concordant barrier defined over Q, we may conclude that the rank of every underlying Euclidean Jordan algebra is also the self-concordant barrier optimal parameter.

Keywords: Symmetric cones, self-concordant barriers, optimal parameters.

1 Introduction

Faybusovich was the first author to consider interior point methods in the general context of Euclidean Jordan algebras [3, 4, 5]. Tsuchiya also applied a Jordan algebraic approach to the optimization over the Lorentz cone, which is a particular symmetric cone [14, 15]. Recently, Alizadeh and Schmieta have published several papers studying interior point techniques within

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the framework of Euclidean Jordan algebras [1, 12, 13]. Self-concordant barriers, introduced in [10], play an important role in interior point methods applied to conic programming, and the determination of the optimal barrier parameter is crucial to these methods when applied to convex programming over symmetric cones. From [7] we may conclude that this optimal value is equal to the rank of an underlying Euclidean Jordan algebra. However, in this paper, in a more direct and suitable way, following a pure algebraic approach, we prove that the optimal parameter of a self-concordant barrier over a symmetric cone is equal to the rank of every underlying Euclidean Jordan algebra.

A deep study on Euclidean Jordan algebras is done in [2, 9]. However, in order to follow the results of this paper, it is sufficient to read the short introduction recently given in [13] and the next section, where the concepts and theorems more often used in this work are shortly presented. In section 3, the main properties of the log-determinant barrier are analyzed. Finally, in section 4, the main result of this paper, which states that the rank of a Euclidean Jordan algebra is equal to the Carathéodory number of its cone of squares, is proved. Therefore, since a cone is symmetric if and only if it is the interior of the cone of squares of some Euclidean Jordan algebra [2] and, according to [7], the optimal parameter of a self-concordant barrier over a symmetric cone is equal to its Carathéodory number, the rank of a Euclidean Jordan algebra is the optimal parameter of a self-concordant barrier defined over its cone of squares. A more general result was obtained in [8], where it is stated that "Any self-concordant barrier function for a convex set S has self-concordance parameter at least k if there exists an affine subspace U such that $S \cap U$ contains a vertex at which precisely k linearly independent smooth constraints are active".

2 Basic results on Euclidean Jordan algebras

Consider a n dimensional real vector space V with a multiplication $\circ$ such that the map $(x, y) \to x \circ y$ is bilinear. For an element x in V, $L(x)$ is the linear map of V defined by $L(x)y = x \circ y$. If for all $x \in V$ $(x \circ x) \circ x = x \circ (x \circ x)$, then V is called a *power associative* algebra. Let V be a power associative algebra with unit element $\mathbf{e}$, $x \in V$, and let k be the least natural number such that $\{e, x, x^2, \ldots, x^k\}$ is linear dependent. Then k is the *rank* of x and we write $rank(x) = k$. We define the rank of V as being the natural number $r = \text{rank}(V) = \max\{rank(x) : x \in V\}$. An element $x \in V$ is *regular* if its rank is equal to the rank of the algebra. Given a regular element x

in a power associative algebra V with unit element $\mathbf{e}$ and rank r, we may conclude that there are r unique real numbers $a_1(x)$, $a_2(x)$, ..., $a_r(x)$ such that

$$x^r - a_1(x)x^{r-1} + \ldots + (-1)^r a_r(x)\mathbf{e} = 0, \tag{1}$$

where 0 is the null vector of V. Taking into account (1), the polynomial

$$p(x,\lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \ldots + (-1)^r a_r(x) \tag{2}$$

is called the *characteristic polynomial* of x, where each coefficient a_i is a homogeneous polynomial of degree i. The definition of the characteristic polynomial may be extended to any element of V. Indeed, since the set of regular elements of V is a dense set in V [2], if $x \in V$ then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of regular elements of V converging to x. Defining $a_i(x) = \lim_{n\to\infty} a_i(x_n) = a_i(\lim_{n\to\infty} x_n)$, we obtain the characteristic polynomial of a non regular element as being the polynomial (2). Jordan algebras with unit element are examples of power associative real algebras with unit element.

Definition 2.1. *Let V be a finite dimensional real vector space, with the operation of multiplication of vectors $\circ$ determined by the bilinear function $(x,y) \to x \circ y$. We say that V is a Jordan algebra if for all $x, y \in V$*

i) $x \circ y = y \circ x$

ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, *where* $x^2 = x \circ x$.

From now on, a Jordan algebra V is a finite dimensional real algebra with unit element $\mathbf{e}$. Since, as it is proved in [9], a Jordan algebra V is power associative, each element $x \in V$ has its characteristic polynomial.

Thus, if r is the rank of V and the characteristic polynomial of x is the polynomial (2) then we define the *determinant* and the *trace* of x, respectively, by the equalities $\mathrm{tr}(x) = a_1(x)$ and $\det(x) = a_r(x)$. Given a Jordan algebra V we say that the element $x \in V$ is *invertible* if $\exists y \in \mathbb{R}[x]$ [1] such that $x \circ y = \mathbf{e}$, and then y is the inverse of x and it is denoted by x^{-1}.

A real Jordan algebra V is *Euclidean* if there is an inner product $\langle \cdot, \cdot \rangle$ such that $\langle u \circ v, w \rangle = \langle u, v \circ w \rangle$ for all $u, v, w \in V$. Additionally, two elements $c, d \in V$ are *orthogonal* relatively to the algebra V if $c \circ d = 0$. Assuming that $\mathbf{e}$ is the unit element of V, $c \in V$ is an *idempotent* if $c^2 = c$ and

[1] Subalgebra of V spanned by $\mathbf{e}$ and x.

$\{c_1, c_2, \dots, c_k\}$ is a *complete system of orthogonal idempotents* if

$$\begin{array}{ll} (i) & c_i^2 = c_i \;\; \forall i \in \{1, \dots, k\}, \\ (ii) & c_i \circ c_j = 0 \;\; \forall i \neq j, \\ (iii) & c_1 + c_2 + \cdots + c_k = \mathbf{e}. \end{array}$$

An idempotent c is *primitive* if it is not the sum of two non null idempotents. We say that $\{c_1, c_2, \dots, c_k\}$ is a complete system of orthogonal primitive idempotents or a *Jordan frame* if $\{c_1, c_2, \dots, c_k\}$ is a complete system of orthogonal idempotents such that each idempotent is primitive.

Theorem 2.1 ([2], p. 43)**.** *Let V be a Euclidean Jordan algebra and $x \in V$. Then there are k unique real numbers $\lambda_1, \lambda_2, \dots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $\{c_1, c_2, \dots, c_k\}$ such that*

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_k c_k. \tag{3}$$

Additionally, $c_j \in \mathbb{R}[x]$, for $j = 1, \dots, k$.

The numbers λ_j of (3) are called the *eigenvalues* of x and $x = \sum_{i=1}^{k} \lambda_i c_i$ is the *spectral decomposition* of x.

Theorem 2.2 ([2], p. 44)**.** *Let V be a Euclidean Jordan algebra with rank r. Then, for each $x \in V$ there is a Jordan frame $\{c_1, c_2, \dots, c_r\}$ and there are real numbers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that*

$$x = \sum_{j=1}^{r} \lambda_j c_j.$$

The λ_is, together with their multiplicities, are uniquely determined by x. Furthermore, $\det(x) = \Pi_{j=1}^{r} \lambda_j$ and $tr(x) = \sum_{j=1}^{r} \lambda_j$.

If V is a Euclidean Jordan algebra with rank r and $\mathbf{c}$ is a primitive idempotent of V then $\mathrm{tr}(c) = 1$ and, therefore, we may conclude that $\mathrm{tr}(\mathbf{e}) = r$.

Definition 2.2. *A Euclidean Jordan algebra V is simple or irreducible if V does not contain any non-trivial ideal.*

Theorem 2.3 ([2], p. 54)**.** *Every Euclidean Jordan algebra V is, in a unique way, the direct sum of l simple Euclidean Jordan subalgebras V_i of V which are ideals of V.*

If V is a Euclidean Jordan algebra then its cone of squares is the set

$$Q = \{x^2 : x \in V\}.$$

From now on, given a cone C, $int(C)$ will denotes the interior of C.

Theorem 2.4. *Let V be a Euclidean Jordan algebra with rank r and Q its cone of squares. If $x \in V$ is such that $x = \sum_{i=1}^{r} \lambda_i c_i$, where $\{c_1, c_2, \ldots, c_r\}$ is a Jordan frame of V, then*

(i) $\lambda_i \geq 0$ for $i = 1, \ldots, r$, if and only if $x \in Q$.

(ii) $\lambda_i > 0$ for $i = 1, \ldots, r$, if and only if $x \in int(Q)$.

Proof. The proof is a consequence of Theorem III.2.1 in [2]. ∎

If V is a Euclidean Jordan algebra with rank r and $x \in V$ has the spectral decomposition $x = \sum_{i=1}^{k} \lambda_i c_i$ then x is *positive definite* if $\lambda_i > 0$ for all i, and is *positive semidefinite* if $\lambda_i \geq 0$ for all i. According to Theorem 2.4 we may redefine Q and $\text{int}(Q)$, respectively, by $Q = \{x \in V : x \text{ is positive semidefinite}\}$ and $\text{int}(Q) = \{x \in V : x \text{ is positive definite}\}$.

Theorem 2.5. *Let V be a Euclidean Jordan algebra, Q its cone of squares and $x \in int(Q)$. Then x is invertible and $x^{-1} \in int(Q)$.*

Proof. Consider $x \in \text{int}(Q)$. By Theorem III.2.1 in [2], x is invertible. From Theorems 2.2 and 2.4 it follows that $x = \sum_{i=1}^{r} \lambda_i c_i$ with all $\lambda_i > 0$ and $\{c_1, \ldots, c_r\}$ a Jordan frame. Therefore $x^{-1} = \sum_{i=1}^{r} \frac{1}{\lambda_i} c_i$ and $x^{-1} \in \text{int}(Q)$. ∎

The *quadratic representation* P of a Jordan algebra V is the function

$$\begin{aligned} P : V &\mapsto \text{End}(V) \\ x &\mapsto P(x), \end{aligned}$$

where $\text{End}(V)$ denotes the set of endomorphisms of V and P is such that

$$P(x) = 2L^2(x) - L(x^2) \;\; \forall x \in V. \tag{4}$$

Let V be a Euclidean Jordan algebra. Then the quadratic representation P of V has the following properties:

1. If $\bar{x} \in V$ then $\bar{x}$ is invertible if and only if $P(\bar{x})$ is invertible.

2. $\forall x, y \in V \;\; P(P(x)y) = P(x)P(y)P(x)$.

From property 2 of P we may conclude that in a Jordan algebra V

$$P(x^n) = (P(x))^n \;\; \forall x \in V \; \forall n \in \mathbb{N}. \tag{5}$$

Theorem 2.6. *Let Q be the cone of squares of a Euclidean Jordan algebra V and $\langle\cdot,\cdot\rangle$ the inner product defined by $\langle x, y\rangle = tr(x \circ y)$. Then, considering the function*

$$\begin{array}{rcl} F : int(Q) & \to & \mathbb{R} \\ x & \to & -\log\det x, \end{array}$$

for all $x \in int(Q)$ and for all $h \in V$,

$$DF(x)[h] = \langle -x^{-1}, h\rangle, \tag{6}$$

$$D^2F(x)[h,h] = \langle P^{-1}(x)h, h\rangle. \tag{7}$$

Proof. The equality (6) is stated by Proposition III.4.2 in [2]. The equality (7) is a direct consequence of Proposition II.3.3 in [2]. ∎

Let V be a Euclidean Jordan algebra with rank r and $x \in V$ an invertible element whose spectral decomposition is $x = \sum_{i=1}^{k} \lambda_i c_i$. If all $\lambda_i \geq 0$ we define $x^{\frac{1}{2}}$ as being the element $x^{\frac{1}{2}} = \sum_{i=1}^{k} \sqrt{\lambda_i} c_i$. Furthermore, if all $\lambda_i > 0$ we define $x^{-\frac{1}{2}} = (x^{\frac{1}{2}})^{-1}$.

Theorem 2.7. *If V is a Euclidean Jordan algebra and $x \in V$ is positive definite then $P(x)$ is positive definite, $P(x) = P(x^{\frac{1}{2}})P(x^{\frac{1}{2}})$ and $P^{-1}(x^{\frac{1}{2}}) = P^{-\frac{1}{2}}(x)$.*

Proof. Let x be a positive definite element of V. Then, x is in the interior of the cone of squares Q of V, that is, there exists $x^{\frac{1}{2}} \in Q$ such that $x = P(x^{\frac{1}{2}})\mathbf{e}$. Therefore,

$$P(x) = P(x^{\frac{1}{2}})P(\mathbf{e})P(x^{\frac{1}{2}}) = P(x^{\frac{1}{2}})P(x^{\frac{1}{2}})$$

and $P(x)$ is positive definite (see [2], p. 55). Furthermore $x^{\frac{1}{2}}$ is invertible and, since $P^{\frac{1}{2}}(x) = P(x^{\frac{1}{2}})$, it follows that

$$P^{-1}(x^{\frac{1}{2}}) = (P(x^{\frac{1}{2}}))^{-1} = (P^{\frac{1}{2}}(x))^{-1} = P^{-\frac{1}{2}}(x).$$

∎

A symmetric cone is an open, nonempty, self-dual and homogeneous cone (see [2], for details).

Theorem 2.8. *A cone is symmetric if and only if it is the interior of the cone of squares of some Euclidean Jordan algebra.*

Proof. Theorem III.2.1 in [2] states that the interior of the cone of squares of an Euclidean Jordan algebra is a symmetric cone. Conversely, Theorem III.3.1 in [2] states that a symmetric cone is the interior of the cone of squares of some Euclidean Jordan algebra. ∎

3 Some properties of the log-determinant barrier over a symmetric cone

Through this section we will consider the function

$$\begin{aligned} F : int(Q) &\rightarrow \mathbb{R} \\ x &\rightarrow F(x), \end{aligned} \tag{8}$$

where $F(x) = -\log(\det(x))$ and Q is the cone of squares of a Euclidean Jordan algebra with rank r.
We verify easily that the function F admits derivatives of any order. Indeed, since det is a polynomial, the function F is analytic in all points $x \in V$ such that $\det(x) > 0$. Therefore, since for all $x \in int(Q)$ $\quad det(x) > 0$, the conclusion follows. Additionally, we may conclude the following properties:

a) F is a strictly convex function.

b) F is a *barrier* for Q, that is, $\lim_{int(Q) \ni x \rightarrow \partial(Q)} F(x) = \infty$, where $\partial(Q)$ denotes the boundary of Q.

In fact, since $P^{-1}(x) = P(x^{-1})$ for $x \in \text{int}(Q)$ (see Proposition II.3.1 in [2]), using Theorems 2.4, 2.5, 2.6 and 2.7, the property a) follows. The property b) can be concluded, taking into account that by Theorems 2.2 and 2.4 $x \in \text{int}(Q)$ if and only if $\det(x) \neq 0$, F is continuous in $\text{int}(Q)$ and then

$$\lim_{int(Q) \ni x \rightarrow \partial(Q)} F(x) = F(\lim_{int(Q) \ni x \rightarrow \partial(Q)} x).$$

It will now be proved that the function F is a r-self-concordant barrier, where $r = r(V)$ denotes the rank of V. Before that, some concepts are defined as in [10].

Definition 3.1. *Let E be a finite dimensional vector space over $\mathbb{R}$, $K \subset E$ a pointed, closed, convex cone with nonempty interior, and $G : int(K) \rightarrow \mathbb{R}$ a C^3-smooth convex function.*

1. *If for all $x \in int(K)$ and for all $h \in E$ the following inequality holds:*

$$|D^3G(x)[h,h,h]| \leq 2(D^2G(x)[h,h])^{\frac{3}{2}}, \tag{9}$$

then G is called self-concordant function on $int(K)$.

2. *If G is a barrier for K which is a self-concordant function on $int(K)$ and there existes $\vartheta \geq 1$ such that*

$$(DG(x)[h]|)^2 \leq \vartheta D^2G(x)[h,h], \tag{10}$$

then G is called ϑ-self concordant barrier for K.

3. *If G is a barrier for K, $\gamma \geq 1$, and, for each $x \in int(K)$ and each $t > 0$,*

$$G(tx) = G(x) - \gamma \ln t, \tag{11}$$

then G is called γ-logarithmically homogeneous barrier for K.

4. *If G is a γ-logarithmically homogeneous barrier for K which is a self-concordant function on $int(K)$, then G is called γ-normal barrier for K.*

In Theorem 3.1 below, we prove that F is a self-concordant barrier function, and in Theorem 3.2, that F is r-logarithmically homogeneous. A proof that F is a self-concordant function has been also obtained, algebraically, by Schmieta [11] in a different way from the one presented in this paper.

Theorem 3.1. *F is a self-concordant function in $int(Q)$.*

Proof. Taking into account that F is a strictly convex function on $\text{int}(Q)$, we will establish this theorem by proving that for all $x \in \text{int}(Q)$ and for all $h \in V$

$$|D^3F(x)[h,h,h]| \leq 2(D^2F(x)[h,h])^{\frac{3}{2}}.$$

Let $x \in \text{int}(Q)$, $h \in V$ and let the inner product $\langle \cdot, \cdot \rangle$ be defined by $\langle u, v \rangle = \text{tr}(u \circ v)$. By Theorem 2.6 it follows that

$$\begin{aligned} DF(x)[h] &= \langle -x^{-1}, h \rangle \\ D^2F(x)[h,h] &= \langle P^{-1}(x)h, h \rangle. \end{aligned}$$

Thus, for $x \in \text{int}(Q)$, we have

$$\begin{aligned} D^3F(x)[h,h,h] &= \frac{\partial}{\partial t}|_{t=0} \langle P^{-1}(x+th)h, h \rangle \\ &= \lim_{t \to 0} \frac{\langle P^{-1}(x+th)h, h \rangle - \langle P^{-1}(x)h, h \rangle}{t} \\ &= \langle \lim_{t \to 0} \frac{P^{-1}(P(x^{\frac{1}{2}})(e + tP^{-1}(x^{\frac{1}{2}})h))h - P^{-1}(x)h}{t}, h \rangle. \end{aligned}$$

Taking into account property 2 of P and Theorem 2.7, it follows that

$$\begin{aligned} D^3F(x)[h,h,h] &= \langle \lim_{t \to 0} \frac{\left(P\left(P(x^{\frac{1}{2}})(e + tP^{-1}(x^{\frac{1}{2}})h)\right)\right)^{-1} h - P^{-1}(x)h}{t}, h \rangle \\ &= \langle \lim_{t \to 0} \frac{\left(P(x^{\frac{1}{2}})(P(e + tP^{-1}(x^{\frac{1}{2}})h)P(x^{\frac{1}{2}})\right)^{-1} h - P^{-1}(x)h}{t}, h \rangle \\ &= \langle \lim_{t \to 0} \frac{P^{-\frac{1}{2}}(x)P^{-1}(e + tP^{-\frac{1}{2}}(x)h)P^{-\frac{1}{2}}(x)h - P^{-1}(x)h}{t}, h \rangle. \end{aligned}$$

Expanding $P(e+tu)v$, by (4), we obtain

$$\begin{aligned} P(e+tu)v &= (2L(e+tu)L(e+tu) - L((e+tu)^2))v \\ &= 2(L(e)+tL(u))((L(e)+tL(u))v) - L(e^2+2tu+t^2u^2)v \\ &= (I + 2tL(u) + t^2(2L(u)L(u) - L(u^2)))v. \end{aligned}$$

Therefore, $P(e+tu) = I + 2tL(u) + t^2(2L^2(u) - L(u^2)) = I - (-2tL(u) - t^2(2L^2(u) - L(u^2)))$ and then

$$P^{-1}(e+tu) = I + (-2tL(u) - t^2(2L^2(u) - L(u^2))) + (-2tL(u) - t^2(2L^2(u) - L(u^2)))^2 + \cdots .$$

It must be noted that since $e+tu \in \text{int}(Q)$ for small values of t, by Theorem 2.5 and property 1 of the quadratic representation, $P(e+tu)$ is invertible. Thus, for small values of t, it follows that $P^{-1}(e+tu) = I - 2tL(u) + o(t^2)$. Considering $u = P^{-\frac{1}{2}}(x)h$, we conclude that $P^{-1}(e + tP^{-\frac{1}{2}}(x)h) = I - 2tL(P^{-\frac{1}{2}}(x)h) + o(t^2)$ and thus

$$\begin{aligned} D^3F(x)[h,h,h] &= \langle \lim_{t\to 0} \frac{P^{-\frac{1}{2}}(x)P^{-1}(e+tP^{-\frac{1}{2}}(x)h)P^{-\frac{1}{2}}(x)h - P^{-1}(x)h}{t}, h\rangle \\ &= \langle \lim_{t\to 0} \frac{P^{-\frac{1}{2}}(x)(I - 2tL(P^{-\frac{1}{2}}(x)h)P^{-\frac{1}{2}}(x)h}{t}, h\rangle + \\ &\quad + \langle \lim_{t\to 0} \frac{t^2P^{-\frac{1}{2}}(x)(\cdots)P^{-\frac{1}{2}}(x)h}{t}, h\rangle - \langle \lim_{t\to 0} \frac{P^{-1}(x)h}{t}, h\rangle \\ &= -2\langle (L(P^{-\frac{1}{2}}(x)h))P^{-\frac{1}{2}}(x)h, P^{-\frac{1}{2}}(x)h\rangle. \end{aligned}$$

As a consequence, we have $D^3F(x)[h,h,h] = -2\langle u^2, u\rangle = -2tr(u^3)$, where $u = P^{-1/2}(x)h$. By the spectral decomposition of u and the Cauchy-Schwarz inequality, it follows that $tr(u^3) \leq \|u\|^3$ and then, taking into account (7), $|D^3F(x)[h,h,h]| \leq 2(D^2F(x)[h,h])^{3/2}$. ■

Theorem 3.2. *The function F is r-logarithmic homogeneous, in the sense that F verifies (11), that is, for all $x \in int(Q)$ and for all $t > 0$, $F(tx) = F(x) - r\log(t)$.*

Proof. The proof that F is a r-logarithmic homogeneous barrier follows from the equalities:

$$F(tx) = -\log(\det(tx)) = -\log(a_r(tx)) = -\log(t^r \det(x)) = F(x) - r\log t.$$

Note that a_r is a homogeneous polynomial of degree r. ■

Then, taking into account the Definition 3.1-4, from Theorems 3.1 and 3.2, we have the following corollary:

Corollary 3.1. *The function F is a r-normal barrier for Q.*

The next theorem is a direct consequence of Theorems 3.1 and 3.2, combined with Corollary 2.3.2 in [10].

Theorem 3.3. *The function F is a r-self-concordant barrier for Q.*

4 Rank of a Euclidean Jordan algebra and Carathéodory number of the cone of squares

In this section we establish that if V is a simple Euclidean Jordan algebra (i.e., irreducible) and Q is the cone of squares of V, then the Carathéodory number of Q is equal to the rank of the Jordan algebra V. First, we introduce the definition of extreme direction and the definition of Carathéodory number.

Definition 4.1. *Let V be a Euclidean Jordan algebra and Q its cone of squares. We call $x \in Q \setminus \{0\}$ a extreme direction of Q if for all $y, z \in Q \setminus \{0\}$ $x = y + z \Rightarrow y = \lambda_1 x \ \wedge \ z = \beta_1 x$, with $\lambda_1 \geq 0$ and $\beta_1 \geq 0$.*

Definition 4.2. *Let V be a Euclidean Jordan algebra, Q the cone of squares of V and $Ex(Q)$ the set of extreme directions of Q. We define the Carathéodory number of Q, denoted by $k(Q)$, as the smallest number k of extreme directions of Q such that for all $x \in Q$*

$$\exists x_1, \ldots, x_k \in Ex(Q) \ \ \wedge \ \ \exists \lambda_1 \ldots \lambda_k \geq 0 \quad \textit{such that} \quad x = \sum_{i=1}^{k} \lambda_i x_i.$$

Osman Guler and Levent Tunçel prove in [7] that if C is an open, non-empty, self-dual, homogeneous cone, then the smallest parameter $\gamma(C)$ such that there is a $\gamma(C)-$self-concordant barrier over C coincides with $k(C)$. Since F is a $r(V)$-self-concordant barrier and $\text{int}(Q)$ is a symmetric cone, if $r(V) = k(Q)$ then F is a self-concordant barrier over $\text{int}(Q)$ with the best parameter. The equality $r(V) = k(Q)$ is proved in the next sections in a more direct and suitable way than the one presented in [7]. Our proof is simpler, since we do not use the classification of simple Euclidean Jordan algebras, and we need not consider the exceptional cone case separately.

4.1 Case I: Simple Euclidean Jordan algebras

Faraut and Korányi [2], considering a simple Euclidean Jordan algebra V, prove that for each extreme direction d of the cone of squares Q there exists a primitive idempotent c of V such that $d = \lambda c$, with $\lambda > 0$. Since for each $x \in Q$, by Theorems 2.2 and 2.4, there is a Jordan frame $\{c_1, \ldots, c_r\}$ such that $x = \sum_{i=1}^{r} \lambda_i c_i$, where all λ_i are real non-negative numbers. Then we may conclude that the Carathéodory number of Q is not greater than $r(V)$, that is,

$$k(Q) \leq r(V). \tag{12}$$

Theorem 4.1. *Let V be a finite dimensional simple Euclidean Jordan algebra with unit element $\mathbf{e}$ and rank $r(V)$. If Q is the cone of squares of V then $r(V) = k(Q)$.*

Proof. Let $k(Q) = k$. Then, as the extreme directions are just the half-lines $\{\lambda c : \lambda > 0\}$, where c is a primitive idempotent, there are k primitive idempotents and k real non negative numbers ($\alpha_i \geq 0,\ i = 1, \dots, k$) such that $e = \sum_{i=1}^{k} \alpha_i c_i$. Thus, multiplying both members of the previous equality on the right by c_i, we obtain $c_i = \alpha_i(c_i \circ c_i) + \sum_{j \neq i}^{k} \alpha_j(c_j \circ c_i)$ and then, since $\mathrm{tr}(c_i) = 1$,

$$1 = \alpha_i \mathrm{tr}(c_i) + \sum_{j \neq i}^{k} \alpha_j \mathrm{tr}(c_j \circ c_i) = \alpha_i + \sum_{j \neq i}^{k} \alpha_j \langle c_j, c_i \rangle. \tag{13}$$

Taking into account that for all $j, i \in \{1, \dots, k\}$ $\alpha_j \geq 0$ and $\langle c_j, c_i \rangle = \langle c_j^2, c_i^2 \rangle \geq 0$ (since $c_j^2, c_i^2 \in Q$ and Q is self-dual), the inequality $\sum_{j \neq i}^{k} \alpha_j \langle c_j, c_i \rangle \geq 0$ follows. Therefore, according to (13), we conclude that for all $i \in \{1, \dots, k\}$ $0 \leq \alpha_i \leq 1$. Thus

$$r(V) = \mathrm{tr}(\mathbf{e}) = \sum_{i=1}^{k} \alpha_i \mathrm{tr}(c_i) = \sum_{i=1}^{k} \alpha_i \leq k. \tag{14}$$

Finally, by (12) and (14), it follows that $r(V) = k(Q)$. ∎

4.2 Case II: Reducible Euclidean Jordan algebras

Now, we may extend the Theorem 4.1 to the general case of Euclidean Jordan algebras (not necessarily irreducible).

Corollary 4.1. *Let V be a Euclidean Jordan algebra with cone of squares Q. Then $k(Q) = r(V)$.*

Proof. Since this result is already proved when V is irreducible (that is, simple), in the following we will consider that V is a reducible Euclidean Jordan algebra. Therefore, by Theorem 2.3, there are l simple Euclidean Jordan subalgebras V_i of V such that $V = \bigoplus_{i=1}^{l} V_i$ and, additionally, for all $i \neq j$ $u \circ v = 0$ for $u \in V_i$ and $v \in V_j$. Note that $V_i \circ V_j = 0$ for all $i \neq j$ implies that the cone of squares Q may be decomposed in $Q = \bigoplus_{i=1}^{l} Q_i$ where Q_i denotes the cone of squares of V_i.
We have $r(V) = \sum_{i=1}^{l} r(V_i)$ and $k(Q) = \sum_{i=1}^{l} k(Q_i)$, see [1]. Since every V_i is a simple Euclidean Jordan algebra, by Theorem 4.1, $k(Q_i) = r(V_i)$ for all i. Therefore, $k(Q) = \sum_{i=1}^{l} k(Q_i) = \sum_{i=1}^{l} r(V_i) = r(V)$. ∎

Since it was proved in [7] that the best parameter of a self-concordant barrier defined over a symmetric cone is equal to the Charathéodory number of the cone and, according to Theorem 2.8, every symmetric cone is the interior of the cone of squares of some Euclidean Jordan algebra, Corollary 4.1 implies that the rank of an underlying Euclidean Jordan algebra of a symmetric cone is also the best self-concordant barrier parameter.

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