Envelope Theorems For Finite Choice Sets

by

Somdeb Lahiri
School of Economic and Business Sciences,
University of Witwatersrand at Johannesburg,
Private Bag 3, WITS 2050,
South Africa.
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Email: lahiris@sebs.wits.ac.za
Or
Lahiri@webmail.co.za

Abstract

This paper is concerned with the study of envelope theorems for finite choice sets. More specifically, we consider the problem of differentiability of the value function arising out of the maximization of a parametrized objective function, when the set of alternatives is non-empty and finite. The parameter is confined to the closed interval [0,1] and for each alternative, the objective function is assumed to be a continuous function of the parameter.

1. Introduction: This paper is concerned with the study of envelope theorems for finite choice sets. More specifically, we consider the problem of differentiability of the value function arising out of the maximization of a parametrized objective function, when the set of alternatives is non-empty and finite. The parameter is confined to the closed interval [0,1] and for each alternative, the objective function is assumed to be a continuous function of the parameter.

A rigorous statement of an envelope theorem when the set of alternatives is a convex subset of an Euclidean space, and the objective function is concave in both the decision variable and the parameter can be found in Benveniste and Scheinkman [1979]. This envelope theorem concerns the differentiability of the optimal value function that usually arise in non-linear as well as dynamic optimization problems. Similar problems are investigated in the mathematical literature on "sensitivity analysis", which however rely on topological assumptions on the choice set and the continuity of the objective function in both the decision variable as well as the parameter. Bonnans and Shapiro [2000] provides a survey of some of these results.

There is a considerable body of literature in game theory and economics, which rely on parametrized optimization where the underlying set of alternatives is finite. A paper by Milgrom and Segal [2002], discusses some of these problems and provides generalized envelope theorems, which are applicable for both finite as well as infinite choice sets at the same time. However, the curious thing about any generalization (and the ones by Milgrom and Segal [2002] are no exception!), is that the price payed for it by way of simplicity, is often quite exorbitant. Had it been the case, that envelope theorems for finite choice sets were not of sufficient interest or applicability by themselves, a separate investigation of such results, even if it were to lead to simpler statements and conceptualizations, would be devoid of merit. Clearly that is not the case, and the main reason behind our inability to provide an "appropriate" evidence to substantiate the position, is the overwhelming academic output that goes in its favor.

The corner-stone of the envelope theorems that we establish in this paper is the following simple observation: if for a parameter value t, the solution set for a parametrized optimization problem is $X^*(t)$, then there is an open interval around t (possibly small!), such that for all values of the parameter within the interval, the corresponding solution sets are subsets of $X^*(t)$.

An immediate consequence of this observation, is that if for some t in the open interval (0,1), $X^*(t)$ is a singleton say $\{x\}$, then the optimal value function is differentiable at t, if and only if the partial derivative of the objective function with respect to t, exists at (x,t). Further, in such a situation the two derivatives are equal. We also record the observation that if the solution set is a singleton for all values of the parameter, then the solution set must be a constant.

The main result reported here, is a considerable strengthening of the envelope theorem that is valid when the solution set at a parameter value is a singleton. The stronger result says, that if [(i) at a particular parameter value t, the partial derivatives of the objective function with respect to the parameter, is the same for all the alternatives in the corresponding solution set for which it exists, and (ii) those alternatives in the solution set at t for which the partial derivative does not exist, it is the case that they are non-optimal in an interval around t (: other than t itself!)] then, the value function is differentiable at t, and its derivative is equal to the common value of the partial derivatives. This result along with its converse, which is also established in this paper, is similar to Theorem 1 in Sah and Zhao [1998], except that unlike their result, which was stated in the context of integer variables, we do not require any concavity assumption for the objective function. Further, unlike Milgrom and Segal [2002], we do not require the continuity of the partial derivatives of the objective function with respect to the parameter at 't', for any alternative in the solution set, to obtain the desired result.

It may be worth noting, that figure 1 of Milgrom and Segal [2002] (: the only one there which is used to motivate their main theoretical results), gives rise to a value function, whose differentiability properties can be derived from this latter envelope theorem in the present paper.

2. The Model and preliminary results: Let $\aleph$ denote the set of natural numbers and $\Re$ the set of real numbers. Let X be a non-empty finite set of alternatives. Let the closed interval [0,1] denote the set of parameters. A parametrized objective function is a function $f : X\times [0,1] \rightarrow \Re$ such that for all $x\in X$, $f(x,.) : [0,1] \rightarrow \Re$ is continuous.

Given, $t\in[0,1]$, let $X^*(t) = \{x\in X/ f(x,t) \geq f(y,t) \text{ for all } y\in X\}$. Clearly $X^*(t)$ is a non-empty subset of X for all $t \in[0,1]$. Further, for $t\in[0,1]$ and $x,y\in X^*(t)$, $f(x,t) = f(y,t)$.

Let $V: [0,1] \rightarrow \Re$ be defined thus: for $t\in[0,1]$, $V(t) = f(x,t)$, where $x\in X^*(t)$.

Proposition 1: Let $<t_k/k\in\aleph>$ be a sequence in [0,1] converging to t. Let $x^*\in X^*(t_k)$ for all $k\in\aleph$. Then, $x^*\in X^*(t)$.

Proof. Let $x\in X$. Thus, $f(x^*,t_k) \geq f(x,t_k)$ for all $k\in\aleph$. By the continuity of both $f(x^*,.)$ and $f(x,.)$ and by taking limits on both sides of the inequality we get $f(x^*,t) \geq f(x,t)$. Thus, $x^*\in X^*(t)$. Q.E.D.

Theorem 1: V is continuous.

Proof: Let $<t_k/k\in\aleph>$ be a sequence in [0,1] converging to $t\in[0,1]$ and let $x_k\in X^*(t_k)$ for $k\in\aleph$. Since X is finite, there exists $x^* \in X$, such that $x_k = x^*$ for infinitely many values of k. Without loss of generality suppose, $x_k = x^*$ for all $k\in\aleph$. By Proposition 1, $x^*\in X^*(t)$. If $<V(t_k)/k\in\aleph>$ does not converge to V(t), then there exists $\varepsilon > 0$, such that $|V(t_k) - V(t)| \geq \varepsilon$ for infinitely many values of 'k'. Let $<s_k/k\in\aleph>$ be a subsequence of $<t_k/k\in\aleph>$ such that $|V(s_k) - V(t)| \geq \varepsilon$ for all $k\in\aleph$. Since, $V(s_k) = f(x^*,s_k)$ for all $k\in\aleph$, $V(t) = f(x^*,t)$ and $<s_k/k\in\aleph>$ converges to 't', by the continuity of $f(x^*,.)$ we get $\lim_{k\to\infty} V(s_k) = V(t)$, contradicting $|V(s_k) - V(t)| \geq \varepsilon$ for all $k\in\aleph$. This proves the theorem. Q.E.D.

Theorem 2: Let $t \in[0,1]$. Then, there exists $\varepsilon > 0$, such that for all $s \in(t-\varepsilon, t + \varepsilon) \cap[0,1]$, $X^*(s) \subset X^*(t)$.

Proof: Let $x\in X^*(t)$. If $X^*(t) = X$, then there is nothing to prove. Hence suppose, $y \in X \setminus X^*(t)$. Thus, $f(x,t) > f(y,t)$. By the continuity of both $f(x,.)$ and $f(y,.)$, there exists $\varepsilon_y > 0$, such that for all $s\in (t - \varepsilon_y, t + \varepsilon_y) \cap [0,1]$, $f(x,s) > f(y,s)$. Let $\varepsilon = \min \{\varepsilon_y / y \in X \setminus X^*(t)\}$. Clearly, $\varepsilon > 0$, since X is a finite set. Further, for all $y\in X \setminus X^*(t)$: $f(x,s) > f(y,s)$ whenever $s \in (t - \varepsilon, t + \varepsilon) \cap [0,1]$ and hence $y \in X \setminus X^*(t)$ for all $s \in (t - \varepsilon, t + \varepsilon) \cap [0,1]$. Thus, $X^*(s) \subset X^*(t)$ for all $s \in (t - \varepsilon, t + \varepsilon) \cap [0,1]$. Q.E.D.

Corollary 1 of Theorem 2: Suppose that for some $t \in[0,1]$, $X^*(t) = \{x^*\}$ for some $x^*\in X$. Then, there exists $\varepsilon > 0$, such that for all $s \in(t-\varepsilon, t + \varepsilon) \cap[0,1]$, $X^*(s) = \{x^*\}$. If in addition, $t\in (0,1)$, then V is differentiable at t if and only if $f(x^*,.)$ is differentiable at t, in which case $DV(t) = \frac{\partial f(x^*,s)}{\partial s}|_{s=t}$.

Proof: The first part of the theorem follows directly from Theorem 2. If $t > 0$, then for all $s \in(t-\varepsilon, t + \varepsilon)$, $V(s) = f(x^*,s)$. This proves the remainder of the theorem. Q.E.D.

Theorem 3: Let $X^*(t)$ be a singleton for all $t \in[0,1]$. Then, there exists $x^*\in X$, such that $X^*(t) = \{x^*\}$ for all $t\in[0,1]$.

Proof: Let $\{x^*\} = X^*(0)$. By Proposition 1 and Corollary 1 of Theorem 2, there exists $\varepsilon > 0$ such that $\{x^*\} = X^*(t)$ for all $t \in[0,\varepsilon]$. Let $\varepsilon^* = \max\{\varepsilon/ \{x^*\} = X^*(t)$ for all $t \in[0,\varepsilon]\}$. Towards a contradiction suppose $\varepsilon^* < 1$. Then by Proposition 1 and Corollary 1 of Theorem 2, there exists $\delta > 0$ with $\varepsilon^*+ \delta < 1$, such that $\{x^*\} = X^*(t)$ for all $t\in[\varepsilon^*, \varepsilon^*+ \delta]$. Thus, $\{x^*\} = X^*(t)$ for all $t\in[0, \varepsilon^*+ \delta]$. Since $\varepsilon^*+ \delta > \varepsilon^*$, it is not true that $\varepsilon^* = \max\{\varepsilon/ \{x^*\} = X^*(t)$ for all $t \in[0,\varepsilon]\}$. This contradiction establishes the theorem. Q.E.D.

3. The Main Envelope Theorem: Condition D(Differentiability) is said to be satisfied at $t \in (0,1)$ if: (i) for all $x \in X^*(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t}$ exists;(ii) for all $x,y \in X^*(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t} = \frac{\partial f(y,s)}{\partial s}|_{s=t}$.

Note: If for $t \in (0,1)$, $X^*(t)$ is a singleton say $\{x\}$ and if $\frac{\partial f(x,s)}{\partial s}|_{s=t}$ exists, then clearly Condition D is satisfied at t.

Condition WD(Weak Differentiability) is said to be satisfied at $t \in (0,1)$ if: (i) $X^0(t) = \{x \in X^*(t) / \frac{\partial f(x,s)}{\partial s}|_{s=t} \text{ exists}\} \neq \phi$;(ii) for all $x,y \in X^0(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t} = \frac{\partial f(y,s)}{\partial s}|_{s=t}$; (iii) there exists $\delta > 0$, such that for all $s \in (t-\delta, t+\delta) \setminus \{t\}$ and $x \in X^*(t) \setminus X^0(t)$, it is the case that $x \notin X^*(s)$.

It is easily observed that Condition D implies Condition WD.

Theorem 4: (a) Suppose Condition WD is satisfied at $t \in (0,1)$. Then V is differentiable at t and $DV(t) = \frac{\partial f(x,s)}{\partial s}|_{s=t}$ for any $x \in X^0(t)$.

(b) Suppose V is differentiable at $t \in (0,1)$. Then for all $x, y \in X^0(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t} = \frac{\partial f(y,s)}{\partial s}|_{s=t}$.

Proof: (a) Suppose Condition WD is satisfied at $t \in (0,1)$. Let $c = \frac{\partial f(x,t)}{\partial s}|_{s=t}$ for all $x \in X^0(t)$.

By, Theorem 2 and part (iii) of Condition WD, there exists $\varepsilon > 0$ with $0 < t - \varepsilon < t + \varepsilon < 1$, such that for all $s \in (t-\varepsilon, t+\varepsilon)$, $X^*(s) \subset X^0(t)$.
Thus, for all $s \in (t-\varepsilon, t+\varepsilon)$, $V(s) \in \{f(x,s)/ x \in X^*(s)\} \subset \{f(x,s)/ x \in X^0(t)\}$.
For $s \in (t-\varepsilon, t+\varepsilon)$, and $x \in X^*(s)$, $V(s) - V(t) = f(x, s) - f(x,t)$.
Let $<t_k/ k \in \aleph>$ be a sequence in $(t-\varepsilon, t) \cup (t, t+\varepsilon)$, such that $\lim_{k\to\infty} t_k = t$. For $k \in \aleph$, let $x(t_k) \in X^*(t_k) \subset X^0(t)$. Consider the sequence $<x(t_k)/ k \in \aleph>$. Without loss of generality, we may assume that every $x \in \{x(t_k)/ k \in \aleph\}$, occurs infinitely often in the sequence. Let $x \in \{x(t_k)/ k \in \aleph\} \subset X^0(t)$. Let $<s_k/ k \in \aleph>$ be a subsequence of $<t_k/ k \in \aleph>$, such that $x(s_k) = x$ for all $k \in \aleph$. Thus,

Thus, $\frac{V(s_k) - V(t)}{s_k - t} = \frac{f(x,s_k) - f(x,t)}{s_k - t}$ for all $k \in \aleph$.

Since, $\lim_{k\to\infty} \frac{f(x,s_k)-f(x,t)}{s_k - t}$ exists and is equal to $\frac{\partial f(x,t)}{\partial s}|_{s=t}$, it must be the case that $\lim_{k\to\infty} \frac{V(s_k)-V(t)}{s_k - t} = \frac{\partial f(x,t)}{\partial s}|_{s=t}$.

Since $\frac{\partial f(x,t)}{\partial s}|_{s=t} = \frac{\partial f(y,t)}{\partial s}|_{s=t}$ for all $x,y \in X^0(t)$, every subsequence $<s_k/ k\in\aleph>$ of $<t_k/ k\in\aleph>$, such that $x(s_k) = x(s_1)$ for all $k\in\aleph$ satisfies the property that $\lim_{k\to\infty} \frac{V(s_k)-V(t)}{s_k - t} =$ c.

Now suppose that there exists a subsequence $<s_k/ k\in\aleph>$ of $<t_k/ k\in\aleph>$ such that $\lim_{k\to\infty} \frac{V(s_k)-V(t)}{s_k - t} \neq c$. Thus, there exists $\delta > 0$, such that $|\frac{V(s_k)-V(t)}{s_k - t} - c| \geq \delta$ infinitely often. Without loss of generality suppose, $|\frac{V(s_k)-V(t)}{s_k - t} - c| \geq \delta$ for all $k\in\aleph$. Hence there exists a subsequence $<r_k/ k\in\aleph>$ of $<s_k/ k\in\aleph>$, such that $x(r_k) = x(r_1)$ for all $k\in\aleph$. Hence, by what has been obtained above, $\lim_{k\to\infty} \frac{V(r_k)-V(t)}{r_k - t} = c$. This contradicts, $|\frac{V(s_k)-V(t)}{s_k - t} - c| \geq \delta$ for all $k\in\aleph$. Thus, $\lim_{k\to\infty} \frac{V(t_k)-V(t)}{t_k - t} = c$. Thus, V is differentiable at t, and $DV(t) = c$.

(b) Suppose V is differentiable at $t\in(0,1)$ and towards a contradiction suppose that for some x, y $\in X^0(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t} \neq \frac{\partial f(y,s)}{\partial s}|_{s=t}$. Let $c^1 = \min\{\frac{\partial f(x,s)}{\partial s}|_{s=t} / x\in X^0(t)\}$ and $c^2 = \max\{\frac{\partial f(x,s)}{\partial s}|_{s=t} / x\in X^0(t)\}$. Thus, $c^2 > c^1$. Suppose, $x^1, x^2 \in X^0(t)$ be such that $\frac{\partial f(x^1,s)}{\partial s}|_{s=t} = c^1$ and $\frac{\partial f(x^2,s)}{\partial s}|_{s=t} = c^2$. Let $< t_k/ k\in\aleph>$ be a sequence in (t, 1) with $\lim_{k\to\infty} t_k = t$ and let $< s_k/ k\in\aleph>$ be a sequence in (0, t) with $\lim_{k\to\infty} s_k = t$. Thus, $DV(t) = \lim_{k\to\infty} \frac{V(t_k)-V(t)}{t_k - t} \geq \lim_{k\to\infty} \frac{f(x^2,t_k)-f(x^2,t)}{t_k - t} = c^2$. Similarly, $DV(t) = \lim_{k\to\infty} \frac{V(s_k)-V(t)}{s_k - t} = \lim_{k\to\infty} \frac{V(t)-V(s_k)}{t - s_k} \leq \lim_{k\to\infty} \frac{f(x^1,t)-f(x^1,s_k)}{t - s_k} = c^1$. Thus, $c^1 \geq DV(t) \geq c^2$. This contradicts $c^2 > c^1$ and proves (b). Q.E.D.

An immediate corollary of Theorem 4 is the following:

Corollary of Theorem 4: (a) Suppose Condition D is satisfied at $t\in(0,1)$. Then V is differentiable at t and $DV(t) = \frac{\partial f(x,s)}{\partial s}|_{s=t}$ for any $x \in X^*(t)$.

(b) Suppose V is differentiable at $t\in(0,1)$ and $\frac{\partial f(x,s)}{\partial s}|_{s=t}$ exists for all $x \in X^*(t)$. Then for all x, $y \in X^*(t)$: $\frac{\partial f(x,s)}{\partial s}|_{s=t} = \frac{\partial f(y,s)}{\partial s}|_{s=t}$.

In view of the note following Condition D, Theorem 4(a) can be invoked to prove the following result contained in Corollary 1 of Theorem 2:

Suppose that for some $t \in(0,1)$, $X^*(t) = \{x^*\}$ for some $x^* \in X$. Then, V is differentiable at t if and only if $f(x^*,.)$ is differentiable at t, in which case $DV(t) = \frac{\partial f(x^*,t)}{\partial s}|_{s=t}$.

<u>**Note**</u>: (1) The differentiability of V at $t\in(0,1)$ does not necessarily imply [there exists $\delta > 0$, such that for all $s\in(t-\delta, t+\delta)\backslash\{t\}$ and $x \in X^*(t)\backslash X^0(t)$, it is the case that $x\notin X^*(s)$]:

Let $X = \{x,y\}$, $f(x,t) = 1 + t$ for all $t \in[0,1]$, $f(y,t) = 3t$ for $t \in[0,\frac{1}{2}]$, $f(y,t) = 1+t$ for $t\in[\frac{1}{2},1]$. Thus, $X^*(t) = \{x\}$ for $t \in[0,\frac{1}{2})$ and $X^*(t) = \{x,y\}$ for $t \in[\frac{1}{2},1]$. $X^0(\frac{1}{2}) = \{x\}$ is a singleton and $V(t) = 1+ t$ for all $t \in[0,1]$. Thus V is differentiable at $\frac{1}{2}$. However, $y\notin X^0(\frac{1}{2})$ but $y \in X^*(s)$ for all $s \in[\frac{1}{2},1]$.

(2) The differentiability of V at $t\in(0,1)$ does not necessarily imply that $X^0(t)$ is non-empty:

Let $X = \{x,y\}$, $f(x,t) = t$ for $t \in[0, \frac{1}{2}]$, $f(x,t) = \frac{1}{2}$ for $t\in[\frac{1}{2},1]$, $f(y,t) = \frac{1}{2}$ for $t \in[0,\frac{1}{2}]$, $f(y,t) = 1- t$ for $t\in[\frac{1}{2},1]$. $V(t) = \frac{1}{2}$ for all $t\in[0,1]$. Thus V is differentiable at $\frac{1}{2}$. However, $X^0(\frac{1}{2}) = \phi$.

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