Inferring efficient weights from pairwise comparison matrices.

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Abstract

Several Multi-Criteria-Decision-Making methodologies assume the existence of weights associated with the different criteria, reflecting their relative importance.

One of the most popular ways to infer such weights is the Analytic Hierarchy Process, which constructs first a matrix of pairwise comparisons, from which weights are derived following one out of many existing procedures, such as the eigenvector method or the least (logarithmic) squares. Since different procedures yield different results (weights) we pose the problem of describing the set of weights obtained by "sensible" methods: those which are efficient for the (vector-) optimization problem of simultaneous minimization of discrepancies.

A characterization of the set of efficient solutions is given, which enables us to assert that the least-logarithmic-squares solution is always efficient, whereas the (widely used) eigenvector solution is not, in some cases, efficient, thus its use in practice may be questionable.
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1 Introduction

Several strategies have been suggested in the literature to associate with a set $\mathcal{D} = \{d_1, \ldots, d_N\}$ of decisions weights $x_1, x_2, \ldots, x_N$ reflecting decision-maker's preferences. In the Analytic Hierarchy Process (AHP), [14, 15, 17, 18], an $N \times N$ matrix

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A,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

is obtained after asking the decision-maker (DM) to quantify the ratio of his/her preferences of one decision over another. In other words, for every pair of decisions d_i, d_j, the term $a_{ij} > 0$ is requested satisfying

$$a_{ij} \approx \frac{x_i}{x_j} \tag{1}$$

The matrix A so obtained must be a *positive reciprocal* matrix, i.e.,

$$a_{ji} = \frac{1}{a_{ij}} > 0 \qquad \text{for all } i, j = 1, 2, \ldots, N.$$

For a given positive reciprocal matrix A, different procedures can be followed in order to obtain weights $x_1, \ldots, x_N$ according to (1), see e.g. [1, 2, 7, 9, 13, 16]. In particular, Saaty proposes the so-called *Eigenvector method* (EM): x is a column vector satisfying the equation

$$Ax = \lambda_{\max} x,$$

where $\lambda_{\max}$ is the dominant eigenvalue of the positive reciprocal matrix A. See e.g. [15] for further details, and [5] for commercial software with it.

Many other choices have been proposed in the literature to derive x according to (1), mostly given as optimal solutions of optimization problems such as

$$\min_{x \in \mathbb{R}^N_{++}} \sum_{i,j=1}^{N} \left(\frac{x_i}{x_j} - a_{ij} \right)^2, \tag{2}$$

or

$$\min_{x \in \mathbb{R}^N_{++}} \sum_{i,j=1}^{N} \left(\log(\frac{x_i}{x_j}) - \log(a_{ij}) \right)^2, \tag{3}$$

where $\mathbb{R}^N_{++}$ denotes the set of strictly positive vectors in $\mathbb{R}^N$.

It should become evident that different procedures, – (EM) or those derived from (2) or (3)–, although following (1), may yield different weights, and even different ranking of decisions may happen, as already shown e.g. in [16].

This naturally leads to the Nonconvex Vector-Optimization problem

$$\min_{x \in \mathbb{R}^N_{++}} (|\frac{x_i}{x_j} - a_{ij}|)_{i \neq j} \tag{X}$$

We recall the reader, e.g. [4, 20, 21], that, given an optimization problem (P),

$$\min_{x \in S} \left(f_1(x), \ldots, f_k(x) \right),$$

$y \in S$ is said to *dominate* $x \in S$ if $f_i(y) \leq f_i(x)$ for all $i = 1, \ldots, k$, with $f_i(y) < f_i(x)$ for some i. Moreover, $x \in S$ is said to be *efficient* for (P) if no $y \in S$ dominates x, and x is said to be locally efficient for (P) if there exists a neighborhood V of x in S such that no $y \in V$ dominates x.

Our aim is to find a full description of the set of locally efficient and efficient solutions for (X), and to explore whether the usual weighting methodologies, are (or are not) efficient for (X).

2 A test for efficiency

Problem (X) is a multiple-objective nonlinear nonconvex problem whose feasible set is the strictly positive orthant $\mathbb{R}^N_{++}$, which is not closed. This makes at first glance (X) very hard to solve. However, it is easy to construct an LP-based test of efficiency. Indeed, one has

Theorem 1 *Let $x^* \in \mathbb{R}^N_{++}$. For each $k, l = 1, \ldots, N$, define $\varepsilon_{kl} = |\frac{x^*_k}{x^*_l} - a_{kl}|$. Then x^* is efficient for (X) if and only if for each $k, l = 1, \ldots, N$, $k \neq l$, ε_{kl} is the optimal value of the Linear Problem*

$$\begin{array}{ll} \min & t \\ s.t. & x_i - (\varepsilon_{ij} + a_{ij})x_j \leq 0 \text{ for all pairs } (i,j) \neq (k,l) \\ & x_i + (\varepsilon_{ij} - a_{ij})x_j \geq 0 \text{ for all pairs } (i,j) \neq (k,l) \\ & x_k - t \leq a_{kl} \\ & x_k + t \geq a_{kl} \\ & x_l = 1 \\ & x_1, \ldots, x_N \geq 0 \\ & t: \text{ unrestricted.} \end{array} \quad (4)$$

Proof

It is a well known result of Vector Optimization, e.g. [4], that x^* is efficient for (X) if and only if for any pair of indices $k, l = 1, \ldots, N$, $k \neq l$, x^* is an optimal solution to the fractional optimization problem (P_{kl}), [19]

$$\begin{array}{ll} \inf & |\frac{x_k}{x_l} - a_{kl}| \\ \text{s.t.} & |\frac{x_i}{x_j} - a_{ij}| \leq \varepsilon_{ij} \text{ for all pairs } (i,j) \neq (k,l) \\ & x_1, \ldots, x_N > 0. \end{array} \quad (P_{kl})$$

Let $k, l \in \{1, \ldots, N\}$, $k \neq l$ be given, and define the vector y as $y = \frac{1}{x^*_l} x^*$.

Then, x^* solves (P_{kl}) if and only if y does, what happens if and only if y solves $(\hat{P}_{kl})$,

$$\begin{array}{ll} \inf & |x_k - a_{kl}| \\ \text{s.t.} & |\frac{x_i}{x_j} - a_{ij}| \leq \varepsilon_{ij} \text{ for all pairs } (i,j) \neq (k,l) \\ & x_l = 1 \\ & x_1, \ldots, x_N > 0. \end{array} \quad (\hat{P}_{kl})$$

Problem $(\hat{P_{kl}})$ is equivalent to the Linear Problem

$$\begin{array}{ll} \inf & t \\ \text{s.t.} & x_i - (\varepsilon_{ij} + a_{ij})x_j \leq 0 \text{ for all pairs } (i,j) \neq (k,l) \\ & x_i + (\varepsilon_{ij} - a_{ij})x_j \geq 0 \text{ for all pairs } (i,j) \neq (k,l) \\ & x_k - t \leq a_{kl} \\ & x_k + t \geq a_{kl} \\ & x_l = 1 \\ & x_1, \ldots, x_N > 0. \end{array} \tag{5}$$

This problem can also be written equivalently by replacing the strict inequalities $x_j > 0$ by non-strict inequalities $x_j \geq 0$. In other words, we claim that (5) is equivalent to (4). Indeed, any x feasible for (5) is also feasible for (4). Conversely, for any x, feasible for (4), we have that

$$\begin{array}{lcl} x_l & = & 1 > 0 \\ x_l - (\varepsilon_{lj} + a_{lj})x_j & \leq & 0 \text{ for all } j, \end{array} \tag{6}$$

from which we deduce that $x_j > 0$ for all j : else, if $x_j = 0$, (6) would yield $x_l = 0$, which contradicts $x_l = 1$. Hence, the result follows. □

Although Theorem 1 enables us to check whether a given $x^* \in \mathbb{R}^N_{++}$ is efficient or not, it does not give insight in the structure of the efficient set. For this reason we devote the remaining of this section to provide alternative characterizations of efficiency for (X).

Given a function $\pi : \mathbb{R}_{++} \longrightarrow \mathbb{R}$, consider the Vector-Optimization Problem (X_π),

$$\min_{x \in \mathbb{R}^N_{++}} \left(\left| \pi\left(\frac{x_i}{x_j}\right) - \pi(a_{ij}) \right| \right)_{i \neq j} \tag{X_π}$$

Lemma 2 *Let $\pi : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ be strictly increasing. Then, $x^* \in \mathbb{R}^N_{++}$ is efficient for (X_π) iff x^* is locally efficient for (X_π).*

Proof

For each $i, j = 1, \ldots, N$, $i \neq j$, the function $x \in \mathbb{R}^N_{++} \longmapsto \frac{x_i}{x_j}$ is quasimonotonous and strictly quasimonotonous, [10] i.e., both lower and upper level sets and strict lower and upper level sets are convex. Since π is strictly increasing, the function $x \in \mathbb{R}^N_{++} \longmapsto \pi(\frac{x_i}{x_j})$ is also quasimonotonous and strictly quasimonotonous. Hence, the function $x \in \mathbb{R}^N_{++} \longmapsto \max\left\{\pi(\frac{x_i}{x_j}) - \pi(a_{ij}), -\pi(\frac{x_i}{x_j}) + \pi(a_{ij})\right\} = |\pi(\frac{x_i}{x_j}) - \pi(a_{ij})|$ is quasiconvex and strictly quasiconvex, i.e., both its lower and strict lower level sets are convex.

By definition, if x^* is efficient then x^* is also locally efficient. Conversely, given x^*, locally efficient, suppose, by contradiction, that it is not efficient. Then there exists $y \in \mathbb{R}^N_{++}$ such that

$$\left|\pi(\frac{x_i^*}{x_j^*}) - \pi(a_{ij})\right| \geq \left|\pi(\frac{y_i}{y_j}) - \pi(a_{ij})\right| \qquad \forall i, j,$$

with at least one inequality strict. Since the function is (strictly) quasiconvex, this property also holds for any z in the open segment with endpoints x^* and y, and, in particular, for z arbitrarily close to x^*. This contradicts the assumption that x^* is locally efficient. Hence x^* must be efficient for (X_π). □

As a first conclusion, taking $\pi(t) = t$, we obtain that

Theorem 3 *$x^* \in \mathbb{R}^N_{++}$ is efficient for (X) iff x^* is locally efficient for (X).*

Theorem 4 *Let $\pi : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ be strictly increasing then, $x^* \in \mathbb{R}^N_{++}$ is efficient for (X) iff x^* is efficient for (X_π).*

Proof

Let x^* be an efficient solution of problem (X); we will show that x^* is also efficient for (X_π). Suppose, by contradiction, that x^* is not efficient for (X_π), thus, by Lemma 2, x^* is not locally efficient for (X_π).

Then there exists $y \in \mathbb{R}^N_{++}$, sufficiently close to x^*, such that

$$\left|\pi\left(\frac{x_i^*}{x_j^*}\right) - \pi(a_{ij})\right| \geq \left|\pi\left(\frac{y_i}{y_j}\right) - \pi(a_{ij})\right| \qquad \forall i, j, \tag{7}$$

with at least one inequality strict, and satisfying

$$\frac{y_i}{y_j} \geq a_{ij} \quad \forall i, j, \text{ such that } \frac{x_i^*}{x_j^*} > a_{ij} \tag{8}$$

$$\frac{y_i}{y_j} \leq a_{ij} \quad \forall i, j, \text{ such that } \frac{x_i^*}{x_j^*} < a_{ij} \tag{9}$$

Moreover, (7) implies

$$\pi\left(\frac{y_i}{y_j}\right) = \pi(a_{ij}), \quad \forall i, j \text{ such that } \pi\left(\frac{x_i^*}{x_j^*}\right) = \pi(a_{ij}). \tag{10}$$

Since π is assumed to be strictly increasing, (8) and (9) can be rephrased as

$$\pi\left(\frac{y_i}{y_j}\right) \geq \pi(a_{ij}), \quad \forall i, j \text{ such that } \pi\left(\frac{x_i^*}{x_j^*}\right) > \pi(a_{ij}), \tag{11}$$

$$\pi\left(\frac{y_i}{y_j}\right) \leq \pi(a_{ij}), \quad \forall i, j \text{ such that } \pi\left(\frac{x_i^*}{x_j^*}\right) < \pi(a_{ij}). \tag{12}$$

By (7) this implies that

$$\left|\frac{y_i}{y_j} - a_{ij}\right| \leq \left|\frac{x_i^*}{x_j^*} - a_{ij}\right| \quad \forall i, j,$$

with at least one inequality strict. This contradicts the assumption that x^* is efficient for (X).

The converse, is shown analogously using Theorem 3, and will not be given here. □

Taking $\pi(t) = \log t$, we have that $x^* \in \mathbb{R}^N_{++}$ is efficient for (X) iff x^* is efficient for $(X_{\log})$,

$$\min_{x \in \mathbb{R}^N_{++}} \left(|\log(x_i) - \log(x_j) - \log(a_{ij})|\right)_{i \neq j} \qquad (X_{\log}).$$

For a given $x \in \mathbb{R}^N_{++}$ let $\log(x)$ denote the vector

$$\log(x) = (\log(x_1), \log(x_2), \ldots, \log(x_N)) \, .$$

The discussion above shows the following

Corollary 5 *$x^* \in \mathbb{R}^N_{++}$ is efficient for (X) iff $\log(x^*)$ is efficient for the piecewise linear convex vector-optimization problem (Y),*

$$\min_{y \in \mathbb{R}^N} \left(|y_i - y_j - \log(a_{ij})|\right)_{i \neq j} . \qquad (Y)$$

Corollary 6 *The set of efficient solutions of (X) is connected.*

PROOF
By [3], the set E_Y of efficient solutions of (Y) is connected. Using Corollary 5 one has that the set of efficient solutions of (X) is the image of E_Y under the continuous mapping

$$(z_1, \ldots, z_N) \mapsto (\exp(z_1), \ldots, \exp(z_N)),$$

showing connectedness. □

Corollary 5 shows that, in particular, any optimal solution to

$$\min_{y \in \mathbb{R}^N} \sum_{i,j=1}^{N} |y_i - y_j - \log(a_{ij})|^2 \qquad (13)$$

is efficient for (Y), yielding

Corollary 7 *The row geometric mean x^*,*

$$x_i^* = \left(\prod_{j=1}^{N} a_{ij} \right)^{\frac{1}{N}} , \; i = 1, \ldots, N \qquad (14)$$

is efficient for (X)

PROOF
Optimality conditions, which are both necessary and sufficient for the unconstrained convex smooth program (13) read

$$\sum_{j \neq k} (y_k - y_i - \log(a_{kj})) - \sum_{i \neq k} (y_i - y_k - \log(a_{ik})) = 0, \qquad k = 1, \ldots, N. \qquad (15)$$

Since A is a positive reciprocal matrix, $\log(a_{ik}) = -\log(a_{ki})$ for all i, k, thus (15) can also be written as

$$N y_k = \sum_{j=1}^{N} y_j + \sum_{j=1}^{N} \log(a_{kj}), \qquad k = 1, \ldots, N,$$

a particular solution of which is given by y,

$$y_k = \frac{1}{N} \sum_{j=1}^{N} \log(a_{kj}) = \log \left(\prod_{j=1}^{N} a_{kj} \right)^{\frac{1}{N}}, \qquad k = 1, \ldots, N$$

Hence, x^* defined in (14) is such that $\log(x^*)$ is efficient for (Y). Hence, x^* is efficient for (X). $\square$

Now we present a geometrical characterization of efficiency.

Definition 8 *Given $y \in \mathbb{R}^N$, let $G(y)$ be the digraph $G(y) := (\{1, 2, \ldots, N\}, E(y))$,*

$$(i, j) \in E(y) \text{ iff } i \neq j \text{ and } y_i - y_j \geq \log(a_{ij})$$

Observe that, by definition, for i, j given, $i \neq j$, either $(i, j) \in E(y)$ or $(j, i) \in E(y)$, or both.

Theorem 9 *y is efficient for (Y) iff $G(y)$ is strongly connected.*

Proof

Let y^* be an efficient solution of (Y). This is equivalent, [3], to the fact that y^* is an optimal solution of the scalar problem (P_λ)

$$\min_{y \in \mathbb{R}^N} \rho_\lambda(y) := \sum_{i \neq j} \lambda_{ij} \left| y_i - y_j - \log(a_{ij}) \right| \qquad (P_\lambda)$$

for some $\lambda = (\lambda_{ij})_{i \neq j}$, with $\lambda_{ij} > 0$ for all i, j, $i \neq j$. Problem (P_λ) is convex, hence, a necessary and sufficient optimality condition for y^* is

$$\mathbf{0} \in \partial \rho_\lambda(y^*). \qquad (16)$$

The objective of (P_λ) can be written as

$$\rho_\lambda(y) = \sum_{i \neq j} \lambda_{ij} \max \left\{ (y_i - y_j - \log(a_{ij})), (y_j - y_i + \log(a_{ij})) \right\}. \qquad (17)$$

hence, every subgradient ξ at y^* has the form

$$\xi = \sum_{\substack{(i,j) \in E(y^*) \\ (j,i) \notin E(y^*)}} \lambda_{ij} \left(\mathbf{e}^i - \mathbf{e}^j \right) + \sum_{\substack{(i,j) \notin E(y^*) \\ (j,i) \in E(y^*)}} \lambda_{ij} \left(-\mathbf{e}^i + \mathbf{e}^j \right) + $$
$$+ \sum_{\substack{(i,j) \in E(y^*) \\ (j,i) \in E(y^*)}} \lambda_{ij} \left[\mu_{ij} \left(\mathbf{e}^i - \mathbf{e}^j \right) + (1 - \mu_{ij}) \left(-\mathbf{e}^i + \mathbf{e}^j \right) \right]$$

where $\mu_{ij} \in [0,1]$ for all i,j such that $\{(i,j),(j,i)\} \subset E(y^*)$. Hence, condition (16) can be rewritten as

$$0 = \sum_{\substack{(i,j)\,\in\,E(y^*)\\(j,i)\,\notin\,E(y^*)}} \lambda_{ij} + \sum_{\substack{(i,j)\,\notin\,E(y^*)\\(j,i)\,\in\,E(y^*)}} -\lambda_{ij} + \sum_{\substack{(i,j)\,\in\,E(y^*)\\(j,i)\,\in\,E(y^*)}} \lambda_{ij}\left[\mu_{ij} - (1-\mu_{ij})\right] \forall i, \qquad (18)$$

that is

$$0 = \sum_{(i,j)\in E(y^*)} \hat{\lambda}_{ij} - \sum_{(j,i)\in E(y^*)} \hat{\lambda}_{ji} \quad \forall i = 1,2,\ldots,N \qquad (19)$$

where $\hat{\lambda}_{ij} > 0$, for all i,j with $i \neq j$.

The homogeneous system (19) has at least one positive solution if and only if there exists a feasible flow in $G(y^*)$ verifying the lower bound on the flow $\hat{\lambda}_{ij} \geq 1$, for every arc (i,j). Following the circulation theorem of Hoffman [8, 11], this is equivalent to the non existence of cuts $(S,\overline{S})$ having positive value $V(S)$, where

$$V(S) = d(S) + l(S,\overline{S}) - u(\overline{S},S), \qquad (20)$$

and $d(S)$ is the sum of the demands at nodes of S, $l()$ and $u()$ are the sums of lower and upper bounds on the corresponding arcs. In our problem every demand is null, then $d(S) = 0$. Moreover

$$l(S,\overline{S}) = \#\{(i,j) \in G(y^*) : i \in S,\ j \notin S\}, \qquad (21)$$

that is, $l(S,\overline{S})$ is the number of arcs from S to its complement. On other hand, there is no upper bounds on the individual flows through the arcs, that is $u(\overline{S},S) = +\infty$ if $(\overline{S},S) \neq \emptyset$.

Hence, $G(y^*)$ cannot contain directed cuts, i.e. cuts satisfying $(\overline{S},S) = \emptyset$, [8, 11], since in other case, there exists S such that $u(\overline{S},S) = 0$ which implies a positive value of $V(S)$ in (20). Finally, note that a directed graph is strongly connected if and only if it has no directed cuts. □

From the previous results one then obtains

Corollary 10 *Vector $x \in \mathbb{R}^N_{++}$ is efficient for (X) iff $G(\log(x))$ is strongly connected.*

Corollary 11 *If $x \in \mathbb{R}^N_{++}$ is efficient for (X) and $x^* \in \mathbb{R}^N_{++}$ is such that $E(\log(x^*)) \supseteq E(\log(x))$, then x^* is efficient for (X).*

Remark 12 *A characterization similar to that obtained in Corollary 10 is possible for weakly efficient solutions. We recall that $x^* \in \mathbb{R}^N_{++}$ is said to be weakly efficient for (X) iff no $x \in \mathbb{R}^N_{++}$ exists with*

$$\left|\frac{x_i}{x_j} - a_{ij}\right| < \left|\frac{x_i^*}{x_j^*} - a_{ij}\right| \qquad \textit{for all } i, j,\ i \neq j$$

With the same scheme of the proof, one can show that $x^* \in \mathbb{R}^N_{++}$ is weakly efficient for (X) iff $G(\log(x^*))$ contains at least one cycle.

Corollary 10 will be the cornerstone of a geometrical characterization of the efficient set for (X). First we have

Lemma 13 *Given $y^* \in \mathbb{R}^N$, the following statements are equivalent:*

1. *y^* is efficient for (Y).*

2. *For all $k = 1, \ldots, N$, the set $B_k(y^*) := \Big\{z \in \mathbb{R}^N : z_k = 0, z_i - z_j - \log(a_{ij}) \geq 0$ $\forall i, j = 1, \ldots, N$ such that $y_i^* - y_j^* - \log(a_{ij}) \geq 0\Big\}$ is bounded.*

Proof
y^* is efficient for (Y) iff for all $k = 1, \ldots, N$, $(y_1^*, \ldots, y_{k-1}^*, y_{k+1}^*, \ldots, y_N^*)$ is efficient for (Y_k)

$$\min \Big(|y_i - y_j - \log(a_{ij})|_{i \neq j,\, j \neq k}, |y_j + \log(a_{kj})|_{j \neq k}, |y_i - \log(a_{ik})|_{i \neq k} \Big) \qquad (Y_k)$$

(Y_k) is a linear multiobjective regression problem with design matrix of maximum rank, $N-1$ (it contains an $(N-1) \times (N-1)$ identity submatrix). Hence, Theorem 1 of [3] applies. Thus, y^* is efficient iff the sets $B_k(y^*)$ are bounded $\square$

Theorem 14 *For $x^* \in \mathbb{R}^N_{++}$ define $C(x^*)$ as*

$$C(x^*) := \Big\{x \in \mathbb{R}^N_+ : x_i - a_{ij}x_j \geq 0,\ \forall i, j = 1, \ldots, N \text{ such that } x_i^* - a_{ij}x_j^* \geq 0\Big\}$$

The following statements are equivalent:

1. *x^* is efficient for (X).*

2. *For all $k = 1, \ldots, N$, the set $\{x \in C(x^*) : x_k = 1\}$ is contained in $\mathbb{R}^N_{++} \cup \{0\}$.*

3. *$C(x^*) \subseteq \mathbb{R}^N_{++} \cup \{0\}$.*

Proof
$(1 \Rightarrow 2)$ Let x^* be an efficient solution for (X) and $k \in \{1, \ldots, N\}$. Then the vector $x^k = \frac{1}{x_k^*}x^*$ is also efficient for (X), thus by Corollary 5, $\log x^k$ is efficient for Problem (Y). By Lemma 13 the set

$$\begin{aligned} B_k(\log x^k) &= \Big\{z \in \mathbb{R}^N : z_k = 0, z_i - z_j - \log(a_{ij}) \geq 0 \\ &\quad \forall i, j = 1 \ldots N \text{ such that } \log(x_i^*) - \log(x_j^*) - \log(a_{ij}) \geq 0\Big\} \\ &= \Big\{z \in \mathbb{R}^N : z_k = 0, z_i - z_j - \log(a_{ij}) \geq 0 \\ &\quad \forall i, j = 1 \ldots N \text{ such that } x_i^* - x_j^* a_{ij} \geq 0\Big\} \end{aligned}$$

is bounded.

Let us suppose by contradiction that there exists $x \in C(x^*) \cap \{x : x_k = 1\}$, with at least one null component, say $x_1 = 0$.

Since $x^* \in \mathbb{R}^N_{++}$, the vector $x^\lambda := (1-\lambda)x + \lambda x^* \in \mathbb{R}^N_{++}$, $\forall \lambda \in (0,1]$ and $x^\lambda \in C(x^*) \cap \{x : x_k = 1\}$ by convexity of such set. Then $\log x^\lambda \in B_k(\log x^*)$, $\forall \lambda \in (0,1]$. Since $B_k(\log x^*)$ is bounded, the limit

$$\lim_{\lambda \downarrow 0} \log(x_1^\lambda)$$

is finite, which is a contradiction with the assumption that $x_1 = 0$.

($2 \Rightarrow 3$) Let $x \in C(x^*)$ and suppose, by contradiction, that at least one of its components, say x_i, is zero. Since $x \neq 0$, there exists at least a nonzero component x_k. Since $C(x^*)$ is a polyhedral cone, the vector $\frac{1}{x_k} x \in C(x^*)$ and has a zero component which contradicts the assumption that $C(x^*) \cap \{x : x_i = 1\} \subseteq \mathbb{R}^N_{++} \cup \{0\}$.

($3 \Rightarrow 1$) Let us assume that $C(x^*) \subseteq \mathbb{R}^N_{++} \cup \{0\}$, and we will show that $B_k(\log x^*)$ is bounded.

$C(x^*)$ can be re-written as

$$C(x^*) = \left\{ \sum_{d \in D} \lambda_d x^d \,:\, \lambda_d \geq 0 \, \forall d \in D \right\} \tag{22}$$

where $\{x^d : d \in D\}$ is the (finite) set of extreme directions of $C(x^*)$.

Since, by assumption, $C(x^*) \subseteq \mathbb{R}^N_{++} \cup \{0\}$, one has that x^d has all its components strictly positive (else, since $0 \in C(x^*)$, one would have some nonzero $x \in C(x^*) \setminus \mathbb{R}^N_{++}$). Hence, for each $k = 1, \ldots, N$,

$$C(x^*) \cap \{x \in \mathbb{R}^N : x_k = 1\} = \left\{ \sum_{d \in D} \frac{\lambda_d}{x_k^d} x^d \,:\, \lambda_d \geq 0, \sum_{d \in D} \lambda_d = 1 \right\}. \tag{23}$$

Thus $C(x^*) \cap \{x \in \mathbb{R}^N : x_k = 1\}$ is bounded (it is the convex combination of a finite set of points). Hence, there exist $0 < L_1^k \leq U_1^k, \ldots, 0 < L_N^k \leq U_N^k$ such that $L_i^k \leq x_i \leq U_i^k$, $\forall i = 1, \ldots, N$, $\forall x \in C(x^*) \cap \{x \in \mathbb{R}^N : x_k = 1\}$. Given $z \in B_k(\log x^*)$, the vector $e^z := (\exp(z_1), \ldots, \exp(z_{k-1}), 1, \exp(z_{k+1}), \ldots, \exp(z_N))$, satisfies

$$\frac{\exp(z_i)}{\exp(z_j)} = \exp(z_i - z_j) \geq \exp(\log(a_{ij})) = a_{ij} \; \forall i, j \text{ such that } (i,j) \in E(\log x^*), \tag{24}$$

thus

$$e^z \in C(x^*) \cap \{x \in \mathbb{R}^N : x_k = 1\}. \tag{25}$$

Hence, we have $0 < L_i^k \leq \exp(z_i) \leq U_i^k$, thus $-\infty < \log(L_i^k) \leq z_i \leq \log(U_i^k) < +\infty \, \forall i$, showing that $B_k(\log x^*)$ is bounded. By Lemma 13, $\log(x^*)$ is efficient for (Y) thus x^* is efficient for (X). □

We summarize with the following

Corollary 15 *Let $\mathcal{E} = \{E \subseteq \{1, \ldots, N\} \times \{1, \ldots, N\}$ such that $\forall i, j$, $i \neq j$, $(i,j) \in E$ or $(j,i) \in E\}$ and, for each $E \in \mathcal{E}$, define C_E as*

$$C_E = \left\{ x \in \mathbb{R}^N_+ : x_i - a_{ij} x_j \geq 0 \forall (i,j) \in E \right\}.$$

Then, for $x^* \in \mathbb{R}^N_{++}$, the following statements are equivalent:

1. x^* is efficient for (X).

2. There exists $E \in \mathcal{E}$ such that $C_E \subseteq \mathbb{R}^N_{++} \cup \{0\}$ satisfying $x^* \in C_E$.

3. There exists $E \in \mathcal{E}$ such that $(\{1, \ldots, N\}, E)$ is strongly connected satisfying $x \in C_E$.

PROOF
($1 \Rightarrow 2, 3$**).** Set $E = E(\log(x^*))$, thus $C_E = C(x^*)$ and $G(\log(x^*)) = (\{1, \ldots, N\}, E)$. Since $x^* \in C(x^*)$, Part 2 follows from Theorem 14, and Part 3 from Corollary 10.

($2 \Rightarrow 1$) Let $E \in \mathcal{E}$ such that $C_E \subseteq \mathbb{R}^N_{++} \cup \{0\}$ satisfies $x^* \in C_E$. By definition $C(x^*) \subseteq C_E$, thus, by Theorem 14 one has that x^* is efficient.

($3 \Rightarrow 1$) Let $E \in \mathcal{E}$ such that $(\{1, \ldots, N\}, E)$ is strongly connected and $x^* \in C_E$. By definition, $E(\log(x^*)) \supseteq E$, thus, $G(log(x^*))$ is strongly connected. Finally, Corollary 10 implies that x^* is efficient. □

3 The Eigenvector Method and efficiency

In this Section we show, by means of an example, that the solution provided by the Eigenvector Method may not be efficient for (X). To do this, consider the 4×4 matrix A,

$$A = \begin{pmatrix} 1 & 2 & 6 & 2 \\ \frac{1}{2} & 1 & 4 & 3 \\ \frac{1}{6} & \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 2 & 1 \end{pmatrix}$$

The eigenvalues of A are the roots of a polynomial function of fourth degree. Hence, they can be calculated analytically. In particular, using the symbolic computation package MAPLE [12] highest-modulus eigenvalue $\lambda_{\max}$,

$$\lambda_{\max} \approx 4.103141140$$

From $\lambda_{\max}$, an associated eigenvector x is obtained exactly. In order to obtain the corresponding x-graph, observe that for $i \neq j$,

$$(i, j) \in E((\log(x))) \text{ iff } \Delta(x)_{ij} := x_i - x_j a_{ij} \geq 0$$

The coefficients $\Delta(x)_{ij}$ were calculated numerically using interval arithmetic, accommodating round-off errors, using the package INTPAK [6]. The results are displayed in Table 1.

This yields

$$E(\log(x)) = \{(1, 3), (1, 4), (2, 1), (2, 3), (4, 2), (4, 3)\}$$

No directed path from 3 to 1. Hence, x is not efficient.

i	j	$\Delta(x)_{ij}$
1	2	(-2.5066080075891861670 , -2.5066080075891843904)
1	3	(.14380576758662222e-1 , .14380576758667095e-1)
1	4	(1.8718973568390277781 , 1.8718973568390377799)
2	1	(1.2533040037945921952 , 1.2533040037945930835)
2	3	(.26049429217392388555 , .26049429217392605112)
2	4	(-1.9532305377055277812 , -1.9532305377055178857)
3	1	(-.23967627931111824e-2 , -.23967627931103703e-2)
3	2	(-.65123573043481512e-1 , -.65123573043480971e-1)
3	4	(-.35620804979908616e-1 , -.35620804979907318e-1)
4	1	(-.93594867841951889 , -.93594867841951389)
4	2	(.65107684590183928 , .65107684590184260)
4	3	(.71241609959814636e-1 , .71241609959817232e-1)

Table 1: Coefficients $\Delta(x)_{ij}$

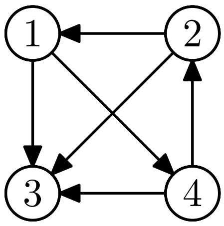

Figure 1: The graph $E(\log(x))$

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