Linear-quadratic control problem with a linear term on semiinfinite interval: theory and applications*

L. Faybusovich
T. Mouktonglang
Department of Mathematics,
University of Notre Dame, Notre Dame, IN 46556 USA

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Abstract

We describe a complete solution of the linear-quadratic control problem on a semiinfinite interval with the linear term in the objective function. Some applications are considered.

1 Introduction

In this paper we consider a standard linear-quadratic control problem on the semiinfinite interval. The only difference is that the problem has an extra linear term in the cost function which makes it time-dependent. The major motivation for this extension comes from [FT] where we consider applications of very efficient primal-dual interior-point algorithms to the computational analysis of multi-criteria linear-quadratic control problems in minimax form. To compute a primal-dual direction it is necessary to solve several linear-quadratic control problems with the same quadratic and different linear parts in the performance index. Another natural motivation comes from the multi-target linear-quadratic control problem (which is, of course, a particular case of multi-criteria LQ problem but admits a much simpler solution than the general problem [FM]). Surprisingly, the solution of LQ problem with an extra term is quite simple and described here in the standard L_2 setting in full generality.

2 Main Result

Denote by $L_2^n[0,\infty)$ the Hilbert space of square integrable functions $f : [0, +\infty) \to R^n$.

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Let $Y = L_2^n[0,\infty) \times L_2^m[0,\infty)$,

$$X = \{(x,u) \in Y : \dot{x} = Ax + Bu, x \text{ is absolutely continuous}, x(0) = 0\},$$

A is an n by n matrix and B is an n by m matrix; Let

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

be a symmetric $(m+n) \times (m+n)$ matrix. Σ_{11} in an n by n matrix and Σ_{22} is an m by m matrix; $(y,v) \in Y$.

Consider the following linear-quadratic control problem:

$$J(x,u) = \frac{1}{2}\int_0^\infty < \begin{bmatrix} x \\ u \end{bmatrix}, \Sigma \begin{bmatrix} x \\ u \end{bmatrix} > dt \tag{1}$$

$$+ \int_0^\infty [< y, x > + < v, u >]\, dt \to \min \tag{2}$$

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0 \tag{3}$$

Here $x_0 \in R^n$ is a fixed vector and $<,>$ is the standard scalar product in a finite-dimensional Euclidean space. We assume that (x,u) is in an affine subspace with the linear part equal to X.

To describe the solution of (1)-(3) we need the following result.

Theorem 1 *Let A be an antistable n by n matrix (i.e. real parts of all eigenvalues of A are positive). Consider the following system of linear differential equations:*

$$\dot{x} = Ax + f, \tag{4}$$

where $f \in L_2^n[0,\infty)$. There exists a unique solution $\mathcal{L}(f)$ of (4) such that $\mathcal{L}(f) \in L_2^n[0,\infty)$. Moreover the map $f \to \mathcal{L}(f)$ is linear and bounded. Explicitly:

$$\mathcal{L}(f)(t) = -\int_0^{+\infty} e^{-A\tau} f(\tau + t)d\tau \tag{5}$$

Proof Since A is antistable, there exist a positive definite symmetric matrix H such that

$$A^T H + HA = I, \tag{6}$$

where I is the identity matrix (see e.g. [Leonov]). Let x be any solution to (4). Consider

$$W(t) =< x(t), Hx(t) > .$$

Then, using (6), we easily obtain:

$$\frac{dW}{dt} = \|x(t)\|^2 + 2 < x(t), Hf(t) >,$$

where $\|x(t)\|^2 =< x(t), x(t) >$. Using Cauchy-Schwarz inequality, we obtain

$$\frac{dW}{dt} \geq \|x(t)\|^2 - 2\|x(t)\|\|Hf(t)\| \geq \|x(t)\|^2 - 2\|x(t)\|\|H\|\|f(t)\|,$$

where $\|H\|$ is the Matrix norm induced by the Euclidean norm on R^n.

It is quite obvious that

$$2\|x(t)\|\|H\|\|f(t)\| \leq \frac{1}{2}\|x(t)\|^2 + 2\|H\|^2\|f(t)\|^2$$

and hence,

$$\frac{dW}{dt} \geq \frac{\|x(t)\|^2}{2} - 2\|H\|^2\|f(t)\|^2. \tag{7}$$

Integrating (7) from 0 to t, we obtain:

$$W(t) - W(0) \ \geq \ \int_0^t \frac{\|x(\tau)\|^2}{2} d\tau - 2\|H\|^2 \int_0^t \|f(\tau)\|^2 d\tau.$$

Consequently,

$$\int_0^t \|x(\tau)\|^2 d\tau \ \leq \ 4\|H\|^2 \int_0^t \|f(\tau)\|^2 d\tau + 2[W(t) - W(0)] \tag{8}$$

Let now

$$x(t) = -\int_0^{+\infty} e^{-A\tau} f(\tau + t) d\tau = -\int_t^{+\infty} e^{-A(\tau - t)} f(\tau) d\tau \tag{9}$$

One can easily see that x is the solution to (4) such that

$$x(0) = -\int_0^{+\infty} e^{-A\tau} f(\tau) d\tau$$

One can easily see from (9) that

$$\|x(t)\|^2 \leq \int_0^{\infty} \|e^{-A\tau}\|^2 d\tau \int_0^{\infty} \|f(\tau)\|^2 d\tau$$

for any $t \geq 0$. Hence x and consequently W are bounded on $[0, +\infty)$. Using (8), we conclude that

$$\int_0^{\infty} \|x(t)\|^2 dt < \infty$$

i.e. $x(t)$ given by (9) is in $L_2^n[0, +\infty)$.

Since x is a solution to (4), we conclude that $\dot{x} \in L_2^n[0, +\infty)$ and hence $x(t) \to 0, t \to +\infty$ (see appendix for the proof). But then using (8) again, we conclude that

$$\int_0^{\infty} \|x(\tau)\|^2 d\tau \ \leq \ 4\|H\|^2 \int_0^{\infty} \|f(\tau)\|^2 d\tau - 2W(0)$$

$$\leq (4\|H\|^2 + 2\|H\|)(\int_0^\infty \|f(\tau)\|^2 d\tau + \int_0^\infty \|e^{-A\tau}\|^2 d\tau \int_0^\infty \|f(\tau)\|^2 d\tau)$$

$$\leq C \int_0^\infty \|f(\tau)\|^2 d\tau \tag{10}$$

for some constant C. Observe now that (4) may have only one solution in $L_2^n[0, +\infty)$. Indeed, if x_1 and x_2 are two such solutions and $x = x_1 - x_2$, then $x \in L_2^n[0, +\infty)$ and

$$\dot{x} = Ax.$$

Since A is antistable and $x \in L_2^n[0, +\infty)$, we immediately conclude that $x \equiv 0$. We can conclude now by (10) that the linear map $f \to \mathcal{L}(f)$ is bounded. $\diamond$

Consider the following linear-quadratic control problem:

$$\frac{1}{2} \int_0^\infty < \begin{bmatrix} x \\ u \end{bmatrix}, \Sigma \begin{bmatrix} x \\ u \end{bmatrix} > dt \to \min \tag{11}$$

$$\begin{bmatrix} x \\ u \end{bmatrix} \in X \tag{12}$$

As is well-known, the following algebraic Riccati equation plays the crucial role in the description of optimal solution to (11),(12):

$$KLK + K\tilde{A} + \tilde{A}^T K - Q = 0 \tag{13}$$

Here

$$\tilde{A} = A - B\Sigma_{22}^{-1}\Sigma_{21}, \;\; Q = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, \;\; L = B\Sigma_{22}^{-1}B^T. \tag{14}$$

Observe that (13) is defined under the assumption that Σ_{22} is invertible. Recall, that a symmetric solution $K = K^T$ to (13) is called stabilizing if the matrix $\tilde{A} + LK$ is stable. the following result is a well-known (for a concise proof see e.g [Lancaster],[Faybusovich]).

Theorem 2 *The following conditions are equivalent:*

i) Σ_{22} *is a positive definite (symmetric) matrix and (13) has a stabilizing solution.*

ii) The pair (A, B) *is stabilizable and there exists* $\epsilon > 0$ *such that*

$$\Gamma(x, u) = \int_0^{+\infty} < \begin{bmatrix} x \\ u \end{bmatrix}, \Sigma \begin{bmatrix} x \\ u \end{bmatrix} > dt \geq \epsilon \int_0^{+\infty} (\|x\|^2 + \|u\|^2) dt$$

for all $(x, u) \in X$.

Remark. The condition ii) means that the quadratic functional $\Gamma(x, u)$ is strictly convex on X.

Theorem 3 *Suppose that the pair* (A, B) *is stabilizable. Then* X *is a closed vector subspace in* Y*. Let*

$$Z = \{ \begin{bmatrix} \dot{p} + A^T p \\ B^T p \end{bmatrix} : p \in L_2^n[0, +\infty), p \text{ is absolutely continuous}, \dot{p} \in L_2^n[0, +\infty) \}$$

Then Z *is an orthogonal complement of* X *in* Y*. (i.e.* $Z = X^{\perp}$*)*

Proof Let $(x, u) \in X$, and $(\dot{p} + A^T p, B^T p) \in Z$. We are going to show that

$$\alpha = \int_0^{+\infty} < \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} \dot{p} + A^T p \\ B^T p \end{bmatrix} > dt = 0$$

Indeed,

$$\begin{aligned} \alpha &= \int_0^{+\infty} [< x, \dot{p} + A^T p > + < u, B^T p >] dt \\ &= \int_0^{+\infty} [< Ax + Bu, p > + < x, \dot{p} >] dt \\ &= \int_0^{+\infty} \left[\frac{d}{dt} < x, p > \right] dt = \lim_{T \to +\infty} < x(T), p(T) > . \end{aligned}$$

Observe that $x(0) = 0$, since $(x, u) \in X$. But $\lim_{T\to\infty} x(T) = \lim_{T\to\infty} p(T) = 0$, since both x and p are in $L_2^n[0, +\infty)$, absolutely continuous and are such that $\dot{x} \in L_2^n[0, +\infty), \dot{p} \in L_2^n[0, +\infty)$ (see Appendix). Hence, $\alpha = 0$.

We next show that any $(\phi, \psi) \in Y$ admits the following representation:

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix} - \begin{bmatrix} \dot{p} + A^T p \\ B^T p \end{bmatrix} \tag{15}$$

with $(x, u) \in X$ and $(\dot{p} + A^T p, B^T p) \in Z$. This will easily imply that $Z = X^{\perp}$ and both X and Z are closed. If Σ is the identity matrix, Theorem 2 easily implies that the corresponding algebraic Riccati equation (13) has a stabilizing solution K_0. Observe that in this case $\tilde{A} = A, \quad Q = I_n, L = BB^T$. We can rewrite (15) in the form:

$$\dot{p} = x - A^T p - \phi \tag{16}$$

$$u = B^T p + \psi \tag{17}$$

We also have

$$\dot{x} = Ax + Bu, \ x(0) = 0 \tag{18}$$

Substituting (17) into (18), we obtain

$$\dot{x} = Ax + BB^T p + B\psi. \tag{19}$$

We are looking for the solution to (16),(19)in the form.

$$p = K_0 x + \rho, \tag{20}$$

where K_0 is the stabilizing solution to the algebraic Riccati equation (13).

Substituting (20) into (16),(19), we obtain:

$$\dot{x} = Ax + BB^T K_0 x + BB^T \rho + B\psi, \tag{21}$$

$$K_0 \dot{x} + \dot{\rho} = x - A^T K_0 x - A^T \rho - \phi. \tag{22}$$

Finally, substituting (21) into (22), we obtain

$$(K_0 A + A^T K_0 + K_0 BB^T K_0 - I)x + \dot{\rho} = -A^T \rho - \phi - K_0 B\psi - K_0 BB^T \rho,$$

i.e.

$$\dot{\rho} = -(A^T + K_0 BB^T)\rho - \phi - K_0 B\psi. \tag{23}$$

Since the matrix $A + BB^T K_0$ is stable, the matrix $-(A + BB^T K_0)$ is antistable hence we can apply Theorem 4. (Observe that $-\phi - K_0 B\psi \in L_2^n[0, +\infty)$) Thus (23) possesses a unique solution $\rho \in L_2^n[0, +\infty)$,

$$\rho = -\mathcal{L}(\phi + K_0 B\psi). \tag{24}$$

Reversing our reasoning, we see that if ρ is defined as in (24), x is defined as (21) with $x(0) = 0$, and p is defined as in (20), we then obtain the representation (15). $\diamond$

We are now in position to describe a solution of the LQ-problem on a semi-infinite interval.

Theorem 4 *Suppose that the conditions of theorem 2 are satisfied. Then the problem (1)-(3) has a unique solution which can be described as follows.*

There exists a stabilizing solution K_0 to the Riccati equation (13). Then the matrix $C = -(\tilde{A} + LK_0)$ is antistable, $(K_0 B - \Sigma_{12})\Sigma_{22}^{-1} v + y \in L_2^n[0, +\infty)$.

Let ρ be a unique solution from $L_2^n[0, +\infty)$ of the system of differential equations

$$\dot{\rho} = C^T \rho + (K_0 B - \Sigma_{12})\Sigma_{22}^{-1} v + y$$

(which exists according to Theorem 1); x is the solution to the system of differential equations

$$\dot{x} = (\tilde{A} + LK_0)x + L\rho - B\Sigma_{22}^{-1} v, \quad x(0) = x_0,$$

$$p = K_0 x + \rho, \quad u = \Sigma_{22}^{-1}(B^T p - v - \Sigma_{21} x).$$

Remark. Observe that by (5) we have the following explicit description of ρ:

$$\rho(t) = -\int_0^{+\infty} e^{C^T \tau}((K_0 B - \Sigma_{12})\Sigma_{22}^{-1} v + y)(t + \tau)d\tau$$

Sketch of the proof Since conditions of Theorem 2 are satisfied, we know that the functional in (1)-(3) is strictly convex, the matrix Σ_{22} is positive definite and the algebraic Riccati equation (13) has (a unique) stabilizing solution K_0.

The necessary and sufficient optimality condition for (1)-(3) obviously takes the form:

$$\Sigma \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} y \\ v \end{bmatrix} \in X^{\perp}, \;\; (x,u) \text{ statifies (3).}$$

Or using the description of $X^{\perp}$ from Theorem 3:

$$\Sigma_{11}x + \Sigma_{12}u + y = \dot{p} + A^T p,$$

$$\Sigma_{21}x + \Sigma_{22}u + v = B^T p$$

for some p satisfying condition of Theorem 3. We are looking for p in the form:

$$p = K_0 x + \rho,$$

where K_0 is the stabilizing solution to the algebraic Riccati equation (13). We finish the proof exactly as in Theorem 3. $\diamond$

Remark. The extension of Theorem 4 to the discrete time case is pretty straightforward.

3 Some Applications

Consider, first, the tracking problem:

$$\int_0^{+\infty} [< x-\phi, x-\phi > + < u-\psi, u-\psi >]dt \to \min, \tag{25}$$

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0. \tag{26}$$

Here $(\phi, \psi) \in Y$. It is quite clear that (25),(26) is equivalent to

$$\frac{1}{2}\int_0^{+\infty} [< x, x > + < u, u >]dt - \int_0^{+\infty} [< x, \phi > + < u, \psi >]dt \to \min,$$

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$

and hence can be solve using Theorem 4. Observe that for $x_0 = 0$,(25),(26) is the problem of finding the orthogonal projection of $(\phi, \psi) \in Y$ onto the closed subspace X.

In [FM] we have considered the following minimax problem. Let $(V, <, >_V)$ be a Hilbert space, T its closed vector subspace, $v_0, v_1, ..., v_l$ be vectors in V. Consider:

$$\max_{1 \le i \le l} \|v - v_i\| \to min, \tag{27}$$

$$v \in v_0 + T. \tag{28}$$

In case, $V = Y$, $T = X$, we arrive at multi-target linear-quadratic control problem on a semiinfinite interval. We have shown in [FM] that the optimal solution of this problem is contained in the convex hull $W = \mathrm{conv}_{\mathbb{R}}(\pi_T v_1, \cdots, \pi_T v_m)$.

Hence, it has the form:

$$v_{opt} = \sum_{i=1}^{m} \mu_i^{opt} \pi_T(v_i),$$

where $\mu_i^{opt} \geq 0$, $\sum_{i=1}^{m} \mu^{opt} = 1$.

Finding μ_i^{opt} can be reduced to solving a finite-dimensional second-order cone programming. Indeed, let

$$v(\mu) = \sum_{i=1}^{m} \mu_i \pi_T v_i$$

We have

$$\|v(\mu) - v_i\| = \sqrt{\|v(\mu) - \pi_T v_i\|^2 + \nu_i^2},$$

where $\nu_i = \|\pi_{T^\perp} v_i\|$ is the norm of the orthogonal projection of the vector v_i onto the orthogonal complement $T^\perp$ of T in Y.

Furthermore,

$$\|v(\mu) - \pi_T v_i\|^2 = \widetilde{\mu}^T(i) \Gamma \widetilde{\mu}(i),$$

where $\widetilde{\mu}_j(i) = \mu_j for j \neq i, \widetilde{\mu}_i(i) = \mu_i - 1$ and $\Gamma = (\langle \pi_T v_i, \pi_T v_j \rangle)$. Let $\Gamma = B^T B$ be the Cholesky decomposition of Γ. Then

$$\|v(\mu) - \pi_T v_i\|^2 = \|B\widetilde{\mu}\|_2^2 = \|B\mu - b_i\|_2^2,$$

where $b_i = Be_i$ the i-th column of B.

Hence,

$$\|v(\mu) - v_i\| = \| \begin{bmatrix} B\mu - b_i \\ \nu_i \end{bmatrix} \|_2$$

and we can rewrite the original problem (27),(28) in the following equivalent form:

$$t \to min, \tag{29}$$

$$\| \begin{bmatrix} B\mu - b_i \\ \nu_i \end{bmatrix} \|_2 \leq t, i = 1, 2, \ldots, m, \tag{30}$$

$$\mu \in \mathbb{R}^m. \tag{31}$$

(See [FM] for details.) The problem (29)- (31) is the second order cone programming problem which can be easily solved using the standard interior-point software.

Here π_T stands for the orthogonal projector of V onto T. In [FM], we have considered linear-quadratic control problem on a finite interval. In the present paper we described the procedure of calculating π_X on a semiinfinite interval. Thus we can solve a multi-target LQ problem on semiinfinite interval. Using Theorem 4, one can easily see that for finding $\pi_T v_i$, $i = 1, 2, ..., m$, it suffices to solve an algebraic Riccati equation $K_0 A + A^T K_0 + K_0 B B^T K_0 - I = 0$ only once.

4 Appendix

Let $H_1^n([0,+\infty)) = \{x \in L_2^n([0,+\infty)),\ \text{is absolutely continuous and } \dot{x} \in L_2^n([0,+\infty))\}$. This is one of the standard Sobolev spaces. Then the proof of the following Lemma is obtained e.g. from a proof of a similar result in [Leonov].

Lemma 1 *Let $x \in H_1^n([0,+\infty))$. Then*

$$\lim_{t\to+\infty} x(t) = 0$$

Proof We have:

$$\int_0^t |< x(\tau), \dot{x}(\tau) > | d\tau \leq \int_0^t \|x(\tau)\|^2 d\tau + \int_0^t \|\dot{x}(\tau)\|^2 d\tau$$

Since, $x(t), \dot{x}(t) \in L_2^n([0,+\infty))$, we have:

$$\lim_{t\to+\infty} \int_0^t < x(\tau), \dot{x}(\tau) > d\tau$$

is finite . But

$$\int_0^t < x(\tau), \dot{x}(\tau) > d\tau = \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|x(0)\|^2.$$

Hence, the limit $\lim_{t\to+\infty} \|x(t)\|^2$ exists, and since $\int_0^{+\infty} \|x(t)\|^2 dt < \infty$, we have

$$\lim_{t\to+\infty} \|x(t)\| = 0. \diamond$$

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