Stable Matchings for Three-Sided Systems: A Comment

by

Somdeb Lahiri
School of Economic and Business Sciences,
University of Witwatersrand at Johannesburg,
Private Bag 3, WITS 2050,
South Africa.
February 8, 2004.
(This version: March 20, 2004.)
Email: lahiris@sebs.wits.ac.za
Or
Lahiri@webmail.co.za

Abstract

Sufficient conditions are provided for the existence of a stable matching for a three-sided system.
JEL Classification Number: C78.
Key Words: three-sided systems, matching, stable.

Introduction: The two-sided matching model of Gale and Shapley(1962) can be thought of as a buyer-seller market with each seller possessing a single indivisible object and each buyer being allocated at most one of those objects. With the introduction of middle-men, each of whom can facilitate at most one transaction, we are naturally lead to the framework of a three-sided system as developed by Alkan(1988). Danilov (2003) provides sufficient conditions for the existence of a stable matching for a three-sided system. Here we provide a weaker sufficient condition which is perhaps more realistic in the context of the manufacturer-vendor-buyer scenario. Our assumption is that manufacturers care first about vendors and then about buyers and vendors care first about buyers and then about manufacturers. This kind of a preference structure is likely in situations where a contract between a manufacturer and a vendor involves sharing profits in a certain proportion, irrespective of who the buyer is. Unlike Danilov (2003) we do not require reciprocity of preferential treatment between any two sides of the market. However, we do require the market to satisfy a certain discrimination property.

The Model: Let M be a no-empty finite set denoting the set of manufacturers, V a non-empty finite set denoting the set of vendors and B a non-empty finite set denoting the set of buyer. Each $m \in M$ has preference over $(V\times B)\cup\{m\}$ defined by a weak order (: reflexive, complete, transitive binary relation) $\geq_m$ whose asymmetric part is denoted $>_m$. Each $v \in V$ has preference over $(M\times B)\cup\{v\}$ defined by a linear order (: anti-symmetric weak order) $\geq_v$ whose asymmetric part is denoted $>_v$. Each $b\in B$ has preference over $(M\times V)\cup\{b\}$ defined by a linear order $\geq_b$ whose asymmetric part is denoted $>_b$.
Given $m\in M$, $v\in V$ and $b\in B$, let $A(m) = \{(m,v,b)\in \{m\}\times V\times B/ (v,b) >_m m\}$, $A(v) = \{(m,v,b)\in M\times\{v\}\times B/ (m,b) >_v v\}$, $A(b) = \{(m,v,b)\in M\times V\times\{b\}/ (m,v) >_b b\}$. A(m) is called the acceptable set of m, A(v) is called the acceptable set of v and A(b) is called the acceptable set of b.
Let $M^* = \{m \in M/ A(m) \neq \phi\}$, $V^* = \{v \in V/ A(v) \neq \phi\}$ *and* $B^* = \{b \in B/ A(b) \neq \phi\}$.

For the sake of expositional simplicity, we assume the following:
For all $b \in B^*$, $v \in V^*$ and $m \in M^*$: $A(b) = M^* \times V^*$, $A(v) = M^* \times B^*$ and $A(m) = V^* \times B^*$.

A matching is a function μ: $M \cup V \cup B \rightarrow (M \times V \times B) \cup (M \cup V \cup B)$ such that:

(i) for all $a \in M \cup V \cup B$: $\mu(a) \in A(a) \cup \{a\}$;

(ii) for all $m \in M$, $v \in V$ and $b \in B$ the following are equivalent: (a) $\mu(m) = (m,v,b)$; (b) $\mu(v) = (m,v,b)$; (c) $\mu(b) = (m,v,b)$.

Given a matching μ, $m \in M$, $v \in V$ and $b \in B$, let

$\mu^M(m) = m$ if $\mu(m) = m$,
$= (v,b)$ if $\mu(m) = (m, v, b)$;

$\mu^V(v) = v$ if $\mu(v) = v$,
$= (m,b)$ if $\mu(v) = (m, v, b)$;

$\mu^B(b) = b$ if $\mu(b) = b$,
$= (m,v)$ if $\mu(b) = (m, v, b)$.

A matching μ is said to be stable if there does not exist $m \in M$, $v \in V$ and $b \in B$ such that: $(v,b) >_m \mu^M(m)$, $(m,b) >_v \mu^V(v)$ and $(m,v) >_v \mu^B(b)$.

The Existence Result: The market is said to satisfy Discrimination Property (DP) if:

(i) For all $m \in M$ there exists a linear order P_m on V such that for all $v,v' \in V$ with $v \neq v'$ and $b,b' \in B$: vP_mv' implies $(v,b) >_m (v',b')$.

(ii) There exists a function $\beta: V^* \times M^* \rightarrow B^*$ such that (a) for all $m,m_1 \in M^*$ and $v,v_1 \in V^*$ with $m \neq m_1$ and $v \neq v_1$: $\beta(v,m) \neq \beta(v_1,m_1)$; (b) for all $m \in M^*$, $v \in V^*$ and $b \in B^*$, $(m,\beta(v,m)) \geq_v (m,b)$.

Clearly a market that satisfies the discrimination property has more buyers than sellers. The following example shows that merely (i) of DP is not enough to guarantee the existence of a stable matching.

Example 1: Let $M = \{m_1,m_2\}$, $V = \{v_1,v_2\}$, $B = \{b_1, b_2\}$.
Suppose preferences are such that for all $m \in M$, $v \in V$ and $b \in B$: m prefers (b,v) to remaining single, v prefers (m,b) to remaining single and b prefers (m,v) to remaining single.
Further assume that the preferences of all $m \in M$ satisfy condition (i) of DP, with both m_1 and m_2 preferring b_1 to b_2 for any given vendor v. Suppose both m_1 and m_2 prefer v_1 to v_2.
Suppose v_1 prefers (m_2,b_1) to (m_1, b_1) to (m_1,b_2) to $(m_2,b_2)$and v_2 prefers (m_1,b_1) to (m_2,b_1) to (m_2,b_2) to (m_1, b_2).
Suppose b_1 prefers (m_1,v_2) to (m_2,v_1) to (m_2, v_2) to (m_1, v_1) and b_2 prefers (m_1,v_1) to (m_2,v_1).
Let us consider the following four matchings:

(1) $\{(m_1, v_1, b_1), (m_2, v_2, b_2)\}$;
(2) $\{(m_1, v_1, b_2), (m_2, v_2, b_1)\}$;
(3) $\{(m_1, v_2, b_1), (m_2, v_1, b_2)\}$;
(4) $\{(m_1, v_2, b_2), (m_2, v_1, b_1)\}$.

Matching (1) is blocked by (m_2, v_2, b_1) since m_2 prefers (v_2,b_1) to (v_2,b_2), b_1 prefers (m_2,v_2) to (m_1, v_1) and v_2 prefers (m_2, b_1) to (m_2, b_2).
Matching (2) is blocked by (m_2, v_1,b_1) since m_2 prefers (v_1,b_1) to (v_2,b_1), v_1 prefers (m_2,b_1) to (m_1,b_2) and b_1 prefers (m_2,v_1) to (m_2,v_2).
Matching (3) is blocked by (m_1,v_1,b_2) since m_1 prefers (v_1,b_2) to (v_2,b_2), v_1 prefers (m_1,b_2) to (m_2,b_2) and b_2 prefers (m_1,v_1) to (m_2,v_1).
Matching (4) is blocked by (m_1,v_2,b_1) since m_1 prefers (v_2,b_1) to (v_2,b_2), v_2 prefers (m_1,b_1) to (m_1,b_2) and b_1 prefers (m_1,v_2) to (m_2,v_1).
Hence none of the four matchings above are stable. A matching where some agents are single can similarly be shown to be unstable, since a matching with any agent single must have either three agents each on a different side of the market or all six agent remaining single.
Further, $\beta(v_1,m_2) = \beta(v_2,m_1) = b_1$. This contradicts requirement (ii) of DP.

Theorem 1: Suppose the market satisfies DP. Then there exists a stable matching.

Proof: Suppose

(i) For all $m \in M$, there exists a linear order P_m on V such that for all $v,v' \in V$ with $v \neq v'$ and $b,b' \in B$: vP_mv' implies $(v,b) >_m (v',b')$.

(ii) There exists a function $\beta: V^* \times M^* \rightarrow B^*$ such that (a) for all $m,m_1 \in M^*$ and $v,v_1 \in V^*$ with $m \neq m_1$ and $v \neq v_1$: $\beta(v,m) \neq \beta(v_1,m_1)$; (b) for all $m \in M^*$, $v \in V^*$ and $b \in B^*$, $(m,\beta(v,m)) \geq_v (m,b)$.

For $v \in V^*$ let P_v be a linear order on M^* such that for all $m, m' \in M^*$: mP_vm' if and only if $(m, \beta(v,m)) \geq_v (m', \beta(v, m'))$
By the applying the algorithm called deferred acceptance procedure with vendors proposing as in Gale and Shapley (1962) (see appendix for details), we get a function $\rho: M^* \cup V^* \rightarrow M^* \cup V^*$ such that:
(i) for all $m \in M^*$, $v \in V^*$: $\rho(m) \in V^* \cup \{m\}$, $\rho(v) \in M^* \cup \{v\}$;
(ii) for all $a \in M^* \cup V^*$: $\rho(\rho(a)) = a$;
(iii) there does not exist $m \in M^*$ and $v \in V^*$ such that $m \neq \rho(v)$, $v \neq \rho(m)$, $mP_v\rho(v)$ and $vP_m\rho(m)$.
The matching μ is defined as follows:
If $m \in M^*$ and $v \in V^*$ are such that $\rho(m) = v \in V^*$, then let $\mu(m) = \mu(v) = \mu(\beta(v,m)) = (m,v,\beta(v,m))$. For any other 'a' belonging to $M \cup V \cup B$, let $\mu(a) = a$.
By Property (ii) of DP, μ is well defined.
Suppose the matching so defined is not stable. Thus, there exists $m \in M^*$, $v \in V^*$ and $b \in B^*$ such that: $(v,b) >_m \mu^M(m)$, $(m,b) >_v \mu^V(v)$ and $(m,v) >_v \mu^B(b)$.
Clearly $(m,\beta(v,m)) \geq_v (m,b) >_v \mu^V(v) = (m', \beta(v, m'))$ say. By the Gale-Shapley (1962) deferred acceptance procedure with vendors proposing, vendor v must have proposed to

m and had subsequently been rejected by m in favor of some other vendor v'. Since as the deferred acceptance procedure evolves, no manufacturer declines down his preference scale, it must be the case that manufacturer m prefers $\mu^M(m)$ to (v, $\beta(v,m)$). By property (i) of DP, it must be the case that $\mu^M(m) >_m (v,b)$, leading to a contradiction. Q.E.D.

Conclusion: The proof of theorem 1, does not depend on the preferences of the buyers at all. This is a consequence of the Discrimination Property. While the Discrimination Property is as meaningful as our Theorem 1 is in the context of three sided matching problems, we are none-the less lead to an extension of the two-sided matching model of Gale and Shapley (1962), where M can be interpreted as a non-empty finite set of workers, V can be interpreted as a non-empty finite set of firms and B can be interpreted as a non-empty finite set of techniques. Further analysis in this framework is available in a related paper (Lahiri (2004)).

Acknowledgment: I would like to thank Ashok Srinivisan for motivating me towards the manufacturer-vendor-buyer interpretation of a three-sided system and for several discussions on the issue. I would also like to put on record a very deep acknowledgment to V.I. Danilov, for pointing out an error in the proof of an earlier version of the above theorem and for the motivation behind the concluding section of this paper. However, none other than me is responsible for errors that remain.

References:

1. Alkan, A., 1988. Non-existence of stable threesome matchings. Mathematical Social Sciences, 16, 207-209.
2. Danilov, V.I., 2003. Existence of stable matchings in some three-sided systems. Mathematical Social Sciences 46, 145 -148.
3. Gale, D., Shaply, L.S., 1962. College admission and the stability of marriage. American Mathematical Monthly 69, 9-14.
4. Lahiri, S., 2004. Pair-wise envy free and stable matchings for two-sided systems with techniques (unpublished).

Appendix

Deferred Acceptance Procedure With Vendors Proposing (due to Gale and Shapley (1962): To start each vendor makes an offer to her favorite manufacturer, i.e. to the manufacturer ranked first according to her preferences. Each manufacturer who receives one or more offers, rejects all but his most preferred of these. Any vendor whose offer is not rejected at this point is kept "pending".

At any step any vendor whose offer was rejected at the previous step, makes an offer to her next choice (i.e., to her most preferred manufacturer, among those who have not rejected her offer), so long as there remains a manufacturer to whom she has not yet made an offer. If at any step of the procedure, a vendor has already made offers to, and been rejected by all manufacturers, then she makes no further offers. Each manufacturer receiving offers rejects all but his most preferred among the group consisting of the new offers together with any vendor that he may have kept pending from the previous step.

The algorithm stops after any step in which no vendor is rejected. At this point, every vendor is either kept pending by some manufacturer or has been rejected by every manufacturer. The matching ρ that is defined now, associates to each vendor the manufacturer who has kept her pending, if there be any. Further, manufacturers who did not receive any offers at all, and vendors who have been rejected by all the manufacturers, remain single.