Cover Inequalities for Binary–Integer Knapsack Constraints

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Abstract

We consider knapsack constraints involving one general integer and many binary variables. We introduce the concept of a cover for such a constraint and we construct a new family of valid inequalities based on this concept. We generalize this idea to extended covers, and we propose a specialized lifting procedure for cover inequalities. Finally, we illustrate the efficiency of our approach on a large-scale real world supply chain optimization problem.

Keywords: cover inequality, general integer variables, extended cover, lifting, supply chain management

1 Introduction

The object of our research are mixed-integer linear programming problems containing binary as well general integer variables in the following form:

$$
\begin{aligned}
&\min c^T x \\
\text{subject to } &Ax \geq b \\
&x \geq 0 \\
&x_j \in \mathcal{R}_+ \quad \text{for} \quad j \in V_R \\
&x_j \in \mathcal{Z}_+ \quad \text{for} \quad j \in V_I \\
&x_j \in \mathcal{B} \quad \text{for} \quad j \in V_{0-1}
\end{aligned}
\tag{1}
$$

where A is an $m \times n$ matrix. The vector b is m-dimensional, while c and x are n-dimensional. We use $\mathcal{R}_+$, $\mathcal{Z}_+$ and $\mathcal{B}$ to denote the set of positive integer, positive real, and binary (taking values 0 or 1) variables, respectively. The index set $\{1, \ldots, n\}$ is partitioned into the three disjoint subsets V_R, V_I, V_{0-1} standing for the index set of real positive, integer positive, and binary (zero-one) variables respectively. The inequality $\geq$ is understood componentwise.

In the existing literature, cutting planes and valid inequalities for mixed-integer programming problems have been essentially derived for 0–1 variables. Although considered by Ceria et al. [3], Owen and Mehrotra [8, 9], Marchand et al. [6], mixed-integer programs comprising general integer variables have been generally disregarded. They are thus to be solved mainly or solely relying on branch-and-bound or branch-and-cut algorithms embedded in commercial solvers, which are very unlikely to succeed even for problems of limited complexity. However, mixed-integer programs with general integer variables are very common in industrial applications. Ceria et al. [3] propose the following typology of mixed-integer programs with general integer variables:

- models with demand constraints arising in electricity generation, transportation or production;

- models with capacity constraints arising when deciding the construction of a new facility and its capacity;
- models with batches of various capacities arising in network design when capacity can be installed by batches.

In this paper, we derive a new family of valid cover inequalities for knapsack constraints involving many binary variables and one integer variable. More precisely, we derive valid cover inequalities for the set X defined below:

$$X = \Big\{(x, y) \in \{0,1\}^n \times \mathcal{Z}_+ : \sum_{j=1}^{n} a_j x_j + a_y y \leq b,\ 0 \leq y \leq M\Big\}$$

where a_j, and a_y are all between 0 and b.

The valid inequalities derived in this paper for the type of knapsack constraints contained in the set X defined above are later in this paper referred to as *Binary–Integer Cover Inequalities*.

The remaining part of the paper is organized as follows.

In Section 2, we formulate the new family of binary–integer cover inequalities for the set X defined above, and we compare the valid inequalities proposed, using a simple example, with some alternative formulations. The comparison is focused upon the valid inequalities for integer programs with general integer variables fixed to their upper bounds derived by Ceria et al. [3], the introduction of disjunctive cuts [8], and the binarization of general integer variables [9].

In section 3, we apply our family to valid inequalities proposed in the context of supply chain management. More precisely, considering a three-stage supply chain, we construct an inventory-production-distribution plan over a planning horizon containing a finite number of periods. Despite the abundant literature on the analysis and optimization of production-distribution systems, this remains a widely open research idea [12]. A great part of the research done so far focusing on a single component of the production-distribution network [13], and few studies have attempted to develop models allowing the simultaneous management of the production and distribution functions. A review of production-distribution models in supply chain management is given by Vidal and Goetschalckx [14], paying special attention to the difficulty of solving the inherent mixed-integer programs. As a test bed, we use the data provided by one of the major chemical companies in North America.

2 Binary–Integer Covers

As for 0–1 knapsack constraints, our idea is to find subsets of variables, which cannot be included in the knapsack simultaneously. However, in addition to the construction of the cover sets, the values assumed by the general integer variable must also be taken into account, this contrasting with the 0–1 knapsack constraints.

Consider the set

$$X = \Big\{(x,y) \in \{0,1\}^n \times \mathcal{Z}_+ : \sum_{j=1}^{n} a_j x_j + a_y y \leq b,\ 0 \leq y \leq M\Big\}. \tag{2}$$

For $W \subseteq \{1, ..., n\}$ and $K \in \{0, 1, ..., M\}$ we say that:

- the pair (W, K) is a *cover* if

$$\sum_{j \in W} a_j + K a_y = b + \lambda, \quad \lambda > 0; \tag{3}$$

- the cover (W, K) is a *minimal cover* if

$$(K-1)a_y + \sum_{j \in W} a_j \leq b, \tag{4}$$

  and for all $i \in W$

$$K a_y + \sum_{j \in W \setminus \{i\}} a_j \leq b. \tag{5}$$

In other words, (W, K) is a cover if the point with coordinates $x_j = 1$, $j \in W$, $x_j = 0$, $j \notin W$, and $y = 1$ is infeasible. The cover is minimal, if decreasing any of the positive values by 1 makes it feasible.

Inequalities (4) and (5) imply that:

$$\lambda \in \left]0, \max\{\max_{j \in W} a_j, a_y\}\right[. \tag{6}$$

A simple way to construct valid inequalities for the set X defined in (2) is to binarize the general integer variable y: $y = \sum_{k=1}^{M} y_k$, where $y_k \in \{0, 1\}$, $k = 1, \ldots, M$, thus transforming the binary–integer knapsack constraint into a 0–1 knapsack constraint:

$$X = \big\{(x, y) \in \{0, 1\}^{n+M} : \sum_{j=1}^{n} a_j x_j + a_y \sum_{k=1}^{M} y_k \le b\big\}.$$

A more compact binary reformulation of the general integer variable takes the following form:

$$y = y_0 + 2y_1 + 4y_2 + ... + 2^r y_r$$

with $y_k \in \{0, 1\}$, $k = 1, ..., r$, $r = \lfloor \log_2 M \rfloor$. However, due to the massive symmetry introduced in that way, the reformulation of the general integer variables as a sum of binary ones is known to lead to problems which are more difficult to solve [11, 9].

3 Binary–Integer Cover Inequalities

Assuming that (W, K) is a minimal cover for the set (2), we observe that in many cases the following inequality, directly generalizing the binary knapsack case, holds true:

$$\sum_{j \in W} x_j + y \le |W| + K - 1, \tag{7}$$

with $|W|$ denoting the cardinality of the subset W.

If $y \le K$ and if $y > K$ and $a_y \ge \max_{j \in W} a_j$, then inequality (7) is valid. However, if $y > K$ and $a_y < \max_{j \in W} a_j$, *inequality* (7) *may be violated by some feasible solutions*.

Below, we propose a new family of cover inequalities for the set (2), which are valid for each case mentioned above.

Theorem 3.1 *If (W,K) is a minimal cover, then the inequality*

$$\gamma \sum_{j \in W} x_j + y \leq \gamma\,|W| + K - 1 \tag{8}$$

with

$$\gamma = \left\lceil \frac{\max_{j \in W} a_j}{a_y} \right\rceil \tag{9}$$

is a valid inequality for X.

Proof. Since (W, K) is a cover, equality (3) holds true. Subtracting from it the knapsack inequality

$$\sum_{j \in W} a_j x_j + a_y y \leq b$$

we obtain:

$$\sum_{j \in W} a_j (1 - x_j) \geq \lambda + (y - K) a_y. \tag{10}$$

Let γ be the smallest integer greater than or equal to 1 such that

$$\max_{j \in W} a_j \leq \gamma a_y.$$

Then

$$\sum_{j \in W} a_j (1 - x_j) \leq \left(\max_{j \in W} a_j\right) \sum_{j \in W} (1 - x_j) \leq \gamma a_y \sum_{j \in W} (1 - x_j).$$

This combined with (10) implies

$$\gamma a_y \sum_{j \in W} (1 - x_j) \geq \lambda + (y - K) a_y.$$

Equivalent transformations of this inequality yield:

$$\gamma \sum_{j \in W} x_j + y \leq \gamma\,|W| + K - \frac{\lambda}{a_y}.$$

The cover is minimal, and thus $0 < \lambda / a_y < 1$. Since γ is integer, we conclude that

$$\gamma \sum_{j \in W} x_j + y \leq \gamma\,|W| + K - 1,$$

which was set out to prove. $\square$

Inequality (8) is referred to as a *valid cover inequality*.

The strength of this inequality depends on the value of γ. If we can make γ smaller, then the inequality becomes sharper. In formula (9) we can decrease γ by making a_y larger or by making $\max_{j\in W} a_j$ smaller. However, we must keep in mind that the knapsack inequality with the new coefficients must not cut off any points of X. The logical way to guarantee that is to keep a_y unchanged, and to decrease some of a_j, $j \in W$. We can decrease these coefficients to some values $a'_j \le a_j$ as far as the pair (W, K) remains a cover for the new set

$$X' = \left\{(x, y) \in \{0,1\}^n \times \mathcal{Z}_+ : \sum_{j=1}^{n} a'_j x_j + a_y y \le b,\ 0 \le y \le M\right\}.$$

Clearly, the largest of the a_j, $j \in W$, should be tried first. We illustrate this in the next section.

4 Comparison with Other Valid Inequalities

Atamtürk and Rajan [1] handle similar knapsack constraints when analyzing the problem of unsplittable single arc-set relaxations of multicommodity flow capacitated network design. These problems are generally formulated using a binary flow variable x_{ja} for each commodity-arc pair (j, a) that takes on a value of 1 if the commodity uses the arc, and 0 otherwise, as well as an integer capacity variable z_a. For each arc of the network there is a capacity constraint of the form:

$$\sum_{j=1}^{n} d_j x_{ja} \le c_{a0} + c_a z_a, \tag{11}$$

where d_j the demand for commodity j, c_{a0} existing capacity of the arc, and c_a the unit capacity to install.

The set

$$F_U = \left\{(x, z) \in \{0,1\}^n \times \mathcal{Z} : \sum_{j=1}^{n} a_j x_j \le a_0 + z\right\}, \tag{12}$$

in which $a_j > 0$, $j = 0, 1, \ldots, n$, is obtained by dividing (11) by c_a, and is called the *unsplittable flow* *arc set*. The $0-1$ knapsack set

$$F_U(v) = \{(x,z) \in F_U : z = v\}$$

is obtained by the projection of the general integer variable z to v. For this projection, Atamtürk and Rajan [1] say that the set C is a cover if

$$r = \sum_{j \in C} a_j - a_0 - v > 0, \tag{13}$$

and it is minimal if

$$a_j \geq r, \quad j \in C. \tag{14}$$

Introducing the variable z in the standard knapsack cover inequality, which is valid for $F_U(v)$, Atamtürk and Rajan [1] propose a lifting procedure for deriving a valid inequality for the set F_U. This inequality reads

$$\sum_{j \in C} x_j + \alpha(v - z) \leq |C| - 1, \tag{15}$$

and is valid for (12) if and only if

$$\underline{\alpha} \leq \alpha \leq \overline{\alpha}, \tag{16}$$

with

$$\overline{\alpha} = \min \left\{ \frac{|C| - 1 - \sum_{i \in C} x_i}{v - z} : z < v,\ (x,z) \in F_U \right\}, \tag{17}$$

and

$$\underline{\alpha} = \max \left\{ \frac{\sum_{i \in C} x_i - |C| + 1}{z - v} : z > v,\ (x,z) \in F_U \right\}. \tag{18}$$

Inequality (15) is equivalent to (7), and *is valid if* $\max_{j \in C} a_j \leq 1$ in (12), i.e., if $\max_{j \in W} a_j \leq a_y$ holds in (2). Moreover, the computation of $\overline{\alpha}$ and $\underline{\alpha}$ involves highly nontrivial combinatorial problems.

If $\max_{j \in C} a_j > 1$ in (12), we have to pass to fractional coefficients $a_j - \lfloor a_j \rfloor$ and construct a valid inequality for the fractional problem. Then we know that there exist valid cuts for the original knapsack set which have identical fractional parts, but they are hard to elicit. All combinations of integer parts of the coefficients have to be tried in some way, which makes this approach less attractive for large scale problems we are interested in.

To illustrate this, we consider the set

$$X = \big\{(x, y) \in \{0, 1\}^2 \times \mathcal{Z}_+ : 4x_1 + 5x_2 + 2y \le 15,\ y \le 5\big\}. \tag{19}$$

Setting $z = -y$ we obtain the set (12) defined by the knapsack constraint as

$$F_U = \big\{(x, z) \in \{0, 1\}^2 \times \mathcal{Z} : 2x_1 + \frac{5}{2}x_2 \le \frac{15}{2} + z\big\}.$$

Consider $C = \{1, 2\}$ and $v = -4$. They define a cover. Formulas (17)–(18) yield $\overline{\alpha} = 0$ and $\underline{\alpha} = 1$. Obviously, the set in (16) is empty. The reason for this is that the assumption of [1] that $\max_{j \in C} a_j \le 1$ is not satisfied.

Let us follow [1] and use fractional parts of knapsack coefficients to get

$$F_{Uf} = \big\{(x, z) \in \{0, 1\}^2 \times \mathcal{Z} : \frac{1}{2}x_2 \le \frac{1}{2} + z\big\}.$$

Only one cover, $C = \{2\}$ and $v = -1$, is of interest for this simple inequality. Formulas (17)–(18) yield $\overline{\alpha} = +\infty$ and $\underline{\alpha} = 1$. This renders the valid inequality for F_{Uf}:

$$x_2 \le \alpha(z + 1),$$

with $\alpha \ge 1$. It is best if $\alpha = 1$. It is obviously not valid for F_U, which allows $x_1 = 0$, $x_2 = 1$ and $z = -5$. We know than some valid inequality for F_U exists, with fractional parts equal to the coefficients in the last displayed inequality, but [1] does not provide any recipe for finding it. We can thus conclude that the restrictive assumption, that the knapsack coefficients associated with binary variables are not larger than the coefficient associated with the integer variable, makes the results of [1] less attractive for our case.

and generates for it valid cuts in the way described above. Then we know that there exists valid cuts for the original knapsack set which have identical fractional parts, but they are hard to elicit. All combinations of integer parts of the coefficients have to be tried in some way.

Another approach for handling knapsack constraints containing binary and general integer variables is this of Ceria et al. [3]. They derive a family of valid, not facet-defining, inequalities by fixing integer variables at their upper bounds.

They consider the following set involving general integer variables:

$$X_0 = \{x \in \mathcal{Z}_+^n : \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq M_j\}.$$

Assuming that all general integer variables are set equal to their upper bounds, M_j, Ceria et al. [3] assert that the set W is a cover if:

$$\sum_{j \in W} M_j a_j = b + \lambda \quad \text{with} \quad \lambda > 0.$$

The underlying idea for the derivation of the valid inequality below is that if W is a cover, then not all the general integer variables in W can be equal to their upper bounds. The resulting valid inequality takes on the form

$$\sum_{j \in W} (M_j - x_j) \geq \alpha, \tag{20}$$

with

$$\alpha = \lceil \lambda / \bar{a} \rceil, \qquad \bar{a} = \max_{j \in W} a_j.$$

By using the restrictive assumption according to which all n general integer variables are to be set equal either to their upper bounds M_j, or to 0,the valid inequalities proposed by Ceria et al.[1998] are derived from covers, not required to be minimal, in which the general integer variables x_j are transformed into binary-type variables: $x_j \in \{0, M_j\}$, $j = 1, ..., n$.

Assuming that $M_j = 1$, $j = 1, ..., n$, for the binary variables, the valid inequalities proposed by Ceria et al. can be used for the set X in (2).

Another group of valid inequalities can be obtained by applying the idea of *disjunctive cuts*. If (W, K) is a minimal cover, the following disjunctive cut can be implemented:

$$\text{If} \quad \sum_{j \in W} x_j = |W| \quad \text{then} \quad y \leq K - 1,$$
$$\text{If} \quad \sum_{j \in W} x_j < |W| \quad \text{then} \quad y \leq K + \left\lfloor \frac{\sum_{j \in W} a_j(1 - x_j) - \lambda}{a_y} \right\rfloor.$$

In the special case when $K = M$, inequality (7) is always valid. This is in the line of research of Owen and Mehrotra [8], whose disjunctive cutting planes are obtained from the generalization of the 0-1 disjunction used by Balas et al. [2].

To illustrate the differences between these approaches let us consider again the set X defined by We apply the three methods described above.

Disjunctive cuts yield:

1. If $x_1 = 1$ and $x_2 = 1$, then the inequality $y \leq 4 - 1 = 3$ is valid;

2. If $x_1 = 0$ or $x_2 = 0$, then 3 possibilities arise:

   (a) $x_1 = 0, x_2 = 1$, then the inequality $y \leq 4 + \lfloor \frac{5-2}{2} \rfloor = 5$ is valid;

   (b) $x_1 = 1, x_2 = 0$, then the inequality $y \leq 4 + \lfloor \frac{4-2}{2} \rfloor = 5$ is valid;

   (c) $x_1 = 0, x_2 = 0$, then the inequality $y \leq 4 + \lfloor \frac{5+4-2}{2} \rfloor = 7$ is valid.

From this example it can be inferred that the number of disjunctions to be implemented for a program containing a moderate-to-high number of knapsack constraints (2) would be excessive, and would not, to say the least, facilitate the solution of the program.

The approach of Ceria et al. [3] proceeds as follows: Since $a_1 M_1 + a_2 M_2 + a_y M_y = 19$, the set $\{x_1, x_2, y\}$ is a cover with $\lambda = 4$, $s = 0$, $\bar{a} = 5$, $\alpha = \lceil 4/5 \rceil = 1$. The valid inequality (20) takes on the form

$$x_1 + x_2 + y \leq 6.$$

Our binary–integer cover inequalities are different. For $K = 4$ the pair $\big(\{1,2\},K\big)$ is a minimal cover. We obtain $\gamma = 3$ and the valid cover inequality (8) reads:

$$3x_1 + 3x_2 + y \leq 9.$$

Moreover, following the remark after Theorem 3.1, we can notice that the pair $\big(\{1,2\},4\big)$ is also a minimal cover for a larger set

$$X = \{(x,y) \in \{0,1\}^2 \times \mathcal{Z}_+ : 4x_1 + 4x_2 + 2y \leq 15,\ y \leq 5\},$$

obtained by decreasing the coefficient in front of x_2 from 5 to 4. This yields $\gamma = 2$ and a stronger version of (8):

$$2x_1 + 2x_2 + y \leq 7.$$

The advantage of the valid inequalities developed in this paper over those proposed by Ceria et al. [3] can be shown by noticing that the point $(x_1, x_2, y) = (1,1,4)$, which is non-feasible for the constraint on-hand (19), is cut off by our valid inequalities, while it satisfies the inequality of Ceria et al.

5 Extended Covers and Extended Cover Inequalities

In this section, an extension of the valid cover inequality (8) is proposed for the extended cover, (W', K), defined below. The pair (W', K) is an *extended cover* if:

$$W' = W \cup \{j' : a_{j'} \geq a_j \text{ for all } j \in W,\ a_{j'} \geq a_y\} \quad \text{and}$$

$$(W,K) \text{ is a cover.}$$

Proposition 5.1 *If (W', K) is an extended cover, then the inequality*

$$\gamma' \sum_{j \in W'} x_j + y \leq \gamma' \left|W'\right| + K - \vartheta$$

is valid, with

$$\gamma' = \left\lceil \frac{\max_{j \in W'} a_j}{a_y} \right\rceil$$

and

$$\vartheta = \left\lceil \frac{\lambda}{a_y} \right\rceil .$$

Proof. The proof is very similar to that for obtaining (8). Let us consider the system:

$$\sum_{j \in W'} a_j x_j + a_y y \leq b, \tag{21}$$

$$\sum_{j \in W'} a_j + K a_y = b + \lambda, \quad \lambda > 0. \tag{22}$$

Subtracting (21) from (22) and defining γ' as the smallest integer such that

$$\max_{j \in W'} a_j \leq \gamma' a_y,$$

we obtain the following inequality:

$$\gamma' a_y \sum_{j \in W'} (1 - x_j) \geq \lambda + (y - K) a_y.$$

This can be equivalently transformed to

$$\gamma' \sum_{j \in W'} x_j + y \leq \gamma' \left| W' \right| + K - \frac{\lambda}{a_y}.$$

Since γ' is integer and $\lambda > 0$, we conclude that

$$\gamma' \sum_{j \in W'} x_j + y \leq \gamma' \left| W' \right| + K - \vartheta,$$

with $\vartheta = \left\lceil \frac{\lambda}{a_y} \right\rceil$. □

The extended cover is not necessarily a minimal cover, thus explaining why we cannot substitute 1 for λ / a_y in the proof.

6 Lifting

In this section, we propose a lifting procedure for the cover inequalities proposed in this paper. The concept of lifting [10], is aimed at strengthening the valid inequalities and possibly obtain facet-defining inequalities. It has been widely applied, especially for 0-1 mixed integer programs (see, *inter alia*, [7, 2, 15]).

Considering the set X defined by (2), and assuming that (W, K) is a minimal cover for X, the lifting procedure proposed is equivalent to searching the largest value for the coefficients β_j such that the inequality

$$\sum_{j \notin W} \beta_j x_j + (\gamma + \beta_j) \sum_{j \in W} (\gamma + \beta_j) x_j + y \leq \gamma |W| + K - 1$$

is valid. Equivalently, the inequality

$$\sum_{j=1}^{n} \beta_j x_j + \gamma \sum_{j \in W} x_j + y \leq \gamma |W| + K - 1$$

should be valid.

With no loss of generality, we assume that the binary variables x_j, $j = 1, ..., n$, are ordered by the magnitude of their coefficients: $a_1 \geq a_2 \geq \cdots \geq a_n$.

The lifting procedure is a recursive process. Initially, we find the largest value for β_1, for which the inequality $\beta_1 x_1 + \gamma \sum_{j \in W} x_j + y \leq \gamma |W| + K - 1$ remains valid. This is done by solving the following auxiliary knapsack problem:

$$\begin{aligned} \eta_1 := \max & \left(\gamma \sum_{j \in W} x_j + y \right) \\ \text{subject to} & \sum_{j \in W} a_j x_j + a_y y \leq b - a_1, \\ & y \in \{1, \ldots, M\}, \\ & x \in \mathcal{B}^n. \end{aligned} \qquad (23)$$

The optimal value for β_1 is given by: $\beta_1 = \gamma |W| + K - 1 - \eta_1$. After having performed $(t-1)$ iterations the valid inequality

$$\sum_{j=1}^{t-1} \beta_j x_j + \gamma \sum_{j \in W} x_j + y \leq \gamma |W| + K - 1$$

is obtained.

Reaching the t^{th} iteration, we must find the largest value of β_t, for which the inequality

$$\beta_t x_t + \sum_{j=1}^{t-1} \beta_j x_j + \gamma \sum_{j \in W} x_j + y \leq \gamma |W| + K - 1$$

is valid. This is done by solving the following knapsack problem:

$$\eta_t := \max \left(\sum_{j=1}^{t-1} \beta_j x_j + \gamma \sum_{j \in W} x_j + y\right)$$
$$\text{subject to} \quad \sum_{j \in W \cup \{1,\ldots,t-1\}} a_j x_j + a_y y \leq b - a_t,$$
$$y \in \{1, \ldots, M\},$$
$$x \in \mathcal{B}^n.$$

The optimal value for β_t is given by $\beta_t = \gamma |W| + K - 1 - \eta_t$. We continue until $t = n$ or until $\beta_t = 0$.

As an illustration, let us consider the set

$$X' = \{(x, y) \in \{0,1\}^3 \times \mathcal{Z}_+ : x_1 + 4x_2 + 6x_3 + 2y \leq 13, y \leq 4\}.$$

For $K = 2$, the pair $\big(\{2,3\}, K\big)$ is a minimal cover. From (8), $\gamma = 2$, and

$$2x_2 + 2x_3 + y \leq 5$$

is a valid cover inequality, that may be lifted. Considering first the variable x_1, which has the smallest coefficient, $a_1 = 1$, we want to find the largest value of β_1, such that the inequality $\beta_1 x_1 + 2(x_2 + x_3) + y \leq 5$ is valid. Solving of (23) yields: $\eta_1 = 4$, $x_2 = 1$, $x_3 = 1$, $y = 0$, which in turns gives: $\beta_1 = 4 + 2 - 1 - 4 = 1$. Therefore, the cover inequality $2x_2 + 2x_3 + y \leq 5$ can be lifted to

$$x_1 + 2x_2 + 2x_3 + y \leq 5.$$

7 Application

7.1 Test laboratory

As a test bed for the valid inequalities proposed in this paper, we use the data provided by one of the three largest chemical North American company, the General Chemical Group, producing soda ash and calcium chloride. The General Chemical Group is part of a three-stage supply chain, that includes one supplier-manufacturer providing raw materials to the manufacturers as well as end products to the distribution centers, one manufacturer and a dozen of distributors (Figure 1). The distribution is carried out with a heterogeneous fleet of ships and barges differing in their speed and loading capacity. A one-year planning horizon with monthly time-periods is considered.

Figure 1: Three-stage supply chain

In Figure1 arcs denote that an upstream stage supplies a downstream stage. The first stage, i.e., the supplier, assures the procurement of a raw material, or a component. The second stage, i.e., production, represents the manufacturing and/or assembly of the finished product. The third stage, i.e., distribution, represents the transportation of the finished product to a distribution center (retailer, wholesaler) or to the end-customer.

The objective pursued by the supply chain is to construct a deterministic and sustainable inventory-production-distribution plan enabling it to minimize its costs while satisfying its customers' demand, and takes the form of a complex multi-dimensional mixed-integer program. More precisely, the inventory-production-distribution plan to be constructed must determine:

- the production scheme, i.e., the quantity of products and raw materials to be produced at each production facility at each period;
- the supply scheme, i.e., the quantity of products and raw materials to be sent from the supplier to the manufacturers, and from the manufacturers to the distributors at each time period;
- the ending inventory level required at each node in the supply chain and at each period;
- the scheduling and routing scheme for each carrier;

- the maintenance schedule for each carrier.

Below, we report in Table 1 the dimensions of the resulting mixed-integer programming problem. For a thorough description of the model, the reader is referred to [5].

Variables	1341
Continuous variables	801
General integer variables	454
Binary variables	56
Constraints	1871

Table 1: Dimensions of the problem

7.2 Binary-integer cover inequalities for distribution constraints

A general characteristic of the distribution scheme in the multi-stage supply chain management setting is that the number of supplier-to-manufacturer shipments (carriers departing from the supplier, and heading to the manufacturer) is very large, possibly occurring numerous times per time-period, while the number of manufacturer-to-distributor shipments (carriers departing from the manufacturer, and heading to one of the distributors) is much lower. Therefore, it can be inferred without loss of generality that the distribution schedule of any carrier v at any period t can comprise many deliveries from the supplier to the manufacturer and at most one delivery from the manufacturer or supplier to some of the distributors.

In the problem on-hand, some preliminary tests attest that the constraints enforcing the time availability of the different ships at each period of the planning horizon can be modelled as $\{(x, y) \in B^{\imath} \times \mathcal{Z}_+\}$-knapsack constraints containing a single general integer variable (supplier-to-manufacturer shipments), and a number of binary variables (manufacturer-to-distributors shipments). More precisely, the latter constraints take the following forms:

$$\sum_{i \in I} \sum_{j \in J} a_{i,j,v} x_{i,j,v,t} + a_{i,i',v} y_{i,i',v,t} \leq b_{v,t}, \quad v \in V, \quad t \in T,$$
$$x_{i,j,v,t} \in \mathcal{Z}_+^{|I| \times |J|}, \quad y_{i,i',v,t} \leq M_{v,t}, \quad y_{i,i'v,t} \in \mathcal{Z}_+,$$

where V is the set of ships used for the distribution, T is the set of periods in the planning horizon, and I and J are the sets of suppliers and distributors respectively. We use $a_{i,j,v}$ ($a_{i,i',v}$) to denote the lead time (defined as the sum of the loading, unloading and travelling times) for a shipment with ship v departing from supplier i, heading to distributor j (or manufacturer i'), while $x_{i,j,v,t}$ and $y_{i,i',v,t}$ are, respectively, the number of manufacturer-to-distributor and supplier-to-manufacturer shipments carried out with carrier v over the period t. Finally, $b_{v,t}$ is the available time of carrier v in period t. In our problem the lead times from the suppliers to the distributors, associated with the binary variables, are much larger than the lead times from the supplier to the manufacturer, associated with general integer variables.

We have 48 constraints of this form, and for each of them we generate, using formulae (3), (4) and (5), all the minimal covers (W, K), and we construct all the valid cover inequalities

$$\gamma \sum_{(i,j)\in W} x_{i,j,v,t} + y_{i,i',v,t} \leq \gamma\,|W| + K - 1, \tag{24}$$

with

$$\gamma = \left\lceil \frac{\max_{(i,j)\in W} a_{i,j,v,t}}{a_{i,i',v}} \right\rceil,$$

associated with each minimal cover. We obtain a total of 2165 such cover inequalities.

7.3 Computational results

To evaluate the added value of the new family of valid cover inequalities proposed in this paper, we solve the two mixed-integer programs, P and P-G', using the AMPL modelling language [4] and the branch-and-bound algorithm embedded in the CPLEX commercial solver, and we compare the best solution for each of them. The first one, later referred to as P, is the original mixed-integer program, whose compact formulation is given by (1). The second one is the original mixed-integer program complemented by the up-front inclusion of the 2165 cover inequalities discussed in Section 3. Denoting by $G'x \leq h'$ the cover inequalities of form (24), and adding them in the mixed-integer program P defined above, we obtain the following mixed-integer program, further referred to by the acronym P-G':

$$\min c^T x$$

$$\text{subject to } Ax \geq b,$$

$$G'x \leq h',$$

$$x \geq 0,$$

$$x_j \in \begin{cases} \mathcal{R}_+ \text{ for } j \in V_R \\ \mathcal{Z}_+ \text{ for } j \in V_I \\ \mathcal{B} \text{ for } j \in V_{0-1} \end{cases}$$

In Table 2, we report the value of the objective function, i.e., the total costs of the supply chain, and the optimality gap for the two mixed-integer programs.

The optimality gap is computed as follows:

$$GAP = \frac{UB - LB}{LB},$$

where UB and LB are respectively the upper bound, i.e., the best feasible integer solution, and the lower bound, i.e., the optimal solution for the continuous relaxation of the considered mixed-integer program.

	P	$P - G'$
Objective function: total costs of the supply chain	$6889150	$5716070
Optimality gap	23.81%	2.90%

Table 2: Contribution of cover inequalities

The solving process is terminated after a predetermined number of nodes have been processed by the branch-and-bound algorithm of the CPLEX solver, since an optimal solution cannot be reached in a reasonable amount of time.

The impact of the valid inequalities on the solution found is considerable, allowing a reduction of the optimality gap equal to 20.91%, and savings for the supply chain up to $1173000. More precisely, it can be seen in Table 3 that the savings allowed by the family of valid cover inequalities proposed in this

	P	$P-G'$	Cost reduction in absolute numbers	Cost reduction in percentage
Total costs	$5483540	$5716070	$1173000	100%
Distribution costs	$3474960	$2891920	$583040	49.70%
Inventory costs	$1146150	$948320	$197830	16.86%
Production costs	$ 2268040	$1875830	$392210	33.44%

Table 3: Cost distribution

paper predominantly originate from a dramatic reduction in the distribution costs (about 50% of the total savings), thus from an improved distribution scheme.

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