

# Unification of lower-bound analyses of the lift-and-project rank of combinatorial optimization polyhedra

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## Abstract

We present a unifying framework to establish a lower-bound on the number of semidefinite programming based, lift-and-project iterations (rank) for computing the convex hull of the feasible solutions of various combinatorial optimization problems. This framework is based on the maps which are commutative with the lift-and-project operators. Some special commutative maps were originally observed by Lovász and Schrijver, and have been used usually implicitly in the previous lower-bound analyses. In this paper, we formalize the lift-and-project commutative maps and propose a general framework for lower-bound analysis, in which we can recapture many of the previous lower-bound results on the lift-and-project ranks.

**Keywords:** lift-and-project, semidefinite lifting, semidefinite programming, integer programming

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## 1 Introduction.

An important subject in both the theory and the practise of combinatorial optimization involves “computing” the convex hull of the integer points lying in a simply described (usually by facets) polytope. Here, “computing” means generating the facets of the convex hull of integer points either explicitly or implicitly.

For many hard combinatorial optimization problems, we lower our goals from computing all the facets of the convex hull to computing a partial, but still useful subset of the facets of the convex hull. Depending on the approach taken, there are many ways of measuring how complicated a facet of the convex hull is. A traditional theoretical approach is to apply Gomory-Chvátal closures to the original polytope and count the number of major iterations needed to derive a particular facet or all facets of the convex hull. The resulting measure, called *Gomory-Chvátal rank*, has been studied, among others, in [7, 8, 12].

A less main stream approach to computing the convex hull is the lift-and-project methods. Such methods have been proposed by Balas [3], Lovász and Schrijver [26], and Sherali and Adams [27]. Sherali and Adams [28] extended their lift-and-project methods to more general nonconvex problems (also see the references therein for many applications). Closely related Lovász-Schrijver procedures were generalized to compute the convex hull of any compact set in [18]. The reference [18] contains the first convergence proof of such a method in that generality. Later Lasserre [19, 20] proposed similar procedures in a less general setting and used results from real-algebraic geometry to establish convergence. While the convergence of lift-and-project methods for 0-1 optimization problems was well-understood, convergence proofs for lift-and-project methods on more general nonconvex optimization problems involve different techniques. The focus of the current paper is Lovász-Schrijver lift-and-project methods for 0-1 combinatorial optimization problems, especially, the method involving positive semidefiniteness constraints. However, we hope that our approach can be generalized to deal with Sherali-Adams procedures, Lasserre-type methods and the more recently proposed methods of Bienstock and Zuckerberg [5]; also see the analysis in [6].

Cook and Dash [10] were the first to make an explicit connection between the tools for lower-bound proving techniques for the Gomory-Chvátal rank and those for the lift-and-project ranks. Here, we slightly generalize their approach and streamline a proof technique for lower-bound analysis. A main feature of the analysis is that the positive semidefiniteness of certain matrix in a lifted relaxation is established inductively by a simple convexity argument on positive semidefiniteness preserving linear maps (therefore avoiding the need to work out algebraically, the eigenspaces, eigenvalues etc. of the matrices of arbitrary size). More specifically, let  $L_p \in \mathbb{R}^{n \times m}$ ,  $n \geq m$ . Then the linear transformation  $L_p \cdot L_p^T$  maps any  $m \times m$  symmetric positive semidefinite matrix to an  $n \times n$  symmetric positive semidefinite matrix. Therefore, the linear map  $\sum_p \lambda_p (L_p \cdot L_p^T)$  also preserves symmetry and positive semidefiniteness (provided  $\lambda_p \geq 0, \forall p$ ). Taking convex combinations preserves some other critical properties in addition.

The next section contains definitions and the basic properties of the lift-and-project methods that are studied in this paper. In Section 3, we introduce the notions of commutativity of linear maps and the lift-and-project operators. Section 4 contains a description of the skeleton of a generic proof via the unified approach. Section 5 is made up from typical elementary examples of the unifying proof technique.

## 2 The Lift-and-Project Methods.

In this section, we review the definitions and some of the previously established, basic properties of the lift-and-project procedures. For details and further related results, see [26, 27, 28, 4, 23, 10, 16, 24, 13].

Let  $P$  be any convex subset of the  $d$ -dimensional hypercube  $[0, 1]^d$ .  $P_I$  denotes the integral hull of  $P$ , namely the convex hull of 0-1 vectors of  $P$ . The lift-and-project methods are general procedures which take  $P$  as input and deliver  $P_I$  as output. In doing so, it is sometimes convenient to homogenize  $P$  to a cone  $K$  in  $\mathbb{R}^{d+1}$  by introducing an additional coordinate which will be referred to as the 0-th coordinate.

$$K := \left\{ \lambda \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in P, \lambda \geq 0 \right\}, \text{ or } P = \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} \in K \right\}. \quad (1)$$

Accordingly,  $K_I$  is the homogenized cone of  $P_I$ . See Figure 1. It is clear that  $K$  is con-

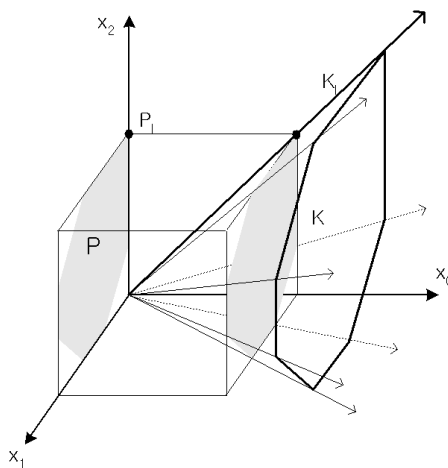


Figure 1:  $P$ ,  $P_I$ ,  $K$ , and  $K_I$ .

tained in  $Q \subseteq \mathbb{R}^{d+1}$ , the homogenization of  $[0, 1]^d$ . The cone  $Q$  has a very simple polyhedral structure. Denote  $H_i(0) := \{x \in \mathbb{R}^{d+1} : x_i = 0\}$  and  $H_i(1) := \{x \in \mathbb{R}^{d+1} : x_i = x_0\}$ . Similarly, for  $J \subseteq \{1, 2, \dots, d\}$ , write  $H_J(0) := \{x \in \mathbb{R}^{d+1} : x_i = 0, i \in J\}$  and  $H_J(1) := \{x \in \mathbb{R}^{d+1} : x_i = x_0, i \in J\}$ . Then, for each  $(d + 1 - k)$ -dimensional face of  $Q$ , there is a set  $J \subseteq \{1, 2, \dots, d\}$  with  $|J| = k$  and its partition  $J = J_0 \cup J_1$  so that the face is given as

$$Q \cap H_{J_0}(0) \cap H_{J_1}(1). \quad (2)$$

Given a set  $S$ , its dual cone is defined as  $S^* := \{x : x^T s \geq 0, s \in S\}$ . Let  $L$  be a linear map. Then, it is easy to see that

$$y \in (LS)^* \Leftrightarrow L^T y \in S^*. \quad (3)$$

It is well known that when  $S$  is polyhedral,  $S^*$  is generated by the vectors determining the facets of  $S$ . Hence, we have

$$Q^* = \text{cone}\{e_1, \dots, e_d, f_1, \dots, f_d\}, \quad (4)$$

where  $e_i$  denotes the  $i$ th unit vector and  $f_i := e_0 - e_i$ . Let  $K_1 \subseteq Q$  and  $K_2 \subseteq Q$  be convex cones such that  $K = K_1 \cap K_2$ . For instance, if  $K$  is polyhedral, then  $K_1$  and  $K_2$  can be obtained by taking proper subsystems of the linear systems determining  $K$ . We are ready to define the lift-and-project operators  $N_0$ ,  $N$  and  $N_+$  in increasing strength. For  $Y \in \mathbb{R}^{(d+1) \times (d+1)}$ , consider the conditions:

$$\text{diag}(Y) = Y e_0, \quad (5)$$

$$u^T Y v \geq 0, \quad \forall u \in K_1^*, v \in K_2^*, \quad (6)$$

where  $\text{diag} : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  maps the diagonal elements of the given matrix onto a vector. Then

$$M_0(K_1, K_2) := \{Y = (y_{ij})_{i,j \in \{0,1,\dots,d\}} : Y \text{ satisfies (5), and (6)}\}.$$

Notice that (5) and (6), respectively, can be restated as follows.

$$\langle Y, f_i e_i^T \rangle := \text{trace}(Y^T f_i e_i^T) = 0, \quad \forall i \in \{1, 2, \dots, d\}, \quad (7)$$

$$Y K_2^* \subseteq (K_1^*)^* = K_1. \quad (8)$$

The additional condition

$$Y \in \Sigma^{d+1}, \text{ the } (d+1) \times (d+1) \text{ symmetric matrices,} \quad (9)$$

yields the stronger operator

$$M(K_1, K_2) := \{Y \in M_0(K_1, K_2) : Y \text{ satisfies (9)}\}.$$

An additional positive semidefiniteness constraint

$$Y \in \Sigma_+^{d+1}, \text{ the } (d+1) \times (d+1) \text{ PSD matrices,} \quad (10)$$

gives

$$M_+(K_1, K_2) := \{Y \in M(K_1, K_2) : Y \text{ also satisfies (10)}\}.$$

We use  $N_{\sharp} \in \{N_0, N, N_+\}$ , and  $M_{\sharp} \in \{M_0, M, M_+\}$ , to state definitions and results for all three operators  $M_0, M, M_+$  and  $N_0, N, N_+$  (defined below) respectively:

$$N_{\sharp}(K_1, K_2) := \{Y e_0 : Y \in M_{\sharp}(K_1, K_2)\}. \quad (11)$$

$N_{\sharp}(K_1, K_2)$  is a relaxation of  $K_I$  tighter than  $K$ . We have

$$K_I \subseteq N_0(K_1, K_2) \subseteq N(K_1, K_2) \subseteq N_+(K_1, K_2) \subseteq K. \quad (12)$$

When  $K_1 := K$ , we can use for  $K_2$  any convex cone such that  $K \subseteq K_2 \subseteq Q$ . While the choice  $K_2 := K$  provides the tightest relaxations, the simplicity of  $Q$  (especially of  $Q^*$ ) allows the usage of more elegant and simpler mathematical tools. Moreover, choosing  $K_2 := Q$  yields a sequence of clearly tractable relaxations from a computational complexity point of view as we explain below. In this case, by (4), (8) is equivalent to

$$Y e_i, Y f_i \in K, \quad i \in \{1, 2, \dots, d\}. \quad (13)$$

For this case, we will adopt the following notation:

$$M_{\sharp}(K) := M_{\sharp}(K, Q), \quad N_{\sharp}(K) := N_{\sharp}(K, Q). \quad (14)$$

Clearly,  $N_{\sharp}$  operators can be applied iteratively:

$$K := N_{\sharp}^0(K), \quad N_{\sharp}^t(K) := N_{\sharp}(N_{\sharp}^{t-1}(K)) \text{ for } t \in \{1, 2, \dots\}. \quad (15)$$

The following conventional notation is also useful.

$$N_{\sharp}(P) := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} \in N_{\sharp}(K) \right\}. \quad (16)$$

Now, we review various facts on the lift-and-project methods.

$$N_0(K) = \bigcap_{i=1}^d [(K \cap H_i(0)) + (K \cap H_i(1))]. \quad (17)$$

For a given set  $J$  with  $|J| = t$ , consider the union  $\bar{F}_{d+1-t}(J)$  of the  $(d+1-t)$ -dimensional faces of  $Q$  determined by the partitions of  $J$  in (2):

$$\bar{F}_{d+1-t}(J) := \bigcup_{J_0 \cup J_1 = J} Q \cap H_{J_0}(0) \cap H_{J_1}(1).$$

Then we can define the following operator:

$$\tilde{N}_0^t(K) := \bigcap_{J \subseteq N, |J|=t} \text{cone} (K \cap \bar{F}_{d+1-t}(J)). \quad (18)$$

Then, the  $N_{\sharp}$ -operators have the following relations:

$$N_+^t(K) \subseteq N^t(K) \subseteq N_0^t(K) \subseteq \tilde{N}_0^t(K). \quad (19)$$

Since  $\tilde{N}_0^d(K) = K_I$ , the above fact implies that the lift-and-project procedures capture the integral hull in at most  $d$  iterations. A remarkable fact is that linear optimization on  $N_{\sharp}(K)$  can be done in polynomial time if  $K$  is polynomially separable. It can be shown that (5), (9), (10) and (13) are polynomially separable constraints if  $K$  is so. Roughly speaking, a separation of  $N_{\sharp}^t(K)$  requires the separation of  $N_{\sharp}^{t-1}(K)$ ,  $O(d)$  times. This implies that  $N_{\sharp}^t(K)$  is polynomially separable when  $t = O(1)$ .

We conclude this section with an elementary fact. For any  $0 < \lambda < 1$ , and sets  $A, B \subseteq \mathbb{R}^d$ , consider the following binary operation:  $\lambda A + (1 - \lambda)B := \{\lambda x + (1 - \lambda)y : x \in A, y \in B\}$ .

**Proposition 1** (“Concavity”)  $M_{\sharp}$  and  $N_{\sharp}$  are concave set operators: For every convex set  $P_1, P_2 \subseteq [0, 1]^d$  and their homogenizations  $K_1, K_2$ , we have

$$\lambda M_{\sharp}(K_1) + (1 - \lambda)M_{\sharp}(K_2) \subseteq M_{\sharp}(\lambda K_1 + (1 - \lambda)K_2), \text{ and} \quad (20)$$

$$\lambda N_{\sharp}(K_1) + (1 - \lambda)N_{\sharp}(K_2) \subseteq N_{\sharp}(\lambda K_1 + (1 - \lambda)K_2). \quad (21)$$

**Proof:** It suffices to prove (20) as it implies (21). Suppose  $Y \in \lambda M_{\sharp}(K_1) + (1 - \lambda)M_{\sharp}(K_2)$ . Then,  $Y = \lambda Y_1 + (1 - \lambda)Y_2$  for some  $Y_i, i \in \{1, 2\}$  that satisfy (5), (9), (10), and (13) for  $K_i, i \in \{1, 2\}$ , respectively. Then, it is easy to see that  $Y$  also satisfies the same conditions for  $\lambda K_1 + (1 - \lambda)K_2$ . Therefore,  $Y \in M_{\sharp}(\lambda K_1 + (1 - \lambda)K_2)$ . ■

### 3 $M_{\sharp}$ - and $N_{\sharp}$ -commutative maps.

**Definition 2** Suppose  $L : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1+k}$  is a linear map. Then,  $L$  is said to be  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative, respectively, if  $LM_{\sharp}(K_1, K_2)L^T \subseteq M_{\sharp}(LK_1, LK_2)$  and  $LN_{\sharp}(K_1, K_2) \subseteq N_{\sharp}(LK_1, LK_2)$  for every pair of closed convex cones  $K_1, K_2 \subseteq Q$  (see Figure 2).

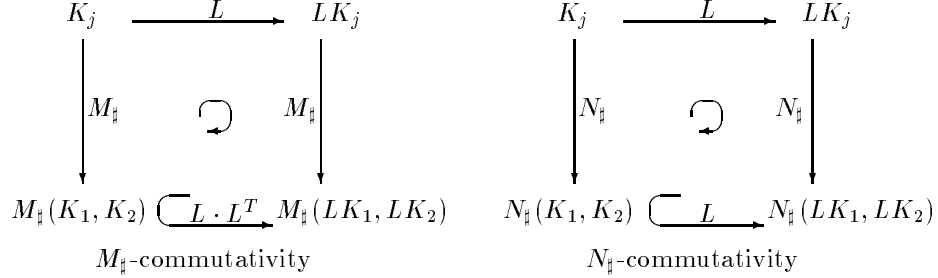


Figure 2:  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative diagram.

Let  $\tilde{e}_i$ 's and  $\tilde{f}_i$ 's be the extreme rays of the dual cone  $\tilde{Q}^*$  of the  $(d+1+k)$ -dimensional cone  $\tilde{Q}$  spanned by the  $(d+k)$ -dimensional hypercube.

**Theorem 3** A linear map  $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$  is  $M_{\sharp}$ -commutative if and only if, for every  $j \in \{1, 2, \dots, d+1+k\}$ ,

$$L^T \tilde{f}_j \tilde{e}_j^T L \in \text{span} \{f_i e_i^T : i \in \{1, 2, \dots, d\}\}. \quad (22)$$

**Proof:** Assume  $Y \in M_{\sharp}(K_1, K_2)$ . We first show that (22) guarantees  $LYL^T \in M_{\sharp}(LK_1, LK_2)$ . First, notice that (6), (9), and (10) are true for  $LYL^T$  regardless of (22): (9) and (10) are clearly satisfied by  $LYL^T$ . Regarding (6), due to (3),  $\tilde{w} \in (LK_j)^*$  if and only if  $L^T \tilde{w} \in K_j^*$  for  $j \in \{1, 2\}$ . Therefore,  $Y \in M_{\sharp}(K_1, K_2)$  implies  $0 \leq (L^T \tilde{u})^T Y (L^T \tilde{v}) = \tilde{u}^T LYL^T \tilde{v}$  for any  $\tilde{u} \in (LK_1)^*$  and  $\tilde{v} \in (LK_2)^*$ . It remains to show (22) guarantees (5) for  $LYL^T$ .

But, by (7) the latter is equivalent to that for all  $j \in \{1, 2, \dots, d+k\}$ ,

$$\text{trace} \left( LYL^T \tilde{f}_j \tilde{e}_j^T \right) = \text{trace} \left( Y (L^T \tilde{f}_j \tilde{e}_j^T L) \right) = 0. \quad (23)$$

From (22),  $L^T \tilde{f}_j \tilde{e}_j^T L = \sum_i \lambda_i f_i e_i^T$  for some  $\lambda_i$ ,  $i \in \{1, 2, \dots, d\}$ . Since  $Y$  satisfies (7), this implies (23).

We will prove the necessity for the  $M$ -operator. The proofs for  $M_0$ - and  $M_+$ -operators are similar. Suppose (22) does not hold for  $j = 1$ :  $L^T \tilde{f}_1 \tilde{e}_1^T L \notin \text{span}\{f_i e_i^T : 1, 2, \dots, d\}$ . Then, there is  $\bar{Y} \in \Sigma^{d+1}$  such that  $\text{trace}(\bar{Y} f_i e_i^T) = 0$  for all  $i \in \{1, 2, \dots, d\}$  and  $\text{trace}(\bar{Y} L^T \tilde{f}_1 \tilde{e}_1^T L) \neq 0$ . Pick  $K_1$  and  $K_2$  so that  $M(K_1, K_2)$  is full dimensional in  $\{Y \in \Sigma^{d+1} : \text{diag}(Y) = Y e_0\}$ . Let  $\hat{Y} \in \text{rel int}(M(K_1, K_2))$ . Then, there is  $\bar{\epsilon} > 0$  such that  $\hat{Y} + \epsilon \bar{Y} \in M(K_1, K_2)$  for all  $\epsilon < \bar{\epsilon}$ .

But, since  $\text{trace}(\bar{Y} L^T \tilde{f}_1 \tilde{e}_1^T L) \neq 0$ ,  $\text{trace}(L(\hat{Y} + \epsilon \bar{Y})L^T \tilde{f}_1 \tilde{e}_1^T) = \text{trace}(\hat{Y} L^T \tilde{f}_1 \tilde{e}_1^T L) + \epsilon \text{trace}(\bar{Y} L^T \tilde{f}_1 \tilde{e}_1^T L)$  can not be constantly 0 on  $0 < \epsilon < \bar{\epsilon}$ . Hence,  $L$  is not  $M$ -commutative. ■

**Remark 4** Notice that in Theorem 3,  $k$  is not necessarily assumed to be nonnegative.

**Corollary 5** *If, in addition,  $L$  is invertible, then the equality holds:  $LM_{\sharp}(K_1, K_2)L^T = M_{\sharp}(LK_1, LK_2)$ .*

**Proof:** It suffices to show that if  $\tilde{Y} \in M_{\sharp}(LK_1, LK_2)$ , then  $L^{-1}\tilde{Y}L^{-T} \in M_{\sharp}(K_1, K_1)$ , namely  $L^{-1}\tilde{Y}L^{-T}$  satisfies (6), (7), (9), and (10). Clearly, (9) and (10) are satisfied. Also, from (3) and invertibility of  $L$ , it follows that  $(LK_i)^* = L^{-T}K_i^*$  for  $i \in \{1, 2\}$ . Therefore,  $\tilde{u}^T\tilde{Y}\tilde{v} \geq 0$  for all  $\tilde{u} \in L^{-T}K_1^*$  and  $\tilde{v} \in L^{-T}K_2^*$ , or equivalently  $u^TL^{-1}\tilde{Y}L^{-T}v \geq 0$  for all  $u \in K_1^*$  and  $v \in K_2^*$ .

Finally, to prove (7), since  $\text{trace}\left(L^{-1}\tilde{Y}L^{-T}f_i e_i^T\right) = \text{trace}\left(\tilde{Y}L^{-T}f_i e_i^T L^{-1}\right)$ , it suffices to show that the linear subspace  $\text{span}\{f_i e_i^T\}$  is invariant under the mapping  $L^{-T} \cdot L^{-1}$ . Or equivalently,  $L^T \text{span}\{f_i e_i^T\} L = \text{span}\{f_i e_i^T\}$ . But, since  $L$  is invertible, the linear map  $L^T \cdot L : \text{span}\{f_i e_i^T\} \rightarrow \text{span}\{f_i e_i^T\}$  is a bijection. Hence the corollary follows. ■

**Corollary 6** *If  $L$  and  $L'$  are  $M_{\sharp}$ -commutative maps, then their composite, if defined, is also  $M_{\sharp}$ -commutative.*

The following facts were observed by Lovász and Schrijver.

**Lemma 7** [26] *If  $L$  is  $M_{\sharp}$ -commutative and  $L^T e_0$  is parallel to  $e_0$ , then  $L$  is also  $N_{\sharp}$ -commutative.*

**Corollary 8** [26] *If  $L : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  is an automorphism of  $Q$ , namely a linear map such that  $LQ = Q$ , then for every pair of closed convex cones  $K_1, K_2 \subseteq Q$ , we have  $LM_{\sharp}(K_1, K_2)L^T = M_{\sharp}(LK_1, LK_2)$  and  $LN_{\sharp}(K_1, K_2) = N_{\sharp}(LK_1, LK_2)$ .*

The proof follows from Corollary 5 and Lemma 7. For if  $L$  is an automorphism of  $Q$  then, there are a permutation  $\sigma : \{1, 2, \dots, d\} \rightarrow \{1, 2, \dots, d\}$  and  $\lambda > 0$  such that  $\{L^T e_i, L^T f_i\} = \{\lambda e_{\sigma(i)}, \lambda f_{\sigma(i)}\}$ , and  $L^T e_0$  is parallel to  $e_0$ . For the details, the reader is referred to the Appendix.

A motivation of Definition 2 is that some previous lower-bound results rely on  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative linear maps that are not necessarily invertible. The followings are  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative maps:

- *Embedding*  $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$  so that, for some  $0 \leq l \leq k$ ,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0, 1, \dots, d\}, \\ 0 & \text{for } i \in \{d+1, \dots, d+l\}, \\ x_0 & \text{for } i \in \{d+l+1, \dots, d+k\}. \end{cases} \quad (24)$$

- *Duplication*  $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$  so that, for a subset  $\{j_1, \dots, j_k\} \subseteq \{1, 2, \dots, d\}$ ,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0, 1, \dots, d\}, \\ x_{j_i-d} & \text{for } i \in \{d+1, \dots, d+k\}. \end{cases} \quad (25)$$

- *Flipping* is an automorphism that maps  $e_j \mapsto f_j, f_j \mapsto e_j$  for each  $j \in J \subseteq \{1, 2, \dots, d\}$ .

In all of the above examples, one can check that for every  $j \in \{1, 2, \dots, d+k\}$ , there is  $i \in \{1, 2, \dots, d\}$  such that

$$\left\{L^T \tilde{e}_j, L^T \tilde{f}_j\right\} = \{e_0, 0\}, \text{ or } \{e_i, f_i\}, \quad (26)$$

that is sufficient for (22). In fact, (26) describes a fairly broad class of linear maps that are both  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative.

**Corollary 9** *Suppose  $L$  satisfies the following conditions: 1) The first row is  $e_0$ , and 2) the rest are either,  $0$ ,  $e_0$ ,  $e_i$ , or  $f_i$  for  $i \in \{1, 2, \dots, d\}$ . Then any positive multiple of  $L$  is both  $M_{\sharp}$ - and  $N_{\sharp}$ -commutative.*

Now, we discuss one of the key properties used in our framework for lower-bound analysis.

**Lemma 10** *Let  $K \subseteq \mathbb{R}^{d+1}$  and  $\tilde{K} \subseteq \mathbb{R}^{d+1+k}$ , respectively, be the homogenizations of the convex sets  $P \subseteq [0, 1]^d$  and  $\tilde{P} \subseteq [0, 1]^{d+k}$ . Assume  $L : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1+k}$  is an  $N_{\sharp}$ -commutative map. If  $L$  is feasible, namely  $LK \subseteq \tilde{K}$ , then for every  $t \geq 0$ , we have  $LN_{\sharp}^t(K) \subseteq N_{\sharp}^t(LK) \subseteq N_{\sharp}^t(\tilde{K})$ .*

**Proof:** By induction on  $t$  using the feasibility of  $L$ . ■

The following fact is potentially useful in a lower bound analysis.

**Theorem 11** *Every convex combination of  $N_{\sharp}$ -commutative maps is also  $N_{\sharp}$ -commutative.*

**Proof:** Let  $L_1$  and  $L_2$  be  $N_{\sharp}$ -commutative maps and  $0 < \lambda < 1$ . Define  $L := \lambda L_1 + (1 - \lambda)L_2$ . Then,

$$\begin{aligned} LN_{\sharp}(K) &= \lambda L_1 N_{\sharp}(K) + (1 - \lambda)L_2 N_{\sharp}(K) \\ &\subseteq \lambda N_{\sharp}(L_1 K) + (1 - \lambda)N_{\sharp}(L_2 K) \\ &\subseteq N_{\sharp}(\lambda L_1 K + (1 - \lambda)L_2 K) = N_{\sharp}(LK). \end{aligned}$$

$N_{\sharp}$ -commutativity of  $L_1$  and  $L_2$  implies the first inclusion. The second inclusion follows from the concavity of  $N_{\sharp}$ -operators. ■

We note that  $M_{\sharp}$ -commutativity, however, is not necessarily preserved under taking convex combinations.

Analogously to the relation of  $P$  and  $K$ , for the notation of (16), we can also define ,

$$LP := \left\{ x : \begin{pmatrix} 1 \\ x \end{pmatrix} \in LK \right\}, \quad LN_{\sharp}(P) := \left\{ x : \begin{pmatrix} 1 \\ x \end{pmatrix} \in LN_{\sharp}(K) \right\}. \quad (27)$$

If  $L$  is  $N_{\sharp}$ -commutative, then it is routine to check that

$$LN_{\sharp}(P) \subseteq N_{\sharp}(LP). \quad (28)$$

## 4 Lower Bound Analysis.

### 4.1 $N_{\sharp}$ -ranks

Let  $\Pi$  be a 0-1 integer programming problem with the instances  $\iota$ . Denote the input size of  $\iota$  by  $\langle \iota \rangle$  and  $\Pi_n := \{\iota \in \Pi : \langle \iota \rangle \leq n\}$ . The rank  $r$  is a function on the quadruples  $(N_{\sharp}, \Pi, P, n)$ , where  $P$  is a initial relaxation scheme of the instances  $\iota$  of  $\Pi$ . For each  $\iota$ , let  $P(\iota) \subseteq Q$  be the relaxation obtained by  $P$  applied to  $\iota$ , and  $\ell_{\iota}$  the minimum  $\ell$  such that  $N_{\sharp}^{\ell}(P(\iota)) \subseteq P_I(\iota)$ , the integral hull of  $P(\iota)$ . Then, the rank function  $r$  is defined as

$$r(N_{\sharp}, \Pi, P, n) := \max\{\ell_{\iota} : \iota \in \Pi_n\}. \quad (29)$$

When  $\Pi$  and  $n$  are clear from the context, we will simply write  $r_{\sharp}(P) := r(N_{\sharp}, \Pi, P, n)$ . Obviously,  $r_{\sharp}(P)$  is a measure of efficiency of the lift-and-project methods for problem  $\Pi$ . However, finding an exact value of  $r$  is usually a difficult task. Therefore, the analyses are focused on finding good lower and/or upper bounds on  $r_{\sharp}(P)$ . The former is equivalent to finding an instance  $\iota \in \Pi_n$ ,



a suitable point  $v(n)$  and the largest  $\ell_n$  and such that  $v(n)$  lies in the gap between  $P_I(\iota)$  and  $N_{\sharp}^{\ell_n}(P(\iota))$ :  $v(n) \in N_{\sharp}^{\ell_n}(P(\iota)) \setminus P_I(\iota)$ . Then, clearly  $r_{\sharp}(P) \geq \ell_n + 1$ .

For lower-bound analysis on various combinatorial optimization problems, see also [2, 17, 21, 22].

## 4.2 Construction of $v(n)$

Many of the existing proofs set up symmetric structures (graphs or polytopes) which allow arguments with convex combinations in the relaxations  $N_{\sharp}^k(P)$ . This in turn, reduces the number of parameters in  $v(n)$ .

We denote by  $\bar{e}$  the vector of all ones of appropriate size. Suppose  $v \in \mathbb{R}_{\sharp}^d$  maximizes  $\bar{e}^T x$  (we assume for this discussion that the underlying combinatorial optimization problem is a maximum cardinality problem) over  $N_{\sharp}^k(P)$ . Thus, if  $N_{\sharp}^k(P)$  is invariant under all permutations  $\mathcal{S}_d$  (represented as permutation matrices), i.e.,

$$\forall R \in \mathcal{S}_d : x \in N_{\sharp}^k(P) \iff Rx \in N_{\sharp}^k(P).$$

Then

$$\left( \frac{1}{|\mathcal{S}_d|} \sum_{R \in \mathcal{S}_d} Rv \right) \in N_{\sharp}^k(P),$$

by the convexity of  $N_{\sharp}^k(P)$ . Therefore, we can assume  $v = \alpha \bar{e}$  for some  $\alpha \geq 0$  (we used  $\bar{e}^T S v = \bar{e}^T v, \forall R \in \mathcal{S}_d$ ).

In other problems, the symmetry might be less pronounced; however, this basic technique can still be useful in reducing the number of parameters in  $v$  from a large function of  $d$  to a constant. Then the conditions of  $N_{\sharp}$  may lead to recursions (as in [16]).

This kind of technique was used in [29, 15, 10, 15, 25, 22, 21]. A recent formal approach is presented in [14].

It turns out, we can summarize conveniently, via  $M_{\sharp}$ -commutative maps, many of the lower-bound analyses available in the literature. They rely on a mathematical induction on the size (suitably defined) of the instances. To facilitate the presentation, we only consider the instances that are symmetric with respect to the variables. Thus, we consider essentially a unique instance of each size. Let  $s_k$  be the size of the instance at the  $k$ -th induction step. For instance,  $s_k$  can be the number of edges, nodes, or variables. Denote by  $P_{s_k}$  and  $K_{s_k}$ , respectively, the initial relaxation and its homogenization for  $\Pi_{s_k}$ . For simplicity, we will write  $M_{\sharp}(s_k) := M_{\sharp}(K_{s_k})$  and  $N_{\sharp}(s_k) := N_{\sharp}(K_{s_k})$ .

## 4.3 Unifying approach

The unifying approach focuses on constructing in a recursive manner, the sequence of proofs  $Y_k \in M_{\sharp}^k(s_k)$  such that  $v(k) = Y_k e_0$  via appropriate  $M_{\sharp}$ -commutative maps  $L_p$ . See Figure 3.

**Scheme 12** *Using the symmetry of  $\iota$ ,  $P$ , and  $v(k)$ , construct  $\{Y_k\}$  so that  $Y_k e_0 = v(k)$ ,  $Y_k \in M_{\sharp}^k(s_k)$  and  $Y_{k+1} = \frac{1}{|S|} \sum_{p \in S} L_p Y_k L_p^T \in M_{\sharp}^{k+1}(s_{k+1})$ , for some set of  $M_{\sharp}$ -commutative maps  $\{L_p : p \in S\}$ .*

The  $M_{\sharp}$ -commutativity of  $L_p$ 's implies that  $L_p Y_k L_p^T \in M_{\sharp}^k(s_{k+1})$  for all  $p$ . Thus, the scheme is based on the intuition that, due to the symmetry, when  $L_p Y_k L_p^T \in M_{\sharp}^k(s_{k+1})$  for  $p \in S$  then their convex combination might lie in the smaller set  $M_{\sharp}^{k+1}(s_{k+1})$ . See Figure 4.

$$\begin{array}{ccc}
\bullet v(k) \in N_{\sharp}^k(s_k) & & \bullet w^p \in N_{\sharp}^k(s_{k+1}) \\
\Downarrow & & \Downarrow \\
Y_k \in M_{\sharp}^k(s_k) & \xrightarrow{L_p \cdot L_p^T} & W^p = L_p Y_k L_p^T \in M_{\sharp}^k(s_{k+1}) \\
\downarrow \text{induction} & & \swarrow \text{proof} \\
\bullet v(k+1) \in N_{\sharp}^{k+1}(s_{k+1}) & & \\
\Downarrow & & \\
\exists? Y_{k+1} \in \text{cone}\{W^p\}. & & \\
\text{s.t } Y_{k+1} e_i, Y_{k+1} f_i \in \text{cone}\{w^p\}. & & 
\end{array}$$

Figure 3: Unifying approach

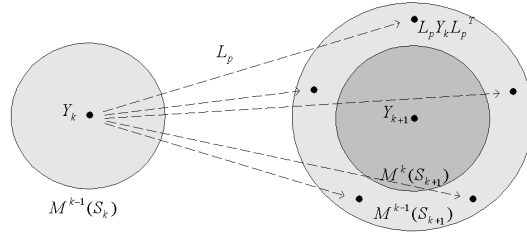


Figure 4: A schematic illustration of the proof technique.

Note that the convex combination of the commutative maps,  $\sum_{p \in S} L_p Y_k L_p^T$  preserves (5), (9), and most importantly, positive semidefiniteness (10) of  $Y_k$ . Hence, due to the manner in which  $Y_k$  is defined, the conditions are automatically met once we establish them in the base step of the induction. Thus, the unifying approach can make the proof more straightforward and systematic.

This approach can be extended to the special structures that a given problem  $\Pi$  may have. Suppose, for instance, some set of integral points  $\{z_q : q \in T\} \subset P_I$  is readily available. Then, for every  $q \in T$  and  $k \geq 0$ ,  $\begin{pmatrix} 1 \\ z_q \end{pmatrix} \begin{pmatrix} 1 \\ z_q \end{pmatrix}^T \in M_{\sharp}^k(K)$ . (An interesting special case is when  $P$  is upper or lower comprehensive.) Thus, in such cases, the recursive definition of  $Y_k$  can be generalized as follows:

$$Y_{k+1} = \mu \left( \frac{1}{|S|} \sum_{p \in S} L_p Y_k L_p^T \right) + (1 - \mu) \left( \frac{1}{|T|} \sum_{q \in T} \begin{pmatrix} 1 \\ z_q \end{pmatrix} \begin{pmatrix} 1 \\ z_q \end{pmatrix}^T \right), \quad (30)$$

for some appropriate set of integral points  $\{z_q : q \in T\}$  and  $0 \leq \mu \leq 1$ .

## 5 Implementations of the Unifying Proof.

### 5.1 Matching polytope

The matching polytope of a graph  $G = (V, E)$  is defined to be the convex hull of the characteristic vectors of the matchings in  $G$ . Then, it is the integral hull,  $P_I$  of

$$P := \{x \in \mathbb{R}^E : x(\delta(v)) \leq 1 \ \forall v \in V, x \geq 0\}, \quad (31)$$

where,  $\delta(v)$  is the set of edges that have  $v$  as an endpoint and  $x(S) := \sum_{j \in S} x_j$ . Denote by  $G_{2k+1}$ , the clique with  $s_k := (2k + 1)$  nodes,  $V = \{1, 2, \dots, 2k + 1\}$ . Then, it has  $t_k := k(2k + 1)$  edges,  $(1, 2), (1, 3), \dots, (2k, 2k + 1)$ . Consider the lexicographic order  $\prec$  on the edges:  $(i, j) \prec (k, l) \Leftrightarrow i < k$ , or  $i = k$  and  $j < l$ . We assume that the edges are numbered to the lexicographic order and denote for simplicity  $E_{2k+1} = \{1, 2, \dots, t_k\}$ . See, e.g.,  $G_{2k+1}$  in Figure 5 and the numbers assigned to the edges. Also, let  $P_{2k+1}$  be the relaxation of (31) for  $G_{2k+1}$  and  $K_{2k+1}$  the homogenization of  $P_{2k+1}$ . Recall that we write  $N_+^t(2k + 1) := N_+^t(K_{2k+1})$ .

Stephen and Tunçel [29] showed that if  $K_{2k+1}$  is used as the initial cone,  $r_+(P_{2k+1})$  is  $k$ . In doing so, they established the lower-bound  $(k - 1) < r_+(P_{2k+1})$  by constructing a uniform point  $v(k)$  in  $N_+^{k-1}(2k + 1) \setminus (P_{2k+1})_I$ : For  $k \in \{1, 2, \dots\}$ ,

$$v(k) := \left( \begin{array}{c} 1 \\ \frac{1}{2k} \bar{e} \end{array} \right) \in \mathbb{R}^{E_{2k+1} \cup \{0\}}. \quad (32)$$

Since the maximum cardinality of a matching on  $G_{2k+1}$  is  $k$ ,  $v(k)$  is not in  $(P_{2k+1})_I$ . Hence, the lower-bound  $k$  on  $r_+(P_{2k+1})$  will follow, if we show, for  $k \in \{1, 2, \dots\}$ ,

$$v(k) \in N_+^{k-1}(2k + 1). \quad (33)$$

Denote by  $G_{2k+1} \setminus p$  the graph obtained by deleting the two endpoints of  $p$  from  $G_{2k+1}$ . Then, a key observation is that for any  $p \in E_{2k+1}$ , the lexicographic orders on  $E_{k-1}$  and  $E(G_{2k+1} \setminus p)$  induces an obvious isomorphism between  $G_{2k-1}$  and  $G_{2k+1} \setminus p$ . Hence, the following map aligns the order of elements of a vector  $v_i, i \in E_{2k-1}$  to the order of edges of the sub-clique  $G_{2k+1} \setminus p$ . For  $p \in \{1, 2, \dots, t_k\}$ , define  $L_p : v \in \mathbb{R}^{E_{2k-1} \cup \{0\}} \mapsto w^p \in \mathbb{R}^{E_{2k+1} \cup \{0\}}$ :

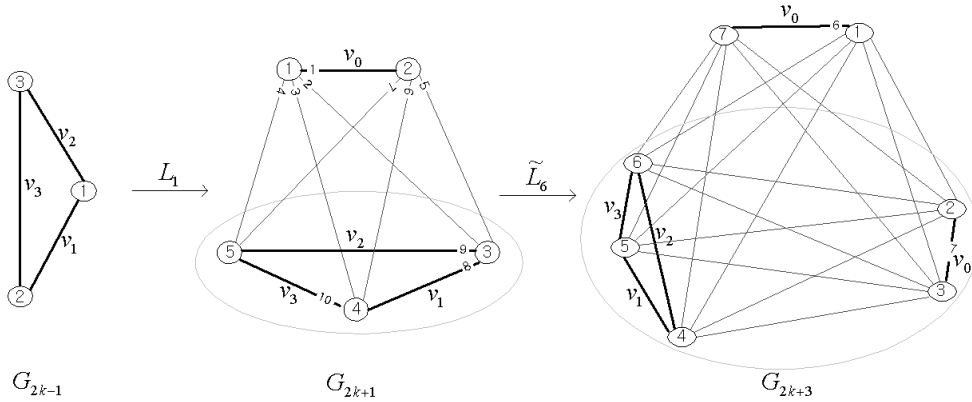
$$w_j^p := \begin{cases} v_0, & j = 0, \\ v_0, & j = p, \\ 0, & j \in \text{Inc}(p), \text{ the set of edges incident to } p, \\ v_i, & j \text{ is the } i\text{-th edge of } G_{2k+1} \setminus p. \end{cases} \quad (34)$$

For  $p \in E_{2k+1}$  and  $q \in E_{2k+3}$ , we can define composite  $\tilde{L}_q L_p$  of the two embeddings,  $L_p : \mathbb{R}^{E_{2k-1} \cup \{0\}} \rightarrow \mathbb{R}^{E_{2k+1} \cup \{0\}}$  and  $\tilde{L}_q : \mathbb{R}^{E_{2k+1} \cup \{0\}} \rightarrow \mathbb{R}^{E_{2k+3} \cup \{0\}}$ . In Figure 5,  $\tilde{L}_6 L_1 v$  with  $v = (v_0, v_1, v_2, v_3)^T$  is illustrated. Notice that for a given vector  $v \in \mathbb{R}^{E_{2k-1} \cup \{0\}}$ , there can be more than one two-level embeddings mapping  $v$  to the same vector.

**Lemma 13** *Let  $p$  and  $q \in E_{2k+3}$  be two non-incident edges of  $G_{2k+3}$ . Suppose  $p$  is the  $p_q$ -th edge of  $G_{2k+3} \setminus q$ . Similarly, define  $q_p$  for  $q$  with respect to  $p$ . Then for every  $v \in \mathbb{R}^{E_{2k-1} \cup \{0\}}$ ,*

$$\tilde{L}_q L_{p_q} v = \tilde{L}_p L_{q_p} v.$$

**Proof:** Trivially, by definition, both  $\tilde{L}_q L_{p_q} v$  and  $\tilde{L}_p L_{q_p} v$  have the same 0-th element,  $v_0$ .

Figure 5: Two-level embed  $\tilde{L}L$ .

Since  $p$  and  $q$  are not incident in  $G_{2k+3}$ ,  $G_{2k+3}\setminus p$  and  $G_{2k+3}\setminus q$  have a common sub-clique with  $t_{k-1} := (k-1)(2k-1)$  edges, namely  $G_{2k+3}\setminus\{p, q\}$ . We will show that,  $\tilde{L}_q L_{p_q}$  assigns 1)  $v_0$  to  $p$  and  $q$ , 2) for each  $i \in E_{2k-1}$ ,  $v_i$  to the  $i$ -th edge of  $G_{2k+3}\setminus\{p, q\}$ , and 3) 0's to the remaining edges in  $E_{2k+3}$ . Then, by the symmetry between  $p$  and  $q$ ,  $\tilde{L}_p L_{q_p}$  does the same and hence the lemma will follow.

By the definition, (34), and the discussion preceding it,  $L_{p_q}$  aligns  $v_i$ ,  $i \in E_{2k-1}$  to  $E(G_{2k+1}\setminus p_q)$ . Now,  $\tilde{L}_q$  assigns  $v_0$  to  $q \in E_{2k+3}$  and aligns  $(L_{p_q} v)_j$ ,  $j \in E_{2k+1}$  to the edges,  $E(G_{2k+3}\setminus q)$ . The  $p_q$ -th element of  $L_{p_q} v$  is  $v_0$  and, from the hypothesis,  $p_q$ -th element of  $E(G_{2k+3}\setminus q)$  is  $p$ . Therefore,  $\tilde{L}_q L_{p_q}$  assigns  $v_0$  to  $p$ . Hence, 1) holds. Furthermore, in this alignment  $(L_{p_q} v)_j$ ,  $j \in E_{2k+1}\setminus p_q$  will be assigned to the edges of  $(G_{2k+3}\setminus q) \setminus p = G_{2k+3}\setminus\{p, q\}$  preserving the order. But,  $(L_{p_q} v)_j$ ,  $j \in E_{2k+1}\setminus p_q$  are ordered the same as  $v_i$ ,  $i \in E_{2k-1}$ . Therefore, 2) also holds. Clearly, the remaining edges of  $E_{2k+3}$  are assigned 0's. Hence the lemma. ■

For example, in Figure 5, let  $p = 7$  and  $q = 6$ . Then,  $p_q = 1$  and  $q_p = 4$ . Therefore, another two-level embedding  $\tilde{L}_7 L_4 v$ , as easily checked, has the same effect as  $\tilde{L}_6 L_1 v$ .

Now, we provide a proof of (33) based on the unifying approach.

**Proof:**

Clearly,  $v(1) \in N_+^0(3) := K_3$  as it satisfies the inequalities of (31). Define  $Y_k \in \mathbb{R}^{E_{2k+1} \cup \{0\} \times E_{2k+1} \cup \{0\}}$  recursively as follows:

$$Y_2 := \left[ \frac{1}{s_2} \sum_{p=1}^{s_2} L_p v(1); \frac{1}{4} L_1 v(1); \dots; \frac{1}{4} L_{s_2} v(1) \right], \quad (35)$$

$$Y_k := \frac{1}{t_k} \sum_{p=1}^{t_k} L_p Y_{k-1} L_p^T, \quad k \in \{3, 4, \dots\}. \quad (36)$$

First, let's prove that, for every  $k \in \{2, 3, \dots\}$ ,

$$Y_k e_0 = v(k), \text{ and} \quad (37)$$

$$Y_k \text{ satisfies (5), (9), and (10)}. \quad (38)$$

It is easily seen from (32) and (35) that  $Y_2$  is given as follows:

$$Y_2 = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \quad (39)$$

Thus,  $Y_2$  satisfies (37). Now, we show it is also true for  $k + 1$ . But,

$$\begin{aligned} Y_{k+1}\tilde{e}_0 &= \frac{1}{t_{k+1}} \sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k \tilde{L}_q^T \tilde{e}_0 = \frac{1}{t_{k+1}} \sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k e_0 \\ &= \frac{1}{t_{k+1}} \sum_{q=1}^{t_{k+1}} \tilde{L}_q v(k) = \frac{1}{t_{k+1}} \begin{pmatrix} t_{k+1} \\ (1 + t_k \frac{1}{2k})e \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2(k+1)}e \end{pmatrix}. \end{aligned}$$

Thus, (37) holds for  $(k + 1)$  and hence for all  $k \geq 2$ . Regarding (38),  $Y_2$  clearly satisfies (5), and (9). It is straightforward to check that  $Y_2$  is positive semidefinite: It has eigenvalues  $0$ ,  $\frac{5}{24}$  and  $\frac{19}{12}$ . Hence, by induction,  $M_+$ -commutativity of the embeddings  $L_p$  implies (38).

To complete the proof, due to (8), it suffices to show that for  $k \in \{2, 3, \dots\}$ ,

$$Y_k e_i, Y_k f_i \in N_+^{k-2}(2k + 1), \forall i \in \{1, 2, \dots, t_k\}. \quad (40)$$

To do so, we will prove the following:

$$Y_k e_i = \frac{1}{2k} L_i v(k - 1), \text{ and} \quad (41)$$

$$Y_k f_i = \frac{1}{4k} \sum_{j \in \text{Inc}(i)} L_j v(k - 1). \quad (42)$$

Then, since  $v(1) \in K_3 := N_+^0(3)$ , using the  $N_+$ -commutativity and the feasibility of  $L_i$ , it is easy to see inductively that (40) follows from (41) and (42). By definition of  $Y_2$ , (41) is satisfied when  $k = 2$ . Now, we show (41) holds for  $k + 1$ . By definition,

$$Y_{k+1}\tilde{e}_j = \frac{1}{t_{k+1}} \sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k \tilde{L}_q^T \tilde{e}_j.$$

**Case 1.** If  $q = j$ , then  $\tilde{L}_q Y_k \tilde{L}_q^T \tilde{e}_j = \tilde{L}_j Y_k e_0 = \tilde{L}_j v(k)$  from (37).

**Case 2.** If  $q$  and  $j$  are incident, then  $\tilde{L}_q^T \tilde{e}_j = 0$ . Therefore, we have  $\tilde{L}_q Y_k \tilde{L}_q^T \tilde{e}_j = 0$ .

**Case 3.** Finally, consider the case when  $q$  and  $j$  are not incident. Suppose  $j$  is the  $j_q$ -th smallest numbered edge of  $G_{2k+3} \setminus q$ . Then, by the definition of  $\tilde{L}$ , we have  $\tilde{L}_q^T \tilde{e}_j = e_{j_q}$ . Therefore,  $\tilde{L}_q Y_k \tilde{L}_q^T \tilde{e}_j = \tilde{L}_q Y_k e_{j_q}$ , which is  $\frac{1}{2k} \tilde{L}_q L_{j_q} v(k - 1)$  from the induction hypothesis, (41). Suppose  $q$  is the  $q_j$ -th edge of  $G_{2k+3} \setminus j$ . Then, by Lemma 13, we get  $\frac{1}{2k} \tilde{L}_q L_{j_q} v(k - 1) = \frac{1}{2k} \tilde{L}_j L_{q_j} v(k - 1)$ . Clearly, for a fixed  $j$ , distinct  $q$ 's have distinct  $q_j$ 's from  $\{1, 2, \dots, t_k\}$ .

Thus, summarizing the cases, we get

$$Y_{k+1}\tilde{e}_j = \frac{1}{t_{k+1}} \left( \tilde{L}_j v(k) + \frac{1}{2k} \sum_{i=1}^{t_k} \tilde{L}_j L_i v(k-1) \right) \quad (43)$$

$$= \frac{1}{t_{k+1}} \tilde{L}_j \left( v(k) + \frac{t_k}{2k} v(k) \right) = \frac{1}{2(k+1)} \tilde{L}_j v(k). \quad (44)$$

The first equality of (44) is from that  $\sum_{i=1}^{t_k} L_i v(k-1) = t_k v(k)$ . Therefore, (41) holds for  $k+1$  and the proof of (41) is completed.

Finally, it is routine to check that (42) is implied by (37) and (41). Thus, (33) follows. ■

**Remark 14** *The usage of the two-level embedding in the above proof can be avoided (making the proof considerably shorter) by a careful counting argument. However, the above linear algebraic proof via the commuting single-level embeddings may be useful in other, more complicated situations. Aguilera, Bianchi and Nasini [1] gave another proof using the work of Doob [11] which in turn has connections to Tutte's much earlier work [30]. While this is a very nice connection found by [1], the underlying proof still relies on working out the eigenspaces and the eigenvalues of the corresponding  $Y$  matrix to establish the positive semidefiniteness.*

## 5.2 Knapsack polytope

Consider the following (reversed) knapsack polytope and its 0-1 integral hull:

$$P_d := \left\{ x \in \mathbb{R}^d : x_1 + \cdots + x_d \geq \frac{1}{2} \right\}, \quad (P_d)_I = \left\{ x \in \mathbb{R}^d : x_1 + \cdots + x_d \geq 1 \right\}. \quad (45)$$

Cook and Dash [10] showed that these are some of the worst-case examples for all  $N_{\frac{1}{2}}$  operators:  $r_0(P_d) = r(P_d) = r_+(P_d) = d$ . To capture such results, it suffices to show that

$$v(d) := \left( \frac{1}{d+1} \bar{e} \right) \in N_+^{d-1}(d). \quad (46)$$

We use the following embeddings. For  $p \in \{1, 2, \dots, d\}$ ,  $L_p : v \in \mathbb{R}^d \mapsto w^p \in \mathbb{R}^{d+1}$  such that

$$w_j^p := \begin{cases} v_0, & j = 0, \\ 0, & j = p, \\ v_i, & j \text{ is the } i\text{th smallest number of } \{1, 2, \dots, d\} \setminus \{p\}. \end{cases} \quad (47)$$

We recursively construct  $Y_d$ , for  $d \in \{1, 2, \dots\}$ , as follows:

$$Y_1 := \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad (48)$$

$$Y_d := \frac{1}{d+1} \sum_{p=1}^d L_p Y_{d-1} L_p^T + \frac{1}{d+1} \text{Arrow}_d \left( \frac{1}{d} \bar{e} \right), \quad \text{for } d \in \{2, 3, \dots\}, \quad (49)$$

where  $\text{Arrow}_d(\cdot) : \mathbb{R}^d \mapsto \Sigma^{d+1}$ , is defined as

$$\text{Arrow}_d(u) := \begin{bmatrix} 1 & u_1 & u_2 & \cdots & u_d \\ u_1 & u_1 & 0 & \cdots & 0 \\ u_2 & 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_d & 0 & 0 & \cdots & u_d \end{bmatrix}.$$

Note that,  $\text{Arrow}_d(\frac{1}{d}\bar{e})$  fits the recursion of (30), since it corresponds to a convex combination (with uniform coefficients  $\frac{1}{d}$ ) of the matrices corresponding to known integer points of  $P$ :

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Clearly,  $Y_1$  satisfies (5), (9), and (10). Also, we have

$$Y_d = \begin{bmatrix} 1 & \frac{1}{d+1} & \frac{1}{d+1} & \cdots & \frac{1}{d+1} \\ \frac{1}{d+1} & \frac{1}{d+1} & 0 & \cdots & 0 \\ \frac{1}{d+1} & 0 & \frac{1}{d+1} & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ \frac{1}{d+1} & 0 & 0 & 0 & \frac{1}{d+1} \end{bmatrix}.$$

In particular,  $Y_d e_0 = v(d)$ . Moreover,  $Y_d e_j = \frac{1}{d+1}(e_0 + e_j)$ . Hence,  $(d+1)Y_d e_j \in P_I(d)$ . Finally,

$$Y_d f_j = \frac{d}{d+1} \begin{pmatrix} 1 \\ \frac{1}{d} \\ \vdots \\ \frac{1}{d} \\ 0 \\ \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{pmatrix} = \frac{d}{d+1} L_j v(d-1) \in N_+^{d-2}(d).$$

Thus,  $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$  for all  $j \in \{1, 2, \dots, d\}$  and (46) follows.

### 5.3 An Empty $\ell_1$ -ball

Consider the following polytope:

$$P_d := \left\{ x \in \mathbb{R}^d : \sum_{j \in S} x_j + \sum_{j \notin S} (1 - x_j) \geq \frac{1}{2}, \forall S \subseteq \{1, 2, \dots, d\} \right\}. \quad (50)$$

Notice that any 0-1 vector  $x$  with  $x_i = 1$  exactly for  $i \in T$ , does not satisfy the inequality corresponding to  $S = \{1, 2, \dots, d\} \setminus T$ . Thus,  $(P_d)_I = \emptyset$ . Denote by  $K_d$  the homogenization of  $P_d$ . Define, for  $p \in \{1, 2, \dots, d\}$ ,

$$\begin{aligned} L_{0p} &: (x_0, x_1, \dots, x_{d-1})^T \mapsto (x_0, x_1, \dots, x_{p-1}, 0, x_p, \dots, x_{d-1})^T, \\ L_{1p} &: (x_0, x_1, \dots, x_{d-1})^T \mapsto (x_0, x_1, \dots, x_{p-1}, x_0, x_p, \dots, x_{d-1})^T, \end{aligned} \quad (51)$$

$$Y_1 := \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \text{ and} \quad (52)$$

$$(53)$$

$$Y_d := \frac{1}{2} (L_{01}Y_{d-1}L_{01}^T + L_{11}Y_{d-1}L_{11}^T) \text{ for } d \in \{2, 3, \dots\}. \quad (54)$$

Then, it is easily seen that  $Y_d$  defined in (54) coincides  $Y_d$  given below:

$$Y_d := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}. \quad (55)$$

For instance,

$$Y_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, L_{01}Y_1L_{01}^T = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \text{ and } L_{11}Y_1L_{11}^T = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \quad (56)$$

Hence,

$$Y_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \quad (57)$$

Clearly,  $v(d) = Y_d e_0$ . Trivially,  $Y_1$  satisfies (5), (9), and (10). Therefore, the same conditions are met by  $Y_d$  for  $d \in \{2, 3, \dots\}$ . Thus, to prove  $v(d) \in N_+^{d-1}(d)$ , it remains to show that  $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$  for  $j \in \{1, 2, \dots, d\}$ . First,  $v(1) \in N_+^0(1) = K_1$ . Notice that for  $j \in \{1, 2, \dots, d\}$ ,

$$Y_d e_j = \frac{1}{2} L_{1j} v(d-1), \text{ and} \quad (58)$$

$$Y_d f_j = \frac{1}{2} L_{0j} v(d-1). \quad (59)$$

From the induction hypothesis,  $v(d-1) \in N_+^{d-2}(d-1)$ . But, for every  $j \in \{1, 2, \dots, d\}$ ,  $L_{i1}$   $i \in \{0, 1\}$  are  $N_+$ -commutative and feasible. Thus, for  $i \in \{0, 1\}$ ,  $L_{i1}N_+^{d-2}(d-1) := L_{i1}N_+^{d-2}(K_{d-1}) \subseteq N_+^{d-2}(L_{i1}K_{d-1}) \subseteq N_+^{d-2}(K_d)$ . Therefore, we have  $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$ . Hence, the proof.

**Remark 15** *Instead of (54), we can also use  $\frac{1}{2d} \sum_{p=1}^d (L_{0p}Y_{d-1}L_{0p}^T + L_{1p}Y_{d-1}L_{1p}^T)$  to get the same results.*

#### 5.4 TSP: (4/3)-conjecture related lower-bounds

Since our techniques generalize those of Cook and Dash [10], their main results concerning  $N_+$ -rank of the TSP polytope fits into the framework.

Recently, Cheung [9] obtained, through an elegant analysis, some lower-bound results for the  $N_+$ -rank of various relaxations obtained from the subtour elimination polytope. Motivation of [9] is the so-called (4/3)-conjecture. The proof analyzes the eigenspaces of the individual matrices to prove that the claimed fractional vector indeed lies in  $N_+^k(P)$ . Our unifying approach is also directly applicable to this situation since the original proof only uses the well established embeddings all of which are  $M_+$ -commutative.



## 6 Conclusion

We presented a unified proof technique for establishing lower-bounds on the SDP-based lift-and-project rank of combinatorial optimization polyhedra. There are two obvious future research directions opened by our approach:

- Clearly, the lower-bound established for a stronger operator directly applies to the weaker operator. However, to obtain better lower-bound results for the weaker operators, one needs to focus on those  $N_0$ - and  $N$ - commutative maps that are not  $N_+$ -commutative. In particular, complete characterization of  $N$ -commutative maps can be very useful in settling many open questions.
- It seems that our technique has a lot of potential for generalization to the so-called “optimization over lattices interpretation” of the lift-and-project methods (see [26] and [21]). Such generalization would help analyze Sherali-Adams operator and Lasserre-type methods. Indeed, in the Appendix of [21], Laurent sketches “a tentative iterative proof” which has similarities to our framework.

## References

- [1] N. E. Aguilera, S. M. Bianchi and G. L. Nasini. Lift and project relaxations for the matching and related polytopes. *Discrete Appl. Math.*, to appear.
- [2] S. Arora, B. Bollobás and L. Lovász. Proving integrality gaps without knowing the linear program. *Proc. 43rd IEEE Symp. FOCS 2002*, 313–322.
- [3] E. Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Management Science Research Report 348* GSIA, Carnegie Mellon University, Pittsburgh, PA, USA, 1974. (Appeared as: E. Balas. Disjunctive programming: properties of the convex hull of feasible points. *Discrete Appl. Math.* 89 (1998), no. 1-3, 3–44.)
- [4] E. Balas, S. Ceria and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Prog. A* **58** (1993) 295–323.
- [5] D. Bienstock and M. Zuckerberg. Subset algebra lift operators for 0-1 integer programming. *CORC Report 2002–01*, Columbia University, NY, NY, USA.
- [6] D. Bienstock and N. Ozbay. Tree-width and the Sherali-Adams operator. *CORC Report 2003–09*, Columbia University, NY, NY, USA.
- [7] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Math.* 4 (1973) 33–41.
- [8] V. Chvátal, W. Cook and M. Hartmann. On cutting plane proofs in combinatorial optimization. *Linear Algebra and Its Applications* 114/115 (1989) 455–499.
- [9] K. K. H. Cheung. The Lovász-Schrijver lift-and-project procedures on the Dantzig-Fulkerson-Johnson relaxation of the TSP. *Research Report CORR 2003-22*, Dept. of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, July 2003.

- [10] W. Cook and S. Dash. On the matrix-cut rank of polyhedra. *Math. Oper. Res.*, 26(1):19–30, 2001.
- [11] M. Doob. An interrelation between line graphs, eigenvalues and matroids. *J. Combinatorial Theory B* 15 (1973) 40–50.
- [12] F. Eisenbrand and A. S. Schulz. Bounds on the Chvátal rank of polytopes in the 0/1 cube. *Combinatorica* 23 (2003) 245–261.
- [13] U. Feige and R. Krauthgamer. The probable value of the Lovasz-Schrijver relaxations for maximum independent set. *SIAM J. Comput.* 32 (2003) 345–370.
- [14] K. Gatermann and P. A. Parrilo. Symmetry groups, semidefinite programs, and sums of squares. *manuscript* November 2002.
- [15] M. X. Goemans and F. Rendl. Semidefinite programs and association schemes. *Computing* 63 (1999), no. 4, 331–340.
- [16] M. X. Goemans and L. Tunçel. When does the positive semidefiniteness constraint help in lifting procedures? *Math. Oper. Res.*, 26(4):796–815, 2001.
- [17] D. Grigoriev, E. A. Hirsch and D. V. Pasechnik. Complexity of semi-algebraic proofs. *Elec. Coll. Computational Complexity*, revision 3 of report no. 103 (2001).
- [18] M. Kojima and L. Tunçel. Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM J. Optim.* 10 (2000) 750–778.
- [19] J. B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. in *Lecture Notes in Computer Science* (K. Aardal and A. M. H. Gerards, eds.) 2001, pp. 293–303.
- [20] J. B. Lasserre. Polynomials nonnegative on a grid and discrete optimization. *Trans. Amer. Math. Soc.* 354 (2002) 631–649.
- [21] M. Laurent. Lower bound for the number of iterations in semidefinite hierarchies for the cut polytope. *Math. Oper. Res.* 28 (2003) 871–883.
- [22] M. Laurent. Tighter linear and semidefinite relaxations for max-cut based on the Lovász-Schrijver lift-and-project procedure. *SIAM J. Optim.* 12 (2002) 345–375.
- [23] L. Lipták. *Critical Facets of the Stable Set Polytope*. Ph.D. Thesis, Yale University, 1999.
- [24] L. Lipták and L. Tunçel. The stable set problem and the lift-and-project ranks of graphs. *Mathematical Programming B*, to appear .
- [25] L. Lovász. Semidefinite programs and combinatorial optimization. *Recent advances in algorithms and combinatorics*, B. A. Reed and C. L. Sales eds., CMS Books Math./Ouvrages Math. SMC, 11, Springer, New York, 2003, pp. 137–194.
- [26] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optimization*, 1(2):166–190, 1991.

- [27] H. D. Sherali and W. P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM J. Discrete Mathematics* **3** (1990) 411–430.
- [28] H. D. Sherali and W. P. Adams. *A reformulation-linearization technique for solving discrete and continuous nonconvex problems*. Nonconvex Optimization and its Applications, 31. Kluwer Academic Publishers, Dordrecht, 1999.
- [29] T. Stephen and L. Tunçel. On a representation of the matching polytope via semidefinite liftings. *Math. Oper. Res.*, 24(1):1–7, 1999.
- [30] W. T. Tutte. *Lectures on Matroids*. *J. Res. Nat. Bur. Standards Sect. B* 69B (1965) 1–47.

## A Proof of Corollary 8.

Assume that  $L$  is an automorphism of  $Q$ . Then, from (3) it follows that  $L^T$  is also an automorphism of  $Q^*$ . Hence,  $L^T$  preserves the set of extreme rays as well as the interior of  $Q^*$ . As  $e_0$  is an interior ray, so is  $L^T e_0$ . But, since  $e_0 = e_i + f_i \forall i \in \{1, 2, \dots, d\}$ , there exist  $\lambda_{i1}, \lambda_{i2} > 0$  such that

$$L^T e_0 = \lambda_{i1} v_{i1} + \lambda_{i2} v_{i2}, \forall i \in \{1, 2, \dots, d\},$$

where  $v_{ij}$  are distinct extreme rays  $\{e_1, \dots, e_d; f_1, \dots, f_d\}$  of  $Q^*$ . Now, it is easy to see that for each  $i$ ,  $v_{i1}$  and  $v_{i2}$  must be “complementary,” that is, there exists  $j$  such that if  $v_{i1} = e_j$  then  $v_{i2} = f_j$  and vice versa. Furthermore,  $\lambda_{i1} = \lambda_{i2}$  should hold. Hence,  $L^T e_0$  is a positive multiple of  $e_0$ . Also, we have shown that there are  $\lambda > 0$  and a permutation  $\sigma : \{1, 2, \dots, d\} \rightarrow \{1, 2, \dots, d\}$  such that

$$\{L^T e_i, L^T f_i\} = \{\lambda e_{\sigma(i)}, \lambda f_{\sigma(i)}\}. \blacksquare \quad (60)$$

The automorphism  $L$  of  $Q$  is a very special type of linear transformation. Since  $e_i$ 's are  $d$  linearly independent vectors,  $L$  is unique up to the constant  $\lambda$ . In fact, (60) implies that  $L^T$  is of a very special form. For instance, suppose  $d = 3$ ,  $\lambda = 1$ , and  $L^T$  maps  $e_1 \mapsto f_3$ ,  $e_2 \mapsto e_1$ , and  $e_3 \mapsto f_2$ . Then,

$$L^T = [ e_0 \quad f_3 \quad e_1 \quad f_2 ] = \left[ \begin{array}{c|cccc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array} \right]. \quad (61)$$

The generalization of (61) for general  $d$  is obvious. It says that an automorphism  $L^T$  is totally unimodular and a composite of permutation, flipping and scaling.  $L^T$  in (61), for example, can be rewritten as,

$$L^T = \left[ \begin{array}{c|cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{c|cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \right]. \quad (62)$$