On the globally convexized filled function method for box constrained continuous global optimization *

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Abstract: In this paper we show that the unconstrained continuous global minimization problem can not be solved by any algorithm. So without loss of generality we consider the box constrained continuous global minimization problem. We present a new globally convexized filled function method for the problem. The definition of globally convexized filled function is adapted from that by Ge and Qin [7] for unconstrained continuous global minimization problems to the box constrained case. A new class of globally convexized filled functions are constructed. These functions contain only one easily determinable parameter. A randomized algorithm is designed to solve the box constrained continuous global minimization problem basing on these globally convexized filled functions. The asymptotic convergence of the algorithm is established. Preliminary numerical experiments show that the algorithm is practicable.

Key Words: Box constrained continuous global minimization problem, globally convexized filled function, asymptotic convergence, stopping rule.

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1 Introduction

Continuous global optimization is very important in many practical applications. Many approaches have been developed for finding global minimizers of continuous global optimization problems. Usually these approaches can be divided into two classes: stochastic global optimization methods and deterministic global optimization methods.

One popular deterministic global optimization method is to use gradient-type methods coupled with certain auxiliary functions to move from one local minimizer to another better one. This includes the filled function method [4-7, 9, 12-13, 15], the tunnelling method [3, 8, 14], and the bridging method [10]. These methods rely heavily on the successful construction of a filled function, a tunnelling function, or a bridging function to by-pass previously converged local minimizers. Another interesting method is to apply the cutting angle method to by-pass local minimizers. This involves solving subproblems of the minimax-type [1].

The filled function method was initially introduced by Ge and Qin [4-7], and has been extended in recent years [9, 12, 13, 15] to solve unconstrained global minimization problems. The idea behind the method is to construct a filled function $U(x)$ and by minimizing $U(x)$ to escape from a given local minimizer x_1^* of the original objective function $f(x)$. The constructed filled functions up to now can be classified into global concavized [4-6, 9, 13, 15] and globally convexized [7, 12] filled functions.

Suppose that x_1^* is the current minimizer of the objective function $f(x)$. The definition of the globally convexized filled function by Ge and Qin [7] is as follows.

Definition 1([7]) A continuous function $U(x)$ is called a globally convexized filled function of $f(x)$ at its minimizer x_1^* if $U(x)$ has the following properties:

(1) $U(x)$ has no stationary point in the region $S_1 = \{x \in \Omega : f(x) \geq f(x_1^*)\}$ except a prefixed point $x_0 \in S_1$, that is a minimizer of $U(x)$, where Ω is bounded closed domain such that it contains all global minimizers of $f(x)$, and the function

value of $f(x)$ on the boundary of Ω is greater than every minimum value of $f(x)$ in Ω.

(2) $U(x)$ does have a minimizer in the region $S_2 = \{x \in \Omega : f(x) < f(x_1^*)\}$.

(3) $U(x) \to +\infty$ as $\|x\| \to +\infty$, *i.e.*, $U(x)$ is globally convex.

A globally convexized filled function is considered better than a globally concavized filled function, since the latter one can not guarantee to have a minimizer but only a "minimizer" along the direction $x - x_1^*$ in the set $\{x \in \Omega : f(x) < f(x_1^*)\}$ and is not very efficient in practice [7].

Ge and Qin [7] constructed some nonsmooth globally convexized filled functions for the unconstrained global minimization problem. Lucidi and Piccalli [12] constructed several new smooth globally convexized filled functions for the same problem and the box constrained global minimization problem. But their globally convexzied filled functions contains two parameters, one of which must satisfied a condition that relies on the diameter of Ω and Lipschitz constant of the original objective function $f(x)$ to ensure the constructed functions to be globally convexzied filled functions. Moreover, the convergence property of the globally convexized filled function method is not proved up to now.

In section 2 of this paper, from complexity issue we prove that the unconstrained continuous global minimization problem can not be solved by any algorithm. Such property has not been demonstrated theoretically up to now. Hence without loss of generality we consider the box constrained continuous global minimization problem. In section 3 we give a definition of the globally convexized filled function for the problem, and analyze their properties. In section 4 we construct a new class of globally convexized filled functions, which contains only one parameter that is more easy to adjust. We present an algorithm to solve the box constrained continuous global minimization problem basing on the globally convexized filled functions in section 5. Also in section 5 we prove the asymptotic convergence with probability one of the algorithm, and present a stopping rule for it. In the last section of this paper the algorithm is tested on 10 standard testing problems to demonstrate its practicability.

2 Solvability of unconstrained continuous global minimization problem

In this section, we prove that there can not be any algorithm for continuous global optimization over unbounded domain.

Consider the following diophantine equation problem:

Problem 1 Given a polynomial function $P(x_1, \cdots, x_n)$ with integer coefficients, whether the following equation has a solution

$$\begin{cases} P(x_1, \cdots, x_n) = 0 \\ x_i : \text{integer}, i = 1 \cdots, n? \end{cases}$$

This is the $10th$ problem of Hilbert 23 problems. Matijasevic [11] gave a solution to this problem.

Theorem 1([11]) There exists no recursive function to decide whether the diophantine equation problem has a solution.

Since all recursive functions are computable, Theorem 1 shows that we can not construct an algorithm to find a solution of the diophantine equation problem or decide that the problem has no solution, *i.e.*, the diophantine equation problem is undecidable.

Now we consider the following continuous global minimization problem:

Given a continuously differentiable function with n variables $f(x_1, \cdots, x_n)$, find a global minimal solution of the following unconstrained continuous global minimization problem

$$\begin{cases} \min \quad f(x_1, \cdots, x_n) \\ s.t. \quad x_i \in R, i = 1, \cdots, n. \end{cases}$$

A decision problem to the above problem is

Problem 2 Given a constant B, whether there exists $x_i \in R$, $i = 1, 2 \cdots, n$ such that $f(x_1, \cdots, x_n) \leq B$?

For the above decision problem, we have

Theorem 2 There can not be any algorithm for the decision problem of the unconstrained continuous global minimization problem.

Proof. Let

$$Q^2(x_1, \cdots, x_n) = P^2(x_1, \cdots, x_n) + \sum_{i=1}^{n} \sin^2 \pi x_i.$$

Obviously, the equation of Problem 1 is equivalent to

$$Q^2(x_1, \cdots, x_n) \leq 0.$$

By Theorem 1, Problem 1 is undecidable. Hence to decide wether the above inequality has a solution is undecidable. Furthermore, since to decide wether the above inequality has a solution is a subproblem of Problem 2, we can conclude that Problem 2 is undecidable, *i.e.*, there can not be any algorithm for the decision problem of the unconstrained continuous global minimization problem.□

Hence there can not be any algorithm for continuous global optimization over unbounded domain.

3 Definition of globally convexized filled function for box constrained problem

Consider the following box constrained global minimization problem

$$(P) \quad \begin{cases} \min & f(x) \\ s.t. & x \in X, \end{cases}$$

where $f(x)$ is continuously differentiable on X, X is a bounded closed box in R^n, *i.e.*, $X = \{x | l_i \leq x_i \leq u_i, i = 1, 2, \cdots, n\}$ with $l_i, u_i \in R$.

Suppose that x_1^* is the current minimizer of problem (P). Before solving problem (P), we can get x_1^* using any local minimization method to minimize $f(x)$ on X.

Construct the following auxiliary global minimization problem

$$(AP) \quad \begin{cases} \min & U(x) \\ s.t. & x \in X, \end{cases}$$

where $U(x)$ is a globally convexized filled function of problem (P) defined as follows.

Definition 2 The function $U(x)$ is called a globally convexized filled function of problem (P) at its minimizer x_1^* if $U(x)$ is a continuous function and has the following properties:

(1) Problem (AP) has no stationary point in the region $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$ except a prefixed point $x_0 \in S_1$ that is a minimizer of $U(x)$.

(2) Problem (AP) does have a minimizer in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$.

It must be remarked that the globally convexized filled function of Definition 1 is for the unconstrained global minimization problem, but Definition 2 is for the box constrained global minimization problem (P). Moreover, in property 2 of Definition 2, a stationary point of problem (AP) is a point $y \in X$ which satisfies the following necessary conditions:

$$\begin{aligned} &\tfrac{\partial U(y)}{\partial x_i} \geq 0, \quad \text{if} \;\; y_i = l_i; \\ &\tfrac{\partial U(y)}{\partial x_i} \leq 0, \quad \text{if} \;\; y_i = u_i; \\ &\tfrac{\partial U(y)}{\partial x_i} = 0, \quad \text{if} \;\; l_i < y_i < u_i. \end{aligned}$$

And a minimizer of problem (AP) is also a stationary point of problem (AP) which satisfies the above conditions. Furthermore, the third property of Definition 1 is not included in Definition 2, since the feasible set X is compact, and we do not need the globally convex property again to ensure that there exists a compact set which contains the global minimizers of the function to be solved.

Similar to [7], we have one question that whether or not we can construct a sequence of globally convexized filled functions for problem (P) without a prefixed minimizer x_0. The following two theorems present a negative answer to this problem.

Theorem 3 Suppose that $U(x)$ is a globally convexized filled function of problem (P), and x_1^* is already a global minimizer of problem (P). Then x_0 is the unique minimizer of problem (P).

Proof. If x_1^* is already a global minimizer of problem (P), then the set S_2 defined in the second property of Definition 2 is empty. Thus $U(x)$ only have the prefixed minimizer x_0.$\square$

Theorem 4 Without a prefixed minimizer x_0 of problem (AP), it is impossible to find a globally convexized filled function $U(x)$ of problem (P) which satisfies all properties of Definition 2.

Proof. We prove this theorem by contradiction. Suppose that such a globally convexized filled function $U(x)$ exists. That is, Problem (AP) has no stationary point in S_1, but it does have a minimizer in S_2. When x_1^* is already a global minimizer of problem (P), the set S_2 is empty, and problem (AP) has no minimizer. But the continuity of $U(x)$ and the boundedness and closedness of the feasible domain X imply that problem (AP) has at least one minimizer. This contradicts the above conclusion about problem (AP).$\square$

4 New class of globally convexized filled functions

In this section, we construct a new class of globally convexized filled functions for problem (P). As before, suppose that x_1^* is the current minimizer of problem (P), which can be found by any local minimization method. Moreover, suppose that x^* is a global minimizer of problem (P), and $x_0 \in X$ is a prefixed point such that $f(x_0) \geq f(x_1^*)$.

Assumption 1 Assume that $u(x) \geq 0$ is a continuously differentiable function on X which has only one minimizer x_0 and for any $x \in X$, $x_0 - x$ is a descent direction of $u(x)$ at x.

Example 1 This kind of $u(x)$ could be a nonnegative continuously differentiable convex function with the unique minimizer x_0 on the domain X, or $u(x) = \eta(\|x - x_0\|^2)$, where $\eta(t)$ is a continuously differentiable univariate function such that $\eta(0) = 0$, $\eta'(t) > 0$ for $t \geq 0$.$\square$

Assumption 2 Assume that $v(x) \geq 0$ is a continuously differentiable function on X such that

$$\begin{cases} v(x) = 0, \quad \text{and} \quad \bigtriangledown v(x) = 0, \quad \text{for all} \quad x \in S_1, \\ v(x) > 0, \quad \text{for all} \quad x \in S_2. \end{cases}$$

Example 2 An example of this kind of $v(x)$ is as follows. Let $\gamma(t)$ be a continuously differentiable univariate function, $\gamma(0) = 0$, and $\gamma'(t) > 0$, for all $t > 0$. Then let $v(x) = \gamma((\min\{f(x) - f(x_1^*), 0\})^2)$. Obviously, this $v(x)$ is a continuously differentiable function, and by choosing $\gamma(t)$ appropriately we can make $v(x)$ as smooth as $f(x)$.$\square$

Let C be a constant such that $C \geq \max_{x \in X} u(x)$. Let $w(t)$ be a continuously differentiable univariate function defined on $[0, +\infty)$, $w(0) = 0$, and there exits $t_0 > 0$ such that $w(t_0) > C \geq \max_{x \in X} u(x)$. Then we construct the following auxiliary function

$$U(x) = u(x) - w(A \cdot v(x)),$$

where $u(x)$ and $v(x)$ satisfy Assumptions 1 and 2, A is a nonnegative parameter.

In the following we will prove that the above constructed auxiliary function is a globally convexized filled function of problem (P) if parameter A is large enough.

Theorem 5 The prefixed point x_0 is a local minimizer of problem (AP) if $f(x_0) > f(x_1^*)$, or if x_0 is a local minimizer of problem (P) with $f(x_0) \geq f(x_1^*)$.

Proof. If $f(x_0) > f(x_1^*)$, or x_0 is a local minimizer of problem (P) with $f(x_0) \geq f(x_1^*)$, then there exists a neighborhood $B(x_0)$ of x_0 such that $\forall x \in B(x_0) \cap X$, $f(x) \geq f(x_1^*)$. Thus $\forall x \in B(x_0) \cap X$, $v(x) = 0$, and

$$U(x) = u(x) - w(A \cdot v(x)) = u(x) - w(0) = u(x).$$

Since $u(x)$ has only one local minimizer x_0 on X, we have

$$U(x) = u(x) \geq u(x_0) = U(x_0), \quad \text{for all} \quad x \in B(x_0) \cap X,$$

i.e., x_0 is a local minimizer of problem (AP).$\square$

Theorem 6 Except the prefixed point x_0, problem (AP) has no stationary point in the region $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$.

Proof. The derivative of $U(x)$ is

$$\bigtriangledown U(x) = \bigtriangledown u(x) - Aw'(Av(x)) \cdot \bigtriangledown v(x).$$

For all $x \in S_1$, by the assumption on $v(x)$ we have $\bigtriangledown v(x) = 0$. Thus

$$\bigtriangledown U(x) = \bigtriangledown u(x), \quad \text{for all} \quad x \in S_1.$$

Hence if $x \in S_1$ and $x \neq x_0$, then $\bigtriangledown U(x) \neq 0$. Moreover for such x, let $d = x_0 - x$, by the assumption on the function $u(x)$ we have $d^T \bigtriangledown u(x) < 0$. Thus $d^T \bigtriangledown U(x) = d^T \bigtriangledown u(x) < 0$. So d is a descent direction of $U(x)$ at any point $x \in S_1$ with $x \neq x_0$. Hence Problem (AP) has no stationary point in the region $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$ except x_0.$\square$

By Theorems 5 and 6, we know that the constructed function $U(x)$ satisfies the first property of Definition 2 without any assumption on the parameter A.

Since $S_1 \cup \{x \in X : f(x) < f(x_1^*)\} = X$, Theorem 6 implies the following corollary.

Corollary 1 Any stationary point of problem (AP) except x_0 must be in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$.

However, if $A = 0$, then $U(x) = u(x)$, and $U(x)$ has a unique minimizer x_0 in X. Since $f(x_0) \geq f(x_1^*)$, that is, $x_0 \in S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$, $U(x)$ has no local minimizers in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$, and $U(x)$ is not a globally convexized filled function of problem (P). So we have one question that how large the parameter A would be such that $U(x)$ has a local minimizer in the region S_2. In fact, we have the following theorem.

Theorem 7 Let S^* be the set of global minimizers of problem (P). Suppose that x_1^* is not a global minimizer of problem (P), and suppose that parameter A satisfies that $A > \frac{t_0}{\max_{x\in S^*} v(x)}$. Then problem (AP) does have a minimizer in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$. Especially, all global minimizers of problem (AP) are in the region $\{x \in X : f(x) < f(x_1^*)\}$.

Proof. Since $U(x)$ is a continuous function on the bounded closed box X, it has a global minimizer in X.

For any $x \in S_1$, we have $f(x) \geq f(x_1^*)$, $v(x) = 0$ and

$$U(x) = u(x) - w(A \cdot v(x)) = u(x) - w(0) = u(x) \geq 0.$$

Since x_1^* is not a global minimizer of problem (P), *i.e.*, $f(x_1^*) > f(x)$ for all $x \in S^*$, it holds that $S_2 \neq \emptyset$, and $S^* \subset S_2$. Moreover, since $v(x) > 0$ for all $x \in S_2$ and S^* is a compact set, we can suppose that $\max_{x\in S^*} v(x)$ takes its maximal value at $x^* \in S^*$, *i.e.*, $v(x^*) = \max_{x\in S^*} v(x) > 0$. Then by $A > \frac{t_0}{\max_{x\in S^*} v(x)} = \frac{t_0}{v(x^*)}$, it holds that

$$Av(x^*) > \frac{t_0}{v(x^*)} \cdot v(x^*) = t_0.$$

Furthermore, since $v(x)$ is a continuous function on X, the value of $Av(x)$ on the domain S_2 can run continuously between 0 and $Av(x^*)$. Thus there exists $y \in S_2$ such that $Av(y) = t_0$, and

$$U(y) = u(y) - w(Av(y)) \leq C - w(t_0) < 0.$$

Hence the global minimal value of problem (AP) is less than 0, and all global minimizers of problem (AP) are in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$. Since all global minimizers of problem (AP) are local minimizers of problem (AP), the last result implies that problem (AP) does have a local minimizer in the region S_2 if $A > \frac{t_0}{\max_{x\in S^*} v(x)}$. $\square$

Thus by Theorems 5-7, if parameter A is large enough then $U(x)$ is a globally convexized filled functions of problem (P) at x_1^*.

Corollary 2 Suppose that x_1^* is not a global minimizer of problem (P), and suppose that parameter A satisfies that $A > \frac{t_0}{v(x^*)}$, where x^* is a global minimizer of problem (P). Then all global minimizers of problem (AP) are in the region $S_2 = \{x \in X : f(x) < f(x_1^*)\}$, and $U(x)$ is a globally convexized filled function of problem (P) at x_1^*.

Proof. Corollary 2 holds directly by Theorem 7, since $v(x^*) \le \max_{x\in S^*} v(x)$, and $\frac{t_0}{v(x^*)} \ge \frac{t_0}{\max_{x\in S^*} v(x)}$.$\square$

By the above discussions, we can construct many new globally convexized filled functions for problem (P). It is easy to verify that the following 5 simple functions are globally convexized filled functions of problem (P).

1. $U(x) = u(x) - Av(x)$.
2. $U(x) = u(x) - \ln(1 + Av(x))$.
3. $U(x) = u(x) - p \cdot \sin(Av(x))$, where p is a constant and $p > \max_{x\in X} u(x)$.
4. $U(x) = u(x) - p \cdot \mathrm{arctg}(Av(x))$, where p is a constant and $p > \frac{\max_{x\in X} u(x)}{\pi/2}$.
5. $U(x) = u(x) - p \cdot (1 - e^{-Av(x)})$, where p is a constant and $p > \max_{x\in X} u(x)$.

Example 3 Let $u(x)$ and $v(x)$ be as in Examples 1 and 2, and $U(x) = u(x) - Av(x)$ $= u(x) - A\gamma((\min\{f(x) - f(x_1^*), 0\})^2)$. If x_1^* is not a global minimizer of problem (P), then by Theorem 7, to ensure that $U(x)$ be a globally convexized filled function of problem (P) at x_1^*, the parameter A should satisfy that

$$A > \frac{C}{\gamma((\min\{f(x^*) - f(x_1^*), 0\})^2)} = \frac{C}{\gamma((f(x^*) - f(x_1^*))^2)},$$

where $C \ge \max_{x\in X} u(x)$, and x^* is a global minimizer of problem (P).

However, we know the value of $f(x_1^*)$, and generally we do not know the global minimal value or a global minimizer of problem (P), so it is difficult to find the lower bound of parameter A presented in Theorem 7 or the lower bound of parameter A presented in Corollary 2.

But for practical consideration, problem (P) might be considered solved if we can find an $x \in X$ such that $f(x) < f(x^*) + \epsilon$, where $f(x^*)$ is the global minimal value of problem (P), and $\epsilon > 0$ is a given desired optimality tolerance. So we consider the case that the current minimizer x_1^* satisfies that $f(x_1^*) \geq f(x^*) + \epsilon$. In the following two theorems we develop a lower bound of parameter A which depends only on the given desired optimality tolerance ϵ.

Theorem 8 Let $v(x)$ be as in Example 2, *i.e.*, $v(x) = \gamma((\min\{f(x) - f(x_1^*), 0\})^2)$, and $U(x) = u(x) - w(Av(x))$. Suppose that ϵ is a small positive constant, and $A > \frac{t_0}{\gamma(\epsilon^2)}$. Then for any current minimizer x_1^* of problem (P) such that $f(x_1^*) \geq f(x^*) + \epsilon$, where x^* is a global minimizer of problem (P), problem (AP) has a minimizer in the region $\{x \in X : f(x) < f(x_1^*)\}$, and all global minimizers of problem (AP) are in the region $\{x \in X : f(x) < f(x_1^*)\}$.

Proof. Since $\gamma(t)$ is a monotonically increasing function, and $f(x_1^*) - f(x^*) \geq \epsilon$, it holds that

$$\frac{t_0}{\max_{x \in S^*} v(x)} = \frac{t_0}{\max_{x \in S^*} \gamma((\min\{f(x) - f(x_1^*), 0\})^2)}$$
$$= \frac{t_0}{\gamma((f(x^*) - f(x_1^*))^2)} = \frac{t_0}{\gamma((f(x_1^*) - f(x^*))^2)} \leq \frac{t_0}{\gamma(\epsilon^2)}.$$

Hence, if $A > \frac{t_0}{\gamma(\epsilon^2)}$ then $A > \frac{t_0}{\max_{x \in S^*} v(x)}$, and by Theorem 7, the conclusions of this theorem hold. □

Especially, if $w(t)$ is a monotonically increasing function, we have the following theorem.

Theorem 9 Let $v(x)$ be as in Example 2, *i.e.*, $v(x) = \gamma((\min\{f(x) - f(x_1^*), 0\})^2)$, and $U(x) = u(x) - w(Av(x))$. Suppose that $w(t)$ is a monotonically increasing function, and $w'(t) \geq w_0 > 0$, $\gamma'(t) \geq \gamma_0 > 0$ for all $t \geq 0$. Moreover suppose that ϵ is a small positive constant, and $A > \frac{C}{w_0 \gamma_0 \epsilon^2}$. Then for any current minimizer x_1^* of problem (P) such that $f(x_1^*) \geq f(x^*) + \epsilon$, where x^* is a global minimizer of problem (P), problem (AP) does have a minimizer in the region $\{x \in X : f(x) < f(x_1^*)\}$, and all global minimizers of problem (AP) are in the region $\{x \in X : f(x) < f(x^*) + \epsilon\}$.

Proof. For any $x \in \{x \in X : f(x) \geq f(x^*) + \epsilon\}$, if $f(x) \geq f(x_1^*)$, then $v(x) = \gamma((\min\{f(x) - f(x_1^*), 0\})^2) = 0$, and $U(x) = u(x) - w(Av(x)) = u(x) - w(0) = u(x) - 0 = u(x) \geq 0$; if $f(x^*) + \epsilon \leq f(x) < f(x_1^*)$, then

$$f(x^*) - f(x) \leq -\epsilon,$$

$$0 < f(x_1^*) - f(x) \leq f(x_1^*) - f(x^*) - \epsilon,$$

$$(f(x_1^*) - f(x))^2 \leq (f(x_1^*) - f(x^*) - \epsilon)^2,$$

and since $w(t)$ and $\gamma(t)$ are differentiable and monotonically increasing, it holds that

$$\begin{aligned}
&U(x) = u(x) - w(Av(x)) \\
&= u(x) - w(A\gamma((f(x_1^*) - f(x))^2)) \\
&\geq u(x) - w(A\gamma((f(x_1^*) - f(x^*) - \epsilon)^2)) \\
&= u(x) - w(A\gamma((f(x_1^*) - f(x^*))^2 + \epsilon^2 - 2\epsilon(f(x_1^*) - f(x^*)))) \\
&= u(x) - w(A\gamma((f(x_1^*) - f(x^*))^2) + A\gamma'(\xi_0)(\epsilon^2 - 2\epsilon(f(x_1^*) - f(x^*)))) \\
&= u(x) - (w(A\gamma((f(x_1^*) - f(x^*))^2)) + w'(\eta_0) \cdot A\gamma'(\xi_0)(\epsilon^2 - 2\epsilon(f(x_1^*) - f(x^*)))) \\
&= (u(x^*) - w(A\gamma((f(x_1^*) - f(x^*))^2))) + ((u(x) - u(x^*)) - \\
&\quad Aw'(\eta_0)\gamma'(\xi_0)(\epsilon^2 - 2\epsilon(f(x_1^*) - f(x^*)))) \\
&= U(x^*) + ((u(x) - u(x^*)) - Aw'(\eta_0)\gamma'(\xi_0)(\epsilon^2 - 2\epsilon(f(x_1^*) - f(x^*)))) \\
&= U(x^*) + ((u(x) - u(x^*)) + Aw'(\eta_0)\gamma'(\xi_0)(2\epsilon(f(x_1^*) - f(x^*)) - \epsilon^2)) \\
&> U(x^*) + ((u(x) - u(x^*)) + Aw'(\eta_0)\gamma'(\xi_0)(2\epsilon^2 - \epsilon^2)) \\
&= U(x^*) + ((u(x) - u(x^*)) + Aw'(\eta_0)\gamma'(\xi_0)\epsilon^2) \\
&\geq U(x^*) + (Aw_0\gamma_0\epsilon^2 + (u(x) - u(x^*))) \\
&\geq U(x^*) + (Aw_0\gamma_0\epsilon^2 - C).
\end{aligned}$$

Since $A > \frac{C}{w_0\gamma_0\epsilon^2}$, the above inequalities imply that $U(x) > U(x^*)$. Moreover, for $U(x^*)$, by $A > \frac{C}{w_0\gamma_0\epsilon^2}$,

$$
\begin{aligned}
U(x^*) &= u(x^*) - w(A\gamma(v(x^*))) \\
&= u(x^*) - w(A\gamma((f(x_1^*) - f(x^*))^2)) \\
&\leq u(x^*) - w(A\gamma(\epsilon^2)) \\
&\leq C - (w(0) + w'(\eta_1) \cdot A\gamma(\epsilon^2)) \\
&= C - (0 + w'(\eta_1)A \cdot (\gamma(0) + \gamma'(\xi_1)\epsilon^2)) \\
&= C - Aw'(\eta_1)(0 + \gamma'(\xi_1)\epsilon^2) \\
&\leq C - Aw_0\gamma_0\epsilon^2 \\
&< 0.
\end{aligned}
$$

Hence for any $x \in \{x \in X : f(x) \geq f(x^*)+\epsilon\}$, $U(x) > U(x^*)$. So all global minimizers of problem (AP) are in the region $\{x \in X : f(x) < f(x^*) + \epsilon\}$, and problem (AP) has a minimizer in the region $\{x \in X : f(x) < f(x_1^*)\}$. □

According to the above discussions, given any desired tolerance $\epsilon > 0$, for $U(x) = u(x) - w(Av(x))$, where $v(x) = \gamma((\min\{f(x) - f(x_1^*), 0\})^2)$, by Theorems 5, 6 and 8, if $A > \frac{t_0}{\gamma(\epsilon^2)}$, then $U(x)$ is a globally convexized filled function of problem (P) at its current minimizer x_1^* which satisfies that $f(x_1^*) \geq f(x^*) + \epsilon$. Thus if we use a local minimization method to solve problem (AP) from any initial point on X, then by the properties of globally convexized filled functions, it is obvious that the minimization sequence converges either to the prefixed minimizer x_0 of $U(x)$ or to a point $x' \in X$ such that $f(x') < f(x_1^*)$. If we find such an x', then using a local minimization method to minimize $f(x)$ on X from initial point x', we can find a point $x'' \in X$ such that $f(x'') \leq f(x')$ which is better than x_1^*. This is the main idea of the algorithm presented in the next section to find an approximate global minimal solution of problem (P).

However, what will happen if $f(x_1^*) < f(x^*) + \epsilon$? In this case, by Theorem 6 the function $U(x)$ has no stationary point in the region $\{x \in X : f(x \geq f(x_1^*))\}$ except the prefixed minimizer x_0 of $U(x)$ on X, but it does not ensure that problem (AP) has a local minimizer in the region $\{x \in X : f(x) < f(x_1^*))\}$ since the parameter A might not be large enough. In this case, the region $\{x \in X : f(x) < f(x_1^*))\}$ is small, and using any local minimization method to minimize $U(x)$ on X the probability that the minimization sequence converges to a point $x' \in \{x \in X : f(x) < f(x_1^*)\}$ is small. But this is not a problem, since in this case $0 \leq f(x_1^*) - f(x^*) < \epsilon$, and we can stop minimizing $U(x)$ on X.

5 The algorithm and its asymptotic convergence

In this section, we present an algorithm to solve problem (AP) to find a better local minimizer of problem (P) than the current one x_1^*. The algorithm is described as follows.

Algorithm

Step 1. Select randomly a point $x \in X$, and start from which to minimize $f(x)$ on X to get a minimizer x_1^* of problem (P). Let A be a sufficiently large positive number, and N_L be a sufficiently large integer.

Step 2. Construct a globally convexized filled function $U(x)$ with a prefixed minimizer x_0 of problem (P) at x_1^*. Set $N = 0$.

Step 3. If $N \geq N_L$, then go to Step 6.

Step 4. Set $N = N + 1$. Draw an initial point uniformly on the boundary of the bounded box X, and start from which to minimize $U(x)$ on X using any local minimization method. Suppose that x' is an obtained local minimizer. If $x' = x_0$, then go to Step 3, otherwise go to Step 5.

Step 5. Minimize $f(x)$ on X from the initial point x', and obtain a local minimizer x_2^* of $f(x)$. Let $x_1^* = x_2^*$ and go to Step 2.

Step 6. Stop the algorithm, output x_1^* and $f(x_1^*)$ as an approximate global minimal solution and global minimal value of problem (P) respectively.

In Step 1 of the above algorithm, parameter A is set large enough. In fact, if one is satisfied with a solution x such that $f(x) < f(x^*) + \epsilon$, where ϵ is a sufficiently small positive number, then by Theorem 8, the value of A should be greater than $\frac{t_0}{\gamma(\epsilon^2)}$, or by Theorem 9, the value of A should be greater than $\frac{C}{w_0\gamma_0\epsilon^2}$, according to the constructed globally convexized filled function.

In Step 1 of the above algorithm, parameter N_L is the maximal number of minimizing problem (AP) between Steps 3 and 4.

In the following two subsections, we discuss the asymptotic convergence and present a stopping rule of the algorithm.

5.1 Asymptotic convergence

Without loss of generality, suppose that problem (P) has finite local minimal values, and f_L^* is the least local minimal value of problem (P) which is larger than the global minimal value $f(x^*)$ of problem (P). Since $f(x)$ is a continuous function, it is obvious that the Lebesgue measure of the set $S_L^* = \{x \in X : f(x) < f_L^*\}$ is $m(S_L^*) > 0$.

Suppose that the local minimization method used in Step 5 of the algorithm is strictly descent and can converge to a local minimizer of the problem being solved. Thus with an initial point $x' \in S_L^* = \{x \in X : f(x) < f_L^*\}$, the minimization sequence generated by the minimization of $f(x)$ on X will converge to a global minimizer of $f(x)$ on X.

In Step 4 of the algorithm, by the proof of Theorem 6, for any initial point x, $x_0 - x$ is a descent direction of $U(x)$ at x, where x_0 is the prefixed minimizer of $U(x)$. So we suppose that the local minimization method used in Step 4 of the algorithm to minimize $U(x)$ on X is just a line search along the direction $x_0 - x$, in which the initial step length is drawn uniformly in $[0, \|x_0 - x\|]$. Obviously, by the above discussions this line search will converge to the prefixed minimizer x_0 of $U(x)$ or will find a point in $\{x \in X : f(x) < f(x_1^*)\}$, and is this case the algorithm will go to Step 5 and find another local minimizer of $f(x)$ which is lower than x_1^*.

Let x_k be the k-th random point drawn uniformly on the boundary of the bounded closed box X, t_k be the k-th random number drawn uniformly in $[0, 1]$, and x_k^* be a local minimizer of $f(x)$ on X which is such that if the line search from x_k along $x_0 - x_k$ converges to x_0, then $x_k^* = x_{k-1}^*$, otherwise the algorithm goes to Step 5 and x_k^* is the local minimizer found at this step. Thus we have three sequences x_k, t_k, x_k^*, $k = 1, 2, \cdots$, and obviously, $f(x_1^*) \geq f(x_2^*) \geq \cdots \geq f(x_k^*) \geq \cdots \geq f(x^*)$.

Lemma 1 The probability that $x_k + t_k(x_0 - x_k) \notin S_L^*$ satisfies that

$$0 \leq P\{x_k + t_k(x_0 - x_k) \notin S_L^*\} < 1,$$

and

$$P\{x_k + t_k(x_0 - x_k) \notin S_L^*\} = P\{x_{k+1} + t_{k+1}(x_0 - x_{k+1}) \notin S_L^*\}, \quad k = 1, 2, \cdots.$$

Proof. In the algorithm x_k is a random point drawn uniformly on the boundary of the bounded closed box X, and t_k is a random number drawn uniformly on $[0, 1]$. Hence $x_k + t_k(x_0 - x_k)$ is a random point distributed on X. Since $m(S_L^*) > 0$, the probability that $x_k + t_k(x_0 - x_k) \in S_L^*$ satisfies that

$$0 < P\{x_k + t_k(x_0 - x_k) \in S_L^*\} \leq 1.$$

So

$$0 \leq P\{x_k + t_k(x_0 - x_k) \notin S_L^*\} < 1.$$

Furthermore, the random variables x_k, $k = 1, 2, \cdots$ are *i.i.d.*, and t_k, $k = 1, 2, \cdots$ are also *i.i.d.*, so $x_k + t_k(x_0 - x_k)$, $k = 1, 2, \cdots$ are *i.i.d.* too. Thus, $P\{x_k + t_k(x_0 - x_k) \notin S_L^*\} = P\{x_{k+1} + t_{k+1}(x_0 - x_{k+1}) \notin S_L^*\}, \quad k = 1, 2, \cdots.$ □

Lemma 2 Let $q = P\{x_k + t_k(x_0 - x_k) \notin S_L^*\}$. For any $\delta > 0$, the probability that $f(x_k^*) - f(x^*) \geq \delta$ satisfies that

$$P\{f(x_k^*) - f(x^*) \geq \delta\} \leq q^{k-1}, k = 2, 3, \cdots.$$

Proof. x_{k+1}^* is a random variable which is dependent on x_k and t_k, $k = 1, 2, \cdots$. Let E_k be the event that x_k^* is not a global minimizer of problem (P), $k = 1, 2 \cdots$. For any $\delta > 0$, the event that $f(x_k^*) - f(x^*) \geq \delta$ implies the event that x_k^* is not a global minimizer of problem (P). Hence $\{f(x_k^*) - f(x^*) \geq \delta\} \subseteq E_k$, $k = 1, 2, \cdots$.

Let F_k be the event that the line search from x_k along $x_0 - x_k$ converges to a point $x' \in \{x \in X : f(x) < f(x_k^*)\}$ and further the algorithm goes to Step 5 and finds a global minimizer of problem (P) at this step, *i.e.*, $f(x_{k+1}^*) = f(x^*)$. Thus $E_{k+1} = E_k \cap \bar{F}_k$, $k = 1, 2, \cdots$.

Note that during the line search from x_k along $x_0 - x_k$, if $x_k + t_k(x_0 - x_k) \in S_L^*$ happens, then the algorithm goes to Step 5 and finds a global minimizer of problem (P) at this step. Hence $\{x_k + t_k(x_0 - x_k) \in S_L^*\} \subseteq F_k$, and $\bar{F}_k \subseteq \{x_k + t_k(x_0 - x_k) \notin S_L^*\}$. So

$$P\{E_{k+1}\} = P\{E_k \cap \bar{F}_k\} \leq P\{E_k \cap \{x_k + t_k(x_0 - x_k) \notin S_L^*\}\}.$$

Furthermore, since x_k and t_k are drawn independently of E_k, and by Lemma 1 it holds that

$$P\{E_{k+1}\} \leq P\{E_k\} \cdot P\{x_k + t_k(x_0 - x_k) \notin S_L^*\}$$

$$\leq \cdots \leq P\{E_1\} \cdot \Pi_{l=1}^k P\{x_l + t_l(x_0 - x_l) \notin S_L^*\}$$

$$\leq \Pi_{l=1}^k P\{x_l + t_l(x_0 - x_l) \notin S_L^*\} = \Pi_{l=1}^k q = q^k.$$

Hence

$$P\{f(x_k^*) - f(x^*) \geq \delta\} \leq P\{E_k\} \leq q^{k-1}, k = 2, 3, \cdots,$$

and Lemma 2 holds.□

Theorem 10 Let $N_L = +\infty$. x_k^* converges to a global minimizer of problem (P) with probability 1, *i.e.*, $P\{\lim_{k\to\infty} f(x_k^*) = f(x^*)\} = 1$.

Proof. To prove Theorem 10 is equivalent to prove that

$$P\{\cap_{k=1}^\infty \cup_{l=k}^\infty \{f(x_l^*) - f(x^*) \geq \delta\}\} = 0, \forall \delta > 0.$$

By Lemmas 1 and 2, we have

$$P\{\cap_{k=1}^{\infty}\cup_{l=k}^{\infty}\{f(x_l^*)-f(x^*)\geq\delta\}\}$$
$$\leq \lim_{k\to\infty} P\{\cup_{l=k}^{\infty}\{f(x_l^*)-f(x^*)\geq\delta\}\}$$
$$\leq \lim_{k\to\infty}\sum_{l=k}^{\infty} P\{f(x_l^*)-f(x^*)\geq\delta\}$$
$$\leq \lim_{k\to\infty}\sum_{l=k}^{\infty} q^{l-1}$$
$$= \lim_{k\to\infty}\frac{q^{k-1}}{1-q}=0.$$

Hence Theorem 10 holds.□

5.2 Stopping rule

In the algorithm, after getting a current local minimizer x_1^* of problem (P), the algorithm draws an initial point uniformly on the boundary of the bounded box X, and start from which to minimize $U(x)$ on X. By the properties of the globally convexized filled functions, the local minimization will search the solution space X and converges to the prefixed minimizer x_0 of $U(x)$ or to a point in $\{x\in X: f(x)<f(x_1^*)\}$, and if converges to x_0, then the process is repeated again. In this process, parameter N is used to count the times of converging to x_0, and N_L is an upper bound of N.

Roughly speaking, the above process is a way of multistart local search. So we use a Bayesian stopping rule developed by Boender and Rinnooy Kan [2] for multistart local search method to estimate the value of N_L.

In the Bayesian stopping rule, assume that w is the number of different local minimizers of $U(x)$ on X having been discovered, and N is the number of minimizing $U(x)$ for finding these w local minimizers. Then the Bayesian estimate of the total number of local minimizers of $U(x)$ on X is $\frac{w(N-1)}{N-w-2}$. Hence if the value of $\frac{w(N-1)}{N-w-2}$ is very close to w, then one can probabilistically say that $U(x)$ has only w local minimizers on X, which have already been found.

We consider the globally convexized filled function $U(x)$ with parameter A large enough. If in N runs of minimizing $U(x)$ from initial points drawn uniformly on the boundary of the bounded box X, the minimization sequences all converge to the prefixed minimizer x_0 of $U(x)$, then $w = 1$, and the Bayesian estimate of the total number of local minimizers of $U(x)$ on X is approximately $\frac{N-1}{N-3}$. Thus if the value of $\frac{N-1}{N-3}$ is very close to 1, then $U(x)$ has about one local minimizer on X, and we can conclude approximately that a global minimizer of problem (P) has been found.

Note that the Bayesian estimate holds for N large enough, and if the dimension of a global minimization problem is higher, then generally speaking more local searches must be performed to find a global minimizer of the problem. We cope with this situation by considering the dimension of the global minimization problem. Suppose that n is the dimension of the problem to be solved. Then an appealing simple stopping rule is to terminate the algorithm if N satisfies that

$$\frac{N-1}{N-3} \leq 1 + \frac{1}{2n},$$

which leads to $N \geq 4n + 3$.

The above discussion means that after $4n + 3$ minimizations of $U(x)$ with initial points drawn uniformly on the boundary of X, if we can only find the prefixed local minimizer x_0 of $U(x)$, then we can conclude approximately that $U(x)$ has only one local minimizer on X, and a global minimizer of problem (P) has been found. So in the algorithm we set $N_L = 4n + 3$, and terminate the algorithm if $N \geq N_L = 4n + 3$.

6 Test results of the algorithm

Although the focus of this paper is more theoretical than computational, we still test our algorithm on 10 standard global minimization problems to have an initial feeling of the practical interest of the globally convexized filled function method.

We choose $u(x) = \sqrt{(x - x_0)^T (x - x_0)}$, and set the unique minimizer x_0 of $u(x)$ as

$x_0 = x_1^*$. We take $v(x) = (\min\{f(x) - f(x_1^*), 0\})^2$, and $A = 10^4$. We set $N_L = 4n+3$, where n is the dimension of the global minimization problem to be solved. We use the BFGS local minimization method with inexact line search to minimize both the objective function and the globally convexized filled function. The stopping criterion of the local minimization method is that the norm of the derivative is less than 10^{-5}. Each test problem has been solved ten times. The obtained results are reported in Tables 1 and 2.

Problem 1 Branin's function

$$f(x) = (x_2 - \frac{5.1x_1^2}{4\pi^2} + \frac{5x_1}{\pi} - 6)^2 + 10(1 - \frac{1}{8\pi})\cos x_1 + 10$$

has many local minimizers in the domain $-5 \leq x_1 \leq 10$, $0 \leq x_2 \leq 15$, but the global minima are $x^* = (3.141593, 2.275000)^T$, $x^* = (-3.141593, 12.275000)^T$. The global minimal value is $f(x^*) = 0.397667$.

Problem 2 The three hump camel function

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} - x_1x_2 + x_2^2$$

has three local minimizers in the domain $-3 \leq x_i \leq 3$, $i = 1, 2$, and the global minimizer is $x^* = (0, 0)^T$. The global minimal value is $f(x^*) = 0$.

Problem 3 The Treccani function

$$f(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2$$

has two local minimizers $x^* = (-2, 0)^T$ and $x^* = (0, 0)^T$ in the domain $-3 \leq x_1 \leq 3$, $i = 1, 2$. The global minimal value is $f(x^*) = 0$.

Problem 4 The six hump camel function

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{x_1^6}{3} + x_1x_2 - 4x_2^2 + 4x_2^4$$

has six local minimizers in the domain $-3 \leq x_1 \leq 3$, $-1.5 \leq x_2 \leq 1.5$, and two of them are global minimizers: $x^* = (-0.089842, 0.712656)^T$, $x^* = (0.089842, -0.712656)^T$. The global minimal value is $f(x^*) = -1.031628$.

Problem 5 The two dimensional Shubert function

$$f(x) = \{\sum_{i=1}^{5} i\cos[(i+1)x_1 + i]\}\{\sum_{i=1}^{5} i\cos[(i+1)x_2 + i]\}$$

has 760 local minimizers in the domain $-10 \leq x_i \leq 10$, $i = 1, 2$, and eighteen of them are global minimizers. The global minimal value is $f(x^*) = -186.730909$.

Problem 6 The two dimensional Shubert function

$$f(x) = \{\sum_{i=1}^{5} i\cos[(i+1)x_1+i]\}\{\sum_{i=1}^{5} i\cos[(i+1)x_2+i]\}+\frac{1}{2}[(x_1+0.80032)^2+(x_2+1.42513)^2]$$

has roughly the same behavior as the function presented in Problem 5 in the domain $-10 \leq x_i \leq 10$, $i = 1, 2$, but has a unique global minimizer $x^* = (-0.80032, -1.42513)^T$. The global minimal value is $f(x^*) = -186.730909$.

Problem 7 The two dimensional Shubert function

$$f(x) = \{\sum_{i=1}^{5} i\cos[(i+1)x_1+i]\}\{\sum_{i=1}^{5} i\cos[(i+1)x_2+i]\}+[(x_1+0.80032)^2+(x_2+1.42513)^2]$$

in the domain $-10 \leq x_i \leq 10$, $i = 1, 2$ has roughly the same behavior and the same global minimizer and global minimal value as the function presented in Problem 6, but with steeper slope around global minimizer.

Problem 8 Shekel's function

$$f(x) = -\sum_{i=1}^{m} \frac{1}{(x-a_i)^T(x-a_i)+c_i}$$

has m local minimizers in the domain $0 \leq x_i \leq 10$, $i = 1, 2, 3, 4$, but only one global minimizer, where m=5, or 7, or 10. The parameters are presented in the following table.

i	a_i				c_i
1	4	4	4	4	0.1
2	1	1	1	1	0.2
3	8	8	8	8	0.2
4	6	6	6	6	0.4
5	3	7	3	7	0.4
6	2	9	2	9	0.6
7	5	5	3	3	0.3
8	8	1	8	1	0.7
9	6	2	6	2	0.5
10	7	3.6	7	3.6	0.5

Problem 9 The Goldstein-Price function

$$f(x) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)]$$

$$\times[30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]$$

has 4 local minimizers in the domain $-2 \le x_i \le 2$, $i = 1, 2, 3, 4$, but only one global minimizer $x^* = (0, -1)^T$ with the global minimal value $f(x^*) = 3$.

The test results of the above 9 problems are presented in the following table.

Table 1. Numerical results of the algorithm on Problems 1-9.

Problem No.	n	NF	NG	NFF	NFG	LNF	LNG	LNFF	LNFG	Fail
1	2	23	11	1044	167	21	11	971	154	0
2	2	88	24	2147	292	45	13	153	23	0
3	2	125	33	4048	575	29	11	83	11	0
4	2	46	14	1280	178	43	14	544	81	0
5	2	116	31	1488	210	106	29	325	60	0
6	2	113	31	1333	224	109	30	264	57	0
7	2	133	34	1478	236	121	32	366	64	0
8(m=5)	4	75	32	2243	304	72	34	956	85	1
8(m=7)	4	50	19	1760	302	48	18	1501	211	2
8(m=10)	4	100	20	2095	293	98	18	1879	282	2
9	2	72	22	1170	17	69	22	514	75	0

Problem 10. The function

$$f(x) = \frac{1}{10}\{\sin^2(3\pi x_1) + \sum_{i=1}^{n-1}(x_i - 1)^2[1 + \sin^2(3\pi x_{i+1})] + (x_n - 1)^2[1 + \sin^2(2\pi x_n)]\}$$

has roughly 30^n local minimizers in the domain $-10 \leq x_i \leq 10$, $i = 1, \cdots, n$, but only one global minimizer $x^* = (1, 1, \cdots, 1)^T$ with the global minimal value $f^* = 0$.

The test results of Problem 10 are presented in the following table.

Table 2. Numerical results of the algorithm on Problem 10.

Problem No.	n	NF	NG	NFF	NFG	LNF	LNG	LNFF	LNFG	Fail
10	2	177	52	1095	200	175	46	189	41	0
	3	216	55	1381	233	216	55	244	52	0
	4	594	125	2488	396	474	98	754	120	0
	5	617	124	2956	436	534	117	499	78	0
	6	500	107	3705	514	551	125	700	98	0
	7	602	119	2918	401	739	153	1351	226	2
	8	1130	240	4123	737	887	183	1587	225	1
	9	1039	217	3231	485	1418	294	2905	431	2
	10	1092	229	5237	821	1013	259	1086	158	0
	15	1895	404	5578	762	1527	354	1143	2205	1
	20	2258	564	14159	1872	1718	525	1579	201	2
	25	3333	778	11030	1516	2720	584	2564	345	1

In the above two tables,

n= the dimension of the tested problem,

NF= the mean number of objective function evaluations to satisfy the stopping rule,

NG= the mean number of evaluations of the gradient of the objective function to satisfy the stopping rule,

NFF= the mean number of filled function evaluations to satisfy the stopping rule,

NFG= the mean number of evaluations of the gradient of the filled function to satisfy the stopping rule,

LNF= the mean number of objective function evaluations needed to get the global minimal value,

LNG= the mean number of evaluations of the gradient of the objective function needed to get the global minimal value,

LNFF= the mean number of filled function evaluations needed to get the global minimal value,

LNFG= the mean number of evaluations of the gradient of the filled function needed to get the global minimal value,

Fail= the number of times that the stopping rule is satisfied but no global minima are located.

All the mean values have been computed without considering the failures, and have been rounded to integers.

7 Conclusions

The globally convexized filled function methods proposed by [7] and [12] are for the unconstrained global minimization problem, and have no convergence properties. However, we proved in this paper that no algorithm can be designed to solve the unconstrained continuous global minimization problem. So we solved the box constrained continuous global minimization problem by designing a new globally convexized filled function method. The definition of the globally convexized filled function is for the box constrained problem, which was adapted from that by [7]. We constructed a new class of globally convexized filled functions. These functions contain only one parameter which is easier to determine, while the globally convexized filled functions developed by [7] and [12] contain two parameters which are more difficult to adjust. The algorithm proposed in this paper converges asymptotically with probability one to a global minimizer of the box constrained continuous global minimization problem, if we use a line search method with random initial step length to minimize the globally convexized filled function. Preliminary numerical experiments show that the algorithm is practicable.

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