Semidefinite descriptions of cones defining spectral mask constraints

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April 7, 2004

Abstract

We discuss in detail an additive structure of cones of trigonometric polynomials nonnegative on the union of finite number of pairwise disjoint segments of the unit circle. We derive new descriptions of these cones in terms of semidefinite constraints. We explain the results of M. Krein and A. Nudelman providing a description of dual cones in terms of the same type of constraints.

1 Introduction

Let $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m < 2\pi$ be given real numbers. Consider

$$\Delta_j = \{e^{i\theta} : \alpha_j < \theta < \beta_j\}, j = 1, 2 \ldots m, i^2 = -1,$$

$$E_k = \mathbf{S^1} \setminus \cup_{j=1}^{k} \Delta_j,$$

$k = 1, \ldots m, E_0 = \mathbf{S^1}$. Here $\mathbf{S^1} \subset \mathbf{C}$ is a unit circle. Introduce a vector space $T_n(\mathbf{C})$ of pseudo-polynomials w of the form:

$$w = \sum_{j=-n}^{n} \rho_j z^j, \rho_{-j} = \bar{\rho}_j,$$

$j = 0, 1, \ldots n$,(here $\bar{x}$ stands for the complex conjugate of $x \in \mathbf{C}$) and corresponding cones

$$K_{n,k} = \{w \in T_n(\mathbf{C}) : w(\xi) \geq 0, \xi \in E_k\}, \quad (1)$$

$k = 0, 1 \dots m$. We consider $W = T_n(\mathbf{C})$ as a real Euclidean vector space with the scalar product

$$\langle w^{(1)}, w^{(2)} \rangle_W = \sum_{j=-n}^{n} \rho_j^{(1)} \rho_j^{(2)}, \tag{2}$$

$$w^{(j)} = \sum_{k=-n}^{n} \rho_k^{(j)} z^k, j = 1, 2.$$

Given $a, b \in W$, and a real vector subspace $X \subset W$, consider the optimization problem of the form:

$$\langle a, w \rangle \to \min, \tag{3}$$

$$w \in (b + X) \cap K_{n,k} \tag{4}$$

and its dual

$$\langle b, u \rangle \to \min, \tag{5}$$

$$u \in (a + X^{\perp}) \cap K^*_{n.k}. \tag{6}$$

Here $X^{\perp}$ is the orthogonal complement of X in W with respect to $\langle , \rangle_W$ and $K^*_{n,k}$ is the cone dual to $K_{n,k}$:

$$K^*_{n,k} = \{u \in W : \langle u, w \rangle \geq 0, \forall w \in K_{n,k}\}.$$

Problems of the type (3),(4) and (5),(6) and some of their natural generalizations arise naturally in filter design (see e.g. [1], [4] and references therein),speech recognition ([3]) and many other applications. The constraints of the type $w \in K_{n,k}$ are usually called spectral mask constraints in signal processing literature. Several numerical schemes have been proposed recently for solving (3), (4) and (5),(6) using modern interior-point technique ([1],[4],[6]).

In the present paper we suggest to use some classical results from the moment theory (see [8]) for the reduction of (3), (4) to semidefinite programming problem. In particular, we provide the description of cones $K_{n,k}$ in terms of semidefinite constraints . The plan of the paper is as follows. In section 2 we give a complete proof of classical results describing $K_{n,k}$ as a sum of weighted "cones of squares" and corresponding results for $K^*_{n,k}$. In section 3 , based on these results, we outline semidefinite reductions mentioned above. In section 4 we consider corresponding results for even trigonometric polynomials and in Section 5 we compare our approach with others.

2 Description of $K_{n,k}$ and $K^*_{n,k}$

. Let

$$\omega_k = 2\cos(\frac{\beta_k - \alpha_k}{2}) - e^{i(\frac{\alpha_k + \beta_k}{2})} z^{-1} - e^{-i(\frac{\alpha_k + \beta_k}{2})} z \in T_1(\mathbf{C}). \tag{7}$$

Theorem 1

$$K_{n,k} = K_{n,k-1} + \omega_k K_{n-1,k-1},$$

$n \geq 1, k = 1, \dots m.$

Remark 1 *By definition* $K_{n,k} = 0$ *if* $n < 0$ *or* $k < 0$.

A brief sketch of the proof of this result is outlined in [8] and contains some inaccuracies. Here we provide a detailed proof of this important theorem. Applying Theorem 1 recursively, we immediately obtain the following.

Corollary 1

$$K_{n,k} = \sum_{\vec{\epsilon}\in\{0,1\}^k} \omega_1^{\epsilon_1}\omega_2^{\epsilon_2}\dots\omega_k^{\epsilon_k} K_{r(\vec{\epsilon}),0}, \tag{8}$$

$$\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_k), r(\vec{\epsilon}) = n - \sum_{j=1}^{k} \epsilon_j;$$

$$K^*_{n,k} = \cap_{\vec{\epsilon}\in\{0,1\}^k} (\omega_1^{\epsilon_1}\omega_2^{\epsilon_2}\dots\omega_k^{\epsilon_k} K_{r(\vec{\epsilon}),0})^*. \tag{9}$$

The remaining part of this section is devoted to the proof of Theorem 1.

Lemma 1 *Let*

$$v(z) = \prod_{j=1}^{l}(z - e^{i\xi_j}), i^2 = -1, \xi_j \in \mathbf{R}, j = 1, \dots l.$$

Then

$$v(e^{i\theta}) = 2^l \exp(\frac{i}{2}(l\pi + l\theta + \sum_{j=1}^{l}\xi_j)) \prod_{j=1}^{l} \sin(\frac{\theta - \xi_j}{2}), \theta \in \mathbf{R}. \tag{10}$$

Proof Let $\alpha_j = \theta - \xi_j, j = 1, \dots l$. We have:

$$\delta_j = e^{i\theta} - e^{i\xi_j} = e^{i\xi_j}[e^{i\alpha_j} - 1] =$$
$$e^{i\xi_j}((\cos\alpha_j - 1) + i\sin\alpha_j).$$

Using elementary trigonometric formulas, we obtain:

$$(\cos\alpha_j - 1) + i\sin\alpha_j = -2\sin^2(\frac{\alpha_j}{2}) + 2i\sin(\frac{\alpha_j}{2})\cos(\frac{\alpha_j}{2}) =$$

$$2\sin(\frac{\alpha_j}{2})(\cos(\frac{\pi + \alpha_j}{2}) + i\sin(\frac{\pi + \alpha_j}{2})) = 2\sin(\frac{\alpha_j}{2})e^{i(\frac{\pi}{2} + \frac{\alpha_j}{2})}.$$

Hence,

$$\delta_j = 2\sin(\frac{\alpha_j}{2})e^{\frac{i}{2}(\pi + \theta + \xi_j)}.$$

The result follows.

Corollary 2 *Let*

$$w(z) = (-1)^l e^{-\frac{i}{2}(\sum_{j=1}^{2l} \xi_j)} \frac{\prod_{j=1}^{2l}(z - e^{i\xi_j})}{z^l}.$$

Then

$$w(e^{i\theta}) = 2^{2l} \prod_{j=1}^{2l} \sin(\frac{\theta - \xi_j}{2}), \tag{11}$$

$$w(\frac{1}{\bar{z}}) = \overline{w(z)}, z \in \mathbf{C} \setminus \{0\}. \tag{12}$$

Proof Immediately follows from Lemma 1.

Lemma 2 *Let* $\alpha, \beta \in \mathbf{R}, \gamma = e^{-i(\frac{\alpha+\beta}{2})}$,

$$w_{\alpha,\beta}(z) = \frac{1}{4}(2\cos(\frac{\beta - \alpha}{2}) - \gamma z - \gamma^{-1} z^{-1}).$$

Then

$$w_{\alpha,\beta}(e^{i\theta}) = \sin(\frac{\theta - \alpha}{2}) \sin(\frac{\theta - \beta}{2}), \theta \in \mathbf{R}. \tag{13}$$

Proof. A direct computation.

Lemma 3 *Let*

$$w(z) = \sum_{k=-n}^{n} \rho_k z^k.$$

Then

$$\overline{w(z)} = w(\frac{1}{\bar{z}}), z \in \mathbf{C} \setminus \{0\},$$

if and only if $\bar{\rho}_k = \rho_{-k}, k = 0, 1, \ldots n$.

Proof A direct computation.

Corollary 3 *Given* $w \in T_n(\mathbf{C})$, *we have:*

$$w(e^{i\theta}) \in \mathbf{R}, \forall \theta \in \mathbf{R}. \tag{14}$$

Proof of the Theorem 1
Let $w \in K_{n,k}$,

$$w(z) = \sum_{j=-n}^{n} \rho_j z^j, \rho_n \neq 0.$$

By Lemma 3

$$w(\frac{1}{\bar{z}}) = \overline{w(z)}, z \in \mathbf{C} \setminus \{0\}.$$

We have:

$$w(z) = \frac{P(z)}{z^n}, \tag{15}$$

where $P(z)$ is a polynomial of degree $2n$. If z^* is a root of P, then $1/\bar{z^*}$ is also a root by (15). Observe that $z^* \neq 0$, since $P(0) = \rho_{-n} = \bar{\rho}_n \neq 0$. Let $z_1, \dots z_k$ be all roots of P (counting multiplicities) such that $| z_j |< 1$. Then

$$\frac{1}{\bar{z}_1}, \dots \frac{1}{\bar{z}_n}$$

are all roots of P with absolute value greater than one. If

$$\zeta_1, \dots \zeta_m$$

are all roots of P on the unit circle, then:

$$w(z) = \rho_n z^{-n} \prod_{j=1}^{k} (z - z_j)(z - \frac{1}{\bar{z_j}}) \prod_{s=1}^{m} (z - \zeta_s). \tag{16}$$

Hence, $2k + m = 2n$, which implies that $m = 2(n-k)$ is even. We can assume that $\zeta_1, \dots \zeta_l$ are all roots of P in $\overline{\Delta_k} \subset \mathbf{S^1}$ counting multiplicities ($l \leq m$). Thus $\zeta_j = e^{i\xi_j}, \alpha_k \leq \xi_j \leq \beta_k, j = 1, \dots m$. By Lemma 1

$$w(e^{i\theta}) = \alpha(\theta) \prod_{j=1}^{l} \sin(\frac{\theta - \xi_j}{2}), 0 \leq \theta \leq 2\pi,$$

where α is a continuous real-valued function such that $\alpha(\theta) \neq 0$ for $\alpha_k \leq \theta \leq \beta_k$. A real-valued function $\theta \to w(e^{i\theta})$ changes the sign an even number of times on the interval $[\alpha_k, \beta_k]$, since it is nonnegative on the intervals adjacent to $[\alpha_k, \beta_k]$ from the left and from the right (we count 0 as an even number). But $w(e^{i\theta})$ may change the sign only at points $\xi_j, j = 1, \dots, l$. It changes the sign at ξ_j if and only if ξ_j has an odd multiplicity. This means that l is an even number. Let $l = 2p$. Using (16), we can rewrite w in the form:

$$w(z) = w_1(z) w_2(z),$$

where

$$w_1(z) = \delta \frac{\prod_{j=1}^{2p} (z - \zeta_j)}{z^p},$$

$$w_2(z) = \frac{\rho_n}{\delta} z^{p-n} \prod_{j=2p+1}^{2(n-k)} (z - \zeta_j) \prod_{s=1}^{k} (z - z_s)(z - \frac{1}{\bar{z}_s}),$$

$$\delta = (-1)^p e^{-\frac{i}{2}(\sum_{j=1}^{2p} \xi_j)}.$$

By (11)

$$w_1(e^{i\theta}) = 2^{2p} \prod_{j=1}^{2p} \sin(\frac{\theta - \xi_j}{2}). \tag{17}$$

By (12)

$$w_1(\frac{1}{\bar{z}}) = \overline{w_1(z)}. \tag{18}$$

Since the same holds true for w, we conclude that

$$w_2(\frac{1}{\bar{z}}) = \overline{w_2(z)}, z \in \mathbf{C} \setminus \{0\}. \tag{19}$$

We claim that

$$w_2(e^{i\theta}) \geq 0, e^{i\theta} \in E_{k-1}.$$

Indeed, $E_{k-1} = E_k \cup \Delta_k, E_k \cap \Delta_k = \emptyset$. Consider, first, the case $e^{i\theta} \in E_k$. We need to consider two cases:a) $0 < \theta < \alpha_k$ and b) $2\pi > \theta > \beta_k$. In the case a) $-2\pi < \theta - \xi_j < 0, j = 1, \ldots, 2p$, i.e. $\sin(\frac{\theta-\xi_j}{2}) < 0, j = 1, \ldots 2p$. In the case b) $2\pi > \theta - \xi_j > 0, j = 1, \ldots 2p$, i.e. $\sin(\frac{\theta-\xi_j}{2}) > 0, j = 1, \ldots 2p$. We conclude that in both cases

$$\prod_{j=1}^{2p} \sin(\frac{\theta - \xi_j}{2}) > 0.$$

By (17) $w_1(e^{i\theta}) > 0$. Since $w(e^{i\theta}) \geq 0, e^{i\theta} \in E_k$, and $w(e^{i\theta}) = w_1(e^{i\theta})w_2(e^{i\theta})$, we conclude that $w_2(e^{i\theta}) \geq 0$ for $e^{i\theta} \in E_k \setminus \{e^{i\alpha_k}, e^{i\beta_k}\}$. It remains to consider the case $\alpha_k \leq \theta \leq \beta_k$. Consider , first, the case where $\theta \neq \xi_j, j = 1, 2, \ldots, 2p$. If $w(e^{i\theta}) > 0$, then the interval $(\theta, \beta_k]$ should contain an even number of ξ_j among $\xi_1, \ldots \xi_{2p}$ counting multiplicities, since $w(e^{i\theta}) \geq 0$ on the interval adjacent to β_k from the right. But then

$$\prod_{j=1}^{2p} \sin(\frac{\theta - \xi_j}{2}) > 0$$

and hence $w_2(e^{i\theta}) > 0$. Similarly, if $w(e^{i\theta}) < 0$, then the interval $(\theta, \beta_k]$ contains an odd number of ξ_j among $\xi_1, \ldots \xi_{2p}$. But then $[\alpha_k, \theta)$ also contains an odd number of ξ_j. Hence,

$$\prod_{j=1}^{2p} \sin(\frac{\theta - \xi_j}{2}) < 0$$

and consequently $w_2(e^{i\theta}) > 0$. Finally, using the continuity of w_2 (or the fact that it does not take the value zero on the interval $[\alpha_k, \beta_k]$), we conclude that $w_2(e^{i\theta}) \geq 0$ on $[\alpha_k, \beta_k]$. Thus, $w_2(e^{i\theta}) \geq 0$ on E_{k-1}. Observe now that

$$\sin(\frac{\theta - \xi_j}{2}) = A_i \sin(\frac{\theta - \alpha_k}{2}) + B_i \sin(\frac{\theta - \beta_k}{2}),$$

$$A_j = \frac{\sin(\frac{\beta_k - \xi_i}{2})}{\sin(\frac{\beta_k - \alpha_k}{2})} \geq 0,$$

$$B_j = \frac{\sin(\frac{\xi_i - \alpha_k}{2})}{\sin(\frac{\beta_k - \alpha_k}{2})} \geq 0, j = 1, 2, \ldots 2p,$$

which can be easily verified by a direct computation. Hence,

$$\prod_{j=1}^{2p} \sin(\frac{\theta - \xi_j}{2}) = \sum_{\vec{\epsilon} \in \{0,1\}^{2p}} C(\vec{\epsilon})\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon}),$$

where $C(\vec{\epsilon}) \geq 0$ and

$$\gamma_j(\vec{\epsilon}) = \sin(\frac{\theta - \alpha_k}{2}),$$

if $\epsilon_j = 1$ and

$$\gamma_j(\vec{\epsilon}) = \sin(\frac{\theta - \beta_k}{2}),$$

if $\epsilon_j = 0$. In other words, each term $\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon})$ has the form:

$$\sin(\frac{\theta - \alpha_k}{2})^{s_1} \sin(\frac{\theta - \beta_k}{2})^{s_2}$$

with $s_1 + s_2 = 2p$. Since $s_1 + s_2$ is even the only two cases are possible: a) both s_1 and s_2 are even; b) both s_1 and s_2 are odd. It leads to the partition of $\{0, 1\}^{2p}$ into two disjoint subsets Γ_1 and Γ_2 according to whether a) or b) holds. We then have:

$$w(e^{i\theta}) = w_2(e^{i\theta})(2^{2p} \sum_{\vec{\epsilon} \in \Gamma_1} C(\vec{\epsilon}\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon})+ \tag{20}$$

$$+\sin(\frac{\theta - \alpha_k}{2})\sin(\frac{\theta - \beta_k}{2})w_2(e^{i\theta})(\sum_{\vec{\epsilon} \in \Gamma_2} \frac{2^{2p}C(\vec{\epsilon})\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon})}{\sin(\frac{\theta - \alpha_k}{2})\sin(\frac{\theta - \beta_k}{2})}), \theta \in [0, 2\pi].$$

By Lemma 2

$$\sin(\frac{\theta - \alpha_k}{2})\sin(\frac{\theta - \beta_k}{2}) = \frac{1}{4}\omega_k(e^{i\theta}), 0 \leq \theta \leq 2\pi,$$

(see (13)). By definition of Γ_1 and Γ_2 we see that

$$\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon}), \vec{\epsilon} \in \Gamma_1$$

and

$$\frac{\gamma_1(\vec{\epsilon}) \ldots \gamma_{2p}(\vec{\epsilon})}{\sin(\frac{\theta - \alpha_k}{2})\sin(\frac{\theta - \beta_k}{2})}, \vec{\epsilon} \in \Gamma_2$$

have the form

$$\sin(\frac{\theta - \alpha_k}{2})^{s_1} \sin(\frac{\theta - \beta_k}{2})^{s_2}$$

with s_1, s_2 to be even.In particular, all these terms are nonnegative for $\theta \in \mathbf{R}$. Applying Lemma 2 again we see that each of these terms has the form $v(e^{i\theta})$ with $v(\frac{1}{\bar{z}}) = \overline{v(z)}, z \in \mathbf{C} \setminus \{0\}$, where v belongs to an appropriate space $T(\mathbf{C})$. We thus conclude that

$$w(e^{i\theta}) = w^{(1)}(e^{i\theta}) + \omega_k(e^{i\theta})w^{(2)}(e^{i\theta}),$$

where $w^{(1)} \in K_{n,k-1}, w^{(2)} \in K_{n-1,k-1}$. But then

$$w(z) = w^{(1)} + \omega_k(z)w^{(2)}(z), z \in \mathbf{C} \setminus \{0\}, \tag{21}$$

since all functions in (21) are holomorphic in a connected open set $\mathbf{C} \setminus \{0\}$.

3 Semidefinite reductions

Let $(V_j, \langle,\rangle_{V_j}), j = 1, \ldots, l$, be Euclidean vector spaces and

$$B_j : V_j \times V_j \to W, j = 1, \ldots, l,$$

are $\mathbf{R}-$bilinear symmetric maps. The maps B_i induce linear maps

$$\tilde{\Lambda}_j : V_j \otimes V_j \to W, j = 1, \ldots l.$$

See , e.g. ([5]) for details. If we restrict $\tilde{\Lambda}_j$ to the vector subspaces $S(V_j)$ of symmetric tensors in $V_j \otimes V_j$, we obtain linear maps

$$\Lambda_j : S(V_j) \to W.$$

We can identify $S(V_j)$ with the vector space of symmetric (with respect to $\langle,\rangle_{V_j}$) linear maps from V_j into itself. In this way, we obtain a linear map

$$\Lambda : S(V_1) \times \ldots \times S(V_l) \to W,$$

$$\Lambda(Y^{(1)}, \ldots Y^{(l)}) = \sum_{j=1}^{l} \Lambda_j(Y^{(j)}). \tag{22}$$

For details of this construction see ([5]). Let K_j be the convex cone in W generated by vectors of the form $B_j(v^{(j)}, v^{(j)}), v^{(j)} \in V_j, j = 1, \ldots, l$ and

$$K = \sum_{j=1}^{l} K_j.$$

If we denote by $S_+(V_j)$ the cone of nonnegative definite symmetric maps from V_j to V_j (with respect to $\langle,\rangle_{V_i}$), we obtain.

Proposition 1

$$K = \Lambda(S_+(V_1) \times S_+(V_2) \times \dots S_+(V_l))$$

.

Introduce a dual map $M_j : W \to S(V_j)$ to the map Λ_j as follows:

$$Tr(M_j(w)Y^{(j)}) = \langle w, \Lambda_j(Y^{(j)})\rangle_W, \tag{23}$$

for any $w \in W, Y^{(j)} \in S(V_j), j = 1, \dots, l$. The map M_j is defined uniquely by conditions (23).

Proposition 2 *We have:*

$$K_j^* = M_j^{-1}(S_+(V_j)), j = 1, \dots, l,$$

$$K^* = \cap_{j=1}^l K_j^*.$$

For proofs of Propositions 1,2 see e.g. ([5]). We see that under very mild assumptions the function

$$F(w) = -\sum_{j=1}^{l} \ln\det(M_j(w)), w \in int(K^*), \tag{24}$$

is the so-called self-concordant homogeneous barrier for the cone K^* (see e.g. [9]). Consider an optimization problem of the form:

$$\langle a, w\rangle_W \to \min, \tag{25}$$

$$w \in (b + X) \cap K \tag{26}$$

and its dual

$$\langle b, u\rangle_W \to \min, \tag{27}$$

$$u \in (a + X^{\perp}) \cap K^*. \tag{28}$$

Compare with (3)-(6). It is then obvious that if $(Y_1^*, \dots, Y_l^*)$ is an optimal solution to the semidefinite programming problem

$$\sum_{j=1}^{l} Tr(M_j(a)Y_j) \to \min, \tag{29}$$

$$(Y_1, \dots Y_l) \in [S_+(V_1) \times S_+(V_2) \times \dots S_+(V_l)] \cap \Lambda^{-1}(b + X), \tag{30}$$

then $\Lambda(Y_1^*, \dots Y_l^*)$ is an optimal solution to (25),(26). Furthermore, an explicit form for the self-concordant barrier (24), enables one to apply polynomial interior-point algorithms to the dual problem (27),(28) (see e.g. [6]). Our goal here is to implement the construction outlined above for problems (3),(4) and (5),(6) using Theorem 1.

Let $U_n(\mathbf{C})$ be the vector space of polynomials with complex coefficients of degree less or equal than n. If

$$v^{(k)} = \sum_{j=0}^{n} \xi_j^{(k)} z^j \in U_n(\mathbf{C}), k = 1, 2,$$

then by definition:

$$\langle v^{(1)}, v^{(2)} \rangle_{U_n(\mathbf{C})} = \Re(\sum_{j=0}^{n} \xi_j^{(1)} \xi_j^{(2)}). \tag{31}$$

We consider $U_n(\mathbf{C})$as a (real) Euclidean vector space of dimension $2(n+1)$. Let

$$B : U_n(\mathbf{C}) \times U_n(\mathbf{C}) \to T_n(\mathbf{C}),$$

$$B(v_1, v_2) = \frac{v_1(z)\overline{v_2(\frac{1}{\bar{z}})} + v_2(z)\overline{v_1(\frac{1}{\bar{z}})}}{2}. \tag{32}$$

Let us calculate the maps Λ and M for this situation. Observe that the vectors $1, z, \dots z^n, i, iz, \dots, iz^n (i^2 = -1)$ form an orthonormal basis in $U_n(\mathbf{C})$ with respect to $\langle , \rangle_{U_n(\mathbf{C})}$. Denote z^j by f_j and iz^j by $g_j, j = 0, 1, \dots n$. Using (32), we immediately obtain:

$$\Lambda(f_s \otimes f_t) = B(f_s, f_t) = \frac{z^{s-t} + z^{t-s}}{2}, \tag{33}$$

$$\Lambda(g_s \otimes g_t) = \frac{z^{s-t} + z^{t-s}}{2}, \tag{34}$$

$$\Lambda(f_s \otimes g_t) = -i\frac{(z^{s-t} - z^{t-s})}{2}, \tag{35}$$

$$\Lambda(g_s \otimes f_t) = i\frac{(z^{s-t} - z^{t-s})}{2}, \tag{36}$$

$s, t = 0, 1, \dots n$. Let

$$Y = \sum_{s,t=0}^{n} (Y_{st}^{ff} f_s \otimes f_t + Y_{st}^{gg} g_s \otimes g_t + Y_{st}^{fg} f_s \otimes g_t + Y_{st}^{gf} g_s \otimes f_t) \tag{37}$$

is in $S(U_n(\mathbf{C}))$.

Observe that Y in the form (37) belongs to $S(U_n(\mathbf{C}))$ if and only if

$$Y_{st}^{ff} = Y_{ts}^{ff}, Y_{st}^{gg} = Y_{ts}^{gg}, Y_{st}^{fg} = Y_{ts}^{gf}, \tag{38}$$

, $s, t = 0, 1, \dots n$. Using (33)-(36), we immediately obtain:

$$\Lambda(Y) = \sum_{k=-n}^{n} (\sum_{s-t=k} Z_{st}) z^{s-t}, \tag{39}$$

where

$$Z_{st} = Y^{ff}_{st} + Y^{gg}_{st} + i(Y^{gf}_{st} - Y^{fg}_{st}), \tag{40}$$

$s, t = 0, 1, \ldots n, i^2 = -1$. If $Z = (Z_{st})$, we see that $Z^* = Z$(here Z^* stands for the conjugate transpose of Z). Let

$$S_{\mathbf{C}}(U_n(\mathbf{C})) = \{Y \in S(U_n(\mathbf{C})) : Y^{ff}_{st} = Y^{gg}_{st}, Y^{fg}_{st} = -Y^{fg}_{ts}\}.$$

We see from (40) that

$$\Lambda(S_{\mathbf{C}}(U_n(\mathbf{C})) = \Lambda(S(U_n(\mathbf{C}))).$$

The following Lemma enables us to identify the cone of nonnegative definite symmetric matrices from $S_{\mathbf{C}}(U_n(\mathbf{C}))$ with the cone of $(n+1)\times(n+1)$ Hermitian nonnegative definite matrices over $\mathbf{C}$.

Lemma 4 *The matrix*

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

with $A = A^T, B = -B^T, A, B \in Mat_n(\mathbf{R})$ is nonnegative definite if and only if the matrix $A + iB$ is Hermitian nonnegative definite matrix over $\mathbf{C}$.

Proof
A direct computation.

Remark 2 *With this identification the map Λ takes the form (see (39)):*

$$\Lambda(Z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^n \end{bmatrix}^T Z \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix}.$$

Let us now calculate the map M. Let

$$w = \sum_{j=-n}^{n} \rho_j z^j \in T_n(\mathbf{C}) \tag{41}$$

and

$$M(w) = \sum_{s,t=0}^{n} m^{ff}_{st} f_s \otimes f_t + m^{gg}_{st} g_s \otimes g_t + m^{fg}_{st} f_s \otimes g_t + m^{gf}_{st} g_s \otimes f_t).$$

Using (23),(33)-(36), we immediately obtain:

$$m^{ff}_{st} = Tr(M(w) f_s \otimes f_t) = \langle w, \Lambda(f_s \otimes f_t)\rangle_W = \langle w, \frac{z^{s-t} + z^{t-s}}{2}\rangle_W = \Re\rho_{s-t} \tag{42}$$

and similarly

$$m_{st}^{fg} = \Re\rho_{s-t}, m_{st}^{gf} = \Im\rho_{s-t}, m_{st}^{fg} = -\Im\rho_{s-t}. \tag{43}$$

This enables us to identify $M(w)$ with the Hermitian Toeplitz matrix $(\rho_{s-t}), s, t = 0, 1, \dots n$, over $\mathbf{C}$. Moreover, $M(w) \in S_+(U_n(\mathbf{C}))$ if and only if the corresponding Toeplitz matrix (ρ_{s-t}) is a Hermitian Toeplitz nonnegative definite matrix over $\mathbf{C}$.

According to the classical Fejer-Riesz theorem (see e.g. [7]), the "cone of squares" K corresponding to the bilinear map B, coincides with the cone

$$\{w \in T_n(\mathbf{C}) : w(e^{i\theta}) \geq 0, \forall\theta \in [0, 2\pi]\}.$$

Let now

$$\omega = \sum_{j=-m}^{m} \omega^{(j)} z^j. \tag{44}$$

We assume that $\omega^{(j)}m \neq 0$ and $\omega \in T_m(\mathbf{C})$. Consider a linear map $L_\omega : T_n(\mathbf{C}) \rightarrow T_{n+m}(\mathbf{C}), L_\omega(w) = \omega w$ and a bilinear form

$$B_\omega : U_n(\mathbf{C}) \times U_n(\mathbf{C}) \rightarrow T_{n+m}(\mathbf{C}),$$

$$B_\omega(v_1, v_2) = \omega B(v_1, v_2),$$

where B is given by (32). In other words , $B_\omega = L_\omega \circ B$. It is obvious that the corresponding map Λ_ω has the form:

$$\Lambda_\omega = L_\omega \circ \Lambda. \tag{45}$$

Let us now calculate M_ω. By (20):

$$Tr(M_\omega(w)Y) = \langle w, L_\omega \Lambda(Y)\rangle_W =$$

$$\langle L_\omega^T(w), \Lambda(Y)\rangle_W = \langle M(L_\omega^T(w)), Y\rangle_W.$$

Here $L_\omega^T : T_{n+m}(\mathbf{C}) \rightarrow T_n(\mathbf{C})$ is the map dual to L_ω. Thus, we obtain:

$$M_\omega(w) = M(L_\omega^T(w)). \tag{46}$$

It remains to evaluate L_ω^T. Let

$$w = \sum_{j=-(n+m)}^{n+m} \rho_j z^j. \tag{47}$$

Lemma 5 *Let w, ω be given by (44) and (47), respectively.Introduce the sequence $\omega(\rho), j = 0, \pm 1, \dots, \pm n$ by the following formula:*

$$\omega(\rho)_j = \sum_{s=-m}^{m} \omega^{(s)} \rho_{s+j}. \tag{48}$$

Then

$$L^T_\omega(w) = \sum_{j=-n}^{n} \omega(\rho)_j z^j. \qquad (49)$$

Proof .

It suffices to check that

$$\langle L^T_\omega(w), w_1\rangle_{T_n(\mathbf{C})} = \langle w, \omega w_1\rangle_{T_{n+m}(\mathbf{C})},$$

$\forall w \in T_{n+m}(\mathbf{C}), w_1 \in T_n(\mathbf{C})$. Let

$$w_1 = \sum_{t=-n}^{n} \rho^{(1)}_t z^t.$$

Then

$$\langle w, \omega w_1\rangle_{T_{n+m}(\mathbf{C})} = (\sum_{l=-m}^{m} \sum_{t=-n}^{n} \omega^{(l)} \rho^{(1)}_t \rho_{t+l}) =$$

$$\sum_{t=-n}^{n} \rho^{(1)}_t \omega(\rho)_t = \langle L^T_\omega(w), w_1\rangle_{T_n(\mathbf{C})}.$$

Here we used the definition (2) of the scalar product in $T_n(\mathbf{C})$.

Let us now return to our original problem (3), (4) and its dual (5),(6). We wish to apply our reduction scheme using decompositions (8) and (9). Each cone in the decomposition (8) has the form ωK, where K is the "cone of squares" generated by a bilinear form of the type (32). We calculated the corresponding maps Λ_ω and M_ω in (45),(46),(39),(43),(49). The map Λ is then described in (22). The corresponding semidefinite reduction will have the form (29),(30). The dual cone $K^*_{n,k}$ will be described by the requirement that 2^k Hermitian Toeplitz matrices of the form:

$$M(L^T_{\omega_1^{\epsilon_1}\omega_2^{\epsilon_2}\dots\omega_k^{\epsilon_k}}(w)),$$

$\epsilon_i \in \{0,1\}$ are nonnegative definite. For example, take $k = 1, \beta_1 = 2\pi - \tau, \alpha_1 = \tau, 0 < \tau < \pi$. Using (2), we obtain:

$$\omega_1(z) = z^{-1} + z - 2\cos\tau.$$

If

$$w = \sum_{j=-n}^{n} \rho_j z^j \in T_n(\mathbf{C}),$$

then

$$\omega_1(\rho)_s = -2\cos\tau\rho_s + \rho_{s-1} + \rho_{s+1},$$

$s = 0, 1, \dots, n-1$, see (48). Then $w \in K^*_{n,1}$ if and only if the Toeplitz forms

$$\sum_{s,t=0}^{n} \rho_{s-t}\xi_s\bar{\xi}_t, \sum_{s,t=0}^{n-1} (-2\cos\tau\rho_{s-t} + \rho_{s-t-1} + \rho_{s-t+1})\xi_s\bar{\xi}_t$$

are nonnegative definite.

Remark 3 *These formulas (for $k = 1$) are already in [8].*

4 Pseudo-polynomials with real coefficients

Let

$$T_n(\mathbf{R}) = \{ \sum_{j=-n}^{n} \rho_j z^j : \rho_j \in \mathbf{R}\}.$$

If $w \in T_n(\mathbf{R})$, then

$$w(e^{i\theta}) = \rho_0 + 2\sum_{j=1}^{n} \rho_j(j\theta).$$

Let $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots \alpha_m < \beta_m < \pi$. Consider

$$\Delta_j = \{e^{i\theta} : \alpha_j < \theta < \beta_j\}.$$

Let,further,

$$\mathbf{S^1_+} = \{e^{i\theta} : 0 \le \theta \le \pi\},$$

$$F_k = \mathbf{S^1_+} \setminus (\cup_{j=1}^{k} \Delta_j).$$

Define

$$L_{n,k} = \{w \in T_n(\mathbf{R}) : w(\xi) \ge 0, \xi \in F_k\},$$

$$\eta_k(z) = \frac{1}{4}(z^2 - 2(\cos\alpha_k + \cos\beta_k)z + 4\cos\alpha_k\cos\beta_k + 2 - 2(\cos\alpha_k + \cos\beta_k)z^{-1} + z^{-2}).$$

Theorem 2 *We have*

$$L_{n,k} = L_{n,k-1} + \eta_k L_{n-2,k-1},$$

$n \ge 1, k = 1, \dots m$.

Remark 4 *By definition $L_{n,k} = 0$ if $n < 0$ or $k < 0$.*

Remark 5 *Observe that*

$$\eta_k(e^{i\theta}) = (\cos\theta - \cos\alpha_k)(\cos\theta - \cos\beta_k).$$

Proof Let Ω be a subset of $[-1,1]$ obtained by removal from $[-1,1]$ a finite number of disjoined open subintervals. Let $(\cos\beta, \cos\alpha) \subset \Omega$ for some $0 < \alpha < \beta < \pi$. Denote by N_n (respectively, $N_n(\alpha,\beta)$) the cone of polynomials of degree less or equal to n which are nonnegative on Ω (respectively, on $\Omega \backslash (\cos\beta, \cos\alpha)$). We have

$$N_n(\alpha,\beta) = N_n + p_{\alpha,\beta} N_{n-2}, \tag{50}$$

where $p_{\alpha,\beta}(t) = (t-\cos\alpha)(t-\cos\beta)$ (see [8] and for a detailed proof [5]).

Let now $U_n(\mathbf{R})$ be the vector space of polynomials of degree less or equal than n with real coefficients. It is well-known that the map $\varphi : U_n(\mathbf{R}) \to T_n(\mathbf{R})$ such that

$$\varphi(p)(e^{i\theta}) = p(\cos\theta)$$

defines a linear isomorphism of $U_n(\mathbf{R})$ onto $T_n(\mathbf{R})$. It is , moreover, obvious that $p(\cos\theta) \geq 0$ if and only if $\varphi(p)(e^{i\theta}) \geq 0$. Take

$$\Omega = [-1,1] \setminus (\cup_{j=1}^{k-1} (\cos\beta_j, \cos\alpha_j)), \beta = \beta_k, \alpha = \alpha_k.$$

Then, given $w \in L_{n,k}$, we see that (50) provides the representation

$$w(e^{i\theta}) = w^{(1)}(e^{i\theta}) + \eta_k(e^{i\theta}) w^{(2)}(e^{i\theta})$$

for some $w^{(1)} \in L_{n,k-1}, w^{(2)} \in L_{n-2,k-1}$. The result follows.

Corollary 4

$$L_{n,k} = \sum_{\vec{\epsilon} \in \{0,1\}^k} \eta_1^{\epsilon_1} \eta_2^{\epsilon_2} \ldots \eta_k^{\epsilon_k} L_{r(\vec{\epsilon}),0},$$

$$\vec{\epsilon} = (\epsilon_1, \ldots \epsilon_k), r(\vec{\epsilon}) = n - 2\sum_{j=1}^{k} \epsilon_j,$$

$$L_{n,k}^* = \cap_{\vec{\epsilon}} \in \{0,1\}^k (\eta_1^{\epsilon_1} \eta_2^{\epsilon_2} \ldots \eta_k^{\epsilon_k} L_{r(\vec{\epsilon}),0})^*.$$

Observe now that the semidefinite reduction of optimization problems of the type (3),(4) and (5,6) (with cones $K_{n,k}, K_{n,k}^*$ substituted by cones $L_{n,k}, L_{n,k}^*$,respectively) follows exactly the same scheme as it was outlined in the previous section with the following obvious modifications: instead of Hermitian (respectively, Toeplitz Hermitian) matrices one should consider real symmetric (respectively, real Toeplitz) matrices. The definition of $\Lambda, \Lambda_\omega, M, M_\omega$ does not change but instead of ω_k one should use η_k.

5 Comparison with other approaches and concluding remarks

The fact that the cones $K_{n,k}, L_{n,k}$ can be described by semidefinite constraints is, in principle, known. In [2],[4],[6] various descriptions of cones of trigonometric polynomials nonnegative on a segment of the unit circle are given. In [6] a particular case of Theorem 1 for $k = 1$ is rediscovered. Since cones $K_{n,k}$ are obtained as intersection of finite number of cones of trigonometric polynomials nonnegative on segments of the unit circle, they are also can be described, in principle, by semidefinite constraints [2]. A significant advantage of the scheme suggested in this paper (at least for small k) is that corresponding Hermitian (symmetric) matrices have the block structure. If $k \leq n$, according to Theorem 1 and the subsequent description of the reduction scheme, for each $0 \leq l \leq k$, there will be exactly C_k^l blocks of the size $(n-l) \times (n-l)$ in the semidefinite parametrization of the cone $K_{n,k}$. In particular, the standard barrier function $F(X) = -\ln \det(X)$ for the cone of nonnegative definite Hermitian block diagonal matrices with the above structure of blocks, will have the barrier parameter

$$\theta(F) = 2^{k-1}(2n-k),$$

which is of order n for small k. For example, if $n = 8, k = 3$, we obtain one block of the size 8×8, three blocks of the size 7×7, three blocks of the size 6×6 and one block of the size 5×5. This block structure provides significant computational advantages at least for small k.

The case of even trigonometric polynomials, i.e. $T_n(\mathbf{R})$, has been considered in some detail in [1]. The explicit reduction of the cone of even trigonometric polynomials nonnegative on a segment of the unit circle to the cone of usual polynomials nonnegative on the corresponding interval of $\mathbf{R}$ with the help of recurrence structure of Chebyshev polynomials, has been used to provide a semidefinite description of dual cones of the type $L_{n,1}^*$. Again, the approach suggested in Theorem 2 provides an explicit description of cones $L_{n,k}$ in terms of semidefinite constraints and is more economic and direct (at least for small k) than the one of [1].

I would like to thank Michael Gekhtman for valuable discussions. This work was supported in part by NSF grant DMS01-02628.

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