Sensitivity analysis for linear optimization problem with fuzzy data in the objective function

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Abstract

Linear programming problems with fuzzy coefficients in the objective function are considered. Emphasis is on the dependence of the optimal solution from linear perturbations of the membership functions of the objective function coefficients as well as on the computation of a robust solution of the fuzzy linear problem if the membership functions are not surely known.

Keywords: Fuzzy linear programming, sensitivity analysis, robust optimization

1 Fuzzy linear optimization

We consider fuzzy linear optimization problems given by

$$\left.\begin{array}{l} F(x) = \sum\limits_{j=1}^{n} \widetilde{c}_j x_j \to \max \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \qquad (1)$$

The coefficients $\widetilde{c} = (\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n)$ of the objective function are fuzzy numbers of the type $L - L$ [4]:

$$\widetilde{c_j} = (\underline{c}_j; \overline{c}_j; \alpha_j; \beta_j)_{L-L}, j = 1, 2, ..., n, \qquad (2)$$

where $\underline{c}_j, \overline{c}_j$ - are the left and right borders of the fuzzy number $\widetilde{c}_j$ corresponding to the maximal reliability level ($\lambda = 1$) and α_j and β_j are non-negative real numbers (see Fig. 1).

A fuzzy number $\widetilde{c}_j, (j = 1, 2, ..., n)$ is defined as a fuzzy set in the space of real numbers with the following membership function [4]:

$$\mu_{\widetilde{c}}(z) = \begin{cases} 1 & \text{if } \underline{c} \leq z \leq \overline{c}, \\ L\left(\dfrac{\underline{c} - z}{\alpha}\right) & \text{if } z \leq \underline{c}, \\ L\left(\dfrac{z - \overline{c}}{\beta}\right) & \text{if } z \geq \overline{c}, \end{cases} \qquad (3)$$

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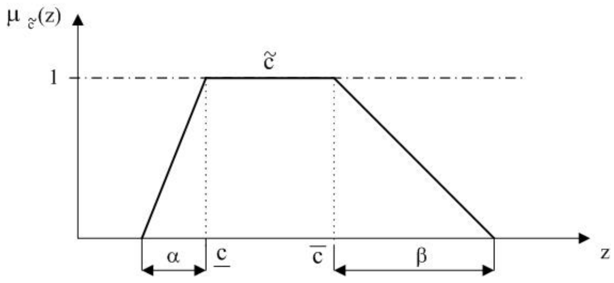

Figure 1: A membership function

where L is a shape function, which satisfies to following conditions:
- L is a continuous non-increasing function on $[0, \infty)$ with $L(0) = 1$;
- L is strictly decreasing on that part of $[0, \infty)$ on which it is positive.

Problem (1) can be associated with a set of the following problems (4), which depend on a parameter $\lambda \in (0, 1)$ [2]:

$$\left.\begin{array}{l} F(x) = \sum\limits_{j=1}^{n} c_j^\lambda x_j \to \max \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \quad (4)$$

where the coefficients c_j^λ in the objective function represent intervals corresponding to the λ-level of the fuzzy number $\widetilde{c}_j$, $j = 1, 2, ..., n$:

$$c_j^\lambda := \{y : \mu_{\widetilde{c}_j}(y) \geq \lambda\} = [\underline{c}_j - L^{-1}(\lambda)\alpha_j, \overline{c}_j + L^{-1}(\lambda)\beta_j].$$

Define the abbreviations

$$\underline{c}_j^\lambda := \underline{c}_j - L^{-1}(\lambda)\alpha_j, \overline{c}_j^\lambda := \overline{c}_j + L^{-1}(\lambda)\beta_j.$$

Similar as in [2] we use the following definition of an optimal solution of problem (4).

Definition 1 *A point $\overline{x} \geq 0$ with $A\overline{x} = b$ is called an t_0, t_1-optimal solution of the problem (4) iff there is no $x' \geq 0$ with $Ax' = b$ satisfying the following inequalities*

$$\sum_{j=1}^{n} (\underline{c}_j^\lambda + t_0(\overline{c}_j^\lambda - \underline{c}_j^\lambda))\overline{x}_j \leq \sum_{j=1}^{n} (\underline{c}_j^\lambda + t_0(\overline{c}_j^\lambda - \underline{c}_j^\lambda))x'_j$$

$$\sum_{j=1}^{n} (\underline{c}_j^\lambda + t_1(\overline{c}_j^\lambda - \underline{c}_j^\lambda))\overline{x}_j \leq \sum_{j=1}^{n} (\underline{c}_j^\lambda + t_1(\overline{c}_j^\lambda - \underline{c}_j^\lambda))x'_j$$

with at least one strict inequality.

Using this definition it is easy to see that $\overline{x}$ is an t_0, t_1-optimal solution iff it is a Pareto-optimal solution of the following bicriterial optimization problem, where

$\theta = L^{-1}(\lambda)$ is used [2]:

$$\left.\begin{array}{l} f_1(x) = \sum\limits_{j=1}^{n} \left(p_1^1(j) + p_2^1(j)\theta\right) x_j \rightarrow \max \\ f_2(x) = \sum\limits_{j=1}^{n} \left(p_1^2(j) + p_2^2(j)\theta\right) x_j \rightarrow \max \\ \sum\limits_{j=1}^{n} a_{ij}x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n, \end{array}\right\} \tag{5}$$

where

$$\begin{array}{l} p_1^1(j) = \underline{c}_j + t_0(\overline{c}_j - \underline{c}_j), \\ p_2^1(j) = t_0(\alpha_j + \beta_j) - \alpha_j, \\ p_1^2(j) = \underline{c}_j + t_1(\overline{c}_j - \underline{c}_j), \\ p_2^2(j) = t_1(\alpha_j + \beta_j) - \alpha_j, \end{array}$$

To compute t_0, t_1-optimal solutions for all λ-levels of the problem (1) we have to compute Pareto-optimal solutions of problem (5) for all $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} = L^{-1}(1),\ \overline{\theta} = L^{-1}(0)$.

The selection of different values for t_0, t_1 corresponds to different preference relations between intervals [3].

For simplicity of our investigations we shall be limited only to the consideration of the following preference relation between intervals $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$:

$$a \leq b \Longleftrightarrow \underline{a} \leq \underline{b} \wedge \overline{a} \leq \overline{b}, \tag{6}$$

$$a < b \Longleftrightarrow a \leq b \wedge a \neq b, \tag{7}$$

In accordance with [3] this case corresponds to the problem (5) with $t_0 = 0$ and $t_1 = 1$.

Substituting $t_0 = 0$ and $t_1 = 1$ in (5) we obtain the following model:

$$\left.\begin{array}{l} f_1(x) = \sum\limits_{j=1}^{n} \left(\underline{c}_j - \alpha_j\theta\right) x_j \rightarrow \max \\ f_2(x) = \sum\limits_{j=1}^{n} \left(\overline{c}_j + \beta_j\theta\right) x_j \rightarrow \max \\ \sum\limits_{j=1}^{n} a_{ij}x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{8}$$

Summing up we have to compute the sets of Pareto-optimal solutions of problem (8) for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

Definition 2 *A point $\overline{x} \geq 0$ with $A\overline{x} = b$ is a Pareto-optimal solution [5] of problem (8) iff there does not exist $x' \geq 0$ with $Ax' = b$ satisfying*

$$f_1(\overline{x}) \leq f_1(x'),\ f_2(\overline{x}) \leq f_2(x')$$

with at least one strict inequality.

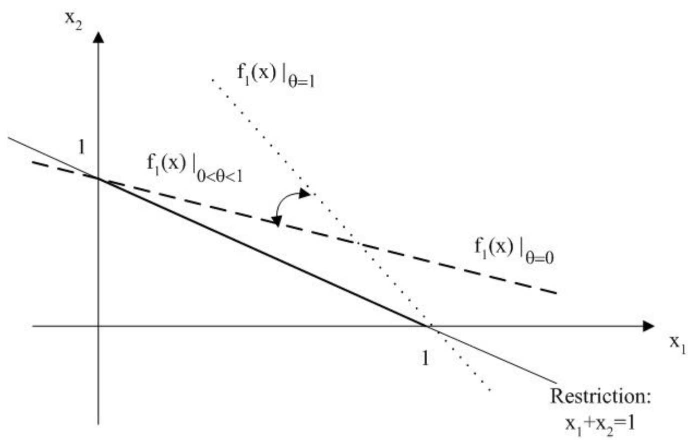

Figure 2: First objective function for $\theta \in [0, 1]$

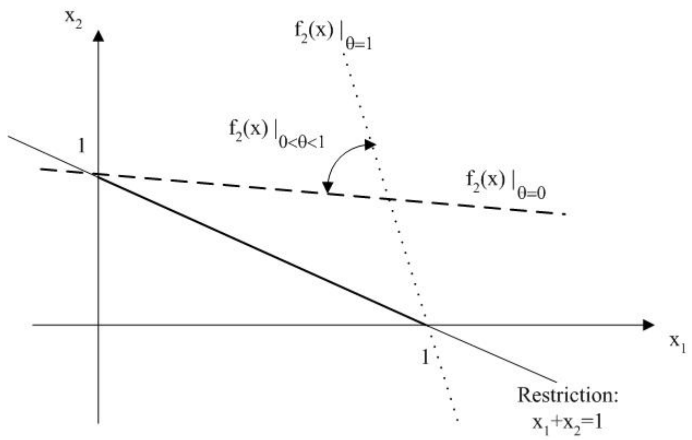

Figure 3: Second objective function for $\theta \in [0, 1]$

We denote the set of Pareto-optimal solutions of problem (8) for fixed θ by $\Psi(\theta)$.

Since the objective functions in model (8) depend on the parameter θ, each objective function in (8) is one element in a set of functions which are located between two borders, according to $\theta = 0$ and $\theta = 1$. This is shown in Fig. 2 and 3.

Coming back to problem (1) and using the above approach for treating fuzziness we see that it is necessary to find all the feasible points which are Pareto-optimal for problem (8) for at least one value of $\theta \in [0, 1]$.

Definition 3 *A point $\overline{x}$ which is Pareto-optimal for problem (8) for at least one value of $\theta \in [0, 1]$ is called essential.*

The solution of a fuzzy optimization problem is again a fuzzy set in the set of feasable solutions. To define a membership function for this solution, we compute the set of all θ for which one essential point is Pareto-optimal for problem (8) [2]. Then,

$$\mu_{FS}(x) = \left| \left\{ \begin{array}{cc} \lambda \in [0,1]: & x \text{ is a Pareto-optimal solution} \\ & \text{of the problem (8) for } \theta = L^{-1}(\lambda) \end{array} \right\} \right| = $$
$$= \sum_{i=1}^{l} (L(\theta_{2i-1}) - L(\theta_{2i})) \quad (9)$$

where $\{\theta_i\}_{i=1}^{2l}$ is such that x is Pareto-optimal for problem (8) for all $\theta \in [\theta_{2i-1}, \theta_{2i}],\ i = 1, \dots, l$. Here, $|Q|$ means the geometric measure of the set Q.

2 Computing Pareto-optimal points

In the following we will assume for simplicity that the set $M := \{x \geq 0 : Ax = b\}$ is not empty and bounded. Then,

$$M = \text{conv } \{x^1, x^2, ..., x^r\}.$$

For the motivation, we will restrict us to the computation of the set of all vertices of M belonging to the set of Pareto-optimal solutions $\Psi(\theta)$ for problem (8) for fixed θ. Clearly, this can be done since the set of Pareto-optimal solutions itself can easily be determined if we know the subset of vertices in this set. Let $\Psi_v(\theta)$ denote the subset of vertices of M in $\Psi(\theta)$. Then,

$$\Psi_v(\theta) = \{x^{i1}, ..., x^{ip}\} \subseteq \{x^1, ..., x^r\}. \quad (10)$$

We sort these Pareto-optimal points such that

$$\begin{array}{l} f_1(x^{i1}) > f_1(x^{i2}) > ... > f_1(x^{ip}), \\ f_2(x^{i1}) < f_2(x^{i2}) < ... < f_2(x^{ip}). \end{array} \quad (11)$$

This means that the solution x^{i1} is optimal for the first objective function without consideration of the second one and x^{ip} is an optimal solution for the second objective function if the first function is not taken into account.

For simplicity assume that the optimization problems of maximizing the function $f_1(x)$ respectively the function $f_2(x)$ on the set M are non-degenerate, i.e. that they have unique optimal solutions. If this would not be the case then we have to select certain optimal solutions out of the sets of optimal ones. Then, to compute $\Psi_v(\theta)$ we can proceed as follows. We solve linear optimization problems

$$\left. \begin{array}{l} f_2(x) \to \max \\ f_1(x) \leq z \\ x \in M \end{array} \right\} \quad (12)$$

with parameter z on the right-hand side of the first constraint. It follows from the Charnes-Cooper observation [7] that we obtain $\Psi(\theta)$ as

$$\Psi(\theta) = \bigcup_{z \in [f_1(x^{ip}), f_1(x^{i1})]} \Phi(z),$$

where $\Phi(z)$ is the set of optimal solutions of problem (12). Let $\varphi(z)$ denote the optimal value function of the parametric optimization problem (12). The function $\varphi(z)$ is piecewise linear and concave [6]. The kinks of this function correspond to vertices of M, hence, to compute $\Psi_v(\theta)$ we have to find the kinks of the function $\varphi(z)$.

Theorem 4 *Let the function $\varphi(z)$ be (affine) linear on both the intervals $[z_0, z_1]$ and $[z_1, z_2]$ with different slopes. Then, there exists a vertex $\overline{x}$ of the set M with $\overline{x} \in \Phi(z_1)$. Vice versa, if $|\Phi(z)| = 1$ for all z and if there is a vertex $\overline{x}$ of the set M in $\Phi(z_1)$ then the function $\varphi(z)$ has a kink at $z = z_1$.*

Proof: If the constraint $f_1(x) = z$ is not active, the optimal solution of the problem (12) is equal to x^{ip} independent of z, i.e. $\varphi(z)$ is constant. Hence, assume that $f_1(x) = z$ is active.

For right-hand side perturbed linear programming problems there exists a piecewise (affine-) linear solution function $x(z)$ (which can be an arbitrary selection function of $\Phi(\cdot)$ in case of nonuniquely optimal solutions) [6]. To verify this consider the optimality conditions of problem (12) and use that the characteristic index set is piecewise constant [6]. This means that $\varphi(z) = f_2(x(z))$ is linear whenever $x(z)$ is linear and that this function is not differentiable whenever $x(z)$ is not.

If the function $x(z)$ does not go through a vertex of M, i.e. if no vertex of M belongs to $\Phi(z_1)$, the functions $x(z)$ and $\varphi(z)$ are locally (affine-) linear. The assumption that no vertex of M belongs to $\Phi(z_1)$ equivalently means that the characteristic index set is constant for z near z_1.

Now let the characteristic index set change at the point z_1. Since the constraint $f_1(x) = z$ is active this means that the characteristic index set corresponding to the inequalities $x \geq 0$ changes. Due to the assumption $|\Phi(z_1)| = 1$ this means that a unique vertex $\overline{x}$ of M is the optimal solution of (12) at $z = z_1$. Hence, the function $x(z)$ is not differentiable at the point $z = z_1$ and so is the function $\varphi(z)$. q.e.d.

3 Sensitivity analysis

Now we assume that the fuzzy numbers $\widetilde{c}_j$ are perturbed by $\delta_j, j = 1, ..., n$. Applying these perturbations to the problem (8) with the two objective functions $f_1(x)$ and $f_2(x)$, we obtain the following model:

$$\left.\begin{array}{l} f_1(x) = \sum\limits_{j=1}^{n} \left[\left(\underline{c}_j - \alpha_j\theta\right) + \delta_j\right] x_j \to \max \\ f_2(x) = \sum\limits_{j=1}^{n} \left[\left(\overline{c}_j + \beta_j\theta\right) + \delta_j\right] x_j \to \max \\ \sum\limits_{j=1}^{n} a_{ij}x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \qquad (13)$$

Here we are interested to determine how much the fuzzy numbers $\widetilde{c}_j, j = 1, ..., n$ can be perturbed such that one Pareto-optimal solution of the initial fuzzy linear optimization problem remains optimal. Problem (13) is called a *changed problem* and the set

$$\mathcal{R}(x, \theta) := \{\delta : x \in \Psi^{\delta}(\theta)\}$$

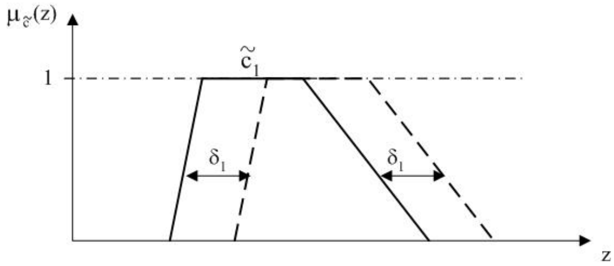

Figure 4: Perturbation of a membership function

is called *region of stability* of the feasible point x, where $\Psi^\delta(\theta)$ denotes the set of Pareto-optimal solutions of problem (13). The corresponding perturbation of the membership function of one fuzzy number is illustrated in Fig. 4.

Theorem 5 *For fixed θ and each feasible point x the set* $\mathrm{cl}\,\mathcal{R}(x)$ *is a convex polyhedron.*

Proof: Let $N_M(x)$ denote the normal cone to the feasible set M of problem (13) at a feasible point x:

$$N_M(x) = \{z : \exists u, \ \exists v \geq 0 \text{ with } z = A^\top u - Ev, \ x^\top v = 0\}.$$

Then, x is Pareto-optimal for problem (13) iff there exists $w \in (0,1)$ such that x is an optimal solution of the problem

$$\left.\begin{array}{l} wf_1(x) + (1-w)f_2(x) \rightarrow \max \\ \sum\limits_{j=1}^{n} a_{ij}x_j = b_i, i = 1,2,...,m, \\ x_j \geq 0, j = 1,2,...,n, \end{array}\right\} \quad (14)$$

cf.e.g. [7]. Now, x is an optimal solution of problem (14) if and only if

$$w\nabla f_1(x) + (1-w)\nabla f_2(x) \in N_M(x)$$

by linear programming where $\nabla f_i(x)$ is constant since $f_i(x)$ is a linear function of x. Evaluating this condition we get

$$w(\underline{c} - \alpha\theta + \delta) + (1-w)(\overline{c} + \beta\theta + \delta) \in N_M(x). \quad (15)$$

Here, $\underline{c} = (\underline{c}_1, \ldots, \underline{c}_n)^\top$, and $\overline{c}, \alpha, \beta$ are accordingly determined vectors. Now, putting the equations (15) and $w \in (0,1)$ and the definition of $N_M(x)$ together we see that the closure of the set of all solutions (w, δ, u, v) of the resulting system is equal to the set of solutions of a linear system of equations and inequalities. Hence, this set is a convex polyhedron and this is also true for the projection of this set onto the δ-space. q.e.d.

To illustrate this theorem let us consider in the following example the special case that $\delta_i = 0$ for all $i > 1$.

$$\left.\begin{array}{l} F(x) = \widetilde{c}_1 x_1 + \widetilde{c}_2 x_2 \ \rightarrow \max \\ x_1 + 2x_2 \leq 6 \\ -x_1 + x_2 \leq 2 \\ 2x_1 + x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{array}\right\} \quad (16)$$

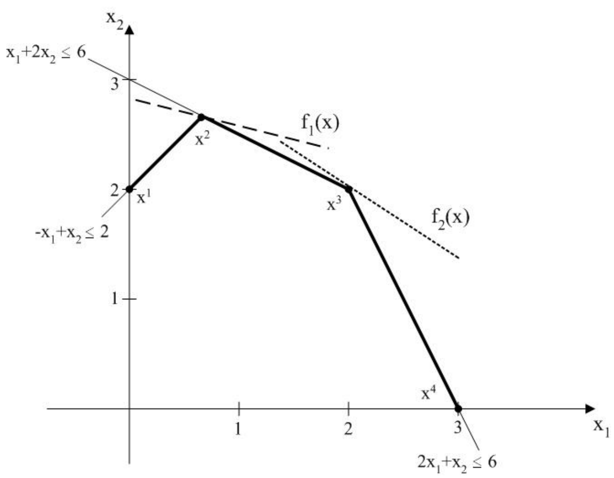

Figure 5: Set of solutions for the perturbed bicriterial problem

where $\widetilde{c}_1 = (2; 5; 1; 2)$ and $\widetilde{c}_2 = (8; 9; 2; 5)$ (see (2)).

If we change this problem by perturbing the membership function of the first fuzzy number $\widetilde{c}_1$ by δ_1 we obtain the following problem:

$$\left.\begin{array}{l} f_1(x) = [(2-\theta)+\delta_1]\, x_1 + \ (8-2\theta)\, x_2 \rightarrow \max \\ f_2(x) = [(5+2\theta)+\delta_1]\, x_1 + \ (9+5\theta)\, x_2 \rightarrow \max \\ x_1 + 2x_2 \le 6 \\ -x_1 + x_2 \le 2 \\ 2x_1 + x_2 \le 6 \\ x_1, x_2 \ge 0. \end{array}\right\} \quad (17)$$

Consider the bicriterial problem (17) with $\theta = 0$. Then the set of Pareto-optimal solutions for $\delta = 0$ is $\Psi^0(\theta = 0) = \text{conv}\,\{x^2, x^3\} = \text{conv}\,\{(2/3, 8/3)^\top, (2, 2)^\top\}$ (see Fig. 5).

Now we consider a changed problem (17) with the parameter δ. We intend to find all such δ_1 for which x^2 is no longer Pareto-optimal, i.e. we need to find $\{\delta_1 : \ x^2 \notin \Psi^{\delta_1}(\theta = 0)\}$.

Now we transform the problem (17) into the problem (14).

$$\left.\begin{array}{l} w((2+\delta_1)x_1 + 8x_2) + (1-w)((5+\delta_1)x_1 + 9x_2) \rightarrow \max \\ x_1 + 2x_2 \le 6 \\ -x_1 + x_2 \le 2 \\ 2x_1 + x_2 \le 6 \\ x_1, x_2 \ge 0. \end{array}\right\} \quad (18)$$

Applying the optimality conditions (15) to problem (18) we obtain

$$\left.\begin{array}{l} -w \cdot (2+\delta_1) - (1-w) \cdot (5+\delta_1) + u_1 - u_2 = 0 \\ -w \cdot 8 - (1-w) \cdot 9 + 2u_1 + u_2 = 0 \\ u_1, u_2 \ge 0, w \in [0, 1]. \end{array}\right\} \quad (19)$$

Note that the Lagrange multipliers to the constraints $x_i \ge 0$ and to the last inequality $2x_1 + x_2 \le 6$ are zero since these conditions are not active. After the transformation of equations (19) the following equalities are obtained:

$$\left.\begin{array}{l} \delta_1 = 3w - 5 + u_1 - u_2 \\ w + 2u_1 + u_2 = 9 \\ u_1, u_2 \geq 0, w \in [0, 1]. \end{array}\right\} \qquad (20)$$

This implies that the following inequality gives an upper bound for δ_1 such that the Pareto-optimal point x^2 remains Pareto-optimal, i.e. it is no longer Pareto-optimal for the problem (17) if δ_1 is larger than the right-hand side of the following inequality:

$$\delta_1 \leq d, \qquad (21)$$

where d is the optimal objective function value of the problem

$$\left.\begin{array}{l} 3w - 5 + u_1 - u_2 \rightarrow \max \\ w + 2u_1 + u_2 = 9 \\ u_1, u_2 \geq 0, w \in [0, 1]. \end{array}\right\}$$

The optimal function value of the last problem is equal to two. Hence,

$$\mathcal{R}(x^2) = (-\infty, 2]$$

which can easily be verified in figure 5. It should be remarked that the region of stability is closed here since the problem (17) is nondegenerate.

Remark 6 *By the same way as in the proof of Theorem 5 we get the conditions for the computation of the bounds θ_i for evaluating the membership function (9) of the solution of problem (1). For this, set $\delta = 0$ in equation (15). If this condition is satisfied then the feasible solution x is Pareto-optimal for problem (8) and, hence, belongs to the solution of problem (1) with positive membership function value. The θ_i are the bounds of θ for which x enters the set of Pareto-optimal solutions respectively leaves this set. Note that the equation (15) is no longer linear if θ is not constant which results in a nonconvex region of stability. The latter result is reflected also by the investigations in the paper [2].*

4 Robust solution

Let us assume now that the coefficients in the objective function in problem (1) are unknown. This is another situation than that in the previous section where we analysed the dependency of one Pareto-optimal solution and hence of one feasible solution with positive value of the membership function on the fuzzy objective function coefficients.

In [1] linear optimization problems with unkown coefficients in the constraints have been considered. To treat the uncertainty resulting from the unknown coefficients the authors developed a robust counterpart of the linear programming problem demanding that a robust feasible solution has to satisfy all the constraints resulting from all possible realizations of the coefficients. And a robust optimal solution is a best robust feasible solution with respect to the original objective function.

Here the situation is slightly different in that we assume that the membership functions of the fuzzy coefficients in the objective function are uncertain. Again we solve the fuzzy optimization problem (1) using the union of the sets of Pareto-optimal solutions of the problem (8) for $\theta \in [0, 1]$. Then, to adopt the approach in [1] we move the objective functions of this problem for fixed θ into the constraint set. This results in

$$\left.\begin{array}{l} z_1 \longrightarrow \max \\ z_2 \longrightarrow \max \\ f_1(x) = \sum\limits_{j=1}^{n} \left(\underline{c}_j - \alpha_j \theta\right) x_j \geq z_1 \\ f_2(x) = \sum\limits_{j=1}^{n} \left(\overline{c}_j + \beta_j \theta\right) x_j \geq z_2 \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{22}$$

Now the conception in [1] was to call a point x a robust feasible point if it satisfies all the constraints for all the possible realizations. This means for our problem that we get

$$\left.\begin{array}{l} z_1 \longrightarrow \max \\ z_2 \longrightarrow \max \\ \widetilde{f}_1(x) := \min \sum\limits_{j=1}^{n} \left(\underline{c}_j - \alpha_j \theta\right) x_j \geq z_1 \\ \widetilde{f}_2(x) := \min \sum\limits_{j=1}^{n} \left(\overline{c}_j + \beta_j \theta\right) x_j \geq z_2 \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{23}$$

where the minimum in the first two constraints is to be taken with respect to all possible functions obtained for all possible realizations of the fuzzy coefficients in the objective function of (1). This problem is equivalent to

$$\left.\begin{array}{l} \widetilde{f}_1(x) := \min \sum\limits_{j=1}^{n} \left(\underline{c}_j - \alpha_j \theta\right) x_j \longrightarrow \max \\ \widetilde{f}_2(x) := \min \sum\limits_{j=1}^{n} \left(\overline{c}_j + \beta_j \theta\right) x_j \longrightarrow \max \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{24}$$

Hence, if we also construct a robust counterpart of our problem (1) we should demand feasibility of a solution with respect to the constraints of the original problem but call a feasible solution robust optimal if it is “nearly optimal” with respect to all possible realizations of the fuzzy objective function.

A fuzzy coefficient $\widetilde{c}_j$ in the objective function of problem (1) is characterized by $(\underline{c}_j, \overline{c}_j, \alpha_j, \beta_j)$. If the fuzzy number $\widetilde{c}_j$ is uncertain this means that its membership function is uncertain and this is reflected by $(\underline{c}_j, \overline{c}_j, \alpha_j, \beta_j)$ being taken as one element of a certain set. Let P be the set of all possible vectors of tuples $(\underline{c}_j, \overline{c}_j, \alpha_j, \beta_j)$ resulting from all possible realizations of the fuzzy coefficients in (1). Then, the minimum in the objective functions in problem (24) is to be taken w.r.t. $p = \left((\underline{c}_j, \overline{c}_j, \alpha_j, \beta_j)\right)_{i=1}^{n} \in P$. We assume in the following that the set P is a compact polyhedron.

A *robust solution* for problem (1) is a feasible solution of this problem, which is essential for problem (24). Its membership function value can be determined accordingly to (9) with (8) being replaced by (24).

To solve the resulting problem (24) we use a similar approach as in Section 2. For that, let the compact polyhedron P be given as convex hull of its vertices $P = \text{conv}\,\{(\underline{c}_j^k, \overline{c}_j^k, \alpha_j^k, \beta_j^k)_{i=1}^n, k = 1, \ldots, K\}$. Then,

$$\widetilde{f}_1(x) := \min_{p \in P} \sum_{j=1}^{n} \left(\underline{c}_j - \alpha_j \theta\right) x_j = \min_{k=1,\ldots,K} \sum_{j=1}^{n} \left(\underline{c}_j^k - \alpha_j^k \theta\right) x_j$$

and

$$\widetilde{f}_2(x) := \min_{p \in P} \sum_{j=1}^{n} \left(\overline{c}_j + \beta_j \theta\right) x_j == \min_{k=1,\ldots,K} \sum_{j=1}^{n} \left(\overline{c}_j^k + \beta_j^k \theta\right) x_j.$$

To compute the bounds for the variations of both objective functions $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ we again solve the two problems of maximizing only one of both functions on the feasible set M. These problems can be transformed into

$$\left.\begin{array}{l} \xi \to \max \\ \sum\limits_{j=1}^{n} \left(\underline{c}_j^k - \alpha_j^k \theta\right) x_j \geq \xi, k = 1, \ldots, K \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n \end{array}\right\} \tag{25}$$

and

$$\left.\begin{array}{l} \xi \to \max \\ \sum\limits_{j=1}^{n} \left(\overline{c}_j^k + \beta_j^k \theta\right) x_j \geq \xi, k = 1, \ldots, K \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{26}$$

Let x^1 be an optimal solution of (25) and x^p denote an optimal solution of (26), then the first objective $\widetilde{f}_1(x)$ varies between $\widetilde{f}_1(x^1)$ and $\widetilde{f}_1(x^p)$, while $\widetilde{f}_2(x)$ runs from $\widetilde{f}_2(x^p)$ to $\widetilde{f}_2(x^1)$.

Now the problem according to (12) reads as

$$\left.\begin{array}{l} \xi \to \max \\ \sum\limits_{j=1}^{n} \left(\overline{c}_j^k + \beta_j^k \theta\right) x_j \geq \xi, k = 1, \ldots, K \\ \widetilde{f}_1(x) \leq z \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array}\right\} \tag{27}$$

Similarly to Section 2, if we solve this problem for $z \in [\widetilde{f}_1(x^p), \widetilde{f}_1(x^1)]$, then we trace the set of Pareto-optimal solutions of the problem (24). Note, that using the regions of stability

$$\mathcal{R}(k,\theta) := \left\{ x : \sum_{j=1}^{n} \left(\underline{c}_j^k - \alpha_j^k \theta \right) x_j \leq \sum_{j=1}^{n} \left(\underline{c}_j^s - \alpha_j^s \theta \right) x_j \ \forall\ s = 1, \ldots, K \right\}$$

we can further transform problem (27) into a sequence of linear optimization problems.

5 Computation of the membership function

To compute values of the membership function of a robust solution of the problem (1) we go along the lines of (9), i.e. we compute for each feasible point $x \in M$ the set of all θ for which this point is Pareto-optimal with respect to the problem (24). Problem (24) again is a convex bicriterial optimization problem. Hence, for each Pareto-optimal solution $\overline{x}$ of this problem there is $w \in [0,1]$ such that $\overline{x}$ is an optimal solution of the problem

$$\left. \begin{array}{l} w\widetilde{f}_1(x) + (1-w)\widetilde{f}_2(x) \to \max \\ \sum\limits_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m, \\ x_j \geq 0, j = 1, 2, ..., n. \end{array} \right\} \tag{28}$$

Vice versa, if $\overline{x}$ is an optimal solution of (28) for some $w \in (0,1)$ (and this is also true for $w \in \{0,1\}$ if the solution is unique) then it is also Pareto-optimal for problem (24).

Let $\partial \left(w\widetilde{f}_1 + (1-w)\widetilde{f}_2 \right)(x)$ denote the superdifferential (in the sense of convex analysis [8]) of the function $\left(w\widetilde{f}_1 + (1-w)\widetilde{f}_2 \right)(x)$.

Theorem 7 ([8]) *Let all optimal solutions of the problems (25) and (26) be uniquely determined. Then, a feasible solution $\overline{x} \in M$ is Pareto-optimal to (24) if and only if there is $w \in [0,1]$ with*

$$\left(-\partial \left(w\widetilde{f}_1 + (1-w)\widetilde{f}_2 \right)(\overline{x}) \right) \cap N_M(\overline{x}) \neq \emptyset.$$

By convex analysis,

$$\partial \left(w\widetilde{f}_1 + (1-w)\widetilde{f}_2 \right)(\overline{x}) = w\partial\widetilde{f}_1(\overline{x}) + (1-w)\partial\widetilde{f}_2(\overline{x}).$$

To compute $\partial\widetilde{f}_1(\overline{x})$ remember that

$$\widetilde{f}_1(\overline{x}) = \min_{k=1,\ldots,K} \sum_{j=1}^{n} \left(\underline{c}_j^k - \alpha_j^k \theta \right) x_j$$

and let

$$\mathcal{K}(x,\theta) := \left\{ k : \sum_{j=1}^{n} \left(\underline{c}_j^k - \alpha_j^k \theta \right) x_j = \widetilde{f}_1(\overline{x}) \right\}.$$

Then,
$$\partial \widetilde{f}_1(\overline{x}) = \text{conv } \{\underline{c}^k - \alpha^k \theta : k \in \mathcal{K}(x, \theta)\}.$$
Hence, to compute the region of stability of some Pareto-optimal solution $\overline{x}$ for problem (24) we have to solve a system of nonlinear equations resulting from insertion of the last equation into the result of Theorem 7.

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